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ON THE NONSTATIONARY STOKES EQUATIONS IN HALF–SPACE WITH CONTINUOUS INITIAL DATA

1. Introduction

In this paper, in half-space $\mathbb{R}^n_+$, $n \geq 3$, we consider the following initial boundary value problem for Stokes system:

$$\begin{gathered} u_t - \Delta u = -\nabla \pi, \quad \nabla \cdot u = 0, \quad \text{on} \quad \mathbb{R}^n_+ \times (0,T), \\ u(x',t) = 0 \quad \text{on} \quad \mathbb{R}^{n-1} \times (0,T), \lim_{|x|\to\infty} u(x,t) = 0, \\ u(x,0) = u^\circ(x). \end{gathered} \tag{1.1}$$

In system (1.1) $u(x',t)$ is the trace of u on $x_n = 0$ and $u^\circ(x)$ is the initial data; we assume that the *kinetic field* tends to zero at infinity.

In [12, 13] and in [16] the initial boundary value problem for Stokes system in $\mathbb{R}^n_+$ is studied. Both the authors give a representation formula of the solutions. However, the approach of the authors is different. Indeed, in [12, 13] Solonnikov gives a Green matrix for the Stokes problem in half-space, considering null initial data and assuming boundary and *body force* different from zero. Afterwards, starting from the fact that the initial data can be extended in $\mathbb{R}^n_+ \times (0,T)$ in such a way that it can be considered as a body force, Solonnikov achieves the results in a complete form. In [13] problem (1.1) is also solved in the class of Hölder functions. Instead, in [16], for the Stokes problem (1.1), with non zero boundary data, Ukai gives a solution formula in terms of Riesz' operators and fundamental solutions of heat and Laplace's equations. Both the authors give a complete L^p-theory with $L^p - L^q$ estimates for the resolving operator for $p \in (1,\infty)$ and $q \in [p,\infty]$. The importance of $L^p - L^q$ estimates is concerned with the developments of an L^p-theory for Stokes problem in smooth domains Ω of $\mathbb{R}^n$ and of an L^p-theory for Navier–Stokes equations [3, 4, 8, 9, 12, 13, 16, 17].

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In this paper, provided that u° satisfies suitable conditions, we study the existence and uniqueness of smooth solutions to system (1.1) corresponding to an initial data $u^\circ(x)$ only continuous. Moreover, we establish some results of continuous dependence on the data and on the asymptotic behavior for large time of the solutions. Our result is the classical maximum modulus theorem for solutions to system (1.1), which in the case of Cauchy problem has a simple resolution thank to the fact that Stokes system becomes *equivalent* to the heat equation. As far as we known, results in this sense are not known with the exception of paper [14], in which a maximum modulus theorem is proved for the initial boundary value problem in a bounded convex domain $\Omega \subset \mathbb{R}^n$ and in $\mathbb{R}^n_+$. However, this theorem limits itself to the pointwise estimate of the solution.

Before stating our result, we introduce some notations. For any $x \in \mathbb{R}^n_+$ we set $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}_+$. If there is no ambiguity, for any function $G(x)$ defined on $\mathbb{R}^n_+$, $G(x') = G(x', 0)$ means the trace of $G(x)$ on $x_n = 0$. We introduce the set of functions

$$\mathcal{M}(\mathbb{R}^n_+) = \left\{ h(x) \in C(\overline{\mathbb{R}}^n_+),\ h(x') = 0,\ \lim_{|x|\to\infty} h(x) = 0,\ \nabla \cdot h(x) = 0 \right\},$$

where $C(\overline{\mathbb{R}}^n_+)$ is the usual space of functions bounded and uniformly continuous on $\mathbb{R}^n_+$. For any function $G(x,t)$ defined on $\Omega \times (0,T) \subseteq \mathbb{R}^n_+ \times \mathbb{R}$, for any *multi-index* i and integer $j \in \mathbb{N} \cup \{0\}$, by symbol $D^{i,j}_{x,t} G(x,t)$ we mean $\frac{\partial^{|i|+j} G(x,t)}{\partial x_1^{i_1} \cdots \partial x_n^{i_n} \partial t^j}$. We denote $\frac{\partial G(x,t)}{\partial t}$ by the symbol $G_t(x,t)$ also; as well as the symbol $D^2 G(x,t)$ indicates $D^i_x G(x,t)$ for $|i| = 2$. We denote by $C^m(\overline{\mathbb{R}}^n_+)$ the set of all functions $G \in C^m(\mathbb{R}^n_+)$ for which $D^i_x G$ is bounded and uniformly continuous on $\mathbb{R}^n_+$, for any $|i| = 0, \cdots, m$. By $C^{m,\alpha}(\overline{\mathbb{R}}^n_+)$, $\alpha \in (0,1]$, we denote the subsets of $C^m(\overline{\mathbb{R}^n_+})$ whose elements satisfy Hölder condition of exponent α. Finally by symbol $C(a,b;X)$, and $L^r(a,b;X)$, $r \geq 1$, we denote the space of countinuous, and $r-$summable, functions from $(a,b) \subseteq \mathbb{R}$ into Banach space X, respectively.

We are in a position to state our main result

Theorem 1.1. *For any $u^\circ(x)$ belonging to $\mathcal{M}(\mathbb{R}^n_+)$, there exists a unique solution (u, π) of system* (1.1) *such that, for any $T > 0$ and $\eta \in (0,T)$, for $\alpha \in (0,1)$,*

$$u \in C(0,T; C(\overline{\mathbb{R}}^n_+)) \cap C(\eta, T; C^{2,\alpha}(\overline{\mathbb{R}}^n_+)),$$

$$\nabla\pi,\ u_t \in C(\eta, T; C^{0,\alpha}(\overline{\mathbb{R}}^n_+)),$$
$$\lim_{t\to 0^+} u(x,t) = u^\circ(x), \forall x \in \mathbb{R}^n_+. \tag{1.2}$$

Moreover

$$|u(x,t)| \le c \sup_{\mathbb{R}^n_+} |u^\circ(x)|,\ \forall (x,t) \in \mathbb{R}^n_+ \times [0,T);$$
$$|\nabla u(x,t)| \le ct^{-\frac{1}{2}} \sup_{\mathbb{R}^n_+} |u^\circ(x)|,$$
$$\forall (x,t) \in \mathbb{R}^n_+ \times (0,T); \tag{1.3}$$
$$|D^2 u(x,t)| + |u_t(x,t)| + |\nabla\pi(x,t)| \le ct^{-1} \sup_{\mathbb{R}^n_+} |u^\circ(x)|,$$
$$\forall (x,t) \in \mathbb{R}^n_+ \times (0,T);$$
$$|D^2 u(\bar{x},\bar{t}) - D^2 u(\bar{\bar{x}},\bar{\bar{t}})| + |u_t(\bar{x},\bar{t}) - u_t(\bar{\bar{x}},\bar{\bar{t}})|$$
$$\le ct_\circ^{-1-\frac{\alpha}{2}} \sup_{\mathbb{R}^n_+} |u^\circ(x)| (|\bar{x} - \bar{\bar{x}}|^2 + |\bar{t} - \bar{\bar{t}}|)^{\frac{\alpha}{2}},$$
$$\forall\, \bar{x}, \bar{\bar{x}} \in \mathbb{R}^n_+ \quad \textit{and} \quad \bar{t}, \bar{\bar{t}} \in (0,T); \tag{1.4}$$
$$|\pi(\bar{x},t) - \pi(\bar{\bar{x}},t)| \le ct^{-\frac{1}{2}-\frac{\alpha}{2}} \sup_{\mathbb{R}^n_+} |u^\circ(x)| |\bar{x} - \bar{\bar{x}}|^\alpha,$$
$$\forall \bar{x}, \bar{\bar{x}} \in \mathbb{R}^n_+ \quad \textit{and} \quad t \in (0,T);$$
$$|\nabla\pi(\bar{x},\bar{t}) - \nabla\pi(\bar{\bar{x}},\bar{\bar{t}})| t_\circ^{-1-\frac{\alpha}{2}} \sup_{\mathbb{R}^n_+} |u^\circ(x)| (|\bar{x} - \bar{\bar{x}}|^2 + |\bar{t} - \bar{\bar{t}}|)^{\frac{\alpha}{2}},$$
$$\forall\, \bar{x}, \bar{\bar{x}} \in \mathbb{R}^n_+ \quad \textit{and} \quad \bar{t}, \bar{\bar{t}} \in (0,T); \tag{1.5}$$

where c is independent of u°, T and $t_\circ = \min\{\bar{t}, \bar{\bar{t}}\}$. Finally

$$\lim_{t\to\infty} |u(x,t)| = 0. \tag{1.6}$$

We prove Theorem 1.1 by making use of Green function given in [13]. Therefore we express the initial data as *body force*. We construct a solution as follows. We study an initial boundary value problem for Stokes system with slip boundary condition in half-space (cf. system (2.1)). This solution does not satisfy our boundary condition. Then we give a correction of the boundary data studying for any t a steady Stokes system (cf. system (3.1)). Thus we construct a smooth one-parameter ($t > 0$) family of solutions, whose time derivate is considered as *body force* for a new Stokes system (cf. system (4.1)) with zero no slip boundary and initial data.

We apply to this system the Green function given in [13]. Therefore, solution u of Theorem 1.1 is given as sum of the solutions of the systems: (2.1), (3.1) and (4.1). Our construction is slightly different with respect to that proposed by Solonnikov. Indeed, in [13] the construction of a solution is given preserving the boundary data and the steady problem is proposed in order to modify the divergence of the first solution, which is a solution to the heat equation in half space. This approach does not seem useful for our analysis, because the solution of steady problem is expressed in terms of a singular integral for which an application of L^p-theory ($p \in (1,\infty)$) appears essential for the estimates. Instead in our construction estimates on solutions of system (3.1) can be given via a maximum modulus theorem which appears congenial to our aims.

The estimates (1.4)–(1.5) given in Theorem 1.1 cannot be seen as particular cases of the estimates in Hölder spaces given in [13]. Indeed in [13] is fundamental that initial data is in a suitable Hölder space. Instead we must look to estimate (1.2)–(1.5) as the analogous of those given for the resolving operator in L^p-theory, as well as to the limit property (1.6) [9, 13, 16].

The plan of the paper is as follows. In Sec. 2 we study the initial boundary value problem for Stokes system with slip boundary condition. In Sec. 3 we consider steady Stokes system with no slip boundary condition. In Sec. 4 we introduce Green matrix for solutions to Stokes system with a *body force* and zero initial and boundary data (no slip). Any solution of the last system can be considered as sum of two functions, say, V^1 and V^2. In Sec. 5 we analyse term V^1 and in Sec. 7 term V^2, in Sec. 8 the pressure field associated to the kinetic field V^1+V^2. Finally, in Sec. 9, on the basis of the results of the previous sections, we give the proof of Theorem 1.1.

2. The Stokes system with slip boundary condition

Let us consider the Stokes system

$$\begin{gathered}
U_t - \Delta U = -\nabla \Pi, \quad \nabla \cdot U = 0, \quad \text{on} \quad \mathbb{R}^n_+ \times (0,T), \\
U_n(x',t) = 0 \quad \text{and} \quad D_{x_n} U_i(x',t) = 0,\ \forall i = 1,\cdots,n-1, \\
\forall (x',t) \in \mathbb{R}^{n-1} \times (0,T), \\
\lim_{|x|\to\infty} U(x,t) = 0, \quad U(x,0) = u^\circ(x) \in \mathcal{M}(\mathbb{R}^n_+).
\end{gathered} \tag{2.1}$$

For System (2.1) we have

Theorem 2.1. *If $u^\circ \in \mathcal{M}$, then there exists a unique solution (U, Π) to System (2.1) such that, for any $T > 0$ and $\eta \in (0, T)$, for $\alpha \in (0, 1)$,*

$$
\begin{gathered}
U \in C(0, T; C(\overline{\mathbb{R}}^n_+)) \cap C(\eta, T; C^{2,\alpha}(\overline{\mathbb{R}}^n_+)),\\
\Pi = \text{const},\ U_t \in C(\eta, T; C(\overline{\mathbb{R}}^n_+)),\\
\lim_{t \to 0^+} U(x,t) = u^\circ(x)
\end{gathered}
\tag{2.2}
$$

Moreover

$$
\begin{gathered}
|U(x,t)| \le \sup_{\mathbb{R}^n_+} |u^\circ(x)| = M,\ \forall (x,t) \in \mathbb{R}^n_+ \times [0, T);\\
|\nabla U(x,t)| \le c t^{-\frac{1}{2}} M,\ \forall (x,t) \in \mathbb{R}^n_+ \times (0, T);\\
|U_t(x,t)| + |D_{x_i x_j} U(x,t)| \le c t^{-1} M,\ \forall (x,t) \in \mathbb{R}^n_+ \times (0, T);
\end{gathered}
\tag{2.3}
$$

for $\alpha \in (0, 1)$,

$$
\begin{gathered}
|U(\bar{x}, \bar{t}) - U(\bar{\bar{x}}, \bar{\bar{t}})| \le c t_\circ^{-\frac{\alpha}{2}} M (|\bar{x} - \bar{\bar{x}}|^2 + |\bar{t} - \bar{\bar{t}}|)^{\frac{\alpha}{2}},\ \forall\, \bar{x},\ \bar{\bar{x}}, \in \mathbb{R}^n_+\\
\text{and}\quad \bar{t}, \bar{\bar{t}} \in (0, T);\\
|D_{x_i x_j} U(\bar{x}, \bar{t}) - D_{x_i x_j} U(\bar{\bar{x}}, \bar{\bar{t}})| + |U_t(\bar{x}, \bar{t}) - U_t \bar{\bar{x}}, \bar{\bar{t}})|\\
\le c M t_\circ^{-1-\frac{\alpha}{2}} (|\bar{x} - \bar{\bar{x}}|^2 + |\bar{t} - \bar{\bar{t}}|)^{\frac{\alpha}{2}}, \forall\, \bar{x}, \bar{\bar{x}}, \in \mathbb{R}^n_+\\
\text{and}\quad \bar{t}, \bar{\bar{t}} \in (0, T);
\end{gathered}
\tag{2.4}
$$

where c is independent of u°, T and $t_\circ = \min\{\bar{t}, \bar{\bar{t}}\}$. Finally

$$
\lim_{t \to \infty} |U(x,t)| = 0. \tag{2.5}
$$

Proof. As it is well known, a smooth solution U with $\Pi = \text{const}$ is given by means of the kernel of heat equation:

$$
\begin{gathered}
U_i(x,t) = \int_{\mathbb{R}^n_+} [H(x-y,t) + H(x-y^*,t)]\, u_i^\circ(y) dy,\ i = 1, \cdots, n-1;\\
U_n(x,t) = \int_{\mathbb{R}^n_+} [H(x-y,t) - H(x-y^*,t)]\, u_n^\circ(y) dy,
\end{gathered}
\tag{2.6}
$$

where $H(z,t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|z|^2}{4t}}$ and $y^* = (y_1, \cdots, y_{n-1}, -y_n), \forall y \in \mathbb{R}^n_+$. Therefore the regularity is a well known result. Estimates (2.3), taking into account representation (2.6), are consequence of Young theorem on convolution product and of the estimates in L^1 of the heat kernel: for any *multi-index* i and $j \in \mathbb{N} \cup \{0\}$, there exists a constant c such that

$$|D^{i,j}_{z,t} H(t)|_1 \le ct^{-(\frac{|i|}{2}+j)}, \forall t > 0. \tag{2.7}$$

The estimate $(2.3)_1$, with constant $c = 1$, will be proved in Lemma 2.2 as well as the limit property (2.5). The estimate $(2.4)_1$ is an immediate consequence of (2.3). By analogous arguments one proves the estimate $(2.4)_2$. We prove the uniqueness in a class of solutions which contains properly one defined by relations (2.2) and (2.3). Indeed, we assume for the pressure field the following behavior: $|\Pi(x,t)| \le ct^{-\delta_\circ}|x|^{1-\delta}$, for some $\delta_\circ \in [0,1)$ and $\delta \in (0,1]$. We prove the result by making use of a classical technique due to Holmgren. We denote by $(\hat{U}, \hat{\Pi})$ the difference of two solutions corresponding to the same data. We consider a solution (φ, p) ($p =$ const .) of System (2.1) corresponding to a initial data $\varphi^\circ \in \mathcal{C}_\circ(\mathbb{R}^n_+)$.[1] Such a solution is smooth and with its derivatives belongs to $L^q(\mathbb{R}^n_+)$, for any $q \in (1,\infty)$.[2] Multiplying (2.1) by $h^R\varphi(t-\tau)$, where $h^R(x) = h(x/R)$, with $h(r) \in [0,1]$ smooth cut-off function such that $h(r) = 1$ for $r \le 1$, $h(r) = 0$ for $r \ge 2$, and integrating by parts in (x,τ), we deduce the relation

$$(\hat{U}(t), \varphi^\circ h^R) = (\hat{U}(s), \varphi(t-s)h^R) + \int_s^t [2(\nabla h^R \cdot \nabla\varphi, \hat{U})$$
$$+(\Delta h^R \hat{U}, \varphi) + (\hat{\Pi}\nabla h^R, \varphi)]d\tau.$$

In virtue of the regularity of $\hat{U}$ and φ we can perform the limit for $s \to 0$, hence

$$(\hat{U}(t), \varphi^\circ h^R) = \int_0^t \Big[2(\nabla h^R \cdot \nabla\varphi, \hat{U}) + (\Delta h^R \hat{U}, \varphi) + (\hat{\Pi}\nabla h^R, \varphi)\Big]\, d\tau.$$

Moreover, taking into account that $R|\nabla h^R(x)| + R^2|\Delta h^R(x)| \le C$, assuming φ and its derivatives belonging to $C(0,T; L^q(\mathbb{R}^n_+))$, with $q > 1$

[1]The symbol $\mathcal{C}_\circ(\mathbb{R}^n_+)$ denotes the set of all functions $\varphi \in C^\infty_0(\mathbb{R}^n_+)$ with $\nabla \cdot \varphi = 0$.

[2]Of course the existence of a such solution (φ, p) is given by formulas (2.6) and the summability properties are consequence of the application of the Young theorem on convolution product.

such that $(\frac{n}{q'} - \delta) < 0, \frac{1}{q'} + \frac{1}{q} = 1$, after applying Hölder inequality to the integral terms on the right, we easily deduce in the limit of $R \to \infty$

$$(\hat{U}(t), \varphi^\circ) = 0,\ \forall \varphi^\circ \in \mathcal{C}_0(\mathbb{R}^n_+),$$

which implies that $\hat{U} = \nabla H$, a.e. on $\mathcal{R}^n_+$, [11]. On the other hand $\nabla \cdot \hat{U} = 0$, $\hat{U}_n(x', t) = 0$ and $\hat{U}(x, t) \to 0$ for $|x| \to \infty$, thus $H =$ const and $\hat{U} = 0$.

Let us introduce the set of functions:

$$\mathcal{M}_{|x|} = \{h \in \mathcal{M}(\mathbb{R}^n_+) \quad \text{with} \quad |h(x)| \le A/(|x|+1)\}.$$

Lemma 2.1. *Let us assume in* (2.1) $u^\circ \in \mathcal{M}_{|x|}$. *Then the solution to system* (2.1) *determined in Theorem 2.1 is such that*

$$\begin{aligned} |U(x,t)| &\le cA \min\{1, t^{-\frac{1}{2}}, |x|^{-1}\}, \forall x \in \mathbb{R}^n_+, t > 0; \\ |\nabla U(x,t)| &\le cAt^{-\frac{1}{2}} \min\{1, t^{-\frac{1}{2}}, |x|^{-1}\}, \forall x \in \mathbb{R}^n_+, t > 0; \\ |U_t(x,t)| &\le cAt^{-1} \min\{1, t^{-\frac{1}{2}}, |x|^{-1}\}, \forall x \in \mathbb{R}^n_+, t > 0; \end{aligned} \tag{2.8}$$

with a constant c independent of u°. *Moreover*

$$|U(x,t)| \le \sup_{\mathbb{R}^n_+} |u^\circ(x)|, \forall x \in \mathbb{R}^n_+, t > 0. \tag{2.9}$$

Proof. We remark that inequalities (2.8) are well known for solutions to the Cauchy problem of heat equation, we refer to [5] for the details of a proof. Since the solution of system (2.1) can be seen as a suitable linear combination of solutions to the heat equation, in virtue of the above remark, we can consider achieved the estimates (2.8). To obtain (2.9), we multiply equation (2.1) by $U(x,t)|U(x,t)|^{p-2}, p > n$. After an integration by parts we obtain

$$\frac{1}{p}\frac{d}{dt}|U(t)|_p^p + \int_{\mathbb{R}^n_+} \nabla U \cdot \nabla U |U|^{p-2} dx +$$
$$(p-2)\int_{\mathbb{R}^n_+} (\nabla U \cdot U)^2 |U|^{p-4} dx = 0,\ \forall t \in (0,T).$$

Thus an integration on $(0, t)$ gives

$$|U(t)|_p \le |U(0)|_p,\ \forall t \in (0,T).$$

Since $U(x,t) \in L^\infty(\mathbb{R}^n_+), \forall t \geq 0$, in the limit for $p \to \infty$ we deduce

$$|U(t)|_\infty \leq |u^\circ|_\infty,\ \forall t \in (0,T);$$

the continuity of $U(x,t)$ completes the proof of (2.9).

Lemma 2.2. *If* (U, const) *is the solution of Theorem* 2.1, *then estimate* $(2.3)_1$ *and limit property* (2.5) *hold.*

Proof. In virtue of Lemma 9.1 (see Sec. 9) we can consider a sequence $\{u^{\circ n}(x)\} \subset \mathcal{M}_{|x|}(\mathbb{R}^n_+)$ converging to u° in $C(\overline{\mathbb{R}}^n_+)$. The sequence $\{(U^n, \mathrm{const})\}$ denotes the set of solutions to system (2.1) corresponding to $u^{\circ n}$, whose existence is guaranteed by Theorem 2.1. Since system (2.1) is linear, from estimates (2.3) it follows that

$$\begin{aligned}
|U^n(x,t) - U^m(x,t)| &\leq \sup_{\mathbb{R}^n_+} |u^{\circ n}(x) - u^{\circ m}(x)|,\\
|\nabla U^n(x,t) - \nabla U^m(x,t)| &\leq ct^{-\frac{1}{2}} \sup_{\mathbb{R}^n_+} |u^{\circ n}(x) - u^{\circ m}(x)|,\\
|U^n_t(x,t) - U^m_t(x,t)| + |D_{x_ix_j}U^n(x,t) &- D_{x_ix_j}U^m(x,t)|\\
&\leq ct^{-1} \sup_{\mathbb{R}^n_+} |u^{\circ n}(x) - u^{\circ m}(x)|
\end{aligned}$$

$\forall (x,t) \in \mathbb{R}^n_+ \times (0,T)$. Therefore the sequence $\{U^n(x,t)\}$ converges uniformly in (x,t) to a limit $\overline{U}(x,t) \in C((0,T); C(\overline{\mathbb{R}}^n_+))$ and whose sequences of the derivatives are convergent uniformly in $(x,t) \in \mathbb{R}^n_+ \times (\eta, T), \forall \eta > 0$, thus $\overline{U}(x,t) \in C(\eta, T; C^2(\overline{\mathbb{R}}^n_+))$ and $\overline{U}_t(x,t) \in C(\eta,T; C(\overline{\mathbb{R}}^n_+))$. The pair $(\overline{U}(x,t), \mathrm{const})$ is a solution to system (2.1) which corresponds to $u^\circ \in \mathcal{M}$ and satisfies properties (2.2), (2.3) and (2.5) of Theorem 2.1. For the uniqueness of the solution, $(\overline{U}, \mathrm{const})$ coincides with (U, const). Finally, for the uniform converges of $\{U_n\}$ again, we have

$$\lim_{t\to\infty} U(x,t) = \lim_{t\to\infty} \lim_n U^n(x,t) = \lim_n \lim_{t\to\infty} U^n(x,t) = 0,$$

which completes the proof of the lemma.

Lemma 2.3. *If* $u^\circ \in C^1(\overline{\mathbb{R}}^n_+) \cap \mathcal{M}$ *with* $|\nabla u^\circ(x)| \leq A'(1+|x|)^{-1}$, *then the solution* (U, const) *to system* (2.1) *of Theorem 2.1 is such that*

$$\begin{aligned}
|\nabla U(x,t)| &\leq cA'(1+|x|)^{-1},\\
t^{\frac{1}{2}}|D_{x_ix_j}U(x,t)| &\leq cA'(1+|x|)^{-1},\\
t^{\frac{1}{2}}|U_t(x,t)| &\leq cA'(1+|x|)^{-1},
\end{aligned} \tag{2.10}$$

uniformly in $(x,t)\in\mathbb{R}^n_+\times(0,T)$, *with* c *independent of* u°.

Proof. From the representation formula of the solution (2.6), taking into account that $u^\circ(x')=0$, after an integration by parts, we deduce for $i=1,\cdots,n-1$

$$\frac{\partial U_i(x,t)}{\partial x_h}=\int\limits_{\mathbb{R}^n_+}[H(x-y,t)+H(x-y^*,t)]\,\frac{\partial u_i^\circ(y)}{\partial y_h}\,dy,\ \ h=1,\cdots,n-1,$$

$$\frac{\partial U_i(x,t)}{\partial x_n}=\int\limits_{\mathbb{R}^n_+}[H(x-y,t)-H(x-y^*,t)]\,\frac{\partial u_i^\circ(y)}{\partial y_n}\,dy; \qquad (2.11)$$

and for $i=n$

$$\frac{\partial U_n}{\partial x_h}(x,t)=\int\limits_{\mathbb{R}^n_+}[H(x-y,t)-H(x-y^*,t)]\,\frac{\partial u_i^\circ}{\partial y_h}(y)dy,\ \ h=1,\cdots,n-1,$$

$$\frac{\partial U_n}{\partial x_n}(x,t)=\int\limits_{\mathbb{R}^n_+}[H(x-y,t)+H(x-y^*,t)]\,\frac{\partial u_i^\circ}{\partial y_n}(y)dy \qquad (2.12)$$

Taking into account the hypothesis on $\nabla u^\circ(x)$ and applying the same arguments employed for estimates (2.8), we deduce estimates $(2.10)_{1,2}$ and, since the pressure is constant, the last of (2.10) from the equation $(2.1)_1$. Thus the lemma is proved.

3. The steady Stokes system with no slip boundary condition

Let us consider the steady Stokes problem:

$$\begin{aligned}&\Delta W-\nabla Q=0, \quad \nabla\cdot W=0, \text{ in } \mathbb{R}^n_+,\\ &W(x')=a(x'), \quad \lim_{|x|\to\infty}W(x)=0.\end{aligned} \qquad (3.1)$$

A solution of (2.1) can be expressed by formulas, [10],

$$W(x)=\int\limits_{\mathbb{R}^{n-1}}K(x-y')\cdot a(y')dy',$$

$$Q(x)=\int\limits_{\mathbb{R}^{n-1}}q(x-y')\cdot a(y')dy', \qquad (3.2)$$

with

$$K(x-y') = \frac{n}{\omega_n}\frac{x_n(x-y')\otimes(x-y')}{|x-y'|^{n+2}}, \quad q(x-y') = -\frac{2}{\omega_n}\nabla_x\Big(\frac{x_n}{|x-y'|^n}\Big),$$

where ω_n is the measure of unit sphere and $(x-y')=(x'-y',x_n)$. The following result holds for system (3.1):

Theorem 3.1. *If $a(x')$ belongs to $C^{2,\lambda}(\mathbb{R}^{n-1})$, with $|D^i_{x'}a(x')|\le c(1+|x'|)^{-1}$, $|i|=0,1,2$, then there exists a unique solution (W,Q) to system (3.1) such that $(W,Q)\in C^{2,\lambda}(\overline{\mathbb{R}}^n_+)\times C^{1,\lambda}(\overline{\mathbb{R}}^n_+)$ with*

$$\begin{gathered}
|W(x)|\le c\max_{\mathbb{R}^{n-1}}|a(x')|, \quad \forall x\in\mathbb{R}^n_+;\\
|W|_{C^{2,\lambda}(\overline{\mathbb{R}}^n_+)} + |Q|_{C^{1,\lambda}(\overline{\mathbb{R}}^n_+)} \le c|a(x')|_{C^{2,\lambda}(\overline{\mathbb{R}}^{n-1})};\\
\alpha\in(0,1),\ |W(\bar{x})-W(\bar{\bar{x}})|\le c[a]_{0,\alpha}|\bar{x}-\bar{\bar{x}}|^\alpha,\ \forall\bar{x},\bar{\bar{x}}\in\mathbb{R}^n_+;\\
|D^2W(\bar{x})-D^2W(\bar{\bar{x}})|+|\nabla Q(\bar{x})-\nabla Q(\bar{\bar{x}})|\\
\le c[D^2a]_{0,\alpha}|\bar{x}-\bar{\bar{x}}|^\alpha, \forall\bar{x},\bar{\bar{x}}\in\mathbb{R}^n_+;
\end{gathered}\tag{3.3}$$

$$\begin{gathered}
|\nabla W(x)|\le c\Big(\frac{1}{\mu}\sup_{\mathbb{R}^{n-1}}|a(y')| + \sup_{\mathbb{R}^{n-1}}|\nabla a(y')| +\\
\mu^\alpha[\nabla a]_{0,\alpha}\Big), \forall\mu>0, \forall x\in\mathbb{R}^n_+;\\
|Q(x)|\le c\Big(\frac{1}{\mu}\sup_{\mathbb{R}^{n-1}}|\nabla a(y')| + \mu^\alpha[\nabla a]_{0,\alpha}\Big);\\
|D^2W(x)|+|\nabla Q(x)|\le c\Big(\frac{1}{\mu}\sup_{\mathbb{R}^{n-1}}|\nabla a(y')| +\\
\sup_{\mathbb{R}^{n-1}}|D^2a(y')| + \mu^\alpha[D^2a]_{0,\alpha}\Big), \forall\mu>0, \forall x\in\mathbb{R}^n_+,
\end{gathered}\tag{3.4}$$

where $[\cdot]_{0,\alpha}$ is Hölder coefficient and c is a constant independent of a. Moreover, for $\beta\in[0,n-1)$,

$$\begin{gathered}
|a(x')|\le A/(|x'|+1)^\beta,\ \forall x'\in\mathbb{R}^{n-1} \text{ implies}\\
|W(x)|\le cA/(|x|+1)^\beta, \forall x\in\mathbb{R}^n_+.
\end{gathered}\tag{3.5}$$

Proof. Existence and uniqueness of a regular solution to system (3.1) is given by making use of solution formula (3.2). A proof of these results jointly with estimates $(3.3)_{1,2}$ can be found in [15]. The proof of estimate $(3.3)_3$ is quite analogous to that proved for solutions to Laplace equations in half space (cf. [15] p.28). The estimates of type $(3.3)_4$ for D^2W is known for solutions to Laplace equation in half-space (cf. [6] Lemma 2.1 p.118). However, following the argument lines of [6], one proves the same result for solutions to (3.1) also[3]. Estimate $(3.3)_4$ for D^2W via equation $(3.1)_1$ implies that of ∇Q. Moreover, estimate (3.5) is trivial from the representation (3.2). We prove (3.4). For any $j = 1, \cdots, n-1$, the estimate for the $D_{x_j}W$ is a consequence of an integration by parts and by application of maximum principle for *Poisson integral*. Afterwards from the equation of the divergence we also deduce the estimate for the term $D_{x_n}W_n$. So we conclude considering for $j = 1, \cdots, n-1$ the derivative $D_{x_n}W_j$:

$$
\begin{aligned}
D_{x_n}W_j(x) = \frac{n}{\omega_n} & \int\limits_{\mathbb{R}^{n-1}} \frac{(x_j-y_j)(x-y')}{|x-y'|^{n+2}} \cdot a(y')dy' \\
+\frac{n}{\omega_n} & \int\limits_{\mathbb{R}^{n-1}} x_n \frac{(x_j-y_j)}{|x-y'|^{n+2}} a_n(y')dy' \\
-(n+2)\frac{n}{\omega_n} & \int\limits_{\mathbb{R}^{n-1}} x_n^2 \frac{(x_j-y_j)(x-y')}{|x-y'|^{n+4}} \cdot a(y')dy'.
\end{aligned}
$$

By making an integration by parts we have

$$
\begin{aligned}
D_{x_n}W_j(x) = & -\frac{1}{\omega_n}(n-2)^{-1} \int\limits_{\mathbb{R}^{n-1}} D_{x_n^2} \frac{1}{|x-y'|^{n-2}} a_j(y')dy' \\
& -\frac{1}{\omega_n} \int\limits_{\mathbb{R}^{n-1}} \frac{(x-y')}{|x-y'|^{n}} \cdot D_{y_j}a(y')dy' \\
& -\frac{1}{\omega_n} \int\limits_{\mathbb{R}^{n-1}} \frac{x_n}{|x-y'|^{n}} D_{y_j}a_n(y')dy' + \\
& \frac{n}{\omega_n} \int\limits_{\mathbb{R}^{n-1}} x_n^2 \frac{(x-y')}{|x-y'|^{n+2}} \cdot D_{y_j}a(y')dy' = \sum_{i=0}^{4} I_i. \qquad (3.6)
\end{aligned}
$$

[3] Just for completeness, we recall that the proof of Lemma 2.1 of [6] is essentially based on the estimate of Hölder seminorm for the integral transformation $\int_{\mathbb{R}^{n-1}} \frac{\partial}{\partial x_i} \frac{1}{|x-y'|^{n-2}} \rho(y')dy'$. In this connection, we also recall the result of Lemma 8.2 due to Solonnikov [13], that we employ in sec. 8 to give an analogous estimate for the pressure. By making simple computation and employing the quoted lemma, it is not difficult to prove the estimate $(3.3)_4$. However we omit the details for the sake of brevity.

Taking into account that $|z|^{-n+2}$ is harmonic, an integration by parts gives

$$I_1 = \frac{1}{\omega_n}(n-2)^{-1}\sum_{k=1}^{n-1}\int\limits_{\mathbb{R}^{n-1}} D^2_{y_k^2}\frac{1}{|x-y'|^{n-2}}a_j(y')dy'$$

$$= \frac{1}{\omega_n}(n-2)^{-1}\sum_{k=1}^{n-1}\int\limits_{|x'-y'|\geq\mu} D^2_{y_k^2}\frac{1}{|x-y'|^{n-2}}a_j(y')dy'$$

$$+\frac{1}{\omega_n}(n-2)^{-1}\sum_{k=1}^{n-1}\int\limits_{|x'-y'|=\mu} D_{y_k}\frac{1}{|x-y'|^{n-2}}\frac{(x_k-y_k)}{\mu}a_j(y')dy'$$

$$-\frac{1}{\omega_n}(n-2)^{-1}\sum_{k=1}^{n-1}\int\limits_{|x'-y'|\leq\mu} D_{y_k}\frac{1}{|x-y'|^{n-2}}(D_{y_k}a_j(y') - D_{x_k}a_j(x'))dy'$$

Hence, taking into account the Hölder condition for the data, a simple computations furnishes

$$|I_1| \leq c(\frac{1}{\mu}\sup_{\mathbb{R}^{n-1}}|a(y')| + \mu^{\alpha}[\nabla a]_{0,\alpha}). \tag{3.7}$$

We estimate I_2. Again an integration by parts gives

$$I_2 = -\frac{1}{\omega_n}\sum_{k=1}^{n-1}\int\limits_{\mathbb{R}^{n-1}}\frac{(x_k-y_k)}{|x-y'|^n}D_{y_j}a_k(y')dy' - \frac{1}{\omega_n}\int\limits_{\mathbb{R}^{n-1}}\frac{x_n}{|x-y'|^n}D_{y_j}a_n(y')dy'$$

$$= \frac{1}{\omega_n}\sum_{k=1}^{n-1}\int\limits_{|x'-y'|\geq\mu} D^2_{y_k y_j}\frac{1}{|x-y'|^{n-2}}a_j(y')dy'$$

$$-\frac{1}{\omega_n}(n-2)^{-1}\sum_{k=1}^{n-1}\int\limits_{|x'-y'|=\mu} D_{y_k}\frac{1}{|x-y'|^{n-2}}\frac{(x_j-y_j)}{\mu}a_j(y')dy'$$

$$-\frac{1}{\omega_n}(n-2)^{-1}\sum_{k=1}^{n-1}\int\limits_{|x'-y'|\leq\mu} D_{y_k}\frac{1}{|x-y'|^{n-2}}(D_{y_j}a_k(y') - D_{x_j}a_k(x'))dy'$$

$$-\frac{1}{\omega_n}\int\limits_{\mathbb{R}^{n-1}}\frac{x_n}{|x-y'|^n}D_{y_j}a_n(y')dy'.$$

Employing the maximum principle for *Poisson integral* of function $D_{y_j}a_k(y')$ (the last term), taking into account the Hölder condition for the data, again a simple computation gives

$$|I_2| \leqslant c(\frac{1}{\mu}\sup_{\mathbb{R}^{n-1}}|a(y')| + \sup_{\mathbb{R}^{n-1}}|\nabla a(y')| + \mu^{\alpha}[\nabla a]_{0,\alpha}). \qquad (3.8)$$

Finally, noting that $x_n^2|\frac{x-y'}{|x-y'|^{n+2}}| \leq x_n|x-y'|^{-n}$, we majorice I_3, I_4 by Poisson integral $|D_{y_j}a|$, then to estimate I_3+I_4 is sufficient to apply the maximum principle for *Poisson integral*:

$$|I_3+I_4| \leq c\sup_{\mathbb{R}^{n-1}}|\nabla a(y)|. \qquad (3.9)$$

Estimates (3.7)–(3.9) and relation (3.6) complete the proof of $(3.4)_1$. By quite analogous arguments one deduces the estimate for $Q(x)$ and D^2W. To estimate term ∇Q we use equation $(3.1)_1$.

4. The Stokes system with no slip boundary condition

Let us consider the nonstationary Stokes system with homogeneous boundary and initial data:

$$\begin{gathered} V_t - \Delta V = -\nabla P + g_t, \quad \nabla\cdot V = 0, \text{ in } \mathbb{R}^n_+\times(0,T), \\ V(x',t)=0,\ \lim_{|x|\to\infty}V(x,t)=0,\ V(x,0)=0. \end{gathered} \qquad (4.1)$$

We assume that the *body force* g_t satisfies the following

ASSUMPTIONS

i) For any $T>0$ and $\eta\in(0,T)$, $g(x,t)\in C(0,T;C(\mathbb{R}^n_+))$, $g_t(x,t)\in C(\eta,T;C(\mathbb{R}^n_+))$, with $\nabla\cdot g_t(x,t) = 0$, $\forall(x,t)\in\mathbb{R}^n_+\times(0,T)$, $g_{n\,t}(x',t)=0$, $\forall\,(x',t)\in\mathbb{R}^{n-1}\times(0,T)$, $\lim_{t\to0^+}g(x,t)=0$ in $\mathbb{R}^n_+$ and $\lim_{|x|\to\infty}g(x,t)=0$;

ii) $|g(x,t)|\leq B$, $|g_t(x,t)|\leq Bt^{-1}$, $\forall(x,t)\in\mathbb{R}^n_+\times(0,T)$; $\alpha\in(0,1)$,

$$|g(\bar{x},t)-g(\bar{\bar{x}},t)| \leq Bt^{-\frac{\alpha}{2}}|\bar{x}-\bar{\bar{x}}|^{\alpha},\ \bar{x},\bar{\bar{x}}\in\mathbb{R}^n_+ \text{ and } \quad \forall t>0;$$

$$|g_t(\bar{x},\bar{t})-g_t(\bar{\bar{x}},\bar{\bar{t}})| \leq Bt_{\circ}^{-1-\frac{\alpha}{2}}(|\bar{x}-\bar{\bar{x}}|^2+|\bar{t}-\bar{\bar{t}}|)^{\frac{\alpha}{2}},$$

$$\forall\bar{x},\bar{\bar{x}}\in\mathbb{R}^n_+ \text{ and } \bar{t},\bar{\bar{t}}>0, \text{with } t_{\circ}=\min\{\bar{t},\bar{\bar{t}}\};$$

iii) $|g(x,t)|\leq A\min\{1,t^{-\frac{1}{2}},|x|^{-1}\}$ and $|g_t(x,t)|\leq A'(1+|x|)^{-1}t^{-\frac{1}{2}}$, $\forall x\in\mathbb{R}^n_+$, $\forall t>0$.

In [13], for system (4.1) Green matrix is given: provided that

$$\nabla \cdot g_t(x,t) = 0 \text{ and } g_{n\,t}(x',t) = 0,$$

then a solution of system (4.1) formally can be expressed by formulas

$$\begin{aligned} V(x,t) = & \int_0^t \int_{\mathbb{R}^n_+} [H(x-y,t-\tau) - H(x-y^*,t-\tau)]\, g_\tau(y,\tau)dyd\tau \\ & + 4A_n \sum_{j=1}^{n-1} \frac{\partial}{\partial x_j} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \nabla_x \mathcal{E}(x-y)\Phi_j(y,t)dy = V^1(x,t) + V^2(x,t), \end{aligned} \tag{4.2}$$

$$\begin{aligned} P(x,t) = 4A_n \sum_{j=1}^{n-1} \frac{\partial}{\partial x_j} \Big[& \int_{\mathbb{R}^{n-1}} \frac{\partial}{\partial x_n} \mathcal{E}(x-y')\Psi_j(y',t)dy' \\ & + \int_{\mathbb{R}^{n-1}} \mathcal{E}(x-y')\widehat{\Psi}_j(y',t)dy' \Big] \end{aligned} \tag{4.3}$$

where

$$\Phi(y,t) = \int_0^t \int_{\mathbb{R}^n_+} H(y-z^*,t-\tau)g_\tau(z,\tau)dzd\tau, \tag{4.4}$$

$$\begin{aligned} \Psi(y',t) &= \int_0^t \int_{\mathbb{R}^n_+} H(y-z,t-\tau)g_\tau(z,\tau)dzd\tau \Bigg|_{y_n=0}, \\ \widehat{\Psi}(y',t) &= \int_0^t \int_{\mathbb{R}^n_+} \frac{\partial}{\partial z_n} H(y-z,t-\tau)g_\tau(z,\tau)dzd\tau \Bigg|_{y_n=0}, \end{aligned} \tag{4.5}$$

and $A_n = [(n-2)\omega_n]^{-1}, \mathcal{E}(z) = |z|^{2-n}$. Our study on formula (4.2) is twofold. On the one hand under assumptions i)-iii) we give sense to formula (4.2) by function $g(z,\tau)$ instead of its derivative $g_\tau(z,\tau)$, and on the other hand we give estimates of V and of its derivatives with respect the uniform norm by making use only of assumptions i)-ii). In other words, assumption iii) appears just as a *tool* for our aims. This study can not be seen as classical results on the proprieties of solutions to heat equation with body force. The difference between our analysis and the classical results is the following. As already said, first of all we try to obtain the results which involve the function $g(x,t)$ and not properly the *body force* which is $g_t(x,t)$. Moreover, any estimate must involve only the norm of $u^\circ(x)$ in $C(\overline{\mathbb{R}^n_+})$. Of course, our estimates on the derivatives or Hölder seminorm estimates can not be uniform in $t > 0$.

5. Estimates on function V^1

The aim of this section is just to prove that function

$$V^1(x,t) = \int_0^t \int_{\mathbb{R}^n_+} [H(x-y,t-\tau) - H(x-y^*,t-\tau)]\, g_\tau(y,\tau) dy d\tau \quad (5.1)$$

and its derivatives with respect to the time and to the space have sense under assumption i)-ii).

We start with the following

Lemma 5.1. *If data $g(x,t)$ satisfies assumptions* i)–iii), *then*

$$\begin{aligned} &V^1(x,0) = 0, \text{ for any } x \in \mathbb{R}^n_+,\\ &V^1(x',t) = 0, \text{ for any } x' \in \mathbb{R}^{n-1} \text{ and for any } t \in [0,T);\\ &|V^1(x,t)| \leq cB, \text{ for any } (x,t) \in \mathbb{R}^n_+ \times (0,T), \forall T > 0, \end{aligned} \quad (5.2)$$

with c independent of $g(x,t)$.

Proof. In virtue of hypothesis iii), applying Young's theorem on convolution product and taking into account that $|H(t)|_1 = 1$, one deduces that $V^1(x,t)$ is well defined for any $(x,t) \in \mathbb{R}^n_+ \times (0,T)$. Moreover, it is easy to verify that for $t \in (0,T)\, V^1(x',t) = 0, \forall x' \in \mathbb{R}^{n-1}$, and $V^1(x,0) = 0, \forall x \in \mathbb{R}^n_+$. Thus we restrict our consideration to $(x,t) \in \mathbb{R}^n_+ \times (0,T)$. Since $g(y,0) = 0$, by making an integration by parts, we get

$$V^1(x,t) = \lim_{\varepsilon\to 0}\int_0^{t-\varepsilon}\int_{\mathbb{R}^n_+}[H(x-y,t-\tau)-H(x-y^*,t-\tau)]\,g_\tau(y,\tau)dyd\tau$$

$$-\lim_{\varepsilon\to 0}\left(\int_{\mathbb{R}^n_+}[H(x-y,\varepsilon)-H(x-y^*,\varepsilon)]\,g(y,t-\varepsilon)dy\right.$$

$$\left.-\int_0^{t-\varepsilon}\int_{\mathbb{R}^n_+}\frac{\partial}{\partial\tau}\,[H(x-y,t-\tau)-H(x-y^*,t-\tau)]\,g(y,\tau)dyd\tau\right)$$

$$=\lim_{\varepsilon\to 0}\left(I_1(x,t,\varepsilon)+I_2(x,t,\varepsilon)\right). \tag{5.3}$$

Since

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n_+} H(x-y,\varepsilon)g(y,t)dy = g(x,t),$$

and since $H(x-y^*,\varepsilon)\le e^{-x_n^2/4\varepsilon}H(x'-y',y_n,\varepsilon)$ implies

$$\lim_{\varepsilon}\int_{\mathbb{R}^n_+} H(x-y^*,\varepsilon)g(y,t)dy = 0,$$

we deduce

$$\lim_{\varepsilon\to 0} I_1(x,t,\varepsilon) = g(x,t). \tag{5.4}$$

Now, let us consider I_2. Recalling that, for $s>0$, $\frac{\partial}{\partial s}H(z,s)=\Delta H(z,s)$, we get

$$I_2(x,t,\varepsilon) = -\int_0^{t-\varepsilon}\int_{\mathbb{R}^{n-1}}\frac{x_n e^{-\frac{x_n^2}{4(t-\tau)}}}{(4\pi(t-\tau)^3)^{\frac{1}{2}}}H'(x'-y',t-\tau)g(x,\tau)dy'd\tau$$

$$+\int_0^{t-\varepsilon}\int_{\mathbb{R}^n_+}\Delta(H(x-y,t-\tau)-H(x-y^*,t-\tau))\,(g(y,\tau)-g(x,\tau))dyd\tau$$

$$=J_1(x,t,\varepsilon)+J_2(x,t,\varepsilon), \tag{5.5}$$

where $H'(z',s) = (4\pi s)^{-\frac{n}{2}+\frac{1}{2}}e^{-\frac{|z'|^2}{4s}}$. As $|H'(s)|_{L^1(\mathbb{R}^{n-1})} = 1$ and $\int_\varepsilon^t x_n \frac{e^{-\frac{x_n^2}{4s}}}{s^{\frac{3}{2}}}ds \le \int_0^t x_n \frac{e^{-\frac{x_n^2}{4s}}}{s^{\frac{3}{2}}}ds \le c$, we deduce

$$|J_1(x,t,\varepsilon)| \le c \sup_{\mathbb{R}^n\times[0,t]} |g(x,\tau)| \le cB,\ \forall (x,t)\in\mathbb{R}^n_+\times(0,T),$$
$$\text{uniformely in} \quad \varepsilon, \tag{5.6}$$

with c independent of g. Recalling that g satisfies an Hölder condition ii) and recalling that for any *multi-index* ℓ and $m\in\mathbb{N}\cup\{0\}$

$$|D^{\ell,m}_{z,s}H(z,s)| \le c(|z|+s^{\frac{1}{2}})^{-n-|\ell|-2m}, \forall (z,s)\in\mathbb{R}^n\times(0,T), \tag{5.7}$$

we can increase J_2 as follows:

$$|J_2(x,t,\varepsilon)| \le cB \int_0^{t-\varepsilon}\int_{\mathbb{R}^n_+} [|x-y|+(t-\tau)^{\frac{1}{2}}]^{-n-2+\alpha}\tau^{-\frac{\alpha}{2}}dyd\tau$$

$$\le cB\int_0^t \frac{1}{(t-\tau)^{1-\frac{\alpha}{2}}}\frac{1}{\tau^{\frac{\alpha}{2}}}d\tau = cB, \forall (x,t)\in\mathbb{R}^n_+\times(0,T), \text{ uniformly in } \varepsilon, \tag{5.8}$$

with c independent of g. Estimates (5.6) and (5.8) ensure that

$$\lim_{\varepsilon\to 0}|I_2(x,t,\varepsilon)| \le cB,\ \forall (x,t)\in\mathbb{R}^n_+\times(0,T).$$

This last estimate together with (5.4) via (5.3) completes the proof of (5.2).

Lemma 5.2. *If data $g(x,t)$ satisfies assumptions* i)–iii), *then for any $T>0$ and $\eta\in(0,T)$,*

$$V^1(x,t)\in C(0,T;C(\overline{\mathbb{R}}^n_+))\cap C(\eta,T;C^2(\overline{\mathbb{R}}^n_+)),\ V^1_t(x,t)\in C(\eta,T;C(\overline{\mathbb{R}}^n_+));$$

moreover, ($\alpha\in(0,1)$)

$$|\nabla V^1(x,t)| \le cBt^{-\frac{1}{2}},\ \textit{uniformely in } x\in\mathbb{R}^n_+,$$
$$|D^2V^1(x,t)|+|V^1_t(x,t)| \le cBt^{-1},\ \textit{uniformely in } x\in\mathbb{R}^n_+,$$
$$|D^2V^1(\bar{x},\bar{t}) - D^2V^1(\bar{\bar{x}},\bar{\bar{t}})| + |V^1_t(\bar{x},\bar{t}) - V^1_t(\bar{\bar{x}},\bar{\bar{t}})| \le cBt_\circ^{-1-\frac{\alpha}{2}}(|\bar{x}-\bar{\bar{x}}|^2+|\bar{t}-\bar{\bar{t}}|)^{\frac{\alpha}{2}},$$
$$\forall \bar{x},\ \bar{\bar{x}}\in\mathbb{R}^n_+,\ \forall \bar{t},\ \bar{\bar{t}}\in(0,T),\ \textit{with } t_\circ=\min\{\bar{t},\bar{\bar{t}}\}, \tag{5.9}$$

where c is independent of g. Finally,

$$\nabla \cdot V^1(x,t) = -2\sum_{j=1}^{n-1} \frac{\partial}{\partial x_j}\Phi_j(x,t) \tag{5.10}$$

and

$$\lim_{t\to 0^+} V^1(x,t) = 0, \quad \lim_{t\to\infty} V^1(x,t) = 0. \tag{5.11}$$

Proof. Recalling that $g(x,0)=0$, after an integration by parts, we get

$$\begin{aligned}
V^1(x,t) &= \int_0^{\frac{t}{2}}\int_{\mathbb{R}^n_+}[H(x-y,t-\tau)-H(x-y^*,t-\tau)]g_\tau(y,\tau)dyd\tau \\
&\quad + \int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}[H(x-y,t-\tau)-H(x-y^*,t-\tau)]g_\tau(y,\tau)dyd\tau \\
&= \int_{\mathbb{R}^n_+}\left[H(x-y,\frac{t}{2}) - H(x-y^*,\frac{t}{2})\right]g(y,\frac{t}{2})dy \\
&\quad - \int_0^{\frac{t}{2}}\int_{\mathbb{R}^n_+}\frac{\partial}{\partial\tau}[H(x-y,t-\tau)-H(x-y^*,t-\tau)]g(y,\tau)dyd\tau \\
&\quad + \int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}[H(x-y,t-\tau)-H(x-y^*,t-\tau)]g_\tau(y,\tau)dyd\tau \\
&= I_1(x,t) + I_2(x,t) + I_3(x,t).
\end{aligned} \tag{5.12}$$

The regularity of $I_1(x,t)$ and of $I_2(x,t)$ is an immediate consequence of the regularity of kernel $H(z,s)$ and of function $g(y,\tau)$, thus we give no proof. Taking into account (2.7) and recalling that $|g(x,t)| \le B$, we deduce

$$\text{for } |i| = 1,2 \quad |D^i_x(I_1(x,t)+I_2(x,t))|$$
$$\le cBt^{-\frac{|i|}{2}} + cB\int_0^{\frac{t}{2}}(t-\tau)^{-(\frac{|i|}{2}+1)}d\tau \le cBt^{-\frac{|i|}{2}}, \forall (x,t) \in \mathbb{R}^n_+ \times (0,T). \tag{5.13}$$

Analogously one deduces the existence of the time derivative of $I_1(x,t)$ and of $I_2(x,t)$ and

$$|D_t(I_1(x,t)+I_2(x,t))| \le cBt^{-1}, \forall (x,t)\in \mathbb{R}^n_+\times(0,T). \tag{5.14}$$

More involved is the proof of the regularity for term $I_3(x,t)$. Indeed, in the case of D^2 or in the case of D_t, the kernel is not integrable in (x,t). However the difficulties are avoided by a classical technique. Let us consider

$$I_3^\varepsilon(x,t)=\int_{\frac{t}{2}}^{t-\varepsilon}\int_{\mathbb{R}^n_+}[H(x-y,t-\tau)-H(x-y^*,t-\tau)]g_\tau(y,\tau)dyd\tau,$$
$$\forall (x,t)\in\mathbb{R}^n_+\times(0,T).$$

It is easy to verify that $I_3^\varepsilon(x,t)$ and $\nabla I_3^\varepsilon(x,t)$ converge uniformly in (x,t) respectively to $I_3(x,t)$ and $\nabla I_3(x,t)$. Moreover, from (2.7) and hypothesis ii) for $g_\tau(y,\tau)$ we obtain

$$\begin{aligned}|\nabla I_3(x,t)| &\le \int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}|\nabla[H(x-y,t-\tau)-H(x-y^*,t-\tau)]||g_\tau(y,\tau)|dyd\tau\\ &\le c\frac{B}{t}\int_{\frac{t}{2}}^{t}|\nabla H(t-\tau)|_1 d\tau \le c\frac{B}{t^{\frac{1}{2}}}, \forall (x,t)\in\mathbb{R}^n_+\times(0,T).\end{aligned} \tag{5.15}$$

Let us consider now the second derivatives of $I_3^\varepsilon(x,t)$. Since for $s>0$

$$\begin{aligned}&\int_{\mathbb{R}^n_+}D_{x_ix_j}[H(x-y),s)-H(x-y^*,s)]dy\\ &=-\int_{\mathbb{R}^n_+}D_{y_ix_j}[H(x-y,s)-H(x-y^*,s)]dy=0, \forall i=1,\cdots,n-1,\\ &\qquad \forall j=1,\cdots,n,\end{aligned} \tag{5.16}$$

we get, $\forall (x,t)\in\mathbb{R}^n_+\times(0,T)$,

$$\begin{aligned}D_{x_ix_j}I_3^\varepsilon(x,t)=\int_{\frac{t}{2}}^{t-\varepsilon}\int_{\mathbb{R}^n_+}&D_{y_ix_j}[H(x-y,t-\tau)-H(x-y^*,t-\tau)](g_\tau(y,\tau)\\ &-g_\tau(x,\tau))dyd\tau.\end{aligned} \tag{5.17}$$

Recalling (5.7) taking into account the Hölder condition of $g_t(y,t)$ given in assumption ii), we deduce from (5.17)

$$|D_{x_ix_j}I_3^\varepsilon(x,t)-D_{x_ix_j}I_3^\delta(x,t)|$$
$$\leq cB\int\limits_{t-\delta}^{t-\varepsilon}\int\limits_{\mathbb{R}^n_+}(|z|^2+(t-\tau))^{-\frac{n}{2}-1}|z|^\alpha\tau^{-1-\frac{\alpha}{2}}dzd\tau$$
$$\leq cB\int\limits_{t-\delta}^{t-\varepsilon}(t-\tau)^{-1+\frac{\alpha}{2}}\tau^{-1-\frac{\alpha}{2}}d\tau\leq\frac{c}{{t_\circ}^{1+\frac{\alpha}{2}}}B(\delta^{\frac{\alpha}{2}}-\varepsilon^{\frac{\alpha}{2}}),\ \forall t>t_0>0,$$

which ensures the existence of

$$\lim_{\varepsilon\to0}\int\limits_{\frac{t}{2}}^{t-\varepsilon}\int\limits_{\mathbb{R}^n_+}D_{y_ix_j}[H(x-y,t-\tau)-H(x-y^*,t-\tau)]g_\tau(y,\tau)dyd\tau,$$
$$\forall(x,t)\in\mathbb{R}^n_+\times(t_\circ,T),\forall t_\circ>0. \tag{5.18}$$

Since for $\varepsilon\to0$, $\nabla I_3^\varepsilon(x,t)$ is uniformly convergent to $\nabla I_3(x,t)$ also, we have proved that $\forall i=1,\cdots,n-1$ and $j=1,\cdots,n$,

$$D_{x_ix_j}I_3(x,t)=\lim_{\varepsilon\to0}\int\limits_{\frac{t}{2}}^{t-\varepsilon}\int\limits_{\mathbb{R}^n_+}D_{y_ix_j}[H(x-y,t-\tau)$$
$$-H(x-y^*,t-\tau)]g_\tau(y,\tau)dyd\tau,$$
$$\forall(x,t)\in\mathbb{R}^n_+\times(t_\circ,T),\forall t_\circ>0. \tag{5.19}$$

Moreover, from relation (5.17) and estimate (5.7), taking into account the Hölder condition of $g_t(x,t)$ in assumption ii), we deduce

$$|D_{x_ix_j}I_3(x,t)|\leq|\lim_{\varepsilon\to0}\int\limits_{\frac{t}{2}}^{t-\varepsilon}\int\limits_{\mathbb{R}^n_+}D_{y_ix_j}[H(x-y,t-\tau)$$
$$-H(x-y^*,t-\tau)](g_\tau(y,\tau)-g_\tau(x,\tau))dyd\tau|$$
$$\leq\lim_{\varepsilon\to0}\int\limits_{\frac{t}{2}}^{t-\varepsilon}(t-\tau)^{-1+\frac{\alpha}{2}}\tau^{-1-\frac{\alpha}{2}}d\tau\leq c\frac{B}{t},\forall(x,t)\in\mathbb{R}^n_+\times(t_\circ,T),\forall t_\circ>0,$$

this last estimate and estimate (5.13) imply for any $i = 1, \cdots, n-1$ and $j = 1, \cdots, n$

$$|D_{x_i x_j} V^1(x,t)| \le c\frac{B}{t}, \forall (x,t) \in \mathbb{R}^n_+ \times (t_0, T), t_0 > 0. \tag{5.20}$$

Now, let us consider the case of $i = j = n$. In this case we have for $s > 0$

$$\int_{\mathbb{R}^n_+} D_{x_n^2}[H(x-y,s) - H(x-y^*,s)]dy = \int_{\mathbb{R}^n_+} D_{y_n^2}[H(x-y,s)$$

$$-H(x-y^*,s)]dy = -\int_{\mathbb{R}^{n-1}} D_{y_n}[H(x-y,s) - H(x-y^*,s)]|_{y_n=0}dy'$$

$$= -\frac{1}{\sqrt{4\pi}}\frac{x_n}{s^{\frac{3}{2}}}e^{-\frac{x_n^2}{4s}} \int_{\mathbb{R}^{n-1}} H'(x'-y',s)dy'.$$

Therefore

$$D_{x_n^2} I_3^\varepsilon(x,t) = \int_{\frac{t}{2}}^{t-\varepsilon}\int_{\mathbb{R}^n_+} D_{y_n^2}[H(x-y,t-\tau) - H(x-y^*,t-\tau)]$$

$$(g_\tau(y,\tau) - g_\tau(x,\tau))dyd\tau$$

$$-\frac{1}{\sqrt{4\pi}}\int_{\frac{t}{2}}^{t-\varepsilon}\frac{x_n}{(t-\tau)^{\frac{3}{2}}}e^{-\frac{x_n^2}{4(t-\tau)}}\int_{\mathbb{R}^{n-1}} H'(x'-y',s)g_\tau(x,\tau)dy'd\tau$$

$$= J_1^\varepsilon(x,t) + J_2^\varepsilon(x,t) \tag{5.21}$$

By making use of the same arguments already employed for $D_{x_i x_j} I_3^\varepsilon(x,t)$, we obtain

$$|J_1^\varepsilon(x,t) - J_1^\delta(x,t)| \le cB\int_{t-\delta}^{t-\varepsilon}\int_{\mathbb{R}^n_+}(|z|^2 + (t-\tau))^{-\frac{n}{2}-1}|z|^\alpha \tau^{-1-\frac{\alpha}{2}}dzd\tau$$

$$\le cB\int_{t-\delta}^{t-\varepsilon}(t-\tau)^{-1+\frac{\alpha}{2}}\tau^{-1-\frac{\alpha}{2}}d\tau \le \frac{c}{t_\circ^{1+\frac{\alpha}{2}}}B(\delta^{\frac{\alpha}{2}} - \varepsilon^{\frac{\alpha}{2}}),$$

$$\forall (x,t) \in \mathbb{R}^n_+ \times (t_\circ, T), t_\circ > 0,$$

For the term $J_2^\varepsilon(x,t)$ we have

$$|J_2^\varepsilon(x,t) - J_2^\delta(x,t)| \le cB \int\limits_{t-\delta}^{t-\varepsilon} \frac{x_n}{\tau} \frac{e^{-\frac{x_n^2}{4(t-\tau)}}}{(t-\tau)^{\frac{3}{2}}} |H'(t-\tau)|_1 d\tau$$

$$\le cB\frac{1}{t_\circ} \int\limits_{x_n/2\sqrt{\delta}}^{x_n/2\sqrt{\varepsilon}} e^{-h^2} dh,$$

$$\forall (x,t) \in \mathbb{R}_+^n \times (t_\circ, T), t_\circ > 0. \tag{5.22}$$

Therefore estimates (5.21)–(5.22) imply that for $\varepsilon \to 0$ $D_{x_n^2} I_3^\varepsilon(x,t)$ is uniformly convergent in $(x,t) \in \mathfrak{B} \times (t_\circ, T)$, with $\mathfrak{B}$ compact set enclosed in $\mathbb{R}_+^n$. Since $\nabla I_3^\varepsilon(x,t)$ is uniformly convergent in $(x,t) \in \overline{\mathbb{R}}_+^n \times (t_\circ, T)$ to $\nabla I_3(x,t)$, we have that there exists

$$D_{x_n^2} I_3(x,t) = \lim_{\varepsilon \to 0} \int\limits_{\frac{t}{2}}^{t-\varepsilon} \int\limits_{\mathbb{R}_+^n} D_{y_n^2} [H(x-y, t-\tau) - H(x-y^*, t-\tau)] g_\tau(y,\tau) dy d\tau,$$

$$\forall (x,t) \in \mathbb{R}_+^n \times (t_\circ, T), \forall t_\circ > 0. \tag{5.23}$$

Moreover, from relation (5.21), taking into account (5.7) and hypothesis ii), we obtain

$$|D_{x_n^2} I_3(x,t)| = |\lim_{\varepsilon \to 0} D_{x_n^2} I_3^\varepsilon(x,t)|$$

$$\le \lim_{\varepsilon \to 0} [\int\limits_{\frac{t}{2}}^{t-\varepsilon} \int\limits_{\mathbb{R}_+^n} |D_{y_n^2}[H(x-y, t-\tau) - H(x-y^*, t-\tau)]| |(g_\tau(y,\tau)$$

$$-g_\tau(x,\tau))| dy d\tau$$

$$+ \frac{1}{\sqrt{4\pi}} \int\limits_{\frac{t}{2}}^{t-\varepsilon} \frac{x_n}{(t-\tau)^{\frac{3}{2}}} e^{-\frac{x_n^2}{4(t-\tau)}} \int\limits_{\mathcal{R}^{n-1}} |H'(x'-y', s)| |g_\tau(x,\tau)| dy' d\tau]$$

$$\le cB \lim_{\varepsilon \to 0} \left[\int\limits_{\frac{t}{2}}^{t-\varepsilon} (t-\tau)^{-1+\frac{\alpha}{2}} \tau^{-1-\frac{\alpha}{2}} d\tau + c\frac{B}{\sqrt{\pi}} \int\limits_{\frac{t}{2}}^{t-\varepsilon} x_n \frac{e^{-\frac{x_n^2}{4(t-\tau)}}}{\tau(t-\tau)^{\frac{3}{2}}} d\tau \right]$$

$$\le c\frac{B}{t}, \quad \forall (x,t) \in \mathbb{R}_+^n \times (t_\circ, T). \tag{5.24}$$

Therefore we have proved

$$
\begin{aligned}
&D^2V_1(x,t) = D^2I_1(x,t) + D^2I_2(x,t) + \lim_{\varepsilon\to 0} D^2I_3^\varepsilon(x,t),\\
&|D^2V_1(x,t)| \le c\frac{B}{t},\ \forall (x,t)\in \mathbb{R}^n_+\times(t_\circ,T), t_\circ>0. \qquad (5.25)
\end{aligned}
$$

To prove the existence of $V^1_t(x,t)$, we consider $\frac{\partial}{\partial t}I_i(x,t), i=1,2,3$. For $i=1,2$ the regularity of $I_1(x,t)$ and $I_2(x,t)$ has already discussed and we have also deduced estimate (5.14). Let us consider $I_3(x,t)$. We have

$$
\begin{aligned}
D_tI_3^\varepsilon(x,t) = &\int_{\mathbb{R}^n_+}[H(x-y,\varepsilon)-H(x-y^*,\varepsilon)]g_\tau(y,t-\varepsilon)dy\\
&-\frac{1}{2}\int_{\mathbb{R}^n_+}[H(x-y,\frac{t}{2})-H(x-y^*,\frac{t}{2})]g_\tau(y,\frac{t}{2})dy\\
+&\int_{\frac{t}{2}}^{t-\varepsilon}\int_{\mathbb{R}^n_+}D_t[H(x-y,t-\tau)-H(x-y^*,t-\tau)]g_\tau(y,\tau)dyd\tau\\
&= J_1^\varepsilon(x,t)+J_2^\varepsilon(x,t)+J_3^\varepsilon(x,t). \qquad (5.26)
\end{aligned}
$$

In virtue of the definition of the kernels and continuity of function $g_t(x,t)$ it follows that

$$
\lim_{\varepsilon\to 0}J_1^\varepsilon(x,t) = g_t(x,t),\ J_2^\varepsilon(x,t) = J_2(x,t), \forall x\in\mathbb{R}^n_+ \quad \text{and} \quad t>0.
$$

From estimate (2.7) and assumption ii) we deduce

$$
|J_1^\varepsilon(x,t)|+|J_2^\varepsilon(x,t)| \le c\frac{B}{t},\ \forall (x,t)\in\mathbb{R}^n_+\times(t_\circ,T), t_\circ>0.
$$

Finally, for term $J_3^\varepsilon(x,t)$, taking into account that $\frac{\partial}{\partial s}H(z,s) = \Delta H(z,s), \forall z\in\mathbb{R}^n$ and $s>0$, we find that

$$
J_3^\varepsilon(x,t) = \int_{\frac{t}{2}}^{t-\varepsilon}\int_{\mathbb{R}^n_+}\Delta[H(x-y,t-\tau)-H(x-y^*,t-\tau)]g_\tau(y,\tau)dyd\tau.
$$

Thus the convergence of $\frac{\partial}{\partial t}I_3^\varepsilon(x,t)$ uniformly in $(x,t)\in\mathfrak{B}\times(t_\circ,T)$ is a consequence of the convergence proved for the term (second derivatives)

$\Delta I_3^\varepsilon(x,t)$ ($\mathfrak{B}$ is a compact set enclosed in $\mathbb{R}^n_+$). In particular from estimate (5.24) we obtain

$$|\frac{\partial}{\partial t} I_3(x,t)| = |\lim_{\varepsilon\to 0}\frac{\partial}{\partial t} I_3^\varepsilon(x,t)| \le c\frac{B}{t},\ \forall (x,t)\in\mathbb{R}^n_+\times(t_\circ,T), t_\circ>0.$$

Therefore we have proved

$$\begin{aligned} V^1{}_t(x,t) &= I_{1\,t}(x,t) + I_{2\,t}(x,t) + \lim_{\varepsilon\to 0} I^\varepsilon_{3\,t}(x,t),\\ |V^1{}_t(x,t)| &\le c\frac{B}{t},\ \forall (x,t)\in\mathbb{R}^n_+\times(t_\circ,T), t_\circ>0.\end{aligned}$$

Of course formulas and estimates given by (5.25) do not complete the proof of the regularity for V_1. Next, we prove $(5.9)_3$ which implies in particular the uniform continuity of D^2V^1 and $V^1{}_t$. This last property united to the fact that the estimates in (5.25) and (5.27) do not depend on x allow to consider achieved the result completely. To prove $(5.9)_3$ we do not lose the generality assuming that $\bar{t} < \bar{\bar{t}}$. We introduce a smooth cut-off function $k(4s/\bar{t})$ with $k(4s/\bar{t}) = 1$ for $s\le\frac{\bar{t}}{4}$, $k(4s/\bar{t}) = 0$ for $s\ge\frac{\bar{t}}{2}$. Introducing $k(4s/\bar{t})$ and integrating by parts we obtain for any $t>\bar{t}$

$$\begin{aligned}
V^1(x,t) =& \int_{\frac{\bar{t}}{4}}^{t}\int_{\mathbb{R}^n_+}\big[H(x-y,t-\tau)-H(x-y^*,t-\tau)\big](1-k(4\tau/\bar{t}))g_\tau(y,\tau)dyd\tau\\
&+\int_0^{\frac{\bar{t}}{2}}\int_{\mathbb{R}^n_+}\big[H(x-y,t-\tau)-H(x-y^*,t-\tau)\big]k(4\tau/\bar{t})g_\tau(y,\tau)dyd\tau\\
=& \int_{\frac{\bar{t}}{4}}^{t}\int_{\mathbb{R}^n_+}\big[H(x-y,t-\tau)-H(x-y^*,t-\tau)\big](1-k(4\tau/\bar{t}))g_\tau(y,\tau)dyd\tau\\
&-\int_0^{\frac{\bar{t}}{2}}\int_{\mathbb{R}^n_+}\big[H(x-y,t-\tau)-H(x-y^*,t-\tau)\big]k'(4\tau/\bar{t})g(y,\tau)dyd\tau\\
&-\int_0^{\frac{\bar{t}}{2}}\int_{\mathbb{R}^n_+}\big[\frac{\partial}{\partial\tau}[H(x-y,t-\tau)-H(x-y^*,t-\tau)\big]k(4\tau/\bar{t})g(y,\tau)dyd\tau\\
&= Y_1(x,t)+Y_2(x,t)+Y_3(x,t).
\end{aligned} \tag{5.28}$$

Taking into account that

$$|k'(4\tau/\bar{t})g(y,\tau)| \le c\frac{B}{\bar{t}} \quad \text{and} \quad |k(4\tau/\bar{t})g(y,\tau)| \le B,$$

and of the fact that Y_2 and Y_3 are smooth function of (x,t), it is immediate to deduce

$$\left.\begin{array}{l} |D^2(Y_2(\bar{x},\bar{t}) - Y_2(\bar{\bar{x}},\bar{\bar{t}}))| + |D^2(Y_3(\bar{x},\bar{t}) - Y_3(\bar{\bar{x}},\bar{\bar{t}}))| \\ |Y_{2t}(\bar{x},\bar{t}) - Y_{2t}(\bar{\bar{x}},\bar{\bar{t}})| + |Y_{3t}(\bar{x},\bar{t}) - Y_{3t}(\bar{\bar{x}},\bar{\bar{t}})| \end{array}\right\}$$
$$\le c\frac{B}{\bar{t}^{1+\frac{\alpha}{2}}}(|\bar{x}-\bar{\bar{x}}|^2 + |\bar{t}-\bar{\bar{t}}|)^{\frac{\alpha}{2}}. \tag{5.29}$$

Now, let us consider $Y_1(x,t)$. We set $F(y,\tau) = (1-k(\tau))g_\tau(y,\tau)$. We note that $F(y,\frac{\bar{t}}{4}) = 0$ and Hölder condition holds with coefficient $c\frac{B}{\bar{t}^{1+\frac{\alpha}{2}}}$:

$$|F(\bar{x},\bar{t}) - F(\bar{\bar{x}},\bar{\bar{t}})| \le c\frac{B}{\bar{t}^{1+\frac{\alpha}{2}}}(|\bar{x}-\bar{\bar{x}}|^2 + |\bar{t}-\bar{\bar{t}}|)^{\frac{\alpha}{2}}.$$

Since $Y_1(x,t)$ can be seen as a solution of the heat equation in half-space with zero boundary and initial data, and *body force* $F(y,\tau)$, from the result of [7], we deduce

$$|D^2(Y_1(\bar{x},\bar{t})-Y_1(\bar{\bar{x}},\bar{\bar{t}}))|+|Y_{1t}(\bar{x},\bar{t})-Y_{1t}(\bar{\bar{x}},\bar{\bar{t}})| \le c\frac{B}{\bar{t}^{1+\frac{\alpha}{2}}}(|\bar{x}-\bar{\bar{x}}|^2+|\bar{t}-\bar{\bar{t}}|)^{\frac{\alpha}{2}}.$$

This last estimate together (5.29) proves completely $(5.9)_3$. From the definition of V^1 we obtain relation $\nabla\cdot V^1 = -2\sum_{j=1}^{n-1}\frac{\partial}{\partial x_j}\Phi_j(x,t)$ performing an integration by parts. From assumption iii) and (2.7) we deduce

$$|V^1(x,t)| \le cA'\int_0^t |H(t-\tau)|_1 \tau^{-\frac{1}{2}}d\tau \le cA't^{\frac{1}{2}},$$

which implies $(5.11)_1$. Finally to prove $(5.11)_2$ we consider the following decomposition for V^1:

$$\begin{aligned} V^1(x,t) =& \int_0^{t-1}\int_{\mathbb{R}^n_+}[H(x-y,t-\tau) - H(x-y^*,t-\tau)]g_\tau(z,\tau)dyd\tau \\ &+ \int_{t-1}^{t}\int_{\mathbb{R}^n_+}[H(x-y^*,t-\tau) - H(x-y^*,t-\tau)]g_\tau(z,\tau)dyd\tau \\ =& Z_1(x,t) + Z_2(x,t). \end{aligned}$$

Taking into account hypothesis iii) for g_t and (2.7) immediately we deduce $|Z_2(x,t)| \le ct^{-\frac{1}{2}}$. For Z_1 we apply an integration by parts, thus we obtain

$$Z_1(x,t) = \int\limits_{\mathbb{R}^n_+} [H(x-y,1) - H(y-z^*,1)]g(z,t-1)dz$$

$$-\int\limits_0^{t-1}\int\limits_{\mathbb{R}^n_+} \frac{\partial}{\partial\tau}[H(x-z,t-\tau) - H(x-z^*,t-\tau)]g(z,\tau)dzd\tau = Z_{11} + Z_{12}.$$

Taking into account (2.7) and hypothesis iii), we deduce

$$|Z_{11}(x,t)| \le ct^{-\frac{1}{2}}.$$

Taking into account hypothesis iii) for g, applying (2.7) we have

$$|Z_{12}(x,t)| \le \int\limits_0^{t-1} |D_\tau H(t-\tau)|_1 |g(\tau)|_\infty d\tau \le c\int\limits_0^{t-1} (t-\tau)^{-1}\tau^{-\frac{1}{2}}d\tau = \frac{c}{t^{\frac{1}{2}}}\log\frac{t}{2}.$$

The estimates obtained for terms Z_2, Z_{11} and Z_{12} complete the proof of the lemma.

6. Some preliminaries to the analysis of the function V^2

Let us consider function Φ defined in (4.4), which we reproduce here

$$\Phi(y,t) = \int\limits_0^t \int\limits_{\mathbb{R}^n_+} H(y-z^*, t-\tau)g_\tau(z,\tau)dzd\tau. \tag{6.1}$$

Lemma 6.1. *If $g(y,t)$ satisfies hypotheses i)-iii), then for any $T>0$ and $\eta \in (0,T)$*

$$\Phi(x,t) \in C(0,T;C(\overline{\mathbb{R}}^n_+)) \cap C(\eta,T;C^2(\overline{\mathbb{R}}^n_+)),\ \Phi_t(x,t) \in C(\eta,T;C(\overline{\mathbb{R}}^n_+)), \tag{6.2}$$

with

$$|\Phi(x,t)| + t^{\frac{1}{2}}|\nabla\Phi(x,t)| + t(|D^2\Phi(x,t)| + |\Phi_t(x,t)|) \le cB,$$
$$\forall x \in \mathbb{R}^n_+ \quad \text{and} \quad t>0,$$
$$|D^2\Phi(\bar{x},\bar{t}) - D^2\Phi(\bar{\bar{x}},\bar{\bar{t}})| + |\Phi_t(\bar{x},\bar{t}) - \Phi_t(\bar{\bar{x}},\bar{\bar{t}})|$$
$$\le c\frac{B}{t_\circ^{1+\frac{\alpha}{2}}}(|\bar{x}-\bar{\bar{x}}|^2 + |\bar{t}-\bar{\bar{t}}|)^{\frac{\alpha}{2}}, \forall \bar{x},\bar{\bar{x}} \in \overline{\mathbb{R}}^n_+ \quad \text{and} \quad \forall \bar{t},\bar{\bar{t}} > 0, \tag{6.3}$$

where $t_\circ = \min\{\bar{t}, \bar{\bar{t}}\}$, $c = c(\alpha)$, $\alpha \in (0,1)$, *independent of* g. *Moreover, for any* $\delta \in (0,1)$ *and* $j = 1, \cdots, n-1$

$$t|D_{y_h y_j}\Phi(x,t)| + t^{\frac{1}{2}}|\nabla\Phi(x,t)| + |\Phi(x,t)| \leq \\ c\begin{cases} A'(1+|x|)^{-1+\delta} t^{\frac{1}{2}-\frac{\delta}{2}} & \text{for } |x| \geq 1, \\ B \quad \text{for} \quad |x| \leq 1 & \text{and for any} \quad t > 0. \end{cases} \tag{6.4}$$

and

$$\lim_{t\to 0^+} \Phi(x,t) = 0, \quad \lim_{t\to\infty} \Phi(x,t) = 0 \tag{6.5}$$

Proof. We define

$$\hat{g}(y,t) = \begin{cases} g(y,t) & \text{if} \quad y \in \mathbb{R}^n_+, \\ g(y^*,t) & \text{if} \quad y \in \mathbb{R}^n_-. \end{cases}$$

We get

$$\hat{\Phi}(y,t) = \int_0^t \int_{\mathbb{R}^n_+} H(y-z, t-\tau) g_\tau(z,\tau) dz d\tau \\ + \int_0^t \int_{\mathbb{R}^n_+} H(y-z^*, t-\tau) g_\tau(z,\tau) dz d\tau \\ = \int_0^t \int_{\mathbb{R}^n} H(y-z, t-\tau) \hat{g}_\tau(z,\tau) dz d\tau. \tag{6.6}$$

Since $\Phi(x,t) = \frac{1}{2}(\hat{\Phi} - V^1)$, we prove the lemma if we are able to prove that, in the hypotheses i) and ii) for data $g(y,t)$, function $\hat{\Phi}$ has the regularity expressed in (6.2) and verifies estimates (6.3). In this connection, we observe that for the results, with exception of estimate (6.4), it is possible to apply to function $\hat{\Phi}$ expressed by (6.6) the same arguments employed in the case of V^1. Hence we omit the proof. Taking into account hypothesis iii), as far as Φ and $\nabla\Phi$, estimate (6.4) for $|x| \geq 1$ is immediate from the representation (6.1) of Φ, instead a little more complex is the case of second derivatives. Thus we give a short proof of it. We start observing that for $j \neq n$ it is $\int_{\mathbb{R}^n_+} D_{z_j} H(z,s) dz = 0$. Thus we have

$$|D_{x_h x_j}\Phi(x,t)| = |\int_0^t \int_{\mathbb{R}^n_+} D_{y_h y_j} H(x-y^*, t-\tau)(g_\tau(y,\tau) - g_\tau(x,\tau))dyd\tau|$$

$$\leq \int_0^t \int_{\mathbb{R}^n_+} |D_{y_h y_j} H(x-y^*, t-\tau)||g_\tau(y,\tau) - g_\tau(x,\tau)|^\delta |g_\tau(y,\tau)$$
$$-g_\tau(x,\tau)|^{1-\delta} dyd\tau.$$

Taking into account hypotheses ii) and iii) we increase the last integral in the following way

$$|D_{x_h x_j}\Phi(x,t)| \leq \int_0^t \tau^{-\delta(1+\frac{\alpha}{2})-\frac{1}{2}(1-\delta)} \int_{\mathbb{R}^n_+} (|x-y| + (t-\tau)^{\frac{1}{2}})^{-n-2} |x-y|^{\alpha\delta}$$
$$[(1+|y|)^{\delta-1} + (1+|x|)^{\delta-1}]dyd\tau,$$

for any $\delta \in (0,1)$. Therefore, we can estimate the last integral as follows

$$|D_{x_h x_j}\Phi(x,t)| \leq c|x|^{-1+\delta} \int_0^t \tau^a (t-\tau)^b d\tau$$

with $a = \delta(1+\frac{\alpha}{2}) + \frac{1}{2}(1-\delta)$ and $b = 1 - \delta\frac{\alpha}{2}$, which implies the estimate. In the case of $|x| \leq 1$ the estimate is implied by $(6.3)_1$, hence we have completed the proof of the lemma.

In the sequel, the symbol $\mathbb{R}^n_+, x_n$ denotes the set $\mathbb{R}^{n-1} \times (0, x_n)$. Let us consider the integral transformation

$$T(\varphi)(x) = A_n \int_{\mathbb{R}^n_+, x_n} \nabla_x \mathcal{E}(x-y)\varphi(y)dy, \ x \in \mathbb{R}^n_+.$$

By making use of known results related to integral transformation $T(\varphi)$ we can prove the following

Lemma 6.2. *If $\varphi \in L^p(\mathbb{R}^n_+), p > n$, then $T(\varphi) \in C^{0,\beta}(\overline{\mathbb{R}}^n_+), \forall \beta \in (0, 1 - \frac{n}{p})$, with*

$$|T(\varphi)(x)| \leq c x_n^{1-\frac{n}{p}} |\varphi|_p, |T(\varphi)|_{C^{0,\beta}} \leq c|\varphi|_p.$$

If $\varphi \in C^{0,\beta}(\overline{\mathbb{R}}^n_+) \cap L^p(\mathbb{R}^n_+)$, $p > n$, then $T(\varphi)(x) \in C^{1,\beta}(\overline{\mathbb{R}}^n_+)$, with

$$|\nabla T(\varphi)|_{C^{0,\beta}} \leq c(x_n)[\varphi]_{0,\beta}$$

where $[\cdot]_{0,\beta}$ is the Hölder coefficient. Moreover,

$$D_{x_h} T_i(\varphi)(x) = \frac{1}{2}\frac{\delta_{ih}}{n}\varphi(x) - A_n \int^*_{\mathbb{R}^n_+, x_n} D_{y_h y_i} \mathcal{E}(x-y)\varphi(y)dy,$$

$$\text{for any} \quad i = 1, \cdots, n, \text{and} \quad h = 1, \cdots, n-1; \tag{6.7}$$

$$D_{x_n} T_i(\varphi)(x) = -A_n \int^*_{\mathbb{R}^n_+, x_n} D_{y_n y_i} \mathcal{E}(x-y)\varphi(y)dy$$

$$+A_n \int^*_{\mathbb{R}^{n-1}} D_{y_i} \mathcal{E}(x'-y')\varphi(y', x_n)dy', \text{ for any} \quad i = 1, \cdots, n-1; \tag{6.8}$$

$$D_{x_n} T_n(\varphi)(x) = \frac{1}{2}\frac{1}{n}\varphi(x) + A_n \sum_{j=1}^{n-1} \int^*_{\mathbb{R}^n_+, x_n} D^2_{y_j} \mathcal{E}(x-y)\varphi(y)dy. \tag{6.9}$$

7. Estimates on function V^2

Lemma 7.1. *If assumptions i)-iii) hold for $g(x,t)$, then for $\Phi(x,t)$, defined by formula (6.1), the following estimates hold:*

$$|D^k_x \Phi(x-h,t) - D^k_x \Phi(x,t)| \leq c \frac{B}{t^{\frac{|k|}{2}}} \frac{|h|}{x_n}, \ \forall (x,t) \in \mathbb{R}^n_+ \times (0,T),$$

$$|D_t \Phi(x-h,t) - D_t \Phi(x,t)| \leq c \frac{B}{t} \frac{|h|}{x_n}, \ \forall (x,t) \in \mathbb{R}^n_+ \times (0,T), \tag{7.1}$$

$|k| = 0, 1, 2$, for any h such that $h_n \in [0, \frac{x_n}{2}]$ and with c independent of g.

Proof. We recall that for any $x \in \mathbb{R}^n_+$ and $t > 0$ $H(x-z^*, t-\tau)$ is smooth for any $(z, \tau) \in \mathbb{R}^n_+ \times (0, t)$ with $D^j_t H(x - z^*, 0) = 0, j = 0, 1$, $H_t(x -$

$z^*, t-\tau) = \Delta_x H(x-z^*, t-\tau) = \Delta_z H(x-z^*, t-\tau)$. Thus, by making an integration by parts with respect to τ in formula (6.1) and recalling that $g(z,0)=0$, we deduce

$$\begin{aligned}\Phi(x,t) = &\int_0^{\frac{t}{2}}\int_{\mathbb{R}^n_+} \Delta_z H(x-z^*, t-\tau) g(z,\tau)\,dz d\tau \\ &+ \int_{\mathbb{R}^n_+} H(x-z^*, \frac{t}{2}) g(z, \frac{t}{2})\,dz \\ + \int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+} &H(x-z^*, t-\tau) g_\tau(z,\tau)\,dz d\tau = \sum_{i=1}^{3} \Phi^i(x,t).\end{aligned} \tag{7.2}$$

We set

$$\begin{aligned}|D^k\Phi^1(x-h,t) - D^k\Phi^1(x,t)| = |\int_0^1\int_0^{\frac{t}{2}}\int_{\mathbb{R}^n_+} \nabla_x D^k \Delta_z H(x-sh-z^*, t-\tau) \\ \cdot h g(z,\tau)\,dz d\tau ds|.\end{aligned}$$

Hence, by virtue of hypothesis ii) for g and of estimate (5.7), taking into account that $x_n - sh_n \geq \frac{x_n}{2}, \forall s \in [0,1]$, we easily deduce

$$\begin{aligned}|D^k(\Phi^1(x-h,t) - \Phi^1(x,t))| &\leq cB|h| \int_0^1\int_0^{\frac{t}{2}}\int_0^\infty\int_{\mathbb{R}^{n-1}} [|x'-sh'-z'| + (x_n - sh_n + z_n) \\ &\quad + (t-\tau)^{\frac{1}{2}}]^{-n-3-|k|} dz' dz_n d\tau ds \\ &\leq cB|h| \int_0^1\int_0^{\frac{t}{2}}\int_0^\infty [(x_n - sh_n) + z_n + (t-\tau)^{\frac{1}{2}}]^{-4-|k|} dz_n d\tau ds \\ &\leq cB|h| \int_0^{\frac{t}{2}} (t-\tau)^{-1-\frac{|k|}{2}} \int_0^1 (x_n - sh_n)^{-1} ds \leq c\frac{B}{t^{\frac{|k|}{2}}}\frac{|h|}{x_n}.\end{aligned} \tag{7.3}$$

For term Φ^2 we have by similar arguments

$$
\begin{aligned}
&|D^k(\Phi^2(x-h,t)-\Phi^2(x,t))|\\
&\le cB|h|\int_0^1\int_{\mathbb{R}^n_+}|\nabla^{|k|+1}H(x-sh-z^*,\frac{t}{2})||g(z,\frac{t}{2})|dzds\\
&\le cB|h|\int_0^1\int_{\mathbb{R}^n_+}[|x'-z'-sh'|+(x_n-sh_n+z_n)+(\frac{t}{2})^{\frac12}]^{-1-n-|k|}dzds\\
&\le c\frac{B}{t^{\frac{|k|}{2}}}\frac{|h|}{x_n}.
\end{aligned}
\tag{7.4}
$$

Finally, by virtue of hypothesis ii) for g_t and of estimate (5.7), for the last term we get

$$
\begin{aligned}
&|D^k(\Phi^3(x–h,t)-\Phi^3(x,t))|\\
&\qquad=|\int_0^1\int_{\frac{t}{2}}^t\int_{\mathbb{R}^n_+}\nabla_x D^kH(x-z^*,t-\tau)\cdot hg_\tau(z,\tau)dzd\tau ds|\\
&\qquad\le c\frac{B}{t}|h|\int_0^1\int_{\frac{t}{2}}^t\int_0^\infty\int_{\mathbb{R}^{n-1}}[|x'-sh'-z'|+(x_n-sh_n+z_n)\\
&\qquad+(t\ !-\tau)^{\frac12}]^{-n-1-|k|}dz'dz_nd\tau ds\\
&\qquad\le c\frac{B}{t}|h|\int_0^1\int_{\frac{t}{2}}^t\int_0^\infty[(x_n-sh_n)+z_n+(t-\tau)^{\frac12}]^{-2-|k|}dz_nd\tau ds\\
&\qquad\le c\frac{B}{t}|h|\int_0^1\int_{\frac{t}{2}}^t[(x_n-sh_n)+(t-\tau)^{\frac12}]^{-1-|k|}dsd\tau\\
&\qquad\le c\frac{B}{t^{\frac{|k|}{2}}}\frac{|h|}{x_n}.
\end{aligned}
\tag{7.5}
$$

Estimates (7.3)–(7.5) and formula (7.2) imply $(7.1)_1$. Now, let us consider the time derivative. From (7.2) we have

$$\Phi_t(x,t) = \sum_{i=1}^{3} \Phi_t^i(x,t). \tag{7.6}$$

For terms $\Phi_t^i(x,t)$, $i = 1,2$, the estimates are deduced in quite analogous way to those of the terms Φ^i, $i = 1,2$. Thus we get

$$|\Phi_t^i(x-h,t) - \Phi_t^i(x,t)| \le c\frac{B}{t}\frac{|h|}{x_n}, \ i = 1,2. \tag{7.7}$$

Taking into account the proprieties of $H(x-z^*, t-\tau)$, we easily deduce

$$\Phi_t^3(x,t) = -\int_{\mathbb{R}^n_+} H(x-z^*, \frac{t}{2}) g_t(z, \frac{t}{2}) dz + \int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+} H_t(x-z^*, t-\tau) g_\tau(z,\tau) dz d\tau$$

$$= -\int_{\mathbb{R}^n_+} H(x-z^*, \frac{t}{2}) g_t(z, \frac{t}{2}) dz$$

$$-\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+} \Delta H(x-z^*, t-\tau) g_\tau(z,\tau) dz d\tau = \Phi^{31} + \Phi^{32}. \tag{7.8}$$

Taking into account hypothesis ii) for g_t, repeating the same arguments employed to obtain (7.3), we obtain

$$|\Phi^{31}(x-h,t) - \Phi^{31}(x,t)| \le c\frac{B}{t}\frac{|h|}{x_n}. \tag{7.9}$$

The term Φ^{32} can be estimated by making use of (7.5) with $|k| = 2$:

$$|\Phi^{32}(x-h,t) - \Phi^{32}(x,t)| \le c\frac{B}{t}\frac{|h|}{x_n}. \tag{7.10}$$

Thus from (7.8) and estimate (7.9)–(7.10) we get

$$|\Phi_t^3(x-h,t) - \Phi_t^3(x,t)| \le c\frac{B}{t}\frac{|h|}{x_n},$$

which, together with (7.6) and (7.7), completes the proof of the estimate.

Let us consider the integral transformation

$$T^*(\phi)(x) = \int^{*}_{\mathbb{R}^n_+, x_n} D_{y_j y_i} \mathcal{E}(x-y)\phi(y) dy, \quad \text{with} \quad j \ne n. \tag{7.11}$$

Lemma 7.2. *Assume that* $\phi \in C(\overline{\mathbb{R}}^n_+)$ *with*

$$\sup_{\mathbb{R}^n_+} |\phi(x)| \le D_1$$

$$|\phi(x-h) - \phi(x)| \le D_2 \frac{|h|}{x_n}, \forall x \quad \text{and} \quad h \in \mathbb{R}^n_+ \quad \text{with} \quad h_n \le \frac{x_n}{2}. \tag{7.12}$$

Then there exists a constant c *independent of* ϕ *such that*

$$|T^*(\phi)(x)| \le c(D_1 + D_2), \; \forall x \in \mathbb{R}^n_+. \tag{7.13}$$

Proof. We recall that $\int_{\varepsilon \le |z| \le \frac{x}{2}^n \cap \mathbb{R}^n_+} D_{z_j z_i} \mathcal{E}(z) dz = 0$. Thus from (7.11) we have

$$\begin{aligned}
|T^*(\phi)(x)| =& |\int^*_{\mathbb{R}^n_+, x_n} D_{z_i z_j} \mathcal{E}(z) \phi(x-z) dz| \\
=& \Big| \lim_{\varepsilon \to 0} \Big[\int_{\mathbb{R}^n_+, x_n - B^+(0, \frac{x}{2}^n)} D_{z_i z_j} \mathcal{E}(z) \phi(x-z) dz \\
&+ \int_{\varepsilon \le |z| \le \frac{x}{2}^n \cap \mathbb{R}^n_+} D_{z_i z_j} \mathcal{E}(z)(\phi(x-z) - \phi(x)) dz \Big] \Big| \\
=& |\lim_{\varepsilon \to 0} (I_1(x_n) + I_2(x_n, \varepsilon))|.
\end{aligned}$$

We estimate I_1: taking into account $(7.12)_1$ we deduce

$$\begin{aligned}
|I_1(x_n)| \le& c D_1 \int_{\mathbb{R}^n_+, x_n \cap |z| \ge \frac{x}{2}^n} |z|^{-n} dz \\
\le& c D_1 \Big[\int_{\mathbb{R}^n_+, x_n \cap (2x_n \ge |z| \ge \frac{x}{2}^n)} |z|^{-n} dz + \int_0^{x_n} \int_{|z'| \ge \sqrt{3} x_n} |z|^{-n} dz \Big] \le c D_1.
\end{aligned}$$

We estimate I_2: taking into account $(7.12)_2$ we deduce

$$|I_2(x, \varepsilon)| \le c \frac{D_2}{x_n} \int_{\varepsilon \le |z| \le \frac{x}{2}^n} |z|^{-n+1} dz \le c D_2, \quad \text{uniformly in} \quad \varepsilon > 0;$$

hence $|\lim_{\varepsilon \to 0} I_2(x, \varepsilon)| \le c D_2$, which together (7.14) completes the proof of estimate (7.13).

Let us consider the integral transformation

$$T_1^*(\varphi)(x) = A_n \overset{*}{\int\limits_{\mathbb{R}^{n-1}}} D_{y_i}\mathcal{E}(x'-y')\varphi(y',x_n)dy', \qquad \text{for some}$$
$$i = 1,\cdots,n-1, \tag{7.15}$$

where $\mathcal{E}(z') = |z'|^{-n+2}$.

Lemma 7.3. *Let us suppose that $\varphi(x) \in C(\mathbb{R}^n_+, x_n) \cap L^q(\mathbb{R}^n_+, x_n)$, for some $q > 1$ and $\varphi(x) = \frac{\partial}{\partial x_j}\Phi(x)$ for some $j \in (1,\cdots,n-1)$. Moreover, we assume that, for $\alpha \in (0,1)$,*

$$\sup_{\overline{\mathbb{R}}^n} |\varphi(x)| \le D_1 \quad \textit{and} \quad \sup_{\overline{\mathbb{R}}^n} |\Phi(x)| \le D_2$$
$$|\varphi(x'-h',x_n) - \varphi(x)| \le [\varphi]_{0,\alpha}|h'|^\alpha, \forall h' \in \mathbb{R}^{n-1}, \forall x \in \mathbb{R}^n_+, x_n. \tag{7.16}$$

Then there exists a constant c independent of φ such that for

$$|T_1^*(\varphi)(x)| \le c(\frac{1}{\mu}D_2 + \mu^\alpha[\varphi]_{0,\alpha}), \forall \mu > 0. \tag{7.17}$$

Proof. For any $\mu > 0$, from formula (7.15) and hypothesis $\varphi = D_{x_j}\Phi$ we get

$$T_1^*(\varphi)(x) = -\int\limits_{|z'|\ge\mu} D_{z_i}\mathcal{E}(z')D_{z_j}\Phi(x'-z',x_n)dz'$$
$$-\lim_{\varepsilon\to 0}\int\limits_{\varepsilon\le|z'|\le\mu} D_{z_i}\mathcal{E}(z')\varphi(x'-z',x_n)dz' = J_1 + \lim_{\varepsilon\to 0} J_2.$$

An integration by parts gives

$$J_1 = \int\limits_{|z'|=\mu} z_i z_j |z'|^{-n-1}\Phi(x'-z',x_n)d\sigma_{z'} + \int\limits_{|z'|\ge\mu} D_{z_i z_j}\mathcal{E}(z')\Phi(x'-z',x_n)dz'.$$

By making a simple computation we have

$$J_1 \le \frac{c}{\mu}D_2. \tag{7.18}$$

Taking into account that $\int_{\varepsilon<|z'|<\mu} D_{z_i}\mathcal{E}(z')dz' = 0$ and hypothesis $(7.16)_2$ we obtain

$$
\begin{aligned}
|J(\mu,\varepsilon)| =& \Big|\int\limits_{\varepsilon<|z'|<\mu} D_{z_i}\mathcal{E}(z')(\varphi(x'-z',x_n)-\varphi(x',x_n))dz'\Big| \\
\leq& c[\varphi]_{0,\alpha}\int\limits_{\varepsilon<|z'|<\mu}|z|^{1+\alpha-n}dz' \leq c[\varphi]_{0,\alpha}\mu^{\alpha}.
\end{aligned}
\tag{7.19}
$$

Estimates (7.18)–(7.19) imply estimate (7.17).

Now we are in a position to establish the properties of function V^2, which we reproduce here

$$
V^2(x,t) = 4A_n\sum_{j=1}^{n-1}\frac{\partial}{\partial x_j}\int\limits_{\mathbb{R}^n_+,x_n}\nabla_x|x-y|^{2-n}\Phi_j(y,t)dy,
$$

where Φ is defined by formula (4.4):

$$
\Phi(y,t) = \int\limits_0^t\int\limits_{\mathbb{R}^n_+} H(y-z^*,t-\tau)g_\tau(z,\tau)dzd\tau.
$$

Taking into account Lemma 6.1 and applying Lemma 6.2 we deduce the following expression for V^2 :

$$
\begin{aligned}
V_i^2(x,t) =& \frac{2}{n}\Phi_i(x,t) - 4A_n\sum_{j=1}^{n-1}\overset{*}{\int\limits_{\mathbb{R}^n_+,x_n}}\frac{\partial}{\partial y_j}\frac{\partial}{\partial y_i}|x-y|^{2-n}\Phi_j(y,t)dy, \\
& i=1,\cdots,n-1; \\
V_n^2(x,t) =& -4A_n\sum_{j=1}^{n-1}\overset{*}{\int\limits_{\mathbb{R}^n_+,x_n}}\frac{\partial}{\partial y_j}\frac{\partial}{\partial y_n}|x-y|^{2-n}\Phi_j(y,t)dy;
\end{aligned}
\tag{7.20}
$$

moreover, from the representation we deduce

$$
V^2(x,t) = 4A_n\sum_{j=1}^{n-1}\int\limits_{\mathbb{R}^n_+,x_n}\nabla_x|x-y|^{2-n}\varphi_j(y,t)dy,
\tag{7.21}
$$

with $\varphi_j(y,t) = \frac{\partial}{\partial y_j}\Phi_j(y,t)$, $j=1,\cdots,n-1$.

Lemma 7.4. *If g satisfies assumptions i)-iii), then $V^2(x,t) \in C(0,T; C(\overline{\mathbb{R}}^n_+))$, with*

$$|V^2(x,t)| \le cB,\ \forall (x,t) \in \mathbb{R}^n_+. \tag{7.22}$$

Moreover,

$$\lim_{t\to 0^+} V^2(x,t) = 0,\ \lim_{|x|\to\infty} V^2(x,t) = 0,\ V^2(x',t) = 0, \forall t \in (0,T). \tag{7.23}$$

Finally,

$$\nabla \cdot V^2(x,t) = 2\sum_{j=1}^{n-1} \frac{\partial}{\partial x_j} \Phi_j(x,t). \tag{7.24}$$

Proof. In virtue of Lemma 6.1 and Lemma 6.2 applied to (7.21), for any $t \in (\varepsilon, T)$, we deduce $V^2(x,t) \in C^{0,\beta}(\mathbb{R}^n_+)$ with

$$\begin{gathered} |V^2(x,t)| \le c\, x_n^{1-\frac{n}{p}} |\varphi(t)|_p, \\ |V^2(\bar{x},t) - V^2(\bar{\bar{x}},t)| \le c|\varphi(t)|_p |\bar{x} - \bar{\bar{x}}|^\beta, \forall \beta \in (0, 1 - \frac{n}{p}). \end{gathered} \tag{7.25}$$

Estimate $(7.25)_1$ $(7.23)_3$. Applying Lemma 7.2 to (7.20), we prove estimate (7.22). In order to apply Lemma 7.2 let us verify (7.12) for Φ_j, which satisfies (7.12) in virtue of estimate (6.3) and Lemma 7.1 for $k = 0$. Hence estimate $(7.25)_2$ and (7.22) ensure that $V^2(x,t) \in C(\overline{\mathbb{R}}^n_+)$ for any $t \in (\varepsilon, T)$. For $p > n$, from relations (7.20) and estimate (6.4) we obtain

$$|V^2(t)|_p \le c|\Phi(t)|_p \le cA' t^{\frac{1}{2} - \frac{\delta}{2}}, \forall \delta \in (0, 1 - \frac{n}{p}). \tag{7.26}$$

Analogously, from relation (7.21) and estimate (6.4) we obtain

$$|\nabla V^2(x,t)|_p \le c|\nabla \Phi(t)|_p \le cA' t^{-\frac{\delta}{2}}. \tag{7.27}$$

In virtue of the linearity of T, applying Sobolev interpolation inequality, we deduce with $a = \frac{n}{p}$

$$\begin{aligned} \sup_{\mathbb{R}^n_+} |V^2(x,t) - V^2(x,s)| &\le c|\nabla V^2(t) - \nabla V^2(s)|_p^a |V^2(t) - V^2(s)|_p^{1-a} \\ &\le c|\nabla \Phi(t) - \nabla \Phi(s)|_p^a |\Phi(t) - \Phi(s)|_p^{1-a}, \end{aligned}$$

which, in virtue of (6.2), implies $V^2(x,t) \in C(0,T; C(\overline{\mathbb{R}}^n_+))$; moreover from inequalities (7.26) and (7.27), applying Sobolev interpolation inequality again, we obtain with $a = \frac{n}{p}$

$$|V^2(x,t)| \le ct^{\frac{1}{2}(1-a-\delta)}.$$

Choosing $\delta < 1-a$, the last inequality implies $(7.23)_1$. Property $(7.23)_2$ is an immediate consequence of the the above estimates. Finally, the value (7.24) of the divergence of V^2 holds via Lemma 6.2 applied to (7.21). The lemma is completely proved.

Lemma 7.5. *If g satisfies assumptions i)-iii), then for any $T > 0$ and $\eta \in (0,T)$*

$$V^2(x,t) \in C(\eta, T; C^2(\overline{\mathbb{R}}^n_+)), \quad V^2_t(x,t) \in C(\eta, T; C(\overline{\mathbb{R}}^n_+)).$$

Moreover,

$$t|V^2_t(x,t)| \le cB, \ \forall (x,t) \in \mathbb{R}^n_+ \times (0,T); \tag{7.28}$$

$$(t^{\frac{1}{2}}|\nabla V^2(x,t)| + t|D^2V^2(x,t)|) \le cB, \forall (x,t) \in \mathbb{R}^n_+ \times (0,T); \tag{7.29}$$

for $\alpha \in (0,1)$ and $\gamma \in (0,\alpha)$,

$$\begin{aligned} &|D^2V^2(\bar{x},\bar{t}) - D^2V^2(\bar{\bar{x}},\bar{\bar{t}})| + |V^2_t(\bar{x},\bar{x}) - V^2_t(\bar{\bar{x}},\bar{\bar{t}})| \\ &\qquad \le c\frac{B}{t_\circ^\beta}(|\bar{x}-\bar{\bar{x}}|^2 + |\bar{t}-\bar{\bar{t}}|)^{\frac{\alpha}{2}-\frac{\gamma}{2}}, \end{aligned} \tag{7.30}$$

where $t_\circ = \min\{\bar{t},\bar{\bar{t}}\}$ and $\beta = 1 + \frac{\alpha}{2} - \frac{\gamma}{2}$, $\forall \bar{x},\bar{\bar{x}} \in \mathbb{R}^n_+$ and $\forall \bar{t},\bar{\bar{t}} > 0$.

Proof. Function V^2_t is represented by (7.20) with Φ replaced by Φ_t. Hence, applying the same arguments employed for $V^2(x,t)$ we prove that $V^2_t(x,t) \in C^{0,\beta}(\mathbb{R}^n_+)$, for any $t \in (\eta,T)$. Moreover, estimate (7.28) is a consequence of Lemma 6.1 and Lemma 7.1–7.2. Indeed, estimate (6.3) implies for $\Phi_t(x,t)$ assumptions $(7.12)_1$ with $D_1 = c\frac{B}{t}$ and estimate (7.1) implies $(7.12)_2$ with $D_2 = c\frac{B}{t}$. Thus the result follows from Lemma 7.2. In this way we have also proved that, for any $t \in (\eta,T)$, $V^2_t(x,t) \in C(\overline{\mathbb{R}}^n_+)$. Now the regularity with respect to t is deduced proving (7.30). To this end, it is sufficient to apply a classical technique for singular integrals. Indeed, we have

$$\begin{aligned} &|V^2_{it}(\bar{x},\bar{t}) - V^2_{it}(\bar{\bar{x}},\bar{\bar{t}})| \le \frac{2}{n}|\Phi_{it}(\bar{t},\bar{x}) - \Phi_{it}(\bar{\bar{x}},\bar{\bar{t}})| \\ &+4A_n\sum_{j=1}^{n-1}|\int^{*}_{\mathbb{R}^n_{+,x_n}} D_{z_jz_i}|z|^{2-n}[\Phi_{jt}(\bar{x}-z,\bar{t}) - \Phi_{jt}(\bar{\bar{x}}-z,\bar{\bar{t}})]dz| \end{aligned}$$

$$= \frac{2}{n}|\Phi_{it}(\bar{t},\bar{x})-\Phi_{it}(\bar{\bar{x}},\bar{\bar{t}})|+4A_n\sum_{j=1}^{n-1}|\int\limits^{*}_{\mathbb{R}^n_{+},x_n} D_{z_j z_i}|z|^{2-n}[\Phi_{jt}(\bar{x}-z,\bar{t})$$

$$-\Phi_{jt}(\bar{x},\bar{t})+\Phi_{jt}(\bar{\bar{x}},\bar{\bar{t}})-\Phi_{jt}(\bar{\bar{x}}-z,\bar{\bar{t}})]dz|.$$

Taking into account estimates (6.3) and (7.1) we easily deduce that

$$|V^2_{it}(\bar{x},\bar{t})-V^2_{it}(\bar{\bar{x}},\bar{\bar{t}})| \leq c\frac{B}{t_0{}^{1+\frac{\alpha}{2}-\frac{\gamma}{2}}}(|\bar{x}-\bar{\bar{x}}|^2+|\bar{t}-\bar{\bar{t}}|)^{\frac{\alpha}{2}-\frac{\gamma}{2}}$$

$$+c\frac{B}{t_0{}^{1+\frac{\alpha}{2}-\frac{\gamma}{2}}}\frac{1}{x_n{}^{\frac{\gamma}{\alpha}}}\int\limits_0^{\frac{x_n}{2}}\int\limits_{\mathbb{R}^{n-1}}|z|^{\frac{\gamma}{\alpha}}\frac{(|\bar{x}-\bar{\bar{x}}|^2+|\bar{t}-\bar{\bar{t}}|)^{\frac{\alpha}{2}-\frac{\gamma}{2}}}{(|z'|^2+z_n^2)^{\frac{n}{2}}}dz'dz_n,$$

which implies (7.30) for V^2_t. As far as ∇V^2 is concerned, we limit ourself to prove estimate (7.29). Indeed the regularity of ∇V^2 can be deduced by interpolation. Estimate (7.29) is a consequence of Lemma 6.1-6.2 and Lemma 7.2–7.3. In fact, setting $\varphi = \sum\limits_{j=1}^{n-1}\frac{\partial}{\partial y_j}\Phi_j(y,t)$ in formula (7.21), in virtue of (6.7)–(6.9) of Lemma 6.2 ∇V^2 is given by a suitable linear combination of the following terms:

$$\varphi(x), \quad \int\limits^{*}_{\mathbb{R}^n_{+},x_n} D_{y_h y_i}\mathcal{E}(x-y)\varphi(y)dy, \quad \int\limits^{*}_{\mathbb{R}^{n-1}} D_{y_p}\mathcal{E}(x-y)\varphi(y)dy, \tag{7.31}$$

where $h,p \neq n$. Thus we achieve the result if we are able to prove estimate (7.29) for any term of (7.31). Lemma 6.1 implies an estimate of the type (7.29) for φ. The lemma just quoted implies assumption $(7.12)_1$ for φ with $D_1 = c\frac{B}{t^{\frac{1}{2}}}$ and Lemma 7.1 via (7.1) implies assumption $(7.12)_2$ for φ with $D_2 = c\frac{B}{t^{\frac{1}{2}}}$, then Lemma 7.2 implies the estimate for any term expressed by singular integral defined on $\mathbb{R}^n_{+},_{x_n}$ contained in (7.31). Now we estimate the singular integral on $\mathbb{R}^{n-1}$. Recalling the definition of function φ, by virtue of (6.3) of Lemma 6.1, we have that φ and Φ satisfy assumption $(7.16)_1$, uniformly in x_n, with $D_1 = c\frac{B}{t^{1/2}}$ and $D_2 = cB$. Moreover, assumption $(7.16)_2$ for φ is implied by (6.3) and Lagrange theorem with $[\varphi]_{0,\alpha} = c\frac{B}{t^{\frac{1}{2}+\frac{\alpha}{2}}}$. Then we are in a position to apply Lemma 7.3: setting $\mu = t^{\frac{1}{2}}$ we deduce the estimate (7.29) for ∇V^2 from (7.17). We estimate the second derivatives of V^2. From (6.7)–(6.9) of Lemma 6.2

we have

$$D_{x_k x_h} V_i^2 = \frac{1}{2}\frac{\delta_{ih}}{n} D_{x_k}\varphi(x) + 4A_n \overset{*}{\int\limits_{\mathbb{R}^n_{+,x_n}}} D_{y_h y_i}\mathcal{E}(x-y) D_{y_k}\varphi(y)dy,$$
$$i=1,\cdots,n;\, h,k=1,\cdots,n-1; \tag{7.32}$$

$$D_{x_k x_n} V_i^2 = 4A_n \overset{*}{\int\limits_{\mathbb{R}^n_{+,x_n}}} D_{y_h y_i}\mathcal{E}(x-y) D_{y_k}\varphi(y)dy$$
$$+4A_n \overset{*}{\int\limits_{\mathbb{R}^{n-1}}} D_{y_i}\mathcal{E}(x'-y') D_{y_k}\varphi(y',x_n)dy',$$
$$i=1,\cdots,n-1;\, k=1,\cdots,n-1; \tag{7.33}$$

$$D_{x_n x_n} V_i^2 = \frac{1}{2n} D_{x_i}\varphi(x) + 4A_n \sum_{r=1}^{n-1} \overset{*}{\int\limits_{\mathbb{R}^n_{+,x_n}}} D_{y_r^2}\mathcal{E}(x-y) D_{y_i}\varphi(y)dy$$
$$+4A_n \overset{*}{\int\limits_{\mathbb{R}^{n-1}}} D_{y_i}\mathcal{E}(x'-y') D_{x_n}\varphi(y',x_n)dy',\ i=1,\cdots,n-1; \tag{7.34}$$

taking into account relation (7.24) we deduce

$$D_{x_n^2} V_n^2 = -\sum_{i=1}^{n-1} D_{x_n x_i} V_i^2 + D_{x_n}\varphi. \tag{7.35}$$

Now, taking into account Lemma 6.1 and Lemma 7.1-7.2, by analogous considerations already developed for ∇V^2, one deduces estimate (7.29) for D^2V^2. Finally we prove (7.30). From formulas (7.32)–(7.34) we have that the second derivatives of V^2 are a linear combination of the terms

$$D_{x_k}\varphi, \quad \overset{*}{\int\limits_{\mathbb{R}^n_{+,x_n}}} D_{y_h y_i}\mathcal{E}(x-y) D_{y_k}\varphi(y)dy,$$
$$\overset{*}{\int\limits_{\mathbb{R}^{n-1}}} D_{y_p}\mathcal{E}(x'-y') D_{y_k}\varphi(y',x_n)dy', \tag{7.36}$$

where $h, p \neq n$. Hence, taking into account (6.3) and the above arguments employed for V_t^2, it is not difficult to deduce for the first and second term of (7.36) the Hölder property given in (7.30). Thus we restrict our consideration to the last term of (7.30). For any $t_\circ > 0$, recalling that $\varphi = \sum_{j=1}^{n-1} D_{y_j}\Phi_j$, making an integration by parts we have

$$\begin{aligned}
&\overset{*}{\int\limits_{\mathbb{R}^{n-1}}} D_{y_p}\mathcal{E}(x'-y')D_{y_k}\varphi(y',x_n,t)dy' \\
&= \int\limits_{|z'|\geq t_\circ} D_{y_p y_j}\mathcal{E}(z')D_{y_k}\Phi_j(x'-z',x_n,t)dz' \\
&+ \int\limits_{|z'|=t_\circ} \frac{z_j}{t_\circ} D_{y_p}\mathcal{E}(z')D_{y_k}\varphi(x'-z',x_n,t)d\sigma'_z \\
&- \overset{*}{\int\limits_{|z'|\leq t_\circ}} D_{y_p}\mathcal{E}(z')D_{y_k}\varphi(x'-z',x_n,t)dz' = J_1 + J_2 + J_3.
\end{aligned}$$

From estimate (6.3) we deduce

$$\begin{aligned}
|J_1(\bar{x},\bar{t}) - J_1(\bar{\bar{x}},\bar{\bar{t}})| &\leq \int\limits_{|z'|\geq t_\circ} |z'|^{-n}|\nabla\Phi(\bar{x}'-z',\bar{x}_n,\bar{t}) - \nabla\Phi(\bar{\bar{x}}'-z',\bar{\bar{x}}_n,\bar{\bar{t}})|dz' \\
&\leq c\frac{B}{t_\circ^{1+\frac{\alpha}{2}-\frac{\gamma}{2}}}(|\bar{x}-\bar{\bar{x}}|^2 + |\bar{t}-\bar{\bar{t}}|)^{\frac{\alpha}{2}-\frac{\gamma}{2}}.
\end{aligned}$$

By making use of similar arguments, one deduces the estimate

$$|J_2(\bar{x},\bar{t}) - J_2(\bar{\bar{x}},\bar{\bar{t}})| \leq c\frac{B}{t_\circ^{1+\frac{\alpha}{2}-\frac{\gamma}{2}}}(|\bar{x}-\bar{\bar{x}}|^2 + |\bar{t}-\bar{\bar{t}}|)^{\frac{\alpha}{2}-\frac{\gamma}{2}}.$$

Finally, from estimate (6.3) and relation $\int_{\varepsilon\leq|z'|\leq t_0} D_{z_p}\mathcal{E}(z')dz' = 0$ we obtain

$$\begin{aligned}
&|J_3(\bar{x},\bar{t}) - J_3(\bar{\bar{x}},\bar{\bar{t}})| = \\
&|\int\limits_{|z'|\leq t_\circ} D_{z_p}\mathcal{E}(z')[D_{z_k}\varphi(\bar{x}'-z',\bar{x}_n,\bar{t}) - D_{z_k}\varphi(\bar{x},\bar{t}) \\
&+ D_{x_k}\varphi(\bar{\bar{x}},\bar{\bar{t}}) - D_{z_k}\varphi(\bar{\bar{x}}'-z',\bar{\bar{x}}_n,\bar{\bar{t}})]dz'|
\end{aligned}$$

$$\leq c \int\limits_{|z'|\leq t_\circ} |z'|^{1-n}[|\nabla\varphi(\bar{x}'-z',\bar{x}_n,\bar{t})-\nabla\varphi(\bar{x},\bar{t})|^{\bar{\gamma}}$$

$$+|\nabla\varphi(\bar{\bar{x}},\bar{\bar{t}})-\nabla\varphi(\bar{\bar{x}}'-z',\bar{\bar{x}}_n,\bar{\bar{t}})|^{\bar{\gamma}}]$$

$$\cdot[|\nabla\varphi(\bar{x}'-z',\bar{x}_n,\bar{t})-\nabla\varphi(\bar{\bar{x}}'-z',\bar{\bar{x}}_n,\bar{\bar{t}})|^{1-\bar{\gamma}}$$

$$+|\nabla\varphi(\bar{x},\bar{t})-\nabla\varphi(\bar{\bar{x}},\bar{\bar{t}})|^{1-\bar{\gamma}}]dz'$$

$$\leq c\frac{B}{t_\circ^{(1-\bar{\gamma})\frac{\alpha}{2}+\frac{\bar{\gamma}}{2}}}(|\bar{x}-\bar{\bar{x}}|^2+|\bar{t}-\bar{\bar{t}}|)^{(1-\bar{\gamma})\frac{\alpha}{2}} \int\limits_{|z'|\leq t_\circ} |z'|^{1+\bar{\gamma}-n}dz'$$

$$\leq c\frac{B}{t_\circ^{(1-\bar{\gamma})\frac{\alpha}{2}-\frac{\bar{\gamma}}{2}}}(|\bar{x}-\bar{\bar{x}}|^2+|\bar{t}-\bar{\bar{t}}|)^{(1-\bar{\gamma})\frac{\alpha}{2}},$$

which gives the estimate with the correct exponents by choosing $\bar{\gamma}=\frac{\gamma}{1+\alpha}$. The proof of the lemma is completed.

8. Estimates on pressure field P defined by formula (4.3)

We start with the following

Lemma 8.1. *If g satisfies assumption i)-ii), then functions Ψ and $\widehat{\Psi}$ defined by (4.5) belong to $C(0,T;C(\overline{\mathbb{R}}^{n-1})\cap C(\eta,T;C^{1,\alpha}(\overline{\mathbb{R}}^{n-1})$ and $C(\eta,T;C^{0,\alpha}(\overline{\mathbb{R}}^{n-1}))$ respectively, with*

$$|\Psi(x',t)|+t^{\frac{1}{2}}|\widehat{\Psi}(x',t)|\leq cB, \quad \textit{uniformly in} \quad x'\in\mathbb{R}^{n-1},$$

$$|\nabla_{x'}\Psi(\bar{x}',\bar{t})-\nabla_{x'}\Psi(\bar{\bar{x}}',\bar{\bar{t}})|+|\widehat{\Psi}(\bar{x}',\bar{t})-\widehat{\Psi}(\bar{\bar{x}}',\bar{\bar{t}})|$$

$$\leq c\frac{B}{t_\circ^{\frac{1}{2}+\frac{\alpha}{2}}}(|\bar{x}'-\bar{\bar{x}}'|^2+|\bar{t}-\bar{\bar{t}}|)^{\frac{\alpha}{2}}, \tag{8.1}$$

with c independent of g and for any $\bar{x}',\bar{\bar{x}}'\in\mathcal{R}^{n-1}$ and $\bar{t},\bar{\bar{t}}\in(0,T)$, where $t_\circ=\min\{\bar{t},\bar{\bar{t}}\}$. Under assumption iii) we have also, $\delta\in(0,1)$,

$$|\Psi(x',t)|+t^{\frac{1}{2}}|\widehat{\Psi}(x',t)|\leq cB(1+|x'|)^{-1+\delta}t^{-\frac{\delta}{2}}. \tag{8.2}$$

Proof. The proof of the regularity and of estimate (8.1) is similar to those furnished for V^1 in Lemma 5.1 and 5.2. Estimate (8.2) it is immediate from the representation.

Lemma 8.2. *Let us consider the potential*

$$\theta(x)=\int\limits_{\mathbb{R}^{n-1}} D_{x_i}\mathcal{E}(x-y')\rho(y')dy',\ x\in\mathbb{R}^n_+, \tag{8.3}$$

for $\rho(y')$ decreasing at infinity as a power function. Then, for any $\alpha \in (0,1)$,

$$\sup_{\bar{x},\bar{\bar{x}}} |\bar{x}-\bar{\bar{x}}|^{-\alpha}|\theta(\bar{x})-\theta(\bar{\bar{x}})| \leq c \sup_{\bar{x},\bar{\bar{x}}} |\bar{x}-\bar{\bar{x}}|^{-\alpha}|\rho(\bar{x})-\rho(\bar{\bar{x}})|, \tag{8.4}$$

with c independent of ρ.

Proof. See Lemma 7.1 applied to integral transformation (7.7) on p. 503 of [13].

We are in a position to state the result of regularity for the pressure field

Lemma 8.3. *If g satisfies assumptions* i)-iii)*, then for pressure P defined by (4.3) the following estimates hold*

$$\sup_{\bar{x},\bar{\bar{x}}} |\bar{x}-\bar{\bar{x}}|^{-\alpha}|P(\bar{x},t)-P(\bar{\bar{x}},t)| \leq cBt^{-\frac{1}{2}-\frac{\alpha}{2}}, \forall t>0;$$

$$|\nabla P(x,t)| \leq cBt^{-1}, \forall x \in \mathbb{R}^n_+$$

$$|\nabla P(\bar{x},\bar{t})-\nabla P(\bar{\bar{x}},\bar{\bar{t}})| \leq c\frac{B}{t_0^{1+\frac{\alpha}{2}}}(|\bar{x}-\bar{\bar{x}}|^2+|\bar{t}-\bar{\bar{t}}|)^{\frac{\alpha}{2}}, \forall \bar{x},\bar{\bar{x}} \in \overline{\mathbb{R}}^n_+$$
$$\text{and} \quad \bar{t},\bar{\bar{t}}>0, \tag{8.5}$$

with c independent of g, and $t_0 = \min\{\bar{t},\bar{\bar{t}}\}$.

Proof. From formula (4.3) we can consider $P(x,t) = P_1(x,t)+P_2(x,t)$ with

$$P_1(x,t) = 4A_n\sum_{j=1}^{n-1}\frac{\partial}{\partial x_j}\int_{\mathbb{R}^{n-1}}\frac{\partial}{\partial x_n}\mathcal{E}(x-y')\Psi_j(y',t)dy'$$

$$P_2(x,t) = 4A_n\sum_{j=1}^{n-1}\frac{\partial}{\partial x_j}\int_{\mathbb{R}^{n-1}}\mathcal{E}(x-y')\widehat{\Psi}_j(y',t)dy'. \tag{8.6}$$

Unless a multiplicative constant, function P_1 is a linear combination of first derivatives of a *Poisson integral*. Hence, taking into account Lemma 8.1, for function P_1 immediately follows estimate $(8.5)_1$. Moreover, taking into account Lemma 8.1 again and Lemma 8.2 we deduce estimate $(8.5)_1$ for P_2. Thus we have proved the first estimate for P. Finally, to prove the estimates on the gradient of pressure P, we recall that under assumptions i)-iii), as proved in [13], the pair $(V = V^1+V^2, P)$ is a solution to system (4.1). Hence, estimates $(8.5)_{2,3}$ are deduced from the equation and estimates $(5.9)_{2,3}$ for V^1 and (7.29)–(7.30) for V^2.

9. Proof of Theorem 1.1

In order to prove Theorem 1.1 we need of the following

Lemma 9.1. *For any $u^\circ \in \mathcal{M}$ there exists a sequence $\{u^{\circ k}\} \subset \mathcal{M}_{|x|}(\mathbb{R}^n_+) \cap C^1(\overline{\mathbb{R}}^n_+)$ convergent to u° with respect the norm of $C(\overline{\mathbb{R}}^n_+)$, such that*

$$|\nabla u^{\circ k}(x)| \leq A'(1+|x|)^{-1}, \quad \textit{for any} \quad x \in \mathbb{R}^n_+. \tag{9.1}$$

Proof. Let us introduce the smooth cut-off function $h^R(x)$ used in the proof of Theorem 2.1 and define $\widehat{u}^R(x) = u^\circ(x)h^R(x)$. We consider the boundary value problem

$$\nabla \cdot \widetilde{u}^R = -\nabla \cdot \widehat{u}^R, \quad \text{in} \quad \Omega^R, \qquad \text{with } \widetilde{u}^R{}_{|\partial\Omega^R} = 0, \tag{9.2}$$

where $\Omega^R = \{x \in \mathbb{R}^n_+$, such that $R \leq |x| \leq 2R\}$. A field $\widetilde{u}^R$ can be constructed by Bogovskii representation [1, 2]. According to [1], for the particular geometry of Ω^R, we have that for any $p \in (1,\infty)$ problem (9.2) admits a solution such that $\widetilde{u}^R \in W^{1,p}_0(\Omega^R)$ with

$$|\nabla \widetilde{u}^R|_p \leq c|\nabla \cdot \widehat{u}^R|_p = c|u^\circ \cdot \nabla h^R|_p \leq cR^{\frac{n}{p}-1} \max_{R\leq|x|\leq 2R} |u^\circ(x)|, \tag{9.3}$$

where c is a constant independent of R. By using Poincaré inequality

$$|\widetilde{u}^R|_p \leq cR|u^\circ \cdot \nabla h^R|_p \leq cR^{\frac{n}{p}} \max_{R\leq|x|\leq 2R} |u^\circ(x)|. \tag{9.4}$$

We denote again by $\widetilde{u}^R$ the extension to zero in $\mathbb{R}^n_+$ of the above solution of (9.2). We set $\overline{u}^R = \widehat{u}^R + \widetilde{u}^R$. For any $R > 0$, we have $\nabla \cdot \overline{u}^R = 0$ and functions $\overline{u}^R$ tends to u° in $C(\overline{\mathbb{R}}^n_+)$ for large R. Indeed, for any $p > n$, from Sobolev inequality (of Gagliardo–Nirenberg type) and (9.3)–(9.4) we deduce

$$\sup_{\mathbb{R}^n_+} |\widetilde{u}^R| \leq c|\nabla \widetilde{u}^R|^a_p |\widetilde{u}^R|^{1-a}_p \leq c \max_{R\leq|x|\leq 2R} |u^\circ|,$$

where we have taken into account that the interpolation inequality holds with $a = \frac{n}{p}$. As

$$|u^\circ(x) - \overline{u}^R(x)| \leq |u^\circ(x)(1-h^R(x))| + |\widetilde{u}^R(x)| \leq \sup_{|x|\geq R} |u^\circ|$$
$$+c \max_{R\leq|x|\leq 2R} |u^\circ|, \forall x \in \mathbb{R}^n_+,$$

the convergence is a consequence of $\lim\limits_{|x|\to\infty}|u^\circ(x)|=0$. Now let us consider $\overline{u}^R_\delta = J_\delta(\overline{u}^R)$, with $J_\delta(\cdot)$ mollifier by Weierstrass. As it is well known, function $\overline{u}^R_\delta \in C_0^\infty(\mathbb{R}^n)$ and for $\delta \to 0$ we have $\overline{u}^R_\delta \to \overline{u}^R$ in $C(\overline{\mathbb{R}}^n_+)$; moreover, since $\nabla\cdot\overline{u}^R = 0$, $\overline{u}^R_\delta$ also has divergence free. Finally, we consider the pair $(u^R_\delta, \pi^R_\delta)$ solution to system (3.1) with boundary data $a(x') = -\overline{u}^R_\delta(x)_{|x_n=0} = -\overline{u}^R_\delta(x')$. In virtue of Theorem 3.1 u^R_δ is the unique smooth solution corresponding to $-\overline{u}^R_\delta(x')$ and in particular from $(3.3)_1$ we deduce

$$|u^R_\delta(x)| \le c\sup_{\mathbb{R}^{n-1}}|\overline{u}^R_\delta(x')|,\ \forall R,\delta>0, \tag{9.5}$$

with c independent of $\overline{u}^R_\delta$. Moreover, taking into account of the regularity of $\overline{u}^R_\delta(x')$, formula (3.2) and the fact that data $\overline{u}^R_\delta(x')$ has compact support, we easily deduce for any $R,\delta>0$

$$|u^R_\delta(x)|\le A'(1+|x|)^{-1},\quad |\nabla u^R_\delta(x)|\le A'(1+|x|)^{-1}. \tag{9.6}$$

For any $k\in\mathbb{N}$, we set $u^{\circ k} = \overline{u}^k_{\frac{1}{k}} + u^k_{\frac{1}{k}}$. Taking into account of regularity of $\overline{u}^R_\delta$, u^R_δ, of the fact that $\overline{u}^R_\delta$ has compact support and u^R_δ satisfies estimate (9.6), if we prove the convergence of $\{u^{\circ k}\}$ to u°, then set $\{u^{\circ k},\ k\in\mathbb{N}\}$ is the wanted sequence. We can limit ourself to prove that $u^k_{\frac{1}{k}}\to 0$ for $k\to\infty$. For this task, taking into account (9.5), we must prove that

$$\sup_{\mathbb{R}^{n-1}}|\overline{u}^k_{\frac{1}{k}}(x')|\to 0 \quad \text{for} \quad k\to\infty.$$

On the other hand from the uniform continuity of u° we deduce that $\forall\varepsilon>0$ there exists $\delta(\varepsilon) : |\overline{u}^k(x)-\overline{u}^k(y)|<\varepsilon, \forall k\in\mathbb{N}$ and $|x-y|<\delta$. Thus from the definition of $\overline{u}^k_{\frac{1}{k}}$ we have for sufficient large k_1

$$\sup_{\mathbb{R}^{n-1}}|\overline{u}^k_{\frac{1}{k}}(x')| \le \sup_{|x'-y|<\frac{1}{k}}|\overline{u}^k(y)|<\varepsilon, \forall k\ge k_1.$$

Therefore the lemma is proved.

We start the proof of Theorem 1.1 by proving the uniqueness of the solutions. In this connection we prove the following

Lemma 9.2. *Let* (u,π) *be a solution to system* (1.1) *with* $u(x,0)=0$, *satisfying* (1.2) *with*

$$|u(x,t)|\le D \quad \text{and, for} \quad \alpha\in(0,1),\ |\pi(x,t)|\le Dt^{-\frac12-\frac{\alpha}{2}}(|x|^\alpha+1),$$
$$\forall(x,t)\in\mathbb{R}^n_+\times(0,T), \tag{9.7}$$

for some constant D. Then $u(x,t)$ is identically equal zero on $\mathbb{R}^n_+ \times (0,T)$.

Proof. As in the proof of uniqueness result proved for the solutions to system (2.1) we make use of the Holmgren technique. We begin proving that under assumptions (1.2) and (9.7) $u(x,t)w(x) \in L^2(\mathbb{R}^n_+)$, for any $t \in (0,T)$, where function $w(x) = (|x|+1)^{-\frac{n}{2}+1-\delta}$, for an arbitrary $\delta \in (\alpha, 1)$. We define $w^R = wh^R(x)$ where h^R is the smooth cut-off function used in the proof of Theorem 2.1. Multiplying equation (1.1) by $u(w^R)^2$, an integration by parts on $\mathbb{R}^n_+ \times (\eta, T)$ $(\eta > 0)$ furnishes

$$|u(t)w^R|_2^2 + 2\int_\eta^t |\nabla u(\tau)\, w^r|_2^2 d\tau = |u(\eta)w^R|_2^2 +$$

$$2\int_\eta^t \int_{\mathbb{R}^n_+} u^2(x,\tau)\Delta(w^R(x))^2 dx d\tau$$

$$+4\int_\eta^t \int_{\mathbb{R}^n_+} \pi(x,\tau) w^R(x) u(x,\tau)\cdot \nabla w^R(x) dx d\tau. \tag{9.8}$$

Taking into account $(9.7)_1$ we easily obtain that

$$\int_{\mathbb{R}^n_+} u^2(x,t)\Delta(w^R(x))^2 dx \le c. \tag{9.9}$$

Moreover, applying Schwartz inequality and taking into account $(9.7)_2$ we obtain that

$$\int_{\mathbb{R}^n_+} \pi(x,t) w^R(x) u(x,t)\cdot \nabla w^R(x) dx \le \frac{c}{t^{\frac{1}{2}+\frac{\alpha}{2}}} |u(t)w^R|_2. \tag{9.10}$$

From relation (9.8) and estimates (9.9) and (9.10) we deduce

$$|u(t)w^R|_2^2 \le |u(\eta)w^R|_2^2 + c\int_\eta^t \tau^{-\frac{1}{2}-\frac{\alpha}{2}} |u(\tau)w^R|_2 d\tau + c(t-\eta), \forall t \in (\eta, T),$$

which implies

$$|u(t)w^R|_2^2 \le 2|u(\eta)w^R|_2^2 + cT^{1-\alpha} + cT, \forall t \in (\eta, T).$$

Making the limit for $\eta \to 0$ we obtain

$$|u(t)w^R|_2^2 \le cT^{1-\alpha} + cT, \forall t \in (0,T).$$

Making the limit for $R \to \infty$ we obtain

$$|u(t)w|_2^2 < \infty, \forall t \in (0,T).$$

The last estimate and $|u(x,t)| \le D$ imply

$$\int\limits_{\mathbb{R}^n_+} |u(x,t)|^s (w(x))^2 dx < \infty, \forall s \ge 2, \forall t \in (0,T). \tag{9.11}$$

Now, let us consider the solution (φ, p) of system (1.1) corresponding to $\varphi^\circ(x) \in \mathcal{C}_0(\mathbb{R}^n_+)$. In virtue of the results of [8], we can assume that $\varphi(x,t) \in C(0,T; L^q(\mathbb{R}^n_+)) \cap L^r(0,T; W^{2,q} \cap W^{1,q}_\circ(\mathbb{R}^n_+))$, with $\varphi_t(x,t), \nabla p(x,t) \in L^r(0,T; L^q(\mathbb{R}^n_+)), \forall r, q > 1$. Moreover, we can assume $p(x,t) \in L^r(0,T; L^{\frac{nq}{n-q}}(\mathbb{R}^n_+))$, $\forall q \in (1,n)$. We multiply equation $(1.1)_1$ by $\varphi(x, t-\tau)h^R(x)$. An integration by parts on $(\eta, T) \times \mathbb{R}^n_+$ furnishes

$$(u(t), \varphi^\circ h^R) = (u(\eta), \varphi(t-\eta)h^R) + \int\limits_\eta^t (p(t-\tau), u(\tau) \cdot \nabla h^R) d\tau$$

$$+2\int\limits_\eta^t (\nabla h^R \cdot \nabla \varphi(t-\tau), u(\tau)) d\tau + \int\limits_\eta^t (\varphi(t-\tau), u(\tau)\Delta h^R) d\tau$$

$$+\int\limits_\eta^t (\pi(\tau)\nabla h^R, \varphi(t-\tau)) d\tau.$$

In virtue of the regularity of (φ, p) and $u(x,t)$, we can perform the limit for $\eta \to 0$, hence

$$(u(t), \varphi^\circ h^R) = \int\limits_0^t (p(t-\tau), u(\tau) \cdot \nabla h^R) d\tau$$

$$+2\int\limits_0^t (\nabla h^R \cdot \nabla \varphi(t-\tau), u(\tau)) d\tau + \int\limits_0^t (\varphi(t-\tau), u(\tau)\Delta h^R) d\tau$$

$$+\int\limits_0^t (\pi(\tau)\nabla h^R, \varphi(t-\tau)) d\tau = \sum_{i=1}^4 I_i(t,R). \tag{9.12}$$

We estimate $I_i, i = 1, \cdots, 4$ and our aim is to prove that any I_i goes to zero for $R \to \infty$. As estimate (9.11) holds for $\delta \in (\alpha, 1) \cap [\frac{1}{2}, 1)$, then corresponding to a fixed δ we consider $q_1 = \frac{n[n-2(1-\delta)]}{n[n-2(1-\delta)]-2(1-\delta)}$, which belongs to $(1, n)$, $\bar{q} = \frac{nq_1}{n-q_1}, \bar{q}' = n - 2 + 2\delta$. Hence, applying Hölder inequality with exponents $\frac{1}{\bar{q}} + \frac{1}{\bar{q}'} = 1$, we obtain

$$|I_1| \leq \int_0^t |p(t-\tau)|_{\frac{q_1 n}{n-q_1}} ||u(\tau)|x|^{-1}|_{L^{\bar{q}'}(R \leq |x| \leq 2R)} d\tau.$$

As exponent $\bar{q}' = n-2+2\delta \geq 2$, in virtue of (9.11), applying the theorem of dominate convergence, we get

$$\lim_{R \to \infty} I_1(t, R) = 0.$$

Since $|\nabla h^R| + R|\Delta h^R| \leq \frac{c}{R}$ and $|u| \leq D$, and since exponent q is arbitrary, applying Hölder inequality we obtain

$$|I_2 + I_3| \leq c \int_0^t (|\nabla \varphi(t-\tau)|_{L^{q_2}(R \leq |x| \leq 2R)} + |\varphi(t-\tau)|_{L^{q_3}(R \leq |x| \leq 2R)}) d\tau,$$

with $q_2 = \frac{n}{n-1}$ and $q_3 = \frac{n}{n-2}$. Applying the theorem of dominate convergence again, the last inequality implies that

$$\lim_{R \to \infty} (I_2(t, R) + I_3(t, R)) = 0.$$

Finally, taking into account hypothesis $(9.7)_2$ and $|\nabla h^R| \leq \frac{c}{R}$, applying Hölder inequality with exponents $\frac{1}{q_4} + \frac{1}{q_4'} = 1$ and $q_4 = \frac{n}{n-1+\alpha}$ we obtain

$$|I_4| \leq c \int_0^t |\varphi(t-\tau)|_{L^{q_4}(R \leq |x| \leq 2R)} d\tau,$$

hence in this case also we deduce

$$\lim_{R \to \infty} I_4(t, R) = 0.$$

The above estimates related to $I_i, i = 1, \cdots, 4$ and relation (9.12) allow to get

$$(u, \varphi^\circ) = 0, \forall \varphi^\circ \in \mathcal{C}_0(\mathbb{R}^n_+).$$

This last relation implies that $u = \nabla H$ a.e. on $\mathbb{R}^n_+$. As $\nabla \cdot u = 0$, $u(x', t) = 0$ and $u \to 0$ at infinity, then $H = \text{const}$ and $u = 0$. The lemma is proved.

Remark. For solution of Theorem 1.1 we cannot apply the uniqueness result given in [13]. Indeed in [13] the technique requires that the time derivative is summable with some exponent $r > 1$.

Now, we furnish the existence of solutions of Theorem 1.1. Let $\{u^{\circ k}\}$ be the sequence determined in Lemma 9.1. Let us consider the sequence of solutions $\{(U^k(x,t), c^k)\}$ of system (2.1) corresponding for any $k \in \mathbb{N}$ to initial data $\{u^{\circ k}(x)\}$. Then, let us consider solutions $\{(W^k(x,t), Q^k(x,t))\}$ of system (3.1) corresponding for any $k \in \mathbb{N}$ to boundary data $\{-U^k(x',t)\}$. Finally, the sequence of solutions $\{(V^k(x,t), P^k(x,t))\}$ of system (3.1) corresponding, for any $k \in \mathbb{N}$, to body force $\{-{W^k}_t(x,t)\}$, where, of course, for any k, ${W^k}_t(x,t)$ is the solution to system (3.1) corresponding to $-{U^k}_t$. Taking into account the above position, we have that in virtue of the results of section 2 and Theorem 3.1 $-{W^k}_t$ satisfies assumptions of i), $\forall k \in \mathbb{N}$; the estimates required in assumption ii) with constant $B = c \sup_{\mathbb{R}^n_+} |u^{\circ k}(x)|, \forall k \in \mathbb{N}$; assumption iii) in virtue of Lemma 2.3 with constants A, A' which depend on k. Theorem 3.1 furnishes in particular for $W^k(x,t)$ the following estimates:

$$
\begin{gathered}
|\nabla W^k(x,t)| \le c t^{-\frac{1}{2}} \sup_{\mathbb{R}^n_+} |u^{\circ k}(x)|, \; \forall (x,t) \in \mathbb{R}^n_+ \times (0,T); \\
|Q^k(\bar{x},t) - Q^k(\bar{\bar{x}},t)| \le c t^{-\frac{1}{2}-\frac{\alpha}{2}} \sup_{\mathbb{R}^n_+} |u^{\circ k}(x)| \cdot |\bar{x} - \bar{\bar{x}}|^\alpha, \\
\forall \bar{x}, \bar{\bar{x}} \in \mathbb{R}^n_+, \forall t \in (0,T); \\
|D_{x_i x_j} W^k(x,t)| + |\nabla Q^k(x,t)| \le c t^{-1} \sup_{\mathbb{R}^n_+} |u^{\circ k}(x)|, \\
\forall (x,t) \in \mathbb{R}^n_+ \times (0,T);
\end{gathered}
\tag{9.13}
$$

and

$$
\lim_{t \to \infty} |W^k(x,t)| = 0. \tag{9.14}
$$

Indeed, to obtain $(9.13)_1$ we employ $(3.4)_1$ with $\mu = t^{\frac{1}{2}}$ and take into account that Theorem 2.1 gives estimates $|\nabla U^k(y',t)| \le c t^{-\frac{1}{2}} \sup_{\mathbb{R}^n_+} |u^{\circ k}(x)|$ and $[\nabla U^k(y',t)]_{0,\alpha} \le c t^{-\frac{1}{2}-\frac{\alpha}{2}} \sup_{\mathbb{R}^n_+} |u^{\circ k}(x)|$. Analogously, starting from $(3.4)_3$ we deduce $(9.13)_3$. Estimate $(3.4)_2$ (with $\mu = t^{\frac{1}{2}}$) and Theorem 2.1 ensure that $|Q^k(x,t)| \le c t^{-\frac{1}{2}} \sup_{\mathbb{R}^n_+} |u^{\circ k}(x)|$, hence $(9.13)_2$ follows by interpolation taking into account of $(9.13)_3$ for $\nabla Q^k(x,t)$. Finally, limit property (9.14) is an immediate consequence of (3.3) and (2.6). Therefore Theorem 2.1, Theorem 3.1, Lemma 5.1-5.2 and Lemma 7.4-7.5, and the linearity of Stokes system ensure that pair (u^k, π^k), with $u^k = U^k + W^k + V^k$ and $\pi^k = c^k + Q^k + P^k$, is a solution to system (1.1) and, for any $k \in \mathbb{N}$, it satisfies estimates (1.2)-(1.6) with c independent of k. Employing again the linearity of the system and (1.2)-(1.6) we have that for any $t > 0$ sequence $\{u^k, \pi^k\}$, together with the sequence of derivatives, is a Cauchy sequence uniformly convergent in $\mathbb{R}^n_+$. The limit (u, π) is regular in the sense of Theorem 1.1, satisfies estimates (1.2)-(1.6) and it is a solution to system (1.1). The proof of the theorem is completed.

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