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SPLITTING OF SEPARATRICES FOR THE CHIRIKOV STANDARD MAP

Preface of the translator

Approximately 19 years ago, Vladimir Lazutkin wrote two papers on the separatrix splitting of the standard map [2*, 3*]. Both manuscripts were available from a Russian preprint service called VINITI. The second of these was translated in English [7*] and became the most frequently cited Lazutkin's paper. The first one, excepting a Catalan translation done at the end of 1980's, had remained inaccessible outside the Russian-speaking mathematical community. Presenting the translation of the original manuscript [2*], this volume finally covers this gap in the literature.

In his paper, Vladimir Lazutkin derives an asymptotic formula for an angle between stable and unstable separatrices of the standard map. A first problem of this type appeared in the famous Poincaré work [13] written in 1889, where Poincaré first realized that separatrix splitting is responsible for creation of chaotic dynamics. A review of these and other related results can be found in the survey paper [17*].

The difficulty of the problem is related to the exponential smallness of the splitting angle compared to the perturbation parameter. Lazutkin approach to the problem is based on the study of the analytical continuation of the separatrices by means of a specially developed complex matching method. Lazutkin's proof of the asymptotic formula is not complete: the paper contains two conjectures, which are still open in their original form. In spite of that, none of estimates or arguments contained in the paper is seriously wrong. At the end of 1980's and in 1990's, Lazutkin and his group dedicated a lot of efforts to refining the technics [4*–14*]. It was realized that one of the conjectures could be eliminated and a weaker version of another would suffice. Finally a complete proof was published in [16*]. The goal, a proof of the asymptotic formula, was reached but at

The manuscript [2*] translated from Russian by V. Gelfreich.
*– Bibliography added by the translator

the price of introducing cumbersome estimates, which could easily hide the beauty of the method from the reader.

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1. Introduction. Formula for the separatrix splitting

It is known that after a small perturbation of a completely integrable Hamiltonian system a part of Liouville invariant tori is preserved. The corresponding perturbation theory is called KAM-theory (Kolmogorov, Arnold, Moser) [1–3]. Invariant tori of a close-to-integrable system form a smooth family diffeomorphic to a product of a torus and Cantor set of positive Lebesgue measure. We call these tori Kolmogorov tori. The measure of intersection of the set of all Kolmogorov tori with a bounded set has the order of $O(\sqrt{\varepsilon})$ for $\varepsilon \to 0$, where ε is a perturbation parameter. This estimate and the smoothness statements can be found in [4–6]. In the complement to the set of Kolmogorov tori, the system has stochastic behavior [7]. At the present time there is no minimally satisfactory theory which describes behavior of a Hamiltonian system in the complement to the set of Kolmogorov tori. It is known, at least in the case of two-dimensional area-preserving diffeomorphisms, that rests of destroyed Kolmogorov tori have the form of invariant Cantor sets in the phase space [8, 9, 15].

The standard map introduced by Chirikov[1] [10, 11] is a good model for the perturbation theory of completely integrable Hamiltonian systems. The standard map is defined by the formulas:

$$\begin{aligned}
&(x, y) \mapsto (x_1, y_1),\\
&x_1 = x + y_1 \pmod{2\pi},\\
&y_1 = y + \varepsilon \sin x.
\end{aligned} \tag{1.1}$$

For $\varepsilon = 0$ the system (1.1) is completely integrable, the second coordinate y is an integral of motion. For $\varepsilon > 0$ the map has a fixed hyperbolic point at $(0, 0)$. Contracting W_+ and expanding W_- whiskers of separatrices grow from the point $(0, 0)$ and intersect at a point (x_h, y_h) near $(\pi, 2\sqrt{\varepsilon})$, which is shown on Fig. 1.

[1] According to Chirikov, the standard map was introduced by Veksler in 1944. (transl.)

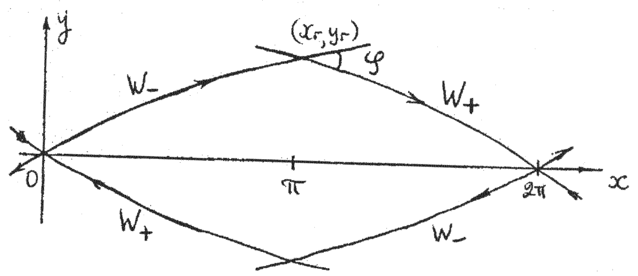

Fig. 1.

The task of the present paper is to derive the following asymptotic for $\varepsilon \longrightarrow 0$ formula for the intersection angle φ:

$$\varphi = \frac{4\pi A_1}{\varepsilon}\, \mathrm{e}^{-\frac{\pi^2}{\sqrt{\varepsilon}}} \left[1 + O\left(\varepsilon^{\frac{1}{8}-\delta}\right)\right]. \tag{1.2}$$

In (1.2) the number A_1 is a constant[2], which is defined from an auxiliary ε-independent nonlinear problem (see Sec. 3). The numerical estimates performed by I. Shakhmanskij give the value $A_1 \approx 260$. The number $\delta > 0$ in (1.2) is arbitrary, the constant in $O\left(\varepsilon^{\frac{1}{8}-\delta}\right)$ term depends on δ. The asymptotic formula (1.2) shows that, for small ε, the angle φ is exponentially small but positive. An exponentially small upper bound was earlier proved by A.I. Neishtadt [12]. It follows from positiveness of φ that (x_h, y_h) is a transversal homoclinic point. The presence of such a point leads to complicated picture of intersections of W_+ and W_-, which was first described by H. Poincaré [13]. Fig. 2 gives an idea about it:

The separatrix W_- is an injective immersion of $\mathbb{R}$ into the phase space $\mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}$. The image of this immersion is a non-compact curve, which contains $(0,0)$ and oscillates increasing amplitude, twists and winds around itself. W_+ has a similar structure. Intersections of the two curves form a complicated net, the closure of which is the stochastic layer near the separatrix. Numerical experiments [10, 11] show existence of this layer. The layer grows with ε and merges with other similar stochastic layers, which are dense in the phase space of the Chirikov standard map (see also [14–16]).

[2] In his later papers Lazutkin used the constant $\Theta_1 = -4iA_1$ instead of A_1. In 1985 M. Tabanov performed a more accurate computation of the Lazutkin constant and obtained $|\Theta_1| = 1118.8277\ldots$ (transl.)

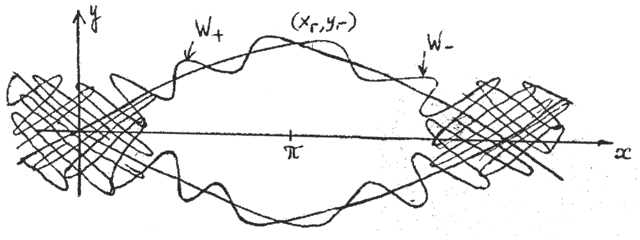

Fig. 2.

The derivation and the proof of formula (1.2) are not completely rigorous. They should be completed by proofs of two conjectures about existence of time-energy coordinates, which are formulated in Secs. 3 and 4, and by a more accurate computation of the constant A_1.

The author thanks I. P. Kornfeld, A. I. Neishtadt, Ya. G. Sinai, B. V. Chirikov, and A. I. Shnirelman for numerous discussions and helpful advises.

The author thanks I. G. Shakhmanskij, who did numerical computations.

2. Approximation of the Chirikov map in a complex domain

The effects related to creation of the stochastic layer, as it was already mentioned in Sec. 1, are exponentially small for small ε. It is difficult to detect them on the background of polynomial in ε errors. A. I. Shnirelman told the author that it would be easier to observe these phenomena in a complex domain, where they grow enough to be comparable with polynomial errors of approximations.

It is convenient to change the coordinates (x, y) to (X, Y) by

$$X = x, \quad Y = \frac{1}{\sqrt{\varepsilon}}\, y. \tag{2.1}$$

In the new variables, the standard map is written in the following way: $(X, Y) \mapsto (X_1, Y_1)$, where

$$\frac{X_1 - X}{\sqrt{\varepsilon}} = Y_1, \quad \frac{Y_1 - Y}{\sqrt{\varepsilon}} = \sin X. \tag{2.2}$$

We will consider (2.2) in the complex phase space $\mathbb{C}/2\pi\mathbb{Z} \times \mathbb{C}$.

If $\operatorname{Im} X$ and $|Y|$ are not too large, the system (2.2) is well approximated for small ε by the system of pendulum differential equations:

$$X' = Y, \quad Y' = \sin X. \tag{2.3}$$

In order to study the standard map for large positive $\operatorname{Im} X$ it is convenient to make the following change of variables:

$$X = -2\mathrm{i}\log\frac{\sqrt{\varepsilon}}{2} - 2\mathrm{i}u, \quad \sqrt{\varepsilon}\,Y = -2\mathrm{i}v. \tag{2.4}$$

The variables u and v will be called superstandard[3]. In these variables our map takes the form

$$\begin{aligned} u_1 &= u + v_1, \\ v_1 &= v - \mathrm{e}^{-2u} + \frac{\varepsilon^2}{16}\mathrm{e}^{2u}. \end{aligned} \tag{2.5}$$

The map (2.5) is defined on the phase space $\mathbb{C}/\mathrm{i}\pi\mathbb{Z} \times \mathbb{C}$. The term $\frac{\varepsilon^2}{16}\mathrm{e}^{2u}$ in (2.5) is small compared e^{-2u} provided $\operatorname{Re} u < \mathrm{const}$. Consequently in this domain the map (2.5) is well approximated by an ε-independent map $(u, v) \mapsto (u_1, v_1)$,

$$u_1 = u + v_1, \quad v_1 = v - \mathrm{e}^{-2u}, \tag{2.6}$$

which we will call superstandard map. The statement about "good approximation" will be made more precise in subsequent.

In the following section we will study invariant curves of the maps. An invariant curve of the map (2.2) with a step h is an analytic map $W : \mathbb{C} \to \mathbb{C}/2\pi\mathbb{Z} \times \mathbb{C}$, $W(t) = (X(t), Y(t))$, which satisfies the following system of finite-difference equations:

$$\begin{aligned} \Delta_h X(t) &= \sqrt{\varepsilon}\,Y(t+h), \\ \Delta_h Y(t) &= \sqrt{\varepsilon}\,\sin X(t). \end{aligned} \tag{2.7}$$

Here

$$\Delta_h f(t) = f(t+h) - f(t). \tag{2.8}$$

A similar definition can be also made for the map (2.6). In order to study W_- in a domain, where it can be approximated by a solution of the differential equation, it is convenient to define h by (4.2). In order to study the invariant curves of the maps (2.5) and (2.6) in the superstandard variables it is convenient to let $h = 1$. In order to come from (2.7) to a corresponding equation in the superstandard variables it is necessary to use the "superstandard time" x defined by

$$t = a + hx. \tag{2.9}$$

[3] The superstandard map is equivalent to the semistandard map $(u, v) \mapsto (u_1, v_1) = (u + v_1, v + \mathrm{e}^u)$, which was independently introduced in [1]. (transl.)

In order to study the separatrix in the domain where $\operatorname{Im} X \geqslant 1$ it is convenient to let $a = \mathrm{i}\frac{\pi}{2}$.

3. Study of the superstandard map

In this section the necessary statements about properties of the superstandard map (2.6) are formulated. An invariant curve of the superstandard map is an analytic map

$$\Gamma : \mathbb{C} \to \mathbb{C}/2\pi\mathbb{Z} \times \mathbb{C}, \quad \Gamma(x) = (u(x), v(x)),$$

which satisfies the equations

$$\begin{aligned} u(x+1) - u(x) &= v(x+1), \\ v(x+1) - v(x) &= -\mathrm{e}^{-2u(x)}. \end{aligned} \tag{3.1}$$

It is easy to check that if $(u(x), v(x))$ is an invariant curve, then $(u_1(x), v_1(x))$, where

$$\begin{aligned} u_1(x) &= u(-x), \\ v_1(x) &= -v(-x+1) \end{aligned} \tag{3.2}$$

is an invariant curve too.

Theorem 1 (existence and uniqueness of the unstable manifold)**.** *There is a unique invariant curve $\Gamma_-(x) = (u_-(x), v_-(x))$ of the superstandard map, which has the following asymptotic behavior for $x \to -\infty$:*

$$\begin{aligned} u_-(x) &= \log(-x) + O\left(\frac{1}{x^2}\right), \\ v_-(x) &= \frac{1}{x} + O\left(\frac{1}{x^2}\right), \end{aligned} \tag{3.3}$$

where the branch of the logarithm is fixed by the condition $\log(-x) > 0$ for $x < 0$. The following asymptotic expansions are uniform in any sector $|\arg x| \geqslant \alpha > 0$

$$\begin{aligned} u_-(x) &= \log(-x) + \frac{1/8}{x^2} - \frac{91/1728}{x^4} + \sum_{k=3}^{\infty} \frac{a_k}{x^{2k}}, \\ v_-(x) &= \frac{1}{x} + \sum_{j=2}^{\infty} \frac{b_k}{x^j}. \end{aligned} \tag{3.4}$$

This theorem is proved in Sec. 7.

Applying the transformation (3.2) to the curve Γ_-, we obtain the stable manifold $\Gamma_+ = (u_+, v_+)$. On the intersection of the sectors $|\arg x| \geqslant \alpha$ and $|\arg(-x)| \geqslant \alpha$ the functions $u_+(x)$ and $v_+(x)$ have the same asymptotic expansions as $u_-(x)$ and $v_-(x)$, respectively.

If $\operatorname{Re} u$ is large positive, the superstandard map is well approximated by the system of differential equations:

$$u' = v, \quad v' = -\mathrm{e}^{-2u}, \tag{3.5}$$

which has a first integral

$$\mathcal{E} = \frac{1}{2}v^2 - \frac{1}{2}\mathrm{e}^{-2u}. \tag{3.6}$$

On the plane of the complex variable x, let us consider a domain $D(R, \alpha)$, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$, defined by the inequality

$$|\arg(x + R/\sin\alpha)| \geqslant \alpha. \tag{3.7}$$

The domain $D(R, \alpha)$ is shown on Fig. 3.

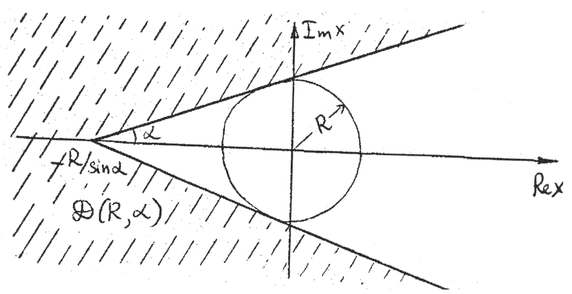

Fig. 3.

Let us denote by $\Gamma_-(R, \alpha)$ the part of Γ_-, which consists of the points $(u_-(x), v_-(x))$ with $x \in D(R, \alpha)$. Let us consider a neighborhood $\Omega(R, \alpha, \beta, m)$ of $\Gamma_-(R, \alpha)$ in the phase space $\mathbb{C}/\mathrm{i}\pi\mathbb{Z} \times \mathbb{C}$:

$$\Omega(R, \alpha, \beta, m) = \bigcup_{x \in \Gamma_-(R,\alpha)} B(x),$$

where $B(x)$ denotes a ball:

$$B(x) = \left\{ (u, v) : |u - u_-(x)| + |v - v_-(x)| \leqslant \frac{\beta}{|x|^m} \right\}.$$

Conjecture A. *There are positive numbers R, α, $\beta, m \geqslant 2$, $m_1 \geqslant 2$, c_1, c_2 and analytical functions $\tau_-(u,v)$ and $\mathcal{E}_-(u,v)$ defined on $\Omega(R,\alpha,\beta,m)$ (time-energy coordinates) such that*

$$\mathcal{E}_-(u_-(x), v_-(x)) = 0, \tag{3.8}$$

$$\tau_-(u_-(x), v_-(x)) = x, \tag{3.9}$$

$$\mathrm{d}\tau_- \wedge \mathrm{d}\mathcal{E}_- = \mathrm{d}u \wedge \mathrm{d}v, \tag{3.10}$$

$$\mathcal{E}_-(u_1, v_1) = \mathcal{E}_-(u,v), \tag{3.11}$$

$$\tau_-(u_1, v_1) = \tau_-(u,v) + 1, \tag{3.12}$$

$$\left|\frac{\partial \mathcal{E}_-}{\partial u}\right| + \left|\frac{\partial \mathcal{E}_-}{\partial v}\right| + \sum_{i,k=1}^{2} \left|\frac{\partial^2 \mathcal{E}_-}{\partial \xi_i \partial \xi_k}\right| \leqslant c_1, \quad (u,v) = (\xi_1, \xi_2), \tag{3.13}$$

$$\left|\frac{\partial \tau_-}{\partial u}\right| + \left|\frac{\partial \tau_-}{\partial v}\right| \leqslant c_2 |\tau_-|^{m_1}. \tag{3.14}$$

In (3.11) and (3.12) the variables (u_1, v_1) are related to (u,v) by (2.6). In its domain, $\mathcal{E}_-$ can be expanded in an asymptotic series of the form

$$\mathcal{E}_-(u,v) = \frac{1}{2}v^2 - \frac{1}{2}p^2 - \frac{1}{2}vp^2 - \frac{1}{12}v^4 + \ldots \tag{3.15}$$

in powers of the variables v and $p = \mathrm{e}^{-u}$ as $(v,p) \to 0$.

The justification of the asymptotic formula for the separatrix splitting essentially depends on the correctness of this conjecture.

Let us assume that the integral $\mathcal{E}_-(u,v)$ does exists. Note that a part of the stable manifold Γ_+, which corresponds to a large x in the sector $\pi - \alpha \leqslant |\arg x| \leqslant \alpha$, belongs to the domain of $\mathcal{E}_-$ due to the asymptotic expansion (3.4) for $u_+(x)$ and $v_+(x)$. Let us consider the function

$$\Theta(x) = \mathcal{E}_-(u_+(x), v_+(x)). \tag{3.16}$$

It follows from (3.11) and (3.12) that $\Theta(x)$ is periodic in x with the period equal to 1. The functions $u_+(x), v_+(x)$ and $u_-(x), v_-(x)$ have a single asymptotic expansion for $\operatorname{Im} x \to -\infty$ and (3.8) holds, consequently $\Theta(x)$ is well defined in a half-plane $-\operatorname{Im} x \geqslant \text{const} > 0$ and vanishes as $\operatorname{Im} x \to -\infty$. Thus, in the lower half-plane, the function $\Theta(x)$ can be expanded in a Fourier series of the form

$$\Theta(x) = \mathrm{i} \sum_{n=1}^{\infty} A_n \mathrm{e}^{-\mathrm{i}n 2\pi x}. \tag{3.17}$$

Numerical computations give $A_1 \approx 260$. This number A_1 enters the asymptotic formula (1.2).

4. Separatrices of the standard map in the complex plane

We consider the standard map $(X, Y) \to (X_1, Y_1)$,

$$X_1 = X + \sqrt{\varepsilon}\, Y_1 \pmod{2\pi}, \quad Y_1 = Y + \sqrt{\varepsilon}\, \sin X \tag{4.1}$$

in the space $\mathbb{C}/2\pi\mathbb{Z} \times \mathbb{C}$. The map (4.1) has a hyperbolic for $\varepsilon > 0$ fixed point at $(0, 0)$. The eigenvalues of the linear part of the map at this point equal $\lambda_1 = \mathrm{e}^h$, $\lambda_2 = \mathrm{e}^{-h}$, where

$$h = \log(1+\sqrt{\varepsilon}\sqrt{1+\varepsilon/4}+\varepsilon/2) = \sqrt{\varepsilon} - \frac{1}{24}\varepsilon^{3/2} + \frac{3}{640}\varepsilon^{5/2} + O(\varepsilon^{7/2}). \tag{4.2}$$

In this section we consider the invariant curves of the map (4.1) with the step h given by (4.2) (comp. with Sec. 2).

Theorem 2. *There is a unique invariant curve $W_-(t) = (X_-(t), Y_-(t))$ of the map (4.1), such that for $t \to -\infty$*

$$X_-(t) = c\,\mathrm{e}^t + o(\mathrm{e}^t), \quad c \in [3, 5]. \tag{4.3}$$

If ε is sufficiently small, the constant c can be uniquely chosen from the condition

$$X_-(0) = \pi. \tag{4.4}$$

In the following we assume that c for W_- is fixed by the condition (4.4). For small ε, this constant satisfies the estimates (8.12).

We study the behavior of the curve W_- in the domain

$$\mathrm{Re}\, t \leqslant h, \quad |\arg t - \mathrm{i}\tfrac{\pi}{2}(2k+1)| \geqslant \alpha, \quad k \in \mathbb{Z},$$

for small ε. It follows from (8.1)–(8.3) that

$$X_-(t + \mathrm{i}\pi) = -X_-(t), \quad X_-(\bar{t}) = \overline{X_-(t)}. \tag{4.5}$$

It is sufficient to consider W_- in a domain

$$D = \left\{ t \in \mathbb{C} : 0 \leqslant \mathrm{Im}\, t \leqslant \frac{\pi}{2},\ \mathrm{Re}\, t \leqslant h,\ \arg t \leqslant -\alpha < 0 \right\}.$$

Theorem 3. *In the domain D,*

$$\Big| X_-(t) - X_0(t) \Big| \leqslant \mathrm{const} \cdot \varepsilon \cdot \left(1 + \frac{1}{|t - \mathrm{i}\frac{\pi}{2}|^2} \right), \tag{4.6}$$

where

$$X_0(t) = 4 \arctan \mathrm{e}^t, \tag{4.7}$$

the constant in the estimate depends on α only.

Remark. The function (4.7) is a first component of the separatrix solution of the system of differential equations (2.3), which approximates the finite-difference equations (2.2).

It is convenient to break the domain D into the 4 parts shown on Fig. 4.

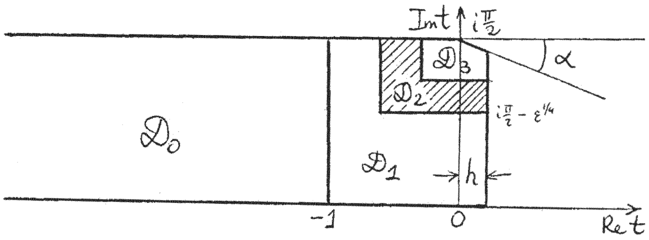

Fig. 4.

The domain D_3 is the intersection of D with the rectangle

$$\Pi_3 = \left\{ t : -\sqrt{\varepsilon}\,\frac{\sigma}{2\pi}\log\frac{1}{\varepsilon} \leqslant \operatorname{Re} t \leqslant h,\ 0 \leqslant \frac{\pi}{2} - \operatorname{Im} t \leqslant \sqrt{\varepsilon}\,\frac{\sigma}{2\pi}\log\frac{1}{\varepsilon} \right\},$$

where

$$\sigma = \frac{1}{8} + \frac{\delta}{2}. \tag{4.8}$$

The domain D_2 is obtained by deleting Π_3 from the rectangle

$$\Pi_2 = \left\{ t : -\varepsilon^{1/4} \leqslant \operatorname{Re} t \leqslant h,\ 0 \leqslant \frac{\pi}{2} - \operatorname{Im} t \leqslant \varepsilon^{1/4} \right\}.$$

The domain D_1 is obtained by deleting Π_2 from the rectangle

$$\Pi_1 = \left\{ t : -1 \leqslant \operatorname{Re} t \leqslant h,\ 0 \leqslant \operatorname{Im} t \leqslant \frac{\pi}{2} \right\}.$$

The domain D_0 is the half-strip

$$\operatorname{Re} t \leqslant -1, \quad 0 \leqslant \operatorname{Im} t \leqslant \frac{\pi}{2}.$$

In order to derive the asymptotic formula for the separatrix splitting, we need an asymptotics of $W_-(t)$ for $t \in D_2$. In this domain, the approximation by $X_0(t)$ is not good. Instead of X_0 we will use the function

$$-2\mathrm{i}\log\frac{h}{2} - 2\mathrm{i}u_-\left(\frac{t - \mathrm{i}\pi/2}{h}\right),$$

where $u_-(x)$ is a first component of the unstable manifold of the superstandard map. It is convenient to use superstandard variables and let

$$X_-\left(\mathrm{i}\tfrac{\pi}{2}+hx\right)=-2\mathrm{i}\log\frac{h}{2}-2\mathrm{i}\widetilde{u}_-(x). \tag{4.9}$$

Since for $hx\to 0$ we have

$$X_0\left(\mathrm{i}\tfrac{\pi}{2}+hx\right)=-2\mathrm{i}\log\frac{h}{2}-2\mathrm{i}\log(-x)+O(\varepsilon x^2), \tag{4.10}$$

taking into account the asymptotic formula (3.4) for the function $u_-(x)$, we obtain that on the boundary between D_1 and D_2 the function $u_-(x)$ approximates $\widetilde{u}_-(x)$ with the same accuracy:

$$|\widetilde{u}_-(x)-u_-(x)|\leqslant \mathrm{const}\,\sqrt{\varepsilon}. \tag{4.11}$$

Theorem 4. *In the domain D_2*

$$|\widetilde{u}_-(x)-u_-(x)|\leqslant \mathrm{const}\,\varepsilon^{1/4-\delta/2}, \tag{4.12}$$

where $\delta>0$ is arbitrary and const *depends on the choice of δ.*

Theorems 3 and 4 are proved in Secs. 9 and 10.

The stable separatrix W_+ is obtained from W_- with the help of the transformation

$$\begin{aligned} X_+(t)&=2\pi-X_-(-t),\\ Y_+(t)&=Y_-(-t+h). \end{aligned} \tag{4.13}$$

Let

$$\widetilde{D}=\left\{t\in\mathbb{C}: |\operatorname{Re}t|\leqslant h,\ |\operatorname{Im}t|\leqslant\frac{\pi}{2}-\sqrt{\varepsilon}\,\frac{\sigma}{2\pi}\log\frac{1}{\varepsilon}\right\}. \tag{4.14}$$

Let $\widetilde{\Omega}\subset\mathbb{C}^2$ denote a union of all straight segments with endpoints at $(X_-(t),Y_-(t))$ and $(X_+(t),Y_+(t))$, where $t\in\widetilde{D}$.

Conjecture B. *There are analytic functions $t(X,Y)$ and $E(X,Y)$ defined in a neighborhood of $\widetilde{\Omega}$, which have the following properties: they are real on real X and Y,*

$$E(X_-(t),Y_-(t))=0, \tag{4.15}$$

$$t(X_-(t),Y_-(t))=t, \tag{4.16}$$

$$\mathrm{d}t\wedge\mathrm{d}E=\mathrm{d}X\wedge\mathrm{d}Y, \tag{4.17}$$

$$E(X_1, Y_1) = E(X, Y), \tag{4.18}$$

$$t(X_1, Y_1) = t(X, Y) + h, \tag{4.19}$$

where (X_1, Y_1) are related to (X, Y) by (4.1). In the superstandard variables defined by (2.4) it is convenient to let

$$E(X, Y) = -\frac{4}{h^2}\widetilde{\mathcal{E}}(u, v). \tag{4.20}$$

The derivatives of the function $\widetilde{\mathcal{E}}$ with respect to u and v up to the second order are bounded by constants, which depend only on δ.

5. Derivation of the formula for the separatrix splitting angle

The intersection of the separatrices segments $W_+(t)$ and $W_-(\widetilde{t})$, which correspond to $-\infty < \widetilde{t} \leqslant h$ and $h \leqslant t \leqslant +\infty$, is not empty. Let (x_h, y_h) be an intersection point with closest to 0 value of $t = t_h$ and let $\widetilde{t}_h$ denote the corresponding value of the parameter $\widetilde{t}$. We evaluate the separatrix splitting angle at this point[4].

Let us define the function

$$\Psi(t) = E(X_+(t), Y_+(t)). \tag{5.1}$$

The function $\Psi(t)$ has the following properties: 1) it is periodic in t with the period equal to h due to (4.18) and (4.19), 2) it vanishes at $t = t_h$ due to (4.15), 3) it is well-defined and analytic in the strip $|\mathrm{Im}\, t| \leqslant \frac{\pi}{2} - \frac{\sigma}{2\pi}\sqrt{\varepsilon}\log\frac{1}{\varepsilon}$ due to Conjecture B and the property 1). Let us evaluate its derivative at the homoclinic point:

$$\Psi'(t_h) = X'_-(\widetilde{t}_h)Y'_+(t_h) - Y'_-(\widetilde{t}_h)X'_+(t_h). \tag{5.2}$$

In order to derive (5.2) we used

$$\begin{aligned} X'_-(t) &= \frac{\partial E}{\partial Y}(X_-(t), Y_-(t)), \\ Y'_-(t) &= -\frac{\partial E}{\partial X}(X_-(t), Y_-(t)), \end{aligned} \tag{5.3}$$

[4] It follows from the normalizing condition (4.4) and the definition (4.13) that $t_h = \widetilde{t}_h = 0$. (transl.)

which can be easily derived from (4.15)–(4.17). If follows from (5.2) that the angle between separatrices at the homoclinic point is given by

$$\varphi \approx \sin\varphi = -\frac{\sqrt{\varepsilon}\,\Psi'(t_h)}{\sqrt{\left(X_-'(\widetilde{t}_h)^2 + Y_-'(\widetilde{t}_h)^2\right)\left(X_+'(t_h)^2 + Y_+'(t_h)^2\right)}} =$$

$$= -\frac{\sqrt{\varepsilon}}{4}\Psi'(t_h)\left(1 + O(\sqrt{\varepsilon})\right). \tag{5.4}$$

Here we used the estimate of Theorem 3 and the equalities (4.13):

$$\begin{aligned} X_-'(\widetilde{t}_h) &= 2 + O(\sqrt{\varepsilon}), & X_+'(t_h) &= 2 + O(\sqrt{\varepsilon}),\\ Y_-'(\widetilde{t}_h) &= O(\sqrt{\varepsilon}), & Y_+'(t_h) &= O(\sqrt{\varepsilon}). \end{aligned} \tag{5.5}$$

In (5.4) the factor $\sqrt{\varepsilon}$ is due to the change of variable (2.1).

In order to estimate the function $\Psi(t)$ we compare it with the function

$$\chi(t) = -\frac{1}{h^2}\Theta\left(\frac{t - \mathrm{i}\frac{\pi}{2}}{h}\right) - \frac{1}{h^2}\overline{\Theta\left(\frac{\overline{t} - \mathrm{i}\frac{\pi}{2}}{h}\right)}, \tag{5.6}$$

where the function $\Theta(x)$ was defined in Sec. 3. Note that $\Theta(x)$ is well defined for $\operatorname{Im} x \leqslant -\text{const} < 0$, where const is independent of ε. It follows that the strip of analyticity of the h-periodic function (5.6) contains the strip

$$|\operatorname{Im} t| \leqslant \frac{\pi}{2} - \frac{\sigma}{2\pi}\sqrt{\varepsilon}\log\frac{1}{\varepsilon}$$

provided ε is sufficiently small. We compare the functions $\Psi(t)$ and $\chi(t)$ in the last strip. Note that both functions are real for real t. Consequently, it is sufficient to compare the functions in the rectangle

$$\Pi = \left\{ t \in \mathbb{C} : |\operatorname{Re} t| \leqslant \frac{h}{2},\ 0 \leqslant \operatorname{Im} t \leqslant \frac{\pi}{2} - \frac{\sigma}{2\pi}\sqrt{\varepsilon}\log\frac{1}{\varepsilon} \right\}.$$

In the domain $\Pi \cap D_1$ (see Sec. 4) the function $\chi(t)$ is exponentially small for small ε, and $\Psi(t)$ is bounded due to Conjecture B and Theorem 3:

$$|\Psi(t)| \leqslant \text{const} \cdot \left(|X_+(t) - X_-(t)| + |Y_+(t) - Y_-(t)|\right) \leqslant \text{const}\,.$$

Consequently in this subdomain

$$|\Psi(t) - \chi(t)| \leqslant \text{const}\,. \tag{5.7}$$

We estimate the difference $\Psi(t)-\chi(t)$ in the domain $\Pi\cap D_2$. It is convenient to use the superstandard variables:

$$\begin{aligned}
X_-(\mathrm{i}\tfrac{\pi}{2}+hx) &= -2\mathrm{i}\log\frac{h}{2}-2\mathrm{i}\widetilde{u}_-(x),\\
Y_-(\mathrm{i}\tfrac{\pi}{2}+hx) &= -\frac{2\mathrm{i}}{\sqrt{\varepsilon}}\,\widetilde{v}_-(x),\\
X_+(\mathrm{i}\tfrac{\pi}{2}+hx) &= -2\mathrm{i}\log\frac{h}{2}-2\mathrm{i}\widetilde{u}_+(x),\\
Y_+(\mathrm{i}\tfrac{\pi}{2}+hx) &= -\frac{2\mathrm{i}}{\sqrt{\varepsilon}}\,\widetilde{v}_+(x).
\end{aligned}\tag{5.8}$$

In $\Pi\cap D_2$, we can neglect the second term in the right-hand side of (5.6). It follows from (4.20) that

$$\Psi(t)=-\frac{4}{h^2}\widetilde{\mathcal{E}}(\widetilde{u}_+(x),\widetilde{v}_+(x)).\tag{5.9}$$

Taking into account (3.16) and (5.3) and boundedness of the second derivatives of the functions $\widetilde{\mathcal{E}}$ and $\mathcal{E}_-$ we obtain in $\Pi\cap D_2$:

$$\begin{aligned}
&\widetilde{\mathcal{E}}(\widetilde{u}_+(x),\widetilde{v}_+(x))-\Theta(x)=\\
&=-\widetilde{v}_-'(x)(\widetilde{u}_+(x)-\widetilde{u}_-(x))+\widetilde{u}_-'(x)(\widetilde{v}_+(x)-\widetilde{v}_-(x))+\\
&+v_-'(x)(u_+(x)-u_-(x))-u_-'(x)(v_+(x)-v_-(x))+\\
&+O\left(|\widetilde{u}_+(x)-\widetilde{u}_-(x)|^2+|\widetilde{v}_+(x)-\widetilde{v}_-(x)|^2\right)+\\
&+O\left(|u_+(x)-u_-(x)|^2+|v_+(x)-v_-(x)|^2\right).
\end{aligned}\tag{5.10}$$

It follows from (3.17) and (4.8) that in this domain

$$|\Theta(x)|\leqslant \mathrm{const}\,\mathrm{e}^{-2\pi\frac{\sigma}{2\pi}\log\frac{1}{\varepsilon}}=\mathrm{const}\,\varepsilon^{\frac{1}{8}+\frac{\delta}{2}}.\tag{5.11}$$

Similar arguments lead to the upper bound:

$$|\tau_-(u_+(x),v_+(x))-x|\leqslant \mathrm{const}\,\varepsilon^{\frac{1}{8}+\frac{\delta}{2}}\tag{5.12}$$

It is not difficult to derive from (5.11) and (5.12), and Conjecture A that

$$\begin{aligned}
|u_+(x)-u_-(x)|+|v_+(x)-v_-(x)| &\leqslant \mathrm{const}\left(\log\frac{1}{\varepsilon}\right)^{m_1}\varepsilon^{\frac{1}{8}+\frac{\delta}{2}}\\
&\leqslant \mathrm{const}\,\varepsilon^{\frac{1}{8}}.
\end{aligned}\tag{5.13}$$

Using the estimate of Theorem 4 we conclude that (5.13) is valid for u, v replaced by $\widetilde{u}$, $\widetilde{v}$:

$$|\widetilde{u}_+(x) - \widetilde{u}_-(x)| + |\widetilde{v}_+(x) - \widetilde{v}_-(x)| \leqslant \operatorname{const} \varepsilon^{\frac{1}{8}}. \tag{5.14}$$

Thus the quadratic term of (5.10) is bounded from above by $O(\varepsilon^{1/4})$. Changing the order of the linear terms in (5.10) and using (3.4) we obtain the following upper bound for the right-hand side of (5.10):

$$\begin{aligned} &|\widetilde{\mathcal{E}}(\widetilde{u}_+(x), \widetilde{v}_+(x)) - \Theta(x)| \\ &\leqslant \operatorname{const} \Big(|\widetilde{u}_+(x) - u_+(x)| + |\widetilde{v}_+(x) - v_+(x)| \\ &\quad + |\widetilde{u}'_+(x) - u'_+(x)| + |\widetilde{v}'_+(x) - v'_+(x)| \\ &\quad + |\widetilde{u}_-(x) - u_-(x)| + |\widetilde{v}_-(x) - v_-(x)| \\ &\quad + |\widetilde{u}'_-(x) - u'_-(x)| + |\widetilde{v}'_-(x) - v'_-(x)| \Big) + O(\varepsilon^{1/4}) \\ &\leqslant \operatorname{const} \varepsilon^{\frac{1}{4} - \frac{\delta}{2}}. \end{aligned} \tag{5.15}$$

It follows from (5.15) and (5.7) that in the strip $|\operatorname{Im} t| \leqslant \frac{\pi}{2} - \frac{\sigma}{2\pi}\sqrt{\varepsilon} \log \frac{1}{\varepsilon}$ the following upper bound holds:

$$|\Psi(t) - \chi(t)| \leqslant \operatorname{const} \varepsilon^{-1 + \frac{1}{4} - \frac{\delta}{2}}. \tag{5.16}$$

The following elementary lemma can be proved using Fourier series.

Lemma 5.1. *Let the function $f(t)$ be analytic in a strip $|\operatorname{Im} t| \leqslant b$, $b > 0$, continuous in the closure of the strip and has the period $a > 0$. If*

$$\mathrm{e}^{-\frac{2\pi b}{a}} \leqslant \frac{1}{2} \quad \textit{and} \quad \int_0^a f(t)\, \mathrm{d}t = 0,$$

then for real t

$$|f(t)| \leqslant 4 \max_{|\operatorname{Im} \tau| \leqslant b} |f(\tau)|\, \mathrm{e}^{-\frac{2\pi b}{a}}. \tag{5.17}$$

We note that the mean value of the functions Ψ and χ vanishes. In the case of Ψ this property[5] follows from (4.17), the area-preservation property of the standard map and existence of KAM invariant curves [3].

[5] In 1991 A. Neishtadt pointed out to the translator that these arguments do not prove that the mean value of Ψ exactly vanishes. Nevertheless they show that the mean value is negligible for the following proof. (transl.)

Applying Lemma 5.1 to the function $\psi(t)-\chi(t)$, we obtain on the real axis

$$\Psi(t) = -\frac{2A_1}{\varepsilon}\mathrm{e}^{-\frac{\pi^2}{\sqrt{\varepsilon}}}\sin\frac{2\pi t}{h} + O\left(\mathrm{e}^{-\frac{\pi^2}{\sqrt{\varepsilon}}}\varepsilon^{-1+\frac{1}{8}-\delta}\right). \tag{5.18}$$

It follows that $t_h = O(h\varepsilon^{\frac{1}{4}-\delta})$. Differentiating (5.18) at $t = t_h$ and substituting the result into (5.4) we obtain the desired formula (1.2).

6. The operator L

In this section we study a linear operator, which is a sufficiently good linearization of a nonlinear operator.

The value of the operator L evaluated on a complex-valued function f of a complex argument x is given by the following expression:

$$(Lf)(x) = \Delta^2 f(x) - \frac{2}{x^2}\, f(x), \tag{6.1}$$

where the second order difference $\Delta^2 f(x)$ is defined by the formula

$$\Delta^2 f(x) = f(x+1) + f(x-1) - 2f(x). \tag{6.2}$$

We consider linear operators defined by expression (6.1) in different functional spaces and construct their inverse. For simplicity, we will use the single letter L for all of them, and L^{-1} for the inverse operators.

First we consider L on all complex-valued functions f, which are possibly not globally defined, of a complex variable x.

Let us state two Lemmas about solutions of the homogeneous equation

$$\Delta^2 u - \frac{2}{x^2}\, u = 0. \tag{6.3}$$

Lemma 6.1. *The general solution of the equation* (6.3) *has the form*

$$u(x) = a(x)u_0(x) + b(x)x^2, \tag{6.4}$$

where $a(x)$ and $b(x)$ are arbitrary periodic functions with period 1,

$$u_0(x) = \frac{1}{2} + x + x^2\sum_{k=1}^{\infty}\frac{1}{(x-k)^2}. \tag{6.5}$$

Lemma 6.2. *In any sector* $|\arg x| \geqslant \alpha > 0$ *the function* u_0 *has the following uniform asymptotic expansion:*

$$u_0(x) = \sum_{m=1}^{\infty} (-1)^m B_m \frac{1}{x^{2m-1}}, \tag{6.6}$$

where B_m *are Bernulli numbers:*

$$B_1 = \frac{1}{6}, \quad B_2 = \frac{1}{30}, \quad B_3 = \frac{1}{42}, \quad B_4 = \frac{1}{30}, \quad B_5 = \frac{5}{66}, \ldots.$$

The function $u_0(x)$ satisfies

$$u_0(x) = -u_0(-x) + \frac{\pi^2 x^2}{\sin^2 \pi x}. \tag{6.7}$$

Let us define a Wronski determinant:

$$W(f,g)(x) = (\Delta f(x)) \cdot g(x) - f(x) \cdot \Delta g(x), \tag{6.8}$$

where $\Delta f(x) = f(x+1) - f(x)$. If f and g are two solutions of the homogeneous equation (6.3), then $W(f,g)$ is periodic with period 1.

Corollary.

$$W(u_0, x^2) = \frac{1}{2}. \tag{6.9}$$

The inverse operator to L is always defined by the equation

$$(L^{-1}f)(x) = 2\sum_{k=1}^{\infty} \left[u_0(x)(x-k)^2 - u_0(x-k)x^2\right] f(x-k). \tag{6.10}$$

It is not too difficult to check that if the series in the right-hand side of (6.10) converges, then $u = L^{-1}f$ satisfies $Lu = f$.

Let us denote $\mathfrak{X}_\gamma(D(R,\alpha))$, $\gamma \geqslant 0$, the Banach space of all functions, which are analytic in $D(R,\alpha)$ (see definition (3.7) and Fig. 3), continuous in the closure of the domain and have the finite norm

$$||f||_\gamma = \sup_{D(R,\alpha)} |x|^\gamma |f(x)|. \tag{6.11}$$

Lemma 6.3. *Let $\gamma > 3$. The equation* (6.10) *defines a continuous operator*

$$L^{-1} : \mathfrak{X}_\gamma(D(R,\alpha)) \to \mathfrak{X}_{\gamma-2}(D(R,\alpha)),$$

the norm of L^{-1} is bounded from above by a constant, which depends only on α and γ.

Proof. Using (6.6), we obtain

$$\|L^{-1}f(x)\|_{\gamma-2} \tag{6.12}$$

$$\leqslant \text{const} \sup_{x\in D(R,\alpha)} \sum_{k=1}^{\infty} \left(|x|^{\gamma-3}|x-k|^{-\gamma+2} + |x|^{\gamma}|x-k|^{-\gamma-1}\right) \|f\|_\gamma .$$

If $\beta > 1$ the following estimate holds:

$$\sum_{k=1}^{\infty} |x-k|^{-\beta} = O(x^{-\beta+1}). \tag{6.13}$$

Bounding the right-hand side of (6.12) using (6.13), we obtain the statement of the lemma.

Now let us consider the Banach space $\mathfrak{X}_\gamma(\ell)$, where ℓ is a segment parallel to the real axis on the complex plane, $\mathfrak{X}_\gamma(\ell)$ consists of all bounded complex-valued functions on ℓ, the norm is defined by

$$\|f\|_\gamma = \sup_{x\in\ell} |x|^\gamma |f(x)|. \tag{6.14}$$

Let us define the operator $L^{-1} : \mathfrak{X}_{\gamma_1}(\ell) \to \mathfrak{X}_{\gamma_2}(\ell)$ by (6.10) assuming that all functions are continued by identical zero onto the "negative prolongation" of the segment ℓ, i.e., onto a ray parallel to the real axis, which starts from the left end of the segment ℓ. All these operators are continuous provided ℓ does not intersect the ray $[1, +\infty[$. The following Lemma can be proved by arguments similar to the proof of Lemma 6.3.

Lemma 6.4. *If $\gamma > 3$ and the segment ℓ does not intersect the ray $[1/2, +\infty[$, then the norm of the operator*

$$L^{-1} \colon \mathfrak{X}_\gamma(\ell) \to \mathfrak{X}_{\gamma-2}(\ell)$$

is bounded by a constant, which depends on γ and $\inf_{x\in\ell} |\arg x|$ only.

Lemma 6.5. *If $1 < \gamma \leqslant 2$ and $\ell \subset D \setminus D_0$ (see definition of Sec. 4 and Fig. 4), then the norm of the operator $L^{-1} : \mathfrak{X}_\gamma(\ell) \to \mathfrak{X}_\gamma(\ell)$ is bounded in the following way:*

$$\|L^{-1}\| \leqslant \operatorname{const} \varepsilon^{-1}, \tag{6.15}$$

where the constant depends only on γ and α. This operator is a Volterra operator in the following sense. If $\Phi \in \mathfrak{X}_0(\ell)$ the operator $I - \varepsilon L^{-1}\Phi$ has a continuous inverse in $\mathfrak{X}_\gamma(\ell)$, and the norm of the inverse operator depends only on the norm of Φ in $\mathfrak{X}_0(\ell)$, γ and α. Here I denotes the identity operator and Φ stands for the operator, which acts by multiplication by the function Φ.

Proof. Similar to the proof of Lemma 6.3, we have

$$\|L^{-1} f\|_\gamma \tag{6.16}$$
$$\leqslant \operatorname{const} \sup_{x\in\ell} \sum_{1\leqslant k \leqslant |\operatorname{Re} x + \frac{1}{\sqrt{\varepsilon}}|} \left(|x|^{-1+\gamma}|x-k|^{2-\gamma} + |x|^{2+\gamma}|x-k|^{-1-\gamma}\right) \|f\|_\gamma .$$

Let us introduce new variables

$$t = 1 + \sqrt{\varepsilon}\, \operatorname{Re} x, \quad \widetilde{t} = 1 + \sqrt{\varepsilon}\,(\operatorname{Re} x - k). \tag{6.17}$$

Then the sum in the right-hand side of (6.16) is bounded from above by the integral

$$\frac{1}{\varepsilon}\int_0^t \left[(1-t)^{\gamma-1}(1-\widetilde{t})^{2-\gamma} + (1-t)^{2+\gamma}(1-\widetilde{t})^{-1-\gamma}\right] \mathrm{d}\widetilde{t}, \tag{6.18}$$

the first statement of the lemma follows directly. The expression under the integral (6.18) is bounded, thus usual Volterra-type estimates for the sequence $f_0 = f$, $f_{n+1} = \varepsilon L^{-1}\Phi f_n$ imply the second statement of the lemma.

7. Proof of Theorem I

It is convenient to replace the system of two first order finite-difference equations (3.1) by a single second order equation

$$\Delta^2 u(x) = -\mathrm{e}^{-2u(x)}. \tag{7.1}$$

It is sufficient to prove existence and uniqueness of an analytical solution to (7.1), which can be represented in the form

$$u(x) = \log(-x) + \xi(x), \tag{7.2}$$

where

$$\xi(x) = O(x^{-2}) \tag{7.3}$$

provided $\operatorname{Re} x \leqslant -R < 0$.

Note that if $\xi(x)$ exists and is unique for $\operatorname{Re} x \leqslant -R < 0$, then $u(x)$ can be uniquely continued onto the entire complex plane by (7.1).

Equation (7.1) takes the form

$$\Delta^2 \xi(x) - \frac{1}{x^2}\,\xi(x) = \Phi(x, \xi(x)), \tag{7.4}$$

where

$$\Phi(x, \xi) = -\log\left(1 - \frac{1}{x^2}\right) - \frac{1}{x^2} - \frac{1}{x^2}\left(\mathrm{e}^{-2\xi} - 1 + 2\xi\right). \tag{7.5}$$

Equation (7.4) and condition (7.3) on the behavior of the function $\xi(x)$ at infinity are equivalent to a single equation

$$\xi = F(\xi), \tag{7.6}$$

where

$$F(\xi) = L^{-1}\Phi(\xi), \quad \Phi(\xi)(x) = \Phi(x, \xi(x)). \tag{7.7}$$

We consider the equation (7.6) in the space $\mathfrak{X}_2(D(R, \alpha))$ (see Sec. 6). Let $B \subset \mathfrak{X}_2(D(R, \alpha))$ denote a unit closed ball centered at 0. If R in $D(R, \alpha)$ is sufficiently large, $\xi, \xi_1, \xi_2 \in B$, then the following upper bounds hold:

$$\|\Phi(\xi)\|_4 \leqslant \operatorname{const} \frac{\|\xi\|_2}{R^2} + \frac{2}{3}, \tag{7.8}$$

$$\|\Phi(\xi_1) - \Phi(\xi_2)\|_4 \leqslant \frac{\operatorname{const}}{R^2}\,\|\xi_1 - \xi_2\|_2. \tag{7.9}$$

The upper bound (7.8) and Lemma 6.3 imply that for a sufficiently large R the sequence ξ_n, $n = 0, 1, 2, \ldots$, $\xi_0 = 0$, $\xi_{n+1} = F(\xi_n)$, is in B. The upper bound (7.9) implies convergence of the sequence in B provided R is sufficiently large.

The statement about the asymptotic expansion can be proved similarly. Let us fixed an arbitrary natural number N and introduce a function $\eta(x)$ by

$$u(x) = \log(-x) + \sum_{k=1}^{N} \frac{a_k}{x^{2k}} + \eta(x). \tag{7.10}$$

We obtain the following equation for $\eta(x)$:

$$L\eta(x) = -\log\left(1 - \frac{1}{x^2}\right) - \sum_{k=1}^{N+1} \frac{1}{k}\frac{1}{x^{2k}} \tag{7.11}$$

$$-\frac{1}{x^2}\left[\mathrm{e}^{-2\sum_{k=1}^{N}\frac{a_k}{x^{2k}} - 2\eta(x)} - 1 + \sum_{k=1}^{N}\frac{1}{k+1}\frac{1}{x^{2k}} + x^2\sum_{k=1}^{N+1} a_k\Delta^2\frac{1}{x^{2k}} + 2\eta(x)\right].$$

We choose the numbers a_k in such a way that the expression in the square brackets is bounded by

$$O(\eta^2) + O\left(\frac{1}{x^2}\eta\right) + O\left(\frac{1}{x^{2N+2}}\right). \tag{7.12}$$

Then we invert the operator L and solve the obtained equation in $\mathfrak{X}_{2N+2}(D(R,\alpha))$.

The uniqueness of the solution of equation (7.6) under the condition (7.3) also follows from (7.8) and (7.9). The first estimate shows the solution belongs to B provided R is sufficiently large, the second estimate implies that F defined by (7.7) is a contracting operator.

8. Proof of Theorem 2

Let us fix the constant C from the interval $[3, 5]$. System (2.7) and condition (4.3) are equivalent to a single equation

$$X_-(t) = C\mathrm{e}^t + F(X_-)(t), \tag{8.1}$$

where

$$F(X)(t) = \frac{\sqrt{\varepsilon}}{\sqrt{1+\varepsilon/4}}\sum_{k=1}^{\infty}\Phi(X(t-kh))\sinh kh, \tag{8.2}$$

$$\Phi(X) = \sin X - X. \tag{8.3}$$

Let us consider Eq. (8.1) in a ball of the radius 10 with the center at the origin in the Banach space of analytic functions in the half-plane $\mathrm{Re}\, t < -R < 0$, continuous in the closure of the half-plane, equipped with the norm

$$\|X\| = \sup_{\mathrm{Re}\, t < -R} |\mathrm{e}^{-t}X(t)|. \tag{8.4}$$

In the ball the nonlinear operator F satisfies the following bounds

$$\|F(X)\| \leqslant \mathrm{const}\, \mathrm{e}^{-2R}\|X\| + 5, \tag{8.5}$$

$$\|F(X_1) - F(X_2)\| \leqslant \mathrm{const}\, \mathrm{e}^{-R} \|X_1 - X_2\|. \tag{8.6}$$

It follows that Eq. (8.1) and, consequently, system (2.7) with condition (4.3) has a unique solution for any $C \in [3,5]$ in this space provided R is sufficiently large. The solution can be continued onto the values of t with $\mathrm{Re}\, t \geqslant -R$ using Eq. (2.7). Let us denote the corresponding solution by $X_-(C,t)$.

We note that the uniqueness of the solution implies

$$X_-(C_1,t) = X_-(C_2, t + \log C_1/C_2) \tag{8.7}$$

for any $C_i \in [3,5]$, $i = 1,2$.

Consider $X_-(4,t)$. Lemma 9.1 to be proved in Sec. 9 implies

$$|X_-(4,t) - 4\arctan \mathrm{e}^t| \leqslant \mathrm{const}\, \varepsilon \tag{8.8}$$

for $\mathrm{Re}\, t \leqslant 1$, $|\,\mathrm{Im}\, t| \leqslant 1$. Consequently for real $t \leqslant 1/2$

$$\left| \frac{\partial}{\partial t} X_-(4,t) - \frac{2}{\cosh t} \right| \leqslant \mathrm{const}\, \varepsilon. \tag{8.9}$$

Equations (8.7) and (8.9) imply

$$\left| \frac{\partial}{\partial C} X_-(C,t) - \frac{2}{\cosh(t + \log \frac{C}{4})} \right| \leqslant \mathrm{const}\, \varepsilon, \tag{8.10}$$

which is valid uniformly for the values $C \in [3,5]$, $t \leqslant \frac{1}{2} - \log\frac{5}{4} \approx 0.277$.

Equations (8.8) and (8.10) imply that for sufficiently small ε the equation

$$X_-(C,0) = \pi \tag{8.11}$$

has a unique solution from the interval $[3,5]$, moreover

$$C = 4 + O(\varepsilon). \tag{8.12}$$

9. Proof of Theorem 3

First, let us prove estimate (4.6) in the domain D_0. It is convenient to prove it in a larger domain $\widetilde{D}_0$, which is obtained as a union of D_0 and the rectangles $|\,\mathrm{Re}\, t| \leqslant 1$, $0 \leqslant \mathrm{Im}\, t \leqslant 1$ and $-1 \leqslant \mathrm{Re}\, t \leqslant -\frac{1}{2}$, $|\,\mathrm{Im}\, t| \leqslant \frac{\pi}{2}$.

Lemma 9.1. *In the domain* $\widetilde{D}_0$

$$|X_-(4,t) - X_0(t)| \leqslant \operatorname{const} \varepsilon, \tag{9.1}$$

where the constant is independent of ε.

We used the notations of Sec. 8 to formulate Lemma 9.1.

Estimate (4.6) in D_0 follows from (9.1), (8.7), and (8.12).

Proof of Lemma 9.1. Let

$$Z(t) = X_-(4,t) - X_0(t). \tag{9.2}$$

Since the function $X_0(t) = 4 \operatorname{arctg} \mathrm{e}^t$ is a solution of the integral equation

$$X_0(t) = 4\mathrm{e}^t + \int\limits_{-\infty}^{t} \Phi(X_0(\tau) \sinh(t-\tau)\, \mathrm{d}\tau, \tag{9.3}$$

where the function $\Phi(X)$ is defined by Eq. (8.3), Eq. (8.1) can be written in the form

$$Z = F(X_0 + Z) - F(X_0) + f, \tag{9.4}$$

where

$$\begin{aligned} f(t) = {} & \frac{\sqrt{\varepsilon}}{\sqrt{1+\varepsilon/4}} \sum_{k=1}^{\infty} \Phi(X_0(t-kh)) \sinh kh \\ & - \int\limits_0^{\infty} \Phi(X_0(t-\varkappa)) \sinh \varkappa \, \mathrm{d}\varkappa, \end{aligned} \tag{9.5}$$

and the operator F is defined by (8.2).

We have

$$|f(t)| \leqslant \operatorname{const} \varepsilon\, \mathrm{e}^{\operatorname{Re} t}. \tag{9.6}$$

Indeed,

$$\begin{aligned} |f(t)| \leqslant \operatorname{const} \varepsilon\, \mathrm{e}^{\operatorname{Re} t} + \sum_{k=0}^{\infty} \sqrt{\varepsilon} \int\limits_{kh}^{(k+1)h} \Big| & \Phi(X_0(t-\varkappa)) \sinh \varkappa \\ & - \Phi(X_0(t-(k+1)h)) \sinh (k+1)h \Big| \mathrm{d}\varkappa \\ \leqslant \operatorname{const} \varepsilon\, \mathrm{e}^{\operatorname{Re} t} + \sum_{k=0}^{\infty} \sqrt{\varepsilon} & \max_{\varkappa \in [kh,(k+1)h]} |\Psi''(\varkappa)| h^2. \end{aligned} \tag{9.7}$$

In the right-hand side of (9.7) the first term is due to replacing of the factor $\frac{\sqrt{\varepsilon}}{\sqrt{1+\sqrt{\varepsilon}}}$ by $\sqrt{\varepsilon}$ and the function $\Psi(\varkappa)$ is defined by

$$\Psi(\varkappa) = \Phi(X_0(t-\varkappa)) \sinh \varkappa.$$

It is obvious that

$$\max_{\varkappa \in [kh,(k+1)h]} |\Psi''(\varkappa)| \leqslant \mathrm{const}\, \mathrm{e}^{\mathrm{Re}\, t} \mathrm{e}^{-kh}. \tag{9.8}$$

Estimate (9.6) follows from (9.7) and (9.8).

Let us solve Eq. (9.4) by iterations: $Z_0 = 0$, $Z_{n+1} = F(X_0 + Z_n) - F(X_0) + f$. An estimate similar to (8.6) can be used to prove by induction the upper bound

$$|Z_{n+1}(t) - Z_n(t)| \leqslant \varepsilon \frac{(\mathrm{const})^{n+1}}{n!} \mathrm{e}^{n\, \mathrm{Re}\, t} \tag{9.9}$$

in $\widetilde{D}_0$. This implies the statement of the lemma.

It follows from (8.7) and (8.12)

$$X_-(t) = X_-\left(4, t + \log\frac{c}{4}\right) = X_-(4, t + O(\varepsilon)). \tag{9.10}$$

Consequently in D_0

$$|X_-(t) - X_-(4,t)| \leqslant \mathrm{const}\, \varepsilon. \tag{9.11}$$

We used here that in the domain $\mathrm{Re}\, t \leqslant -\frac{3}{4}$ the derivatives of the function $X_-(4,t)$ are bounded by a constant independent of ε. The latter follows from (9.1). The required estimate (4.6) in D_0 follows from (9.1) and (9.11). Moreover we have in the domain $\mathrm{Re}\, t \leqslant -3/4$ the inequality

$$|X_-'(t) - X_0'(t)| \leqslant \mathrm{const}\, \varepsilon. \tag{9.12}$$

Let us prove estimate (4.6) in the domain $D \setminus D_0$. Let

$$\widetilde{Z}(t) = X_-(t) - X_0(t). \tag{9.13}$$

This function satisfies the finite-difference equation

$$\Delta^2 \widetilde{Z} = \varepsilon \sin(X_0 + \widetilde{Z}) - \Delta^2 X_0. \tag{9.14}$$

Let us consider (9.14) in the superstandard time x obtained from t by (2.9) with $a = \mathrm{i}\frac{\pi}{2}$. Subtracting $2x^{-2}\widetilde{Z}$ from the both sides of (9.14), we transform this equation to the form

$$L\widetilde{Z} = \widetilde{F}(\widetilde{Z}), \tag{9.15}$$

where

$$\begin{aligned}
\widetilde{F}(\widetilde{Z}) &= \varepsilon \sin(X_0 + \widetilde{Z}) - \Delta^2 X_0 - \frac{2}{x^2}\widetilde{Z} \\
&= \varepsilon \sin X_0 - \varepsilon X_0'' && \text{(A)} \\
&+ \varepsilon X_0'' - \Delta^2 X_0 && \text{(B)} \\
&+ \left(\varepsilon \cos X_0 - \frac{2}{x^2}\right)\widetilde{Z} && \text{(C)} \\
&+ \varepsilon \int_0^1 \sin(X_0 + \tau\widetilde{Z})(1-\tau)\widetilde{Z}^2 \mathrm{d}\tau. && \text{(D)}
\end{aligned} \tag{9.16}$$

The term (A) vanishes. We invert the operator L in (9.15) and obtain

$$\widetilde{Z} = \widetilde{Z}_{\mathrm{init}} + L^{-1}\widetilde{F}(\widetilde{Z}), \tag{9.17}$$

where the function $\widetilde{F}(\widetilde{Z})$ is continued by zero onto the negative infinite ray, which starts from the point $x = -\varepsilon^{-1/2}$, and (comp. Lemma 6.1)

$$\widetilde{Z}_{\mathrm{init}} = a(x)u_0(x) + b(x)x^2. \tag{9.18}$$

The 1-periodic functions $a(x)$ and $b(x)$ (which are not continuous) are defined by the following equations,

$$\left.\begin{aligned}
a(x) &= 2\begin{vmatrix} \widetilde{Z}(x_{\mathrm{init}}), & x_{\mathrm{init}}^2 \\ \widetilde{Z}(x_{\mathrm{init}}-1), & (x_{\mathrm{init}}-1)^2 \end{vmatrix}, \\
b(x) &= 2\begin{vmatrix} u_0(x_{\mathrm{init}}), & \widetilde{Z}(x_{\mathrm{init}}) \\ u_0(x_{\mathrm{init}}-1), & \widetilde{Z}(x_{\mathrm{init}}-1) \end{vmatrix}.
\end{aligned}\right\} \tag{9.19}$$

Here

$$x_{\mathrm{init}} = x - \left[\operatorname{Re} x + \frac{1}{\sqrt{\varepsilon}}\right] \tag{9.20}$$

is a periodic function of the variable x. The square brackets $[\cdot]$ stand for the integer part of a number. This choice of the free term of (9.17) is determined by the condition that $\widetilde{Z}$ coincides with the correct value (9.13) on the boundary between D_0 and D_1.

Let us fix $\operatorname{Im} x = \lambda$ and consider Eq. (9.17) on the segment $\ell = [-\frac{1}{\sqrt{\varepsilon}} + i\lambda, h + i\lambda] \cap D$. It is clear that $\ell \subset D \setminus D_0$. In the space $\mathfrak{X}_4(\ell)$ (see Sec. 6) the norms of the terms in $\widetilde{F}(\widetilde{Z})$ are bounded in the following way

$$\|(\mathrm{B})\|_4 \leqslant \text{const}\,, \tag{9.21}$$

$$\|(\mathrm{D})\|_4 \leqslant \text{const}\,\sqrt{\varepsilon}\,\|\widetilde{Z}\|_2^2 \tag{9.22}$$

provided $|\widetilde{Z}(x)| \leqslant 1$, $x \in \ell$.

Estimate (9.21) is obtained, if we use the Taylor formula with the derivative of the order 4 to estimate $\Delta^2 X_0$ and apply a suitable upper bound for the third derivative of the function $\frac{2}{\sinh \tau}$ in a neighborhood of $\tau = 0$. We have $|(\mathrm{B})| \leqslant \text{const}\,|x|^{-4}$.

In order to estimate the term (D) we take into account $|\sin X_0| \leqslant \frac{\text{const}}{\varepsilon |x|^2}$ in the considered domain. Study the factor before $\widetilde{Z}$ in (C):

$$\begin{aligned} \varepsilon\Phi(x) &= \varepsilon \cos X_0(x) - \frac{2}{x^2} = \varepsilon\left(1 + \frac{2}{\sinh^2\sqrt{\varepsilon}\,x} - \frac{2}{\varepsilon x^2}\right) \\ &= \varepsilon\left[1 + \frac{2(\varepsilon x^2 - \sinh^2\sqrt{\varepsilon}\,x)}{\varepsilon x^2 \sinh^2\sqrt{\varepsilon}\,x}\right] = \varepsilon O(1). \end{aligned} \tag{9.23}$$

We estimate the free term $\widetilde{Z}_{\text{init}}(x)$ in (9.17) defined by (9.18)–(9.20). Using (9.12), we obtain for $\operatorname{Re} t \leqslant -3/4$

$$|\widetilde{Z}(x) - \widetilde{Z}(x-1)| \leqslant \sqrt{\varepsilon}\,|X_0'(\tilde{t}) - X_-'(\tilde{t})| = O(\varepsilon^{3/2}). \tag{9.24}$$

From (9.24), (9.18), and (9.19) and the asymptotics (6.6) for $u_0(x)$ we obtain

$$\begin{aligned} |a(x)| &\leqslant \text{const}\,\Big(|x_{\text{init}}|^2 |\widetilde{Z}(x_{\text{init}}) - \widetilde{Z}(x_{\text{init}} - 1)| \\ &\qquad + |x_{\text{init}}|\,|\widetilde{Z}(x_{\text{init}})|\Big) \leqslant \text{const}\,\varepsilon^{1/2}, \\ |b(x)| &\leqslant \text{const}\,\Big(\frac{1}{|x_{\text{init}}|}\,|\widetilde{Z}(x_{\text{init}}) - \widetilde{Z}(x_{\text{init}} - 1)| \\ &\qquad + \frac{1}{|x_{\text{init}}|^2}\,|\widetilde{Z}(x_{\text{init}})|\Big) \leqslant \text{const}\,\varepsilon^2, \end{aligned} \tag{9.25}$$

$$|\widetilde{Z}_{\text{init}}(x)| \leqslant \text{const}\,\left(\varepsilon^{1/2}\frac{1}{|x|} + \varepsilon^2 |x|^2\right). \tag{9.26}$$

Consider the segments $\ell_n = [-\frac{1}{\sqrt{\varepsilon}} + \mathrm{i}\lambda, -\frac{1}{\sqrt{\varepsilon}} + nh + \mathrm{i}\lambda]$, $n = 1, 2, 3, \ldots$, which are parts of the segment ℓ. Estimating the norms of $\widetilde{Z}_{\rm init}(x)$ in the space $\mathfrak{X}_2(\ell_n)$, denoting these norms by $\|\cdot\|_{2,n}$, we have

$$\|\widetilde{Z}_{\rm init}\|_{2,n} \leqslant \text{const}\,, \tag{9.27}$$

where the constant is independent of ε and n. We invert the operator $I - \varepsilon L^{-1}\Phi$ in $\mathfrak{X}_2(\ell)$, eliminating in this way the term (C), we obtain

$$\widetilde{Z} = (I - \varepsilon L^{-1}\Phi)^{-1}\widetilde{Z}_{\rm init} + (I - \varepsilon L^{-1}\Phi)^{-1}((\mathrm{B}) + (\mathrm{D})). \tag{9.28}$$

Estimate (9.23) and Lemma 6.5 imply $(I - \varepsilon L^{-1}\Phi)^{-1}$ is a continuous operator from the space $\mathfrak{X}_2(\ell_{n-1})$ to $\mathfrak{X}_2(\ell_n)$, and its norm is bounded by a constant independent of ε and n.

From (9.29), (9.21), and (9.22) it follows

$$\|\widetilde{Z}\|_{2,n} \leqslant \text{const} + \text{const}\,\sqrt{\varepsilon}\,\|Z\|^2_{2,n-1}, \tag{9.29}$$

where the constants do not depend on ε and n.

Lemma 9.2. *Let y_n, $n = 1, 2, 3, \ldots$, denote a sequence of positive numbers such that*

$$y_{n+1} \leqslant a + by_n^2,$$

where $a > 0$, $b > 0$, and $ab < 1/4$, moreover

$$y_1 < y^* = \frac{1 - \sqrt{1 - 4ab}}{2b} = a + O(a^2 b).$$

Then $y_n < y^*$ for all n.

We apply this lemma to the sequence $\|Z\|_{2,n}$. We obtain the upper bound

$$\|\widetilde{Z}\|_2 \leqslant \text{const}\,. \tag{9.30}$$

In $D \setminus D_0$, it is equivalent to estimate (4.6) of Theorem 3.

10. Proof of Theorem 4

We let

$$Z(x) = \widetilde{u}_-(x) - u_-(x) \tag{10.1}$$

and use (2.7), (3.1), and (4.9) to rewrite the equation on Z in the form

$$LZ = -\mathrm{e}^{-2u_- - 2Z} + \mathrm{e}^{-2u_-} - \frac{2}{x^2}\,Z + \frac{\varepsilon h^2}{16}\,\mathrm{e}^{2u_- + 2Z}. \tag{10.2}$$

We consider Eq. (10.2) in the space $\mathfrak{X}_{1+\delta'}(\ell)$, where $\delta' = \frac{2}{3}\delta$, $\ell = [\mathrm{i}\lambda + \varepsilon^{-1/4}, \mathrm{i}\lambda + \frac{\sigma}{2\pi}\log\frac{1}{\varepsilon}]$ is a segment in D_2, We estimate the norms of the terms in the right-hand side in the space $\mathfrak{X}_{3+\delta'}(\ell)$ assuming that $\|Z\|_{1+\delta'} \leqslant 1$. We have

$$\left\| -\mathrm{e}^{-2u_- - 2Z} + \mathrm{e}^{-2u_-} - \frac{2}{x^2} Z \right\|_{3+\delta'}$$

$$\leqslant \text{const}\left(\left\| \frac{1}{x^2}|Z(x)|^2 \right\|_{3+\delta'} + \left\| \frac{1}{x^4} Z(x) \right\|_{3+\delta'} \right), \tag{10.3}$$

$$\left\| \frac{1}{x^2} Z^2 \right\|_{3+\delta'} = \max_{x\in\ell} |x|^{1+\delta'} |Z(x)|^2$$

$$\leqslant \text{const} \max_{x\in\ell} |x|^{-1-\delta'} \|Z\|^2_{1+\delta'} \leqslant \frac{\text{const}}{(\log\frac{1}{\varepsilon})^{1+\delta'}} \|Z\|^2_{1+\delta'}, \tag{10.4}$$

$$\left\| \frac{1}{x^4} Z \right\|_{3+\delta'} = \max_{x\in\ell} |x|^{-1+\delta'} |Z(x)| \leqslant \frac{\text{const}}{\log^2\frac{1}{\varepsilon}} \|Z\|_{1+\delta'}, \tag{10.5}$$

$$\left\| \frac{\varepsilon h^2}{16} \mathrm{e}^{2u_- + 2Z} \right\|_{3+\delta'} \leqslant \text{const}\, \|x^2\varepsilon^2\|_{3+\delta'}$$

$$\leqslant \text{const} \max_{x\in\ell} |x|^{5+\delta'} \varepsilon^2 \leqslant \text{const}\, \varepsilon^{2-(5+\delta')/4}. \tag{10.6}$$

Inverting the operator L in (10.2), we obtain an equation on Z, where $Z(x)$ is expressed in terms of $Z(x-k)$ with $k = 1, 2, 3, \ldots$

$$Z = Z_{\text{init}} + L^{-1}(\text{estimated terms}), \tag{10.7}$$

where Z_{init} is defined by an expression similar to (9.18). Instead of (9.24) we obtain

$$|Z(x_{\text{init}}) - Z(x_{\text{init}} - 1)| \leqslant |u'_-(\widetilde{x}) - \widetilde{u}'_-(\widetilde{x})| \leqslant \text{const}\sqrt{\varepsilon}\,\frac{1}{|x_{\text{init}}|}, \tag{10.8}$$

where we used the estimate (4.11), which can be proved in a slightly larger domain with diameter of order of $|x_{\text{init}}|$ on the plane of the variable x.

Instead of (9.25) we have

$$|a(x)| \leqslant \text{const}\Big(|x_{\text{init}}|^2 |Z(x_{\text{init}}) - Z(x_{\text{init}} - 1)|$$

$$+ |x_{\text{init}}|\,|Z(x_{\text{init}})| \Big) \leqslant \text{const}\,\varepsilon^{1/4},$$

$$|b(x)| \leqslant \operatorname{const} \left(\frac{1}{|x_{\text{init}}|} |Z(x_{\text{init}}) - Z(x_{\text{init}} - 1)| + \frac{1}{|x_{\text{init}}|^2} |Z(x_{\text{init}})| \right) \leqslant \operatorname{const} \varepsilon. \tag{10.9}$$

Finally, we obtain bounds for Z_{init}

$$|Z_{\text{init}}| \leqslant \operatorname{const} \left(\varepsilon^{1/4} \frac{1}{|x|} + \varepsilon |x|^2 \right), \tag{10.10}$$

$$\begin{aligned} \|Z_{\text{init}}\|_{1+\delta'} &\leqslant \max_{x \in \ell} \operatorname{const} \left(\varepsilon^{1/4} |x|^{\delta'} + \varepsilon |x|^{3+\delta'} \right) \\ &\leqslant \operatorname{const} \left(\varepsilon^{\frac{1}{4} - \frac{1}{4}\delta'} + \varepsilon^{1 - \frac{3}{4} - \frac{3}{4}\delta'} \right) = O(\varepsilon^{\frac{1}{4} - \frac{\delta}{2}}). \end{aligned} \tag{10.11}$$

Similar to Sec. 9, we consider Eq. (10.7) on a sequence of segments

$$\ell_n = [\mathrm{i}\lambda - \varepsilon^{-1/4}, \mathrm{i}\lambda - \varepsilon^{-1/4} + n].$$

Denoting by $\|\cdot\|_{1+\delta',n}$ the norms in the space $\mathfrak{X}_{1+\delta'}(\ell_n)$, we obtain from (10.7), (10.3), (10.4), (10.5), (10.6), (10.11), and Lemma 6.4 that

$$\begin{aligned} &\|Z\|_{1+\delta',n+1} \\ &\leqslant \operatorname{const} \varepsilon^{\frac{1}{4} - \frac{\delta}{2}} + \frac{\operatorname{const}}{\log^2 \frac{1}{\varepsilon}} \|Z\|_{1+\delta',n} + \frac{\operatorname{const}}{(\log \frac{1}{\varepsilon})^{1+\delta'}} \|Z\|^2_{1+\delta',n}. \end{aligned} \tag{10.12}$$

Applying to the sequence $\|Z\|_{1+\delta',n}$ a lemma similar to Lemma 9.2, we obtain the desired estimate for Z.

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