Записки научных
семинаров ПОМИ
Том 295, 2003 г.

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ON STATIONARY FLOWS WITH ENERGY DEPENDENT NONLOCAL VISCOSITIES

ABSTRACT. A nonlocal constitutive law for an incompressible viscous flow in which the viscosity depends on the total dissipation energy of the fluid is obtained as a limit case of very large thermal conductivity when the viscosity varies with the temperature. A rigorous analysis is illustrated in an Hilbertian framework for unidirectional stationary flows of Newtonian and Bingham fluids with heating by viscous dissipation. The extension to quasi-Newtonian fluids of power law type and with temperature dependent viscosities is obtained in the framework of the heat equation with a L^1-term. The nonlocal model proposed by Ladyzenskaya in 1966 as a modification of Navier-Stokes equations, in particular, may be obtained with this procedure.

Dedicated to O. A. Ladyzenskaya on the occasion of her 80th birthday

1. INTRODUCTION

In [7] Ladyzhenskaya considered, in particular, the nonlocal problem in an open domain $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$

$$-\nu[u_x]\triangle\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in} \quad \Omega \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in} \quad \Omega, \quad \mathbf{u} = 0 \quad \text{on} \quad \partial\Omega \tag{2}$$

for the stationary motion of viscous incompressible fluids. Here $\nabla\cdot\mathbf{u} = \operatorname{div}\mathbf{u}$ and the nonlocal viscosity was defined by

$$\nu[\mathbf{u}_x] = \nu_0 + \nu_1\|\nabla\mathbf{u}\|^2 \quad (\nu_0, \nu_1 > 0) \tag{3}$$

where $\|\nabla\mathbf{u}\|^2 = \int_\Omega |\nabla\mathbf{u}|^2 = \sum_{i,j=1}^n \int_\Omega (\partial u_i/\partial x_j)^2$ and the velocity field is denoted by $\mathbf{u}(x) = (u_1(x), ..., u_n(x))$ at $x \in \mathbb{R}^n$. For any force

Partially supported by Project FCT-POCTI/34471/MAT/2000.

$\mathbf{f} \in \mathbf{H}^{-1}(\Omega) = [\mathbf{H}_0^1(\Omega)]'$, where $\mathbf{H}_0^1(\Omega) = [W_0^{1,2}(\Omega)]^n$ denotes the usual Sobolev space, as a consequence of the general theory [8], there exists at least a solution to (1)-(2) in the class

$$V_2 = \{\mathbf{v} \in \mathbf{H}_0^1(\Omega) : \nabla \cdot \mathbf{v} = 0 \quad \text{in} \quad \Omega\} \quad \text{and} \quad p \in L^2(\Omega).$$

The system (1)-(2) in general has not uniqueness of solution but the corresponding evolution problem exhibits nice properties concerning the existence and uniqueness of weak solutions, as well as regularity properties (see [7]). This is a consequence of the fact that the operator $\mathbf{u} \mapsto -\|\nabla \mathbf{u}\|^2 \Delta \mathbf{u}$ is monotone from $V_2 \to V_2'$ and corresponds to the gradient of the functional $F(\mathbf{v}) = \frac{1}{4}\|\nabla \mathbf{v}\|^4$ (see also [9, pg.218]).

Although a nonlocal viscosity for fluids may seem a paradoxical property, we observe that the nonlocal term in (3) is related to the total dissipation energy of a Newtonian fluid. This remark leads to regard (1) as a particular asymptotic limit case of certain non-isothermal flows with very high thermal conductivity in the fluid.

The stationary flow of an incompressible fluid with energy transfer is described by the equations of motion and of energy, the density being taken $\rho = 1$ for simplicity, respectively

$$-\nabla \cdot \sigma + (\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{f} \tag{4}$$

$$\nabla \cdot \mathbf{q} + \mathbf{u} \cdot \nabla e = \sigma : \mathbf{D} + g. \tag{5}$$

In a Newtonian fluid with the viscosity μ depending on the temperature θ, if p denotes the pressure and

$$\mathbf{D}(\mathbf{u}) = \frac{1}{2}[\nabla \mathbf{u} + (\nabla \mathbf{u})^T], \quad |\mathbf{D}|^2 = \mathbf{D} : \mathbf{D} = \sum_{i,j=1}^{n} D_{ij} D_{ij}$$

the stretching tensor and its second invariant, we have

$$\sigma = -p\mathbf{I} + \mathbf{S} \quad \text{with} \quad \mathbf{S} = 2\mu(\theta)\mathbf{D}$$

and the equation (4) can be writen in the form

$$-\nabla \cdot (2\mu(\theta)\mathbf{D}(\mathbf{u})) + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f}. \tag{6}$$

On the other hand, in (5) we take for the energy e, for the heat flux $\mathbf{q}$ and for the heat source g the following linearizations

$$e = c\theta, \quad \mathbf{q} = -\kappa \nabla \theta \quad \text{and} \quad g = -\theta,$$

where $c > 0$ is the specific heat and $\kappa > 0$ the thermal conductivity. We have then the heat equation

$$-\kappa\Delta\theta + c\mathbf{u}\cdot\nabla\theta + \theta = 2\mu(\theta)|\mathbf{D}(\mathbf{u})|^2 \quad \text{in} \quad \Omega, \tag{7}$$

to which we may associate a homogeneous Neumann boundary condition $\frac{\partial\theta}{\partial n} = 0$ on $\partial\Omega$.

When we let $\kappa \to \infty$ we have $|\nabla\theta_\kappa| \to 0$ and $\theta_\kappa \to \Theta$ where Θ is a constant value, which is easily seen to be also the limit of the spacial average of θ_κ, i.e.,

$$\fint_\Omega \theta_\kappa = \frac{1}{|\Omega|}\int_\Omega \theta_\kappa \to \Theta \quad \text{as} \quad \kappa \to \infty.$$

Therefore by taking the average of equation (7) and the limit $\kappa \to \infty$, assuming a suitable convergence for the respective vector fields $\mathbf{u}_\kappa \to \mathbf{u}$, we find a scalar equation to determine Θ:

$$\frac{\Theta}{\mu(\Theta)} = 2\fint_\Omega |\mathbf{D}(\mathbf{u})|^2 = \fint_\Omega |\nabla\mathbf{u}|^2. \tag{8}$$

Under appropriate assumptions on μ, we may expect to be able to write the constant viscosity in the limit equation (6) in the form

$$\mu(\Theta) = \nu\Big(\int_\Omega |\nabla\mathbf{u}|^2\Big), \tag{9}$$

which corresponds to the Ladyzhenskaya model (1)-(3) if we retain only the first two terms in the limited development of the new viscosity function ν near the rest state.

In this paper we explore this idea by justifying this formal procedure with appropriate assumptions and suitable estimates for the respective elliptic problems. The main difficulty consists of the quadratic term of the dissipation energy arising in the right hand side of the equation (7).

We consider first a scalar model, corresponding to unidirectional flows for a Newtonian fluid and for a Bingham fluid, since it allows to use direct energy methods. In the last section we extend the existence and the asymptotic results in the more general mathematical framework for a class of fluids of power law type, which requires an additional analysis of the heat equation with viscous heating only in L^1. Here we combine and develop some ideas of [2] and [14], where additional references can be found. For a recent introduction to the mathematical analysis to stationary flow of generalized Newtonian incompressible fluids we refer to [6].

2. Unidirectional flow with energy dissipation

We start with the scalar problem corresponding to the unidirectional flow through the cross section of a cylinder of arbitrary shape represented by the open bounded set Ω of $\mathbb{R}^2$. If the x_3-axis denotes the direction of the flow, the Poiseuille type velocity field is given, in the steady-state, by

$$\mathbf{u} = (0, 0, u(x_1, x_2)), \quad (x_1, x_2) \in \Omega.$$

The nonzero components of the stretching tensor $\mathbf{D}$ are only $D_{31} = D_{13} = \frac{1}{2}u_{x_1}$ and $D_{32} = D_{23} = \frac{1}{2}u_{x_2}$ and the viscous stress tensor $\mathbf{S} = (S_{ij})$ is defined in Ω by a vector $\mathbf{s} = (S_{31}, S_{32})$. So the dissipation of energy is simply

$$\sigma : \mathbf{D} = \mathbf{s} \cdot \nabla u.$$

The convection terms vanish and the equations of motion and energy became, respectively, for the velocity u and for the temperature $\theta = \vartheta(x_1, x_2)$:

$$-\nabla \cdot \mathbf{s}(\vartheta, u) = f \quad \text{in} \quad \Omega \tag{10}$$

$$-\kappa\Delta\vartheta + \vartheta = \mathbf{s} \cdot \nabla u \quad \text{in} \quad \Omega, \tag{11}$$

where $f = f(x_1, x_2)$ are the given body forces, incorporating the constant drop in pressure along the flow axis.

The system (10)-(11) for (u, ϑ) is completed with the constitutive relation for the fluid and with homogeneous Dirichlet and Newmann boundary conditions

$$u = 0 \quad \text{and} \quad \frac{\partial\vartheta}{\partial n} = 0 \quad \text{on} \quad \partial\Omega, \tag{12}$$

where we assume the boundary $\partial\Omega$ of class $C^{0,1}$ (Lipschitz).

2.1. Newtonian fluid.

The stress-stretching constitutive relation being linear, we have

$$\mathbf{s} = \mu(\vartheta)\nabla u,$$

where the viscosity μ is given by a continuous function of the temperature under the assumption

$$\mu \in C^0(\mathbb{R}), \quad 0 < \mu_* \leq \mu(\tau) \leq \mu^*, \quad \forall \tau \in \mathbb{R}. \tag{13}$$

The coupled system (10)-(11) reduces now to

$$-\nabla \cdot (\mu(\vartheta)\nabla u) = f \quad \text{in} \quad \Omega \tag{14}$$

$$-\kappa\Delta\vartheta + \vartheta = \mu(\vartheta)|\nabla u|^2 \quad \text{in} \quad \Omega \tag{15}$$

and fixing the boundary conditions (12) we obtain the following existence result

Proposition 1. *Let* (13) *hold,* $\kappa > 0$ *and* $f \in L^r(\Omega), r > 1$. *Then there exists at least a solution* $(u, \vartheta) \in W_0^{1,p}(\Omega) \times H^1(\Omega)$, *for some* $p > 2$, *to the problem* (14)-(15) *with* (12).

Proof. We write the problem in variational form

$$\int_\Omega \mu(\vartheta)\nabla u \cdot \nabla\varphi = \int_\Omega f\varphi, \quad \forall\varphi \in H_0^1(\Omega) \tag{16}$$

$$\kappa\int_\Omega \nabla\vartheta \cdot \nabla\psi + \int_\Omega \vartheta\psi = \int_\Omega \mu(\vartheta)|\nabla u|^2\psi, \quad \forall\psi \in H^1(\Omega) \tag{17}$$

and we apply the Schauder fixed point theorem with standard elliptic estimates. Since $f \in L^r(\Omega) \subset W^{-1,q}(\Omega)$ for some $q > 2 = n$, by Meyer's estimate [12] applied to (16), there exists $p > 2$ ($p \leq q$) and a constant $C_p > 0$, independent of ϑ and of u (C_p depends only on μ^*, μ_* and Ω), such that,

$$||u||_{W_0^{1,p}(\Omega)} \leq C_p||f||_{W^{-1,p}(\Omega)}, \quad \text{for some} \quad p > 2. \tag{18}$$

If we fix $\tau \in B_R = \{v \in L^2(\Omega) : ||v||_{L^2(\Omega)} \leq R\}$ for R large enough, we may solve (16) with ϑ replaced by τ, obtaining $u_\tau \in W_0^{1,p}(\Omega)$ satisfying (18). Solving now (17) with right hand side $\mu(\tau)|\nabla u_\tau|^2 \in L^{p/2}(\Omega)$ and recalling the imbedding $H^1(\Omega) \subset L^s(\Omega)$ for all $s < \infty$ (since $n = 2$), we easily conclude the a priori estimate

$$\kappa\int_\Omega |\nabla\vartheta|^2 + \int_\Omega \vartheta^2 \leq C_1 \tag{19}$$

where the constant $C_1 > 0$ is independent of τ and of $\kappa \geq \kappa_1 > 0$. Hence, the Schauder fixed point theorem yields a solution to (16)-(17), with $\vartheta \in B_R$ for $R \geq \sqrt{C_1}$. •

Remark 1. The argument used in Proposition 1 only holds in two dimensions in view of the Sobolev inequality that allows to set $\psi = \vartheta$ in

(17). For instance, for $n = 3$ we should have $p \geq 12/5$ which cannot a priori be garanteed by Meyer's estimate.

In order to obtain a solution to the nonlocal problem

$$-\nu\Big(\fint_\Omega |\nabla u|^2\Big)\Delta u = f \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega \tag{20}$$

we need to define a new viscosity function ν by

$$\nu = \mu \circ \alpha, \quad \text{i.e.,} \quad \nu(t) = \mu(\alpha(t)), \quad t \geq 0, \tag{21}$$

where $\alpha = \alpha(t)$ is a continuous function given by the assumption

$$\beta(\tau) = \frac{\tau}{\mu(\tau)} \quad \text{is invertible for} \quad \tau \geq 0 \quad \text{with inverse} \quad \alpha(t). \tag{22}$$

Theorem 1. *Let $(u_\kappa, \vartheta_\kappa)$ denote a solution to* (14), (15)-(12) *for each $\kappa > 0$ and assume* (13) *and* (22) *hold. Then there exists a sequence such that*

$$u_\kappa \to u \quad \text{in} \quad H_0^1(\Omega), \quad \vartheta_\kappa \to \alpha\Big(\fint_\Omega |\nabla u|^2\Big) \quad \text{in} \quad H^1(\Omega) \quad \text{as} \quad \kappa \to \infty$$

where u is a solution of the nonlocal problem (20).

Proof. Recalling the a priori estimates (18) and (19) are independent of κ as $\kappa \to \infty$, we may extract a subsequence such that

$$u_\kappa \rightharpoonup u \quad \text{in} \quad W_0^{1,p}(\Omega) - \text{weak and} \quad \nabla\vartheta_\kappa \to 0 \quad \text{in} \quad L^2(\Omega).$$

Then we have as $\kappa \to \infty$

$$\vartheta_\kappa \to \Theta = \text{ constant in} \quad H^1(\Omega)$$

and

$$\mu(\vartheta_\kappa) \to \mu(\Theta) \quad \text{in} \quad L^q(\Omega), \quad \forall q < \infty.$$

Taking the limit $\kappa \to \infty$ in (16) we conclude first that the limit u solves

$$\mu(\Theta)\int_\Omega \nabla u \cdot \nabla\varphi = \int_\Omega f\varphi, \quad \forall\varphi \in H_0^1(\Omega) \tag{23}$$

which is the variational formulation of (20) if $\Theta = \alpha\Big(\fint_\Omega |\nabla u|^2\Big)$.

Comparing (23) with (16) we easily conclude $u_\kappa \to u$ in $H_0^1(\Omega)$-strong and hence

$$\mu(\vartheta_\kappa)|\nabla u_\kappa|^2 \to \mu(\Theta)|\nabla u|^2 \quad \text{in} \quad L^1(\Omega) \quad \text{as} \quad \kappa \to \infty.$$

Observing that $⨍_\Omega \vartheta_\kappa \to \Theta$, taking the limit $\kappa \to \infty$ in (17) with $\psi \equiv 1$ we find

$$\Theta = \mu(\Theta) ⨍_\Omega |\nabla u|^2, \tag{24}$$

which, by the assumption (22), concludes the proof. •

Remark 2. From (13) and the definition (21) we see that

$$0 < \mu_* \le \nu(t) \le \mu^*, \quad t \in \mathbb{R} \tag{25}$$

and if μ is smooth and has a development

$$\mu(\tau) = \mu_0 + \mu_1 \tau + \ldots \quad \text{then} \quad \nu(t) = \mu_0 + \mu_0 \mu_1 t + \ldots \tag{26}$$

From (26) we see that Ladyzhenskaya model (3) corresponds to consider the first two terms of the limited development of the modified viscosity.

Remark 3. Actually the upper bound in (25) is not necessary for the nonlocal limit problem (20), since $⨍_\Omega |\nabla u|^2$ satisfies the a priori estimate

$$⨍_\Omega |\nabla u|^2 \le \frac{C_r^2}{|\Omega|\mu_*^2} \|f\|_{L^r(\Omega)}^2 \equiv M \tag{27}$$

where C_r is the constant of the imbedding $L^r(\Omega) \subset H^{-1}(\Omega)$ $(r > 1 = n/2)$.

Hence, if $u_\tau \in H_0^1(\Omega)$ denotes the solution of $-\Delta u_\tau = f/\nu(\tau)$ we have a continuous mapping $T : [0, M] \ni \tau \mapsto ⨍_\Omega |\nabla u_\tau|^2 \in [0, M]$. Therefore its fixed point $s = T(s)$ yields at least a solution u to (20) under the unique assumption on the continuity and strict uniform positivity of ν, for any $f \in L^r(\Omega)$, $r > 1$.

Remark 4. The assumption (22) on the invertibility of $\beta(\tau) = \tau/\mu(\tau)$ for $\tau \ge 0$ is garanteed, for instance, when $\tau\mu'(\tau) < \mu(\tau)$, in particular, if μ is decreasing. However (22) is not necessary, and there exists always a solution to the nonlocal problem (23) with the implicit condition (24). Of course, uniqueness does not hold in general as in most nonlocal elliptic problems (see, for instance, [1]).

2.2 Bingham Fluid.

The constitutive relation involves an additional parameter γ, the plasticity threshold, which is supposed here to depend also on the temperature (see [3]):

$$\mathbf{s} = \mu(\vartheta)\nabla u + \gamma(\vartheta)\nabla u/|\nabla u| \quad \text{if} \quad |\nabla u| > 0$$
$$|\mathbf{s}| \leq \gamma(\vartheta) \quad \text{if and only if} \quad |\nabla u| = 0.$$

Here μ is given as in (13) and γ satisfies

$$\gamma \in C^0(\mathbb{R}), \quad 0 \leq \gamma(\tau) \leq \gamma^*, \quad \forall \tau \in \mathbb{R}. \tag{28}$$

The heat equation (11) is now given by

$$-\kappa\Delta\vartheta + \vartheta = \mu(\vartheta)|\nabla u|^2 + \gamma(\vartheta)|\nabla u| \quad \text{in} \quad \Omega$$

and the corresponding variational formulation for $(u, \vartheta) \in H_0^1(\Omega)\times H^1(\Omega)$ is given by

$$\int_\Omega \mu(\vartheta)\nabla u \cdot \nabla(\varphi - u) + \int_\Omega \gamma(\vartheta)[|\nabla\varphi| - |\nabla u|] \geq \int_\Omega f(\varphi - u), \qquad \forall \varphi \in H_0^1(\Omega) \tag{29}$$

$$\kappa \int_\Omega \nabla\vartheta \cdot \nabla\psi + \int_\Omega \vartheta\psi = \int_\Omega \Big(\mu(\vartheta)|\nabla u|^2 + \gamma(\vartheta)|\nabla u|\Big)\psi, \qquad \forall \psi \in H^1(\Omega). \tag{30}$$

For any $f \in L^r(\Omega)$, $r > 1$, if we prescribe $\vartheta = \tau$ in $L^2(\Omega)$ we can solve (29) uniquely and we denote by u_τ its solution. Then, as in [4], there exists a Lagrange multiplier $\mathbf{g} \in L^2(\Omega)^2$, such that,

$$|\mathbf{g}(x)| \leq \gamma(\tau(x)) \quad \text{a.e.} \quad x \in \Omega \tag{31}$$
$$\gamma(\tau(x))|\nabla u(x)| = -\mathbf{g}(x) \cdot \nabla u(x) \quad \text{a.e.} \quad x \in \Omega$$
$$-\nabla \cdot (\mu(\tau)\nabla u_\tau) = f - \nabla \cdot \mathbf{g} \quad \text{in} \quad \Omega.$$

By (31) and the assumption (28), we can apply Meyer's estimate to u_τ and so

$$\|u_\tau\|_{W_0^{1,p}(\Omega)} \leq C_p^*, \quad \text{for some} \quad p > 2, \tag{32}$$

where the constant $C_p^* > 0$ is independent of τ. Hence we have $h_\tau \equiv \mu(\tau)|\nabla u_\tau|^2 + \gamma(\tau)|\nabla u_\tau| \in L^{p/2}(\Omega)$, uniformly in $\tau \in L^2(\Omega)$, and we obtain the a priori estimate

$$\kappa \int_\Omega |\nabla \vartheta_\tau|^2 + \int_\Omega \vartheta_\tau^2 \le C_1^* \tag{33}$$

for $\vartheta_\tau \in H^1(\Omega)$ solution of (30) with h_τ in the right hand side.

As in the Proposition 1, the Schauder fixed point theorem yields a solution $(u, \vartheta) \in W_0^{1,p}(\Omega) \times H^1(\Omega)$, for some $p > 2$, to the problem (29)-(30).

On the other hand, since the estimates (32) and (33) are also independent of κ as $\kappa \to \infty$, we may argue as in Theorem 1 and, if

$$u_\kappa \rightharpoonup u \quad \text{in} \quad W_0^{1,p}(\Omega)\text{-weak} \qquad \vartheta_\kappa \to \Theta = \text{const. in} \quad H^1(\Omega),$$

the limit (u, Θ) satisfies

$$\mu(\Theta) \int_\Omega \nabla u \cdot \nabla(\varphi - u) + \gamma(\Theta) \int_\Omega [|\nabla \varphi| - |\nabla u|] \ge \int_\Omega f(\varphi - u), \ \forall \varphi \in H_0^1(\Omega). \tag{34}$$

In this passage we used the lower semicontinuity of convex functionals to show

$$\liminf_{\kappa \to \infty} \Big\{ \int_\Omega \mu(\theta_\kappa)|\nabla u_\kappa|^2 + \int_\Omega \gamma(\theta_\kappa)|\nabla u_\kappa| \Big\} \ge \mu(\Theta) \int_\Omega |\nabla u|^2 + \gamma(\Theta) \int_\Omega |\nabla u|. \tag{35}$$

On the other hand, from (29) we have for all $\varphi \in H_0^1(\Omega)$

$$\limsup_{k \to \infty} \Big\{ \int_\Omega \mu(\vartheta_\kappa)|\nabla u_\kappa|^2 + \int_\Omega \gamma(\vartheta_\kappa)|\nabla u_\kappa| \Big\} \le$$

$$\le \lim_{\kappa \to \infty} \Big\{ \int_\Omega \mu(\vartheta_\kappa) \nabla u_\kappa \cdot \nabla \varphi + \int_\Omega \gamma(\vartheta_\kappa)|\nabla \varphi| + \int_\Omega f(u_\kappa - \varphi) \Big\} =$$

$$= \mu(\Theta) \int_\Omega \nabla u \cdot \nabla \varphi + \gamma(\Theta) \int_\Omega |\nabla \varphi| + \int_\Omega f(u - \varphi).$$

Taking $\varphi = u$ and comparing with (35), we obtain

$$\int_\Omega \mu(\vartheta_\kappa)|\nabla u_\kappa|^2 + \int_\Omega \gamma(\vartheta_\kappa)|\nabla u_\kappa| \to \mu(\Theta) \int_\Omega |\nabla u|^2 + \gamma(\Theta) \int_\Omega |\nabla u|$$

$$\text{as} \quad \kappa \to \infty.$$

Then, by combining (29) with $\varphi = u_\kappa$ and (34) with $\varphi = u$ it is easy to conclude that $u_\kappa \to u$ in $H_0^1(\Omega)$-strong and taking the limit $\kappa \to \infty$ in (30) with $\psi \equiv 1$, the constant Θ must satisfy the condition

$$\Theta = \mu(\Theta) \fint_\Omega |\nabla u|^2 + \gamma(\Theta) \fint_\Omega |\nabla u|, \tag{36}$$

and we have proved the following result.

Theorem 2. *Under the assumptions* (13) *and* (28), *for any* $f \in L^r(\Omega)$, $r > 1$, *there exist solutions* $(u_\kappa, \vartheta_\kappa) \in H_0^1(\Omega) \times H^1(\Omega)$ *to* (29)-(30) *such that*

$$u_\kappa \to u \quad in \quad H_0^1(\Omega), \quad \vartheta_\kappa \to \Theta \quad in \quad H^1(\Omega), \quad as \quad \kappa \to \infty,$$

where $(u, \Theta) \in W_0^{1,p}(\Omega) \times \mathbb{R}$ *is a solution to* (34) *and* (36).

Remark 5. In general, when $\gamma > 0$, (36) is only an implicit condition determining the nonlocal constant $\Theta = \Theta\Big(\fint_\Omega |\nabla u|^2, \fint_\Omega |\nabla u|\Big)$ arising in (34).

Remark 6. For arbitrary continuous functions $\nu, \eta : \mathbb{R}^2 \to \mathbb{R}_+$ such that $\nu(s,t) \geq \mu_* > 0$ and $\eta(s,t) \geq 0$, we may show that there exists at least one solution $u \in H_0^1(\Omega)$ to the nonlocal problem

$$\nu(\fint_\Omega |\nabla u|^2, \fint_\Omega |\nabla u|) \int_\Omega \nabla u \cdot \nabla(\varphi - u) +$$

$$+\eta(\fint_\Omega |\nabla u|^2, \fint_\Omega |\nabla u|) \int_\Omega [|\nabla \varphi| - |\nabla u|] \geq \int_\Omega f(\varphi - u), \quad \forall \varphi \in H_0^1(\Omega).$$

Indeed, taking $\varphi = 0$, we obtain also the a priori estimate (27) and we can argue as in Remark 3, by applying the Brower fixed point theorem as in [1].

3. Quasi-Newtonian flows with viscous heating

In this section we extend the mathematical results to a class of quasi-Newtonian fluids of power law type

$$\mathbf{S} = \mu(\theta) A(\mathbf{D}) = \mu(\theta) \eta(|\mathbf{D}|^2) \mathbf{D} \tag{38}$$

where the viscosity coefficient μ satifies (13) and $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function, satisfying suitable growth and coerciveness properties.

The corresponding generalized Navier-Stokes systems is now given in an open bounded subset $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ by

$$-\nabla \cdot \Big(\mu(\theta)\eta(|\mathbf{D}|^2)\mathbf{D}(\mathbf{u})\Big) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in} \quad \Omega \tag{39}$$

$$-\kappa\Delta\theta + c\mathbf{u}\cdot\nabla\theta + \theta = \mu(\theta)\eta(|\mathbf{D}|^2)|\mathbf{D}(\mathbf{u})|^2 \quad \text{in} \quad \Omega \tag{40}$$

$$\mathbf{u} = 0 \quad \text{and} \quad \partial\theta/\partial n = 0 \quad \text{on} \quad \partial\Omega. \tag{41}$$

Although the model (38) allows different types of shear flows of power law type (see [10] for instance)

$$\eta(d) = 2(\lambda + d^r)^{p-2}, \quad r, \lambda \geq 0, p > 1,$$

including Newtonian fluids $(\lambda = 0, r = 0)$, for simplicity of presentation we shall consider only the case $\lambda = 0, r = 1/2$, first without the inertia term, which allows less restrictions in the power $p > 1$, and afterwards in more general situations.

We shall work with the closed subspace V_p of the Sobolev space $\mathbf{W}_0^{1,p}(\Omega) = [W_0^{1,p}(\Omega)]^n$ of divergence-free functions

$$V_p = \{\mathbf{v} \in \mathbf{W}_0^{1,p}(\Omega) : \quad \nabla\cdot\mathbf{v} = 0\}, \quad 1 < p < \infty$$

and we shall assume a Lipschitz boundary, so that Korn inequality holds, i.e.,

$$\exists K_p > 0 : \quad \|\nabla\mathbf{v}\|_{L^p(\Omega)} \leq K_p\|\mathbf{D}(\mathbf{v})\|_{L^p(\Omega)}, \quad \forall\mathbf{v} \in \mathbf{W}_0^{1,p}(\Omega). \tag{42}$$

3.1. Quasi-static case.

Let $p > 1$ and suppose $\mathbf{f} \in \mathbf{W}^{-1,p'}(\Omega)$, $p' = p/(p-1)$. We shall consider solutions $\mathbf{u} \in V_p$ and $\theta \in W^{1,q}(\Omega)$ for all $q < \frac{n}{n-1}$ to the nonlinear system (39)-(41) without the inertia term in (39)

$$2\int_\Omega \mu(\theta)|\mathbf{D}(\mathbf{u})|^{p-2}\mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{v}) = \int_\Omega \mathbf{f}\cdot\mathbf{v}, \quad \forall\mathbf{v} \in V_p \tag{43}$$

$$\kappa\int_\Omega \nabla\theta\cdot\nabla\psi - c\int_\Omega \theta\mathbf{u}\cdot\nabla\psi + \int_\Omega \theta\psi = 2\int_\Omega \mu(\theta)|\mathbf{D}(\mathbf{u})|^p\psi,$$
$$\forall\psi \in W^{1,q'}(\Omega). \tag{44}$$

Since $q' = q/(q-1) > n$, by Sobolev inequalities ψ is $L^\infty(\Omega)$ and the right hand side of (44) makes sense with just $\mu(\theta)|\mathbf{D}(\mathbf{u})|^p \in L^1(\Omega)$.

On the other hand, the convective term of (44) is well defined provided $\theta\mathbf{u} \in \mathbf{L}^q(\Omega)$. If $n = 2$, $\theta \in L^r(\Omega)$ for all $r < \infty$, and we can take $\mathbf{u} \in V_p \subset \mathbf{L}^s(\Omega)$, for any $p > 1$ since $s = pn/(n-p) > n/(n-1)$.

For $n \geq 3$, we only have $\theta \in L^r(\Omega)$ for $1 < r < n/(n-2)$ and we need to require $\mathbf{u} \in \mathbf{L}^s(\Omega)$ with $s = pn/(n-p) > n$ which is satisfied for $p > n/2$. Hence we have

$$|\int_\Omega \theta\mathbf{u} \cdot \nabla\psi| \leq ||\theta||_{L^r(\Omega)} ||\mathbf{u}||_{L^s(\Omega)} ||\nabla\psi||_{L^{q'}(\Omega)} \tag{45}$$

for $\theta \in W^{1,q}(\Omega), q < n/(n-1), \mathbf{u} \in V_p$ for $p > n/2$ and $\psi \in W^{1,q'}(\Omega)$.

Proposition 2. *Assuming* (13), $\kappa > 0$ *and* $p > n/2$ *for any* $\mathbf{f} \in \mathbf{W}^{-1,p'}(\Omega)$, *there exists at least a solution* $(\mathbf{u}, \theta) \in V_p \times W^{1,q}(\Omega)$, *for all* $q < n/(n-1)$, *to (43)-(44). In addition, there is a constant* $C_q > 0$ *independent of* $\kappa \geq 1$ *such that*

$$||\nabla\theta||_{L^q(\Omega)} \leq \frac{C_q}{\sqrt{\kappa}}. \tag{46}$$

Proof. First we observe that for an arbitrary $\tau \in L^1(\Omega)$ we may solve uniquely (43) with θ replaced by τ. Denoting by $\mathbf{u}_\tau \in V_p$ this solution, using (13) and Korn inequality (42) we obtain

$$||\nabla\mathbf{u}_\tau||_{L^p(\Omega)} \leq C_p ||\mathbf{f}||_{W^{-1,p'}(\Omega)}, \tag{47}$$

where $C_p > 0$ is a constant independent of τ and $\mathbf{u}_\tau$. Moreover the application $\tau \in L^1(\Omega) \mapsto \mathbf{u}_\tau \in V_p$ is continuous for the strong topologies. Indeed, if $\tau_n \to \tau$ in $L^1(\Omega)$, by (47) we have first $\mathbf{u}_{\tau_n} \rightharpoonup \mathbf{u}_\tau$ weakly in V_p and by taking $\mathbf{v} = \mathbf{u}_{\tau_n} - \mathbf{u}_\tau$ in (43) for both solutions we find, after standard calculations,

$$\int_\Omega \{\mu(\tau_n)|\mathbf{D}(\mathbf{u}_{\tau_n})|^{p-2}\mathbf{D}(\mathbf{u}_{\tau_n}) - \mu(\tau)|\mathbf{D}(\mathbf{u}_\tau)|^{p-2}\mathbf{D}(\mathbf{u}_\tau)\} : \mathbf{D}(\mathbf{u}_{\tau_n} - \mathbf{u}_\tau) \to 0 \tag{48}$$

which then implies the strong convergence $\mathbf{u}_{\tau_n} \to \mathbf{u}_\tau$ in V_p.

Let $\delta > 0$ and, by (47) and the assumption (13) the bounded function

$$g_\delta = g_\delta(\tau, \mathbf{u}_\tau) = \frac{2\mu(\tau)|\mathbf{D}(\mathbf{u}_\tau)|^p}{1 + 2\delta\mu(\tau)|\mathbf{D}(\mathbf{u}_\tau)|^p} \quad \text{satisfies} \quad ||g_\delta||_{L^1(\Omega)} \leq C_1 \tag{49}$$

where C_1 is a constant independent of δ, τ and $\mathbf{u}_\tau$.

Clearly, we can solve the linear Newmann problem

$$\theta_\delta \in H^1(\Omega) : \kappa \int_\Omega \nabla\theta_\delta \nabla\psi - c\int_\Omega \theta_\delta \mathbf{u}_\tau \nabla\psi + \int_\Omega \theta_\delta \psi = \int_\Omega g_\delta \psi, \ \forall \psi \in H^1(\Omega), \tag{50}$$

remarking that the convective term still makes sense, since $\theta_\delta \mathbf{u}_\delta \in \mathbf{L}^2(\Omega)$.

Taking $\psi = 1$ in (50) and using (49) we obtain

$$0 \leq \frac{1}{|\Omega|}\int_\Omega \theta_\delta = \frac{1}{|\Omega|}\int_\Omega g_\delta \leq \frac{C_1}{|\Omega|}.$$

As in [13], by using in (50) $\psi = \varphi(\theta_\delta)$, where φ is defined by

$$\varphi(t) = \zeta \operatorname{sign}(t) \int_0^{|t|} \frac{ds}{(1+s)^{\zeta+1}} = \operatorname{sign}(t)\Big[1 - \frac{1}{1+|t|^\zeta}\Big], \tag{51}$$

and observing, with $\Phi' = \varphi$, that

$$\int_\Omega \theta_\delta \mathbf{u}_\tau \cdot \nabla\varphi(\theta_\delta) = -\int_\Omega \varphi(\theta_\delta)\mathbf{u}_\tau \cdot \nabla\theta_\delta = -\int_\Omega \mathbf{u}_\tau \cdot \nabla\Phi(\theta_\delta) = 0,$$

we obtain, remarking $\varphi'(t) = \zeta/(1+|t|)^{\zeta+1}$ and using (49),

$$\kappa \int_\Omega \frac{\zeta|\nabla\theta_\delta|^2}{(1+|\theta_\delta|)^{\zeta+1}} \leq \|g_\delta\|_{L^1(\Omega)} \leq C_1. \tag{52}$$

Using Hölder inequality with exponents $2/q$ and $2/(2-q)$, for $q < 2$, we obtain for small $\varepsilon > 0$

$$\begin{aligned}\int_\Omega |\nabla\theta_\delta|^q &\leq \Big(\int_\Omega \frac{|\nabla\theta_\delta|^2}{(1+|\theta_\delta|)^{\zeta+1}}\Big)^{q/2}\Big(\int_\Omega (1+|\theta_\delta|)^{(\zeta+1)q/(2-q)}\Big)^{(2-q)/2} \\ &\leq \Big(\frac{C_1}{\kappa\zeta}\Big)^{q/2}\Big[\varepsilon\Big(\int_\Omega |\theta_\delta|^{q^*}\Big)^{(2-q)/2} + C(\varepsilon)\Big]\end{aligned} \tag{53}$$

by using (52) and noting that

$$(1+|\theta_\delta|)^{(\zeta+1)q/(2-q)} \leq \varepsilon|\theta_\delta|^{q^*} + C(\varepsilon)$$

if $\zeta > 0$ is choosen such that $(\zeta+1)q/(2-q) < q^* = nq/(n-q)$ which is possible for $q < n/(n-1)$.

Using Sobolev inequality with average $\tilde\theta = \fint_\Omega \theta$,

$$\|\theta_\delta - \tilde\theta_\delta\|_{L^{q^*}(\Omega)} \le C_* \|\nabla\theta_\delta\|_{L^q(\Omega)} \tag{54}$$

we have, using (53) with $\kappa > 1$,

$$\Big(\int_\Omega |\theta_\delta|^{q^*}\Big)^{1/q^*} \le C_*\|\nabla\theta_\delta\|_{L^q(\Omega)} + |\Omega|^{1/q^*}|\tilde\theta_\delta| \le$$
$$\le \varepsilon^{1/q} C_* \Big(\frac{C_1}{\zeta}\Big)^{1/2}\Big(\int_\Omega |\theta_\delta|^{q^*}\Big)^{(2-q)/2q} + C'(\varepsilon).$$

Since $(2-q)/2q \le 1/q^* = (n-q)/nq$, for $n \ge 2$ and ε sufficiently small, we find

$$\|\theta_\delta\|_{L^{q^*}(\Omega)} \le C_{q^*} \quad \text{(independent of } \delta) \tag{55}$$

and, from (53), we see that θ_δ satisfies the estimate (46).

The Schauder fixed point theorem guarantees then the existence of a pair $(\mathbf{u}_\delta, \theta_\delta) \in V_p \times H^1(\Omega)$, where $\mathbf{u}_\delta = \mathbf{u}_{\theta_\delta}$ solves (43) with $\mu(\theta_\delta)$ and θ_δ solves (50) with $g_\delta = g_\delta(\theta_\delta, \mathbf{u}_\delta)$ given by (49). Taking a subsequence $\delta \to 0$, using the estimates (46), (47) and (55) we obtain

$$\mathbf{u}_\delta \rightharpoonup \mathbf{u} \quad \text{in} \quad V_p\text{-weak}$$
$$\theta_\delta \to \theta \quad \text{in} \quad W^{1,q}(\Omega)\text{-weak and } L^1(\Omega)\text{-strong.}$$

Hence also $\mathbf{u}_\delta \to \mathbf{u}$ strongly in V_p by the same argument of (48) and, consequently,

$$g_\delta = g_\delta(\theta_\delta, \mathbf{u}_\delta) \to 2\mu(\theta)|\mathbf{D}(\mathbf{u})|^p \quad \text{in} \quad L^1(\Omega),$$

and it is easy to conclude that $(\mathbf{u}, \theta)$ solve (43)-(44). •

Under the assumption (22) and with the definition (21) we can now obtain the corresponding nonlocal problem

$$\mathbf{u} \in V_p \;:\; 2\nu\Big(2\fint_\Omega |\mathbf{D}(\mathbf{u})|^p\Big)\int_\Omega |\mathbf{D}(\mathbf{u})|^{p-2}\mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{v}) = \int_\Omega \mathbf{f}\cdot\mathbf{v}, \quad \forall \mathbf{v} \in V_p. \tag{56}$$

Indeed, if now we denote by $(\mathbf{u}_\kappa, \theta_\kappa)$ the solution of Proposition 2, we can take a subsequence $\kappa \to +\infty$ such that

$$\mathbf{u}_\kappa \rightharpoonup \mathbf{u} \quad \text{in} \quad V_p\text{-weak}$$
$$\nabla\theta_\kappa \to 0 \quad \text{in} \quad L^q(\Omega)\text{-strong}$$
$$\theta_\kappa \to \Theta \quad \text{in} \quad W^{1,q}(\Omega)\text{-strong.}$$

We may then conclude that $\mathbf{u}$ solves (43) with θ replaced by the constant Θ, which is easily seen to satisfy the equation

$$\Theta = \mu(\Theta)2 \fint_\Omega |\mathbf{D}(\mathbf{u})|^p, \quad \text{i.e.,} \quad \Theta = \alpha(2 \fint_\Omega |\mathbf{D}(\mathbf{u})|^p),$$

since $\mathbf{u}_\kappa \to \mathbf{u}$ in V_p-strong and (44) passes to the limit with $\psi \equiv 1$. We have then proven the following theorem.

Theorem 3. *Under the assumptions* (13), (22), *for* $p > n/2$, *there exists a sequence of solutions of* (43)–(44) *such that*

$$\mathbf{u}_\kappa \to \mathbf{u} \quad in \quad V_p, \quad \theta_\kappa \to \alpha(2 \fint_\Omega |\mathbf{D}(\mathbf{u})|^p) \quad in \quad W^{1,q}(\Omega),$$

as $\kappa \to \infty$ *for all* $q < n/(n-1)$, *where* $\mathbf{u}$ *solves* (56). •

Remark 7. The Remarks 3 and 4 are also applicable to the nonlocal problem (56). Notice that for $p = 2$, by recalling (8) and $\int_\Omega \nabla\mathbf{u} : (\nabla\mathbf{v})^T = 0$, for all $\mathbf{u}, \mathbf{v} \in V_2$, we see that (56) reduces to

$$\mathbf{u} \in V_2 \ : \quad \nu\Big(\fint_\Omega |\nabla(\mathbf{u})|^2\Big) \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v} = \int_\Omega \mathbf{f}\cdot\mathbf{v}, \quad \forall \mathbf{v} \in V_2.$$

3.2 Convective case.

In this section we extend the previous results to the Navier-Stokes system, which in variational form corresponds to replace (43) by looking for $\mathbf{u} \in V_p$, such that,

$$2\int_\Omega \mu(\theta)|\mathbf{D}(\mathbf{u})|^{p-2}\mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{v}) + \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v} = \int_\Omega \mathbf{f}\cdot\mathbf{v}, \quad \forall \mathbf{v} \in V_p \quad (57)$$

where $\theta \in W^{1,q}(\Omega)$ solves (44). If $n > p$, by Sobolev imbedding $\mathbf{u}, \mathbf{v} \in V_p \subset \mathbf{L}^s(\Omega)$ for $s = pn/(n-p)$, and so $(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v} \in L^1(\Omega)$ for $p \geq 3n/(n+2)$, which is a stronger restriction that just $p > n/2$, for $n = 2, 3$, that we have assumed in the previous section. In fact, we shall require $p > 3n/(n+2)$ in order to use the compact imbedding

$$V_p \subset\subset J_r \subset \mathbf{L}^r(\Omega), \quad \text{for} \quad r < pn/(n-p), \qquad (58)$$

where J_r denotes the closure of V_r in $\mathbf{L}^r(\Omega), r > 1$.

Let us fix $\bar{r} = 3n/(n-1)(\bar{r} < pn/(n-p)$ if $p > 3n/(n+2))$. By the theory of monotone coercive operators [9], noting that

$$\int_\Omega (\mathbf{w} \cdot \nabla)\mathbf{v} \cdot \mathbf{v} = 0, \quad \forall \mathbf{v} \in V_p, \mathbf{w} \in J_{\bar{r}}$$

for each $\tau \in L^1(\Omega)$ and $\mathbf{w} \in J_{\bar{r}}$, there exists a unique $\mathbf{u} \in V_p$ such that

$$2\int_\Omega \mu(\tau)|\mathbf{D}(\mathbf{u})|^{p-2}\mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{v}) + \int_\Omega (\mathbf{w} \cdot \nabla)\mathbf{u} \cdot \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v}, \quad \forall \mathbf{v} \in V_p.$$

Moreover, the solution operator

$$(\mathbf{w}, \tau) \in J_{\bar{r}} \times L^1(\Omega) \mapsto \mathbf{u} = \mathbf{u}_{\mathbf{w},\tau} \in V_p$$

is continuous for the strong topologies. Indeed, $\mathbf{u}$ satisfies also the a priori estimate (47), independently of τ and $\mathbf{w}$, and

$$\|\mathbf{u}\|_{L^{\bar{r}}(\Omega)} \le C_{\bar{r}}\|\nabla \mathbf{u}\|_{L^p(\Omega)} \le C_{\bar{r}}C_p\|\mathbf{f}\|_{W^{-1,p'}(\Omega)} \equiv \bar{C}.$$

Then if $\tau_n \to \tau$ in $L^1(\Omega)$ and $\mathbf{w}_n \to \mathbf{w}$ in $J_{\bar{r}}$ we obtain first the convergence $\mathbf{u}_n \rightharpoonup \mathbf{u}$ weakly in V_p and, by (58) also strongly in $J_{\bar{r}}$. Then, observing,

$$\int_\Omega (\mathbf{w}_n \cdot \nabla)\mathbf{u}_n \cdot \mathbf{v} = -\int_\Omega (\mathbf{w}_n \otimes \mathbf{u}_n) : \nabla \mathbf{v}$$

and $\mathbf{w}_n \otimes \mathbf{u}_n \to \mathbf{w} \otimes \mathbf{u}$ in $L^{p'}(\Omega)$-strong, we easily conclude also the strong convergence $\mathbf{u}_n \to \mathbf{u}$ in V_p, by an argument similar to the proof of Proposition 2.

Considering now the unique solution $\theta_{\delta,\mathbf{w},\tau}$ to the linear Neumann problem (50) with $\mathbf{u}_\tau$ replaced by $\mathbf{u}_{\mathbf{w},\tau}$ and $g_\delta = g_\delta(\tau, \mathbf{u}_{\mathbf{w},\tau})$ satisfying (49), we may conclude that it also satisfies the apriori estimate (46), for all $q < n/(n-1)$, and, in particular, is bounded in $L^1(\Omega)$ independently of τ and $\mathbf{w}$. We have then a completely continuous application of $J_{\bar{r}} \times L^1(\Omega)$ into itself defined by $(\mathbf{w}, \tau) \mapsto (\mathbf{u}_{\mathbf{w},\tau}, \theta_{\delta,\mathbf{w},\tau})$ and its fixed point $(\mathbf{u}_\delta, \theta_\delta) \in V_p \times H^1(\Omega)$ is guaranteed by the Schauder theorem. Taking a sequence $\delta \to 0$ and repeating the arguments above, for each $\kappa < \infty$, we obtain a solution $(\mathbf{u}_\kappa, \theta_\kappa) \in V_p \times W^{1,q}(\Omega)$ to (57), (44).

Arguing as in Theorem 3 and controling the convective term as above, letting $\kappa \to \infty$ we can obtain a solution to the nonlocal Navier–Stokes problem

$$\mathbf{u} \in V_p \ : \ 2\nu\Big(2\displaystyle\fint_\Omega |\mathbf{D}(\mathbf{u})|^p\Big) \int_\Omega |\mathbf{D}(\mathbf{u})|^{p-2}\mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{v}) \\ + \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v} = \int_\Omega \mathbf{f}\cdot\mathbf{v}, \ \forall \mathbf{v} \in V_p, \quad (59)$$

and we may state the following result.

Theorem 4. *Under the assumptions* (13) *and* $p > 3n/(n+2)$ *for any* $\mathbf{f} \in \mathbf{W}^{-1,p'}(\Omega)$, *there exist solutions* $(\mathbf{u}_\kappa, \theta_\kappa) \in V_p \times W^{1,q}(\Omega)$ *for all* $q < n/(n-1)$, *to* (57)-(44), *such that, if* (22) *holds*

$$\mathbf{u}_\kappa \to \mathbf{u} \ \text{in} \ V_p, \quad \theta_\kappa \to \alpha(2\fint_\Omega |\mathbf{D}(\mathbf{u})|^p) \ \text{in} \ W^{1,q}(\Omega) \ \text{as} \ \kappa \to \infty$$

where $\mathbf{u}$ *solves the nonlocal problem* (59). •

The problem (50) with ν given by (9) (see Remark 2) corresponds to the Ladyzenskaya model for Navier-Stokes equations with nonlocal viscosity.

Remark 8. In fact, for prescribed $\theta \in L^1(\Omega)$, it is possible to obtain weak solutions to (57) (with smoother test functions $\mathbf{v}$) only for $p \geq 2n/(n+1)$, see [5], which is sufficient to guarantee $(\mathbf{u}\cdot\nabla)\mathbf{u} \in \mathbf{L}^1(\Omega)$. However the problem coupled with the temperature requires the $L^1(\Omega)$ convergence of the energy of dissipation $2\mu(\theta)|\mathbf{D}(\mathbf{u})|^p$, for which we had to assume the stronger condition $p > 3n/(n+2)$.

Remark 9. It is easy to extend Theorem 4 to the case of a Bingham fluid with dissipation of energy given by

$$\Phi = 2\mu(\theta)|\mathbf{D}(\mathbf{u})|^2 + \eta(\theta)|\mathbf{D}(\mathbf{u})|,$$

by replacing (57) by the variational inequality

$$2\int_\Omega \mu(\theta)\mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{v}-\mathbf{u}) + \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v} \\ + \int_\Omega \eta(\theta)\Big(|\mathbf{D}(\mathbf{u})| - |\mathbf{D}(\mathbf{v})|\Big) \geq \int_\Omega \mathbf{f}\cdot(\mathbf{v}-\mathbf{u})$$

for any $\mathbf{v} \in V_2$. The respective nonlocal problem is then given

$$2\mu(\Theta) \int_\Omega \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{v} - \mathbf{u}) + \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{v} + \eta(\Theta) \int_\Omega \Big(|\mathbf{D}(\mathbf{u})| - |\mathbf{D}(\mathbf{v})|\Big) \geq \int_\Omega \mathbf{f} \cdot (\mathbf{v} - \mathbf{u})$$

where Θ satisfies the condition (36).

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November 15, 2002