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SOLVABILITY IN WEIGHTED HÖLDER SPACES FOR A PROBLEM GOVERNING THE EVOLUTION OF TWO COMPRESSIBLE FLUIDS

Abstract. Local (in time) unique solvability of the problem on the motion of two compressible fluids, one of which has a finite volume, is obtained in Hölder spaces of functions with power-like decay at infinity. After the passage to Lagrangian coordinates, we arrive at a nonlinear initial-boundary value problem with a given closed interface between the liquids. We establish the existence theorem for this problem on the basis of the solvability of a linearized one by means of the fixed-point theorem. To obtain the estimates and to prove solvability for the linearized problem, we use the Schauder method and an explicit solution of a model linear problem with a plane interface between the liquids. All results are obtained under some restrictions to the fluid density and viscosities, which mean that the fluids are not so different from each other.

**In honour of the jubilee of
Olga Alexandrovna Ladyzhenskaya**

1. Statement of the problem and formulation of the main result

In this paper, we study the unsteady motion of two compressible fluids separated by a closed interface. We assume that the compressible fluids are barotropic, and the surface tension forces are taken into account on the interface. The main result of this study is a theorem on local unique solvability of the described problem in Hölder spaces with power-like weight at infinity (with respect to the spatial variables). In order to prove this theorem, we develop the technique used by V. A. Solonnikov and I. V. Denisova for investigating the bubble motion in vacuum [1, 2].

Now we formulate our problem. At the initial moment $t = 0$ let a fluid with dynamical viscosities $\mu^+ > 0$, $\lambda^+ \equiv \beta^+\mu^+$ occupy a bounded domain

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Ω_0^+. And let a fluid with dynamical viscosities $\mu^- > 0$, $\lambda^- \equiv \beta^-\mu^-$ be situated in the "exterior" domain $\Omega_0^- \equiv \mathbb{R}^3 \setminus \overline{\Omega_0^+}$.

For every $t > 0$, it is necessary to find the interface $\Gamma_t = \partial\Omega_t^+$ between the liquids, as well as their density functions $\rho(x,t) > 0$ and the velocity vector field $\mathbf{v}(x,t) = (v_1, v_2, v_3)$ satisfying the following initial-boundary value problem for the Navier–Stokes system:

$$\begin{aligned} &\rho(\mathcal{D}_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}) - \nabla\cdot\mathbb{T} = \rho\mathbf{f}, \\ &\mathcal{D}_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0 \quad \text{in} \quad \Omega_t^- \cup \Omega_t^+, \\ &\rho\big|_{t=0} = \rho_0, \quad \mathbf{v}\big|_{t=0} = \mathbf{v}_0 \quad \text{in} \quad \Omega_0^- \cup \Omega_0^+, \quad \mathbf{v} \underset{|x|\to\infty}{\longrightarrow} 0, \end{aligned} \tag{1.1}$$

$$\begin{aligned} &[\mathbf{v}]\Big|_{\Gamma_t} \equiv \lim_{\substack{x\to x_0\in\Gamma_t \\ x\in\Omega_t^+}} \mathbf{v}(x) - \lim_{\substack{x\to x_0\in\Gamma_t \\ x\in\Omega_t^+}} \mathbf{v}(x) = 0, \\ &[\mathbb{T}\mathbf{n}]\Big|_{\Gamma_t} = \sigma H\mathbf{n} \quad \text{on} \quad \Gamma_t. \end{aligned} \tag{1.2}$$

Here $\mathcal{D}_t = \partial/\partial t$, $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$, the stress tensor is

$$\mathbb{T} = \begin{cases} (-p(\rho) + \lambda^+\nabla\cdot\mathbf{v})\,\mathbb{I} + \mu^+\mathbb{S}(\mathbf{v}) & \text{in } \Omega_t^+, \\ (-p(\rho) + \lambda^-\nabla\cdot\mathbf{v})\,\mathbb{I} + \mu^-\mathbb{S}(\mathbf{v}) & \text{in } \Omega_t^-, \end{cases} \tag{1.3}$$

$(\mathbb{S}(\mathbf{v}))_{ik} = \partial v_i/\partial x_k + \partial v_k/\partial x_i$, $i,\ k = 1,2,3$;
$\mathbb{I}$ is the unit matrix; $\mu^- = \nu^-\rho^-$;
$p(\rho)$ is the pressure of the fluids given by a smooth function of the density;
$\mathbf{f}$ is the given vector field of mass forces;
$\mathbf{v}_0$ is the initial value of the velocity vector field;
ρ_0 is the initial density distribution of the fluids;
$\sigma \geq 0$ is the surface tension coefficient,
$\mathbf{n}$ is the outward normal vector to Ω_t^+,
$H(x,t)$ is twice the mean curvature of Γ_t ($H < 0$ at the points where Γ_t is convex towards Ω_t^-);
$\nabla\cdot\mathbb{T}$ means the vector with the components $(\nabla\cdot\mathbb{T})_j = \frac{\partial T_{ij}}{\partial x_i}$, $T_{ij} = (\mathbb{T})_{ij}$, $j = 1,2,3$. We imply the summation from 1 to 3 with respect to repeated indexes. A Cartesian coordinate system $\{x\}$ is introduced in $\mathbb{R}^3$. The central dot denotes the scalar product. We mark the vectors and the vector spaces by boldface letters.

Since we suppose the liquids to be immiscible, it is natural to impose on Γ_t a condition excluding the transportation of mass through this surface.

Mathematically, this condition means that Γ_t consists of the points $x(\xi,t)$ whose radius vector $\mathbf{x}(\xi,t)$ is a solution of the Cauchy problem

$$\mathcal{D}_t\mathbf{x} = \mathbf{v}(x(\xi,t),t), \quad \mathbf{x}(\xi,0)=\xi, \quad \xi\in\Gamma, \quad t>0, \tag{1.4}$$

where $\Gamma \equiv \Gamma_0 = \partial\Omega_0^+$ is a surface given at the initial moment. Hence, $\Omega_t^\pm = \{x = x(\xi,t) | \xi \in \Omega_0^\pm\}$.

Condition (1.4) completes system (1.1)–(1.2).

We transform the Eulerian coordinates $\{x\}$ into the Lagrangian ones $\{\xi\}$ by the formula

$$\mathbf{x}(\xi,t) = \boldsymbol{\xi} + \int_0^t \mathbf{u}(\xi,\tau)d\tau \equiv \mathbf{X}_\mathbf{u}(\xi,t), \tag{1.5}$$

where $\mathbf{u}(\xi,t)$ is the velocity vector field in the Lagrangian coordinates.

The Jacobian of transformation (1.5)

$$\mathcal{J}_\mathbf{u}(\xi,t) = \det\{a_{ij}\}_{i,j=1}^3, \quad a_{ij}(\xi,t) = \delta_j^i + \int_0^t \frac{\partial u_i}{\partial \xi_j}\, d\tau,$$

being a solution of the Cauchy problem

$$\mathcal{D}_t\mathcal{J}_\mathbf{u}(\xi,t) = A_{ij}\frac{\partial u_i}{\partial \xi_j} \equiv \mathcal{J}_\mathbf{u}(\xi,t)(\nabla\cdot\mathbf{v}|_{x=X_\mathbf{v}}), \quad \mathcal{J}_\mathbf{u}(\xi,0)=1,$$

is expressed by the formula

$$\mathcal{J}_\mathbf{u}(\xi,t) = \exp\left(\int_0^t \nabla\cdot\mathbf{v}|_{x=X_\mathbf{u}}\, d\tau\right) \equiv \exp\left(\int_0^t \nabla_\mathbf{u}\cdot\mathbf{u}\, d\tau\right). \tag{1.6}$$

Here we use the standard notation $\{\delta_j^i\}_{i,j=1}^3$ for the Kronecker symbols and also $\nabla_\mathbf{u} \equiv \left\{\frac{\partial\xi_i}{\partial x_k}\frac{\partial}{\partial \xi_i}\right\}_{k=1}^3 = \mathcal{J}_\mathbf{u}^{-1}\mathbb{A}\nabla$; $\mathbb{A}\equiv\{A_{ij}\}_{i,j=1}^3$ is the cofactors matrix of the Jacobi matrix $\{a_{ij}\}$ in (1.5).

After transformation (1.5), we can integrate the second equation in (1.1) and exclude the variable density

$$\widehat{\rho^+}(\xi,t) = \rho_0^+(\xi)\exp\left(-\int_0^t \nabla_\mathbf{u}\cdot\mathbf{u}\, d\tau\right) = \rho_0^+(\xi)\mathcal{J}_\mathbf{u}^{-1}(\xi,t)$$

from the equations.

Next, we use the well-known formula for twice the surface mean curvature:

$$H\mathbf{n} = \Delta(t)\mathbf{x} = \Delta(t)\mathbf{X}_{\mathbf{u}},$$

where $\Delta(t)$ is the Beltrami–Laplace operator on Γ_t. Moreover, we separate the last boundary condition in (1.2) on the tangential and normal components. To this end, we project it first onto the tangent plane of Γ_t and then onto that of Γ by means of projectors Π and Π_0, respectively.

Let $\mathbf{n}_0$ be the outward normal to Γ. It is connected with $\mathbf{n}$ by the relation

$$\mathbf{n} = \frac{\mathcal{J}_{\mathbf{u}}^{-1}\mathbb{A}\mathbf{n}_0}{|\mathcal{J}_{\mathbf{u}}^{-1}\mathbb{A}\mathbf{n}_0|} = \frac{\mathbb{A}\mathbf{n}_0}{|\mathbb{A}\mathbf{n}_0|}.$$

As a result of the above transformations, we arrive at the system:

$$\mathcal{D}_t\mathbf{u} - \frac{1}{\rho(\xi)}\mathbb{A}\nabla\mathbb{T}'_{\mathbf{u}}(\mathbf{u}) = \mathbf{f}(X_{\mathbf{u}},t) - \frac{1}{\rho(\xi)}\mathbb{A}\nabla p(\rho\mathcal{J}_{\mathbf{u}}^{-1}) \quad \text{in} \quad D_T \equiv Q_T^+ \cup Q_T^-,$$

$$\mathbf{u}\big|_{t=0} = \mathbf{v}_0 \quad \text{in} \quad \Omega_0^- \cup \Omega_0^+, \quad \mathbf{u} \underset{|\xi|\to\infty}{\longrightarrow} 0, \tag{1.7}$$

$$[\mathbf{u}]\Big|_{G_T} = 0, \quad [\mu^{\pm}\Pi_0\Pi\mathbb{S}_{\mathbf{u}}(\mathbf{u})\mathbf{n}]\Big|_{G_T} = 0 \quad (G_T \equiv \Gamma\times(0,T)),$$

$$[\mathbf{n}_0\cdot\mathbb{T}'_{\mathbf{u}}(\mathbf{u},q)\mathbf{n}]\Big|_{G_T} - \sigma\mathbf{n}_0\cdot\Delta(t)\mathbf{X}_{\mathbf{u}}\Big|_{G_T} = (\mathbf{n}_0\cdot\mathbf{n})[p(\rho\mathcal{J}_{\mathbf{u}}^{-1})]\Big|_{G_T}$$

which is equivalent to (1.1)–(1.4) provided that $\mathbf{n}\cdot\mathbf{n}_0 > 0$. In (1.7) we have used the notation: $q(\xi,t)$ is the pressure function in the Lagrangian coordinates;

$$(\mathbb{T}'_{\mathbf{u}}(\mathbf{w},q))_{i,j} = \begin{cases} (\lambda^+\nabla_{\mathbf{u}}\cdot\mathbf{w})\delta_j^i + \mu^+(\mathbb{S}_{\mathbf{u}}(\mathbf{w}))_{ij} & \text{in} \quad Q_T^+ \equiv \Omega_0^+\times(0,T), \\ (\lambda^-\nabla_{\mathbf{u}}\cdot\mathbf{w})\delta_j^i + \mu^-(\mathbb{S}_{\mathbf{u}}(\mathbf{w}))_{ij} & \text{in} \quad Q_T^- \equiv \Omega_0^-\times(0,T); \end{cases}$$

$$(\mathbb{S}_{\mathbf{u}}(\mathbf{w}))_{ij} = \mathcal{J}_{\mathbf{u}}^{-1}\left(A_{ik}\frac{\partial w_j}{\partial\xi_k} + A_{jk}\frac{\partial w_i}{\partial\xi_k}\right);$$

$$\Pi_0\omega = \omega - (\mathbf{n}_0\cdot\omega)\mathbf{n}_0, \quad \Pi\omega = \omega - (\mathbf{n}\cdot\omega)\mathbf{n}.$$

The problem (1.7) is uniquely solvable in Sobolev–Slobodetskii spaces of functions for a sufficiently small time interval $(0,T_0)$: $\mathbf{u} \in \bigcup\limits_{i=-,+} \mathbf{W}_2^{2+l,1+l/2}(Q_{T_0}^i)$, $l > 1/2$ (see [3]).

In the present work, we consider this problem in Hölder spaces with power-like weights. These classes of functions were introduced in [4] for a study of a similar problem concerning two incompressible fluids.

The Hölder spaces with power-like weights are defined as follows. Let Ω be a domain in $\mathbb{R}^n$, $n \in \mathbb{N}$; for $T > 0$ we put $\Omega_T = \Omega \times (0,T)$; finally, let $\alpha \in (0,1)$, $\beta \geqslant 0$. By $C_\beta^{\alpha,\alpha/2}(\Omega_T)$ we denote the set of functions f on Ω_T having the norm

$$\|f\|_{\beta,\Omega_T}^{(\alpha,\alpha/2)} = \|f\|_{\beta,\Omega_T} + \langle f\rangle_{\beta,\Omega_T}^{(\alpha,\alpha/2)},$$

where

$$\|f\|_{\beta,\Omega_T} = \sup_{t\in(0,T)}\sup_{x\in\Omega}(1+|x|)^\beta|f(x,t)|,$$

$$\langle f\rangle_{\beta,\Omega_T}^{(\alpha,\alpha/2)} = \langle f\rangle_{x,\beta,\Omega_T}^{(\alpha)} + \langle f\rangle_{t,\beta,\Omega_T}^{(\alpha/2)},$$

$$\langle f\rangle_{x,\beta,\Omega_T}^{(\alpha)} = \sup_{t\in(0,T)}\sup_{x,y\in\Omega}(1+d_{x,y})^\beta|f(x,t)-f(y,t)||x-y|^{-\alpha},$$

$$\langle f\rangle_{t,\beta,\Omega_T}^{(\mu)} = \sup_{x\in\Omega}\sup_{t,\tau\in(0,T)}(1+|x|)^\beta|f(x,t)-f(x,\tau)||t-\tau|^{-\mu},$$

$$\mu\in(0,1), \quad d_{x,y} = \min(|x|,|y|).$$

We introduce the following notation:

$$\mathcal{D}_x^{\mathbf{r}} = \partial^{|\mathbf{r}|}/\partial x_1^{r_1}\dots\partial x_n^{r_n}, \quad \mathbf{r} = (r_1,\dots r_n), \quad r_i \geqslant 0,\ |\mathbf{r}| = r_1+\dots+r_n,$$
$$\mathcal{D}_t^s = \partial^s/\partial t^s, \quad s \in \mathbb{N}\cup\{0\}.$$

Let $k\in\mathbb{N}$. By definition, the space $C_\beta^{k+\alpha,(k+\alpha)/2}(\Omega_T)$ consists of functions f with finite norm

$$\|f\|_{\beta,\Omega_T}^{(k+\alpha,\frac{k+\alpha}{2})} = \sum_{|\mathbf{r}|+2s\leqslant k}\|\mathcal{D}_x^{\mathbf{r}}\mathcal{D}_t^s f\|_{\beta,\Omega_T} + \langle f\rangle_{\beta,\Omega_T}^{(k+\alpha,\frac{k+\alpha}{2})},$$

where

$$\langle f\rangle_{\beta,\Omega_T}^{(k+\alpha,\frac{k+\alpha}{2})} = \sum_{|\mathbf{r}|+2s=k}\langle\mathcal{D}_x^{\mathbf{r}}\mathcal{D}_t^s f\rangle_{x,\beta,\Omega_T}^{(\alpha)} + \sum_{|\mathbf{r}|+2s=k-1}\langle\mathcal{D}_x^{\mathbf{r}}\mathcal{D}_t^s f\rangle_{t,\beta,\Omega_T}^{(\frac{1+\alpha}{2})}.$$

The symbol $\overset{\circ}{C}{}_\beta^{k+\alpha,(k+\alpha)/2}(\Omega_T)$ denotes the subspace of $C_\beta^{k+\alpha,(k+\alpha)/2}(\Omega_T)$ whose elements f have the property: $\mathcal{D}_t^i f\big|_{t=0} = 0$, $i = 0,\dots,\left[\dfrac{k+\alpha}{2}\right]$.

We define $C_\beta^{k+\alpha}(\Omega)$, $k\in\mathbb{N}\cup\{0\}$, as the space of functions $f(x)$, $x\in\Omega$, with the norm

$$\|f\|_{\beta,\Omega}^{(k+\alpha)}=\sum_{|\mathbf{r}|\leqslant k}\|\mathcal{D}_x^{\mathbf{r}}f\|_{\beta,\Omega}+\langle f\rangle_{\beta,\Omega}^{(k+\alpha)}.$$

Here

$$\begin{aligned}\langle f\rangle_{\beta,\Omega}^{(k+\alpha)}&=\sum_{|\mathbf{r}|=k}\langle\mathcal{D}_x^{\mathbf{r}}f\rangle_{\beta,r}^{(\alpha)}\\&=\sum_{|\mathbf{r}|=k}\sup_{x,y\in\Omega}(1+d_{x,y})^\beta|\mathcal{D}_x^{\mathbf{r}}f(x)-\mathcal{D}_y^{\mathbf{r}}f(y)||x-y|^{-\alpha}.\end{aligned}$$

Let us omit the index β in all these norms if it equals zero.

Theorem 1.1. *Suppose that for* $\alpha\in(0,1)$, $\gamma>0$, $T\in(0,\infty)$, $\Gamma\in C^{3+\alpha}$, $\rho_0\in C^{l+\alpha}(\Omega_0^-\cup\Omega_0^+)$, $R_1\geqslant\rho_0(\xi)\geqslant R_0>0$, $p\in C^3(\mathbb{R}_+)$, $\mathbf{f}$, $\mathcal{D}_x\mathbf{f}\in\mathbf{C}_\gamma^{\alpha,\frac{\alpha}{2}}(\mathbb{R}_T^3)$, $\mathbf{v}_0\in\mathbf{C}_\gamma^{2+\alpha}(\Omega_0^-\cup\Omega_0^+)$, $\sigma\in C^{1+\alpha}(\Gamma)$, $\sigma(x)\geqslant0$, $x\in\Gamma$. *In addition, let the dynamical viscosities and the constants* R_1, R_0 *satisfy the inequalities*

$$\frac{\mu^-}{2}\leqslant\mu^+\leqslant2\mu^-,\ 0<\lambda^\pm\leqslant\mu^\pm,\quad\frac{R_1}{R_0}\leqslant2.\tag{1.8}$$

We also assume the compatibility conditions

$$[\mathbf{v}_0]|_\Gamma=0,\quad\Big[\rho_0^{-1}(\xi)(\nabla\cdot\mathbb{T}(\mathbf{v}_0)-\nabla p(\rho_0(\xi)))\Big]\Big|_\Gamma=0,$$

$$[-p(\rho_0)\mathbf{n}_0+\mathbb{T}(\mathbf{v}_0)\mathbf{n}_0]|_\Gamma=\sigma H_0\mathbf{n}_0$$

hold. Here $H_0(\xi)=\mathbf{n}_0\cdot\Delta(0)\boldsymbol{\xi}$ *is the doubled mean curvature of the surface* Γ, $\mathbb{R}_T^3=\mathbb{R}^3\times(0,T)$.

Then the problem (1.7) *is uniquely solvable on some finite time interval* $(0,T_0)$, $T_0\leqslant T$, *whose length depends on the norms of* $\mathbf{f}$, $\mathbf{v}_0$, p, ρ_0 *and on the curvature of* Γ. *The solution* $\mathbf{u}\in\mathbf{C}_\gamma^{2+\alpha,1+\alpha/2}(D_{T_0})$.

This theorem is proved on the basis of the Banach theorem, the problem (1.7) being considered as a perturbation of the linear system (4.1). The existence for problem (4.1) is established by constructing a regularizer, while the uniqueness follows from *a priori* coercive estimates obtained by the Schauder method [2]. Both these methods use an explicit solution of a model problem with a plane interface Γ.

2. The homogeneous model problem with a plane interface between the fluids

In this section, we consider the linear problem with noncoercive boundary conditions

$$\mathcal{D}_t\mathbf{v} - \nu^+\nabla^2\mathbf{v} - (\nu^+ + \kappa^+)\nabla(\nabla\cdot\mathbf{v}) = 0 \quad \text{in} \quad D^+_\infty = \mathbb{R}^3_+ \times (0,\infty),$$

$$\mathcal{D}_t\mathbf{v} - \nu^-\nabla^2\mathbf{v} - (\nu^- + \kappa^-)\nabla(\nabla\cdot\mathbf{v}) = 0 \quad \text{in} \quad D^-_\infty = \mathbb{R}^3_- \times (0,\infty),$$

$$\mathbf{v}\big|_{t=0} = 0 \quad \text{on} \quad \mathbb{R}^3_- \cup \mathbb{R}^3_+, \quad \mathbf{v} \underset{|x|\to\infty}{\longrightarrow} 0, \tag{2.1}$$

$$[\mathbf{v}]\big|_{x_3=0} = 0, \quad -\left[\mu^\pm\left(\frac{\partial v_\beta}{\partial x_3} + \frac{\partial v_3}{\partial x_\beta}\right)\right]\bigg|_{x_3=0} = b_\beta(x',t), \quad \beta = 1,2;$$

$$\begin{aligned}\left[\lambda^\pm\nabla\cdot\mathbf{v} + 2\mu^\pm\frac{\partial v_3}{\partial x_3}\right]\bigg|_{x_3=0} &+ \sigma\Delta'\int_0^t v_3\,d\tau\bigg|_{x_3=0} \\ &= b_3 + \sigma\int_0^t B\,d\tau \equiv b_3' \quad \text{on} \quad \mathbb{R}^2_\infty.\end{aligned} \tag{2.2}$$

Here we have used the notation: $\mathbb{R}^3_\pm = \{\pm x_3 > 0\}$, $\mathbb{R}^2_\infty = \mathbb{R}^2 \times (0,\infty)$, $\nu^+ = \mu^+/\rho_0^+$, $\kappa^+ = \lambda^+/\rho_0^+$, $\nu^- = \mu^-/\rho_0^-$, $\kappa^- = \lambda^-/\rho_0^-$, ρ_0^+, $\rho_0^- > 0$ are constants; $\Delta' = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$, $x' = (x_1, x_2)$.

An explicit solution to this problem after the Fourier–Laplace transform was found in [3]. We will estimate the Hölder norm of this solution by the method from [1] where the unknown vector $\mathbf{v}$ was considered as a solution of a "good" (coercive) problem, the integral term in the boundary condition (2.2) being joined with the given functions in the right-hand side.

So, we take the Fourier transform on the tangent space variables $(x_1, x_2) = x'$ and the Laplace transform on t

$$\widetilde{f}(\xi, x_3, s) = \int_0^\infty \mathrm{e}^{-st}\int_{\mathbb{R}^2} f(x,t)\mathrm{e}^{-ix'\cdot\xi}\mathrm{d}x'\mathrm{d}t, \quad \xi = (\xi_1, \xi_2).$$

The problem (2.1), (2.2) then reduces to the system of ordinary differential equations

$$\begin{aligned} &s\widetilde{v}_\beta - (\nu^\pm + \kappa^\pm) i\xi_\beta \left(i\xi_1 \widetilde{v}_1 + i\xi_2 \widetilde{v}_2 + \frac{d\widetilde{v}_3}{dx_3} \right) \\ &\qquad -\nu^\pm \left(\frac{d^2 \widetilde{v}_\beta}{dx_3^2} - \xi^2 \widetilde{v}_\beta \right) = 0, \quad \beta = 1, 2, \\ &s\widetilde{v}_3 - (\nu^\pm + \kappa^\pm) \frac{d}{dx_3} \left(i\xi_1 \widetilde{v}_1 + i\xi_2 \widetilde{v}_2 + \frac{d\widetilde{v}_3}{dx_3} \right) \\ &\qquad -\nu^\pm \left(\frac{d^2 \widetilde{v}_3}{dx_3^2} - \xi^2 \widetilde{v}_3 \right) = 0 \end{aligned} \tag{2.3}$$

with the boundary conditions

$$\widetilde{\mathbf{v}} \underset{|x_3|\to\infty}{\longrightarrow} 0, \quad [\widetilde{\mathbf{v}}]\big|_{x_3=0} = 0,$$

$$\begin{aligned} &-\left[\mu^\pm \left(\frac{d\widetilde{v}_\beta}{dx_3} + i\xi_\beta \widetilde{v}_3 \right) \right] \Big|_{x_3=0} = \widetilde{b}_\beta, \quad \beta = 1, 2, \\ &\left[\lambda^\pm \left(i\xi_1 \widetilde{v}_1 + i\xi_2 \widetilde{v}_2 + \frac{d\widetilde{v}_3}{dx_3} \right) + 2\mu^\pm \frac{d\widetilde{v}_3}{dx_3} \right] \Big|_{x_3=0} - \frac{\sigma}{s} \xi^2 \widetilde{v}_3 \Big|_{x_3=0} \\ &\qquad = \widetilde{b}_3 + \frac{\sigma}{s} \widetilde{B}. \end{aligned} \tag{2.4}$$

Here $\nu^\pm$, $\kappa^\pm$, $\mu^\pm$, $\lambda^\pm$ are the step-functions of viscosities equal to ν^+, κ^+, μ^+, λ^+ in $\mathbb{R}^3_+$ and to ν^-, κ^-, μ^-, λ^- in $\mathbb{R}^3_-$, respectively; $\xi^2 = \xi_1^2 + \xi_2^2$.

We solve the problem (2.3), (2.4) and we write the solution in the form convenient for the estimates:

$$\widetilde{\mathbf{v}} = \mathbf{W} e_0^\pm + \mathbf{V}^\pm e_1^\pm, \quad \pm x_3 > 0, \tag{2.5}$$

$$e_0^\pm = \mathrm{e}^{\mp r^\pm x_3}, \quad e_1^+ = \frac{\mathrm{e}^{-r^+ x_3} - \mathrm{e}^{-|\xi| x_3}}{r^+ - |\xi|}, \quad e_1^- = \frac{\mathrm{e}^{r^- x_3} - \mathrm{e}^{r_1^- x_3}}{r^- - r_1^-},$$

where $r^\pm = \sqrt{\frac{s}{\nu^\pm} + \xi^2}$, $r_1^\pm = \sqrt{\frac{s}{(2+\beta^\pm)\nu^\pm} + \xi^2}$, $|\xi| = \sqrt{\xi_1^2 + \xi_2^2}$, $|\arg \sqrt{z}| < \pi/2$ for $\forall z \in \mathbb{C}$, $\beta^\pm = \kappa^\pm / \nu^\pm$;

$$\mathbf{W} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}, \quad \mathbf{V}^\pm = -C_3^\pm (r^\pm - r_1^\pm) \begin{pmatrix} i\xi_1 \\ i\xi_2 \\ \mp r_1^\pm \end{pmatrix},$$

$$\begin{aligned}
C_3^+ &= \frac{A}{P}\Big\{\rho_0^-(r^+ + r^-) + \frac{r^- r_1^- - \xi^2}{s}\Big[2(\mu^+ - \mu^-)r^+ + \frac{\sigma}{s}\xi^2\Big]\Big\} \\
&\quad + \frac{\widetilde{b}_3'}{P}\Big\{\rho_0^- r_1^-(r^+ + r^-) + \frac{r^- r_1^- - \xi^2}{s}[\mu^+({r^+}^2 + \xi^2) - \mu^-({r^-}^2 + \xi^2)]\Big\}, \\
C_3^- &= \frac{A}{P}\Big\{\rho_0^+(r^+ + r^-) - \frac{r^+ r_1^+ - \xi^2}{s}\Big[2(\mu^+ - \mu^-)r^- - \frac{\sigma}{s}\xi^2\Big]\Big\} \\
&\quad - \frac{\widetilde{b}_3'}{P}\Big\{\rho_0^+ r_1^+(r^+ + r^-) - \frac{r^+ r_1^+ - \xi^2}{s}[\mu^+({r^+}^2 + \xi^2) - \mu^-({r^-}^2 + \xi^2)]\Big\}, \\
\omega_3 &= \frac{-(r^+ r_1^+ - \xi^2)C_3^+ + (r^- r_1^- - \xi^2)C_3^-}{r^+ + r^-}. \qquad (2.6)
\end{aligned}$$

In these formulas we have used the notation: $A = i\xi_1\widetilde{b}_1 + i\xi_2\widetilde{b}_2$, $\widetilde{b}_3' = \widetilde{b}_3 + \frac{\sigma}{s}\widetilde{B}$,

$$\begin{aligned}
P &= s\left(\frac{\rho_0^+}{r_1^+} + \frac{\rho_0^-}{r_1^-}\right)q + \rho_0^+\rho_0^- s\left(\frac{r_1^+}{r_1^-} + \frac{r_1^-}{r_1^+}\right)\xi^2 \\
&\quad + 4\rho_0^+\left(\mu^+ - \frac{\mu^-}{2}\right)\xi^2\frac{(1+\beta^+)r^+(r^- r_1^- - \xi^2)}{(2+\beta^+)(r^+ + r_1^+)} \\
&\quad + 4\rho_0^-\left(\mu^- - \frac{\mu^+}{2}\right)\xi^2\frac{(1+\beta^-)r^-(r^+ r_1^+ - \xi^2)}{(2+\beta^-)(r^- + r_1^-)} \\
&\quad + 2\rho_0^+\rho_0^- s\xi^2\Big\{\frac{(1+\beta^-)r^+}{(2+\beta^+)(2+\beta^-)(r^+ + r_1^+)} \\
&\qquad + \frac{(1+\beta^+)r^-}{(2+\beta^+)(2+\beta^-)(r_1^- + r^-)} + \frac{\beta^+\beta^-}{(2+\beta^+)(2+\beta^-)} \\
&\qquad + \frac{(1-\beta^+\beta^-)(r^+ r_1^- + r^- r_1^+) + 2(2+\beta^+ + \beta^-)r_1^+ r_1^-}{(2+\beta^+)(2+\beta^-)(r^+ + r_1^+)(r^- + r_1^-)}\Big\} \\
&\quad + \sigma\xi^2 q \equiv \sum_{j=1}^{6} I_j,
\end{aligned}$$

where

$$q = \rho_0^+ r_1^+\frac{(r^- r_1^- - \xi^2)}{s} + \rho_0^- r_1^-\frac{(r^+ r_1^+ - \xi^2)}{s}.$$

We consider the solution (2.5) as a solution of the same system (2.1) but with the following coercive boundary condition:

$$\left[\lambda^\pm \nabla\cdot\mathbf{v} + 2\mu^\pm\frac{\partial v_3}{\partial x_3}\right]\Big|_{x_3=0} = a_3 \quad \text{on} \quad \mathbb{R}^2_\infty \qquad (2.7)$$

instead of (2.2). Then $\widetilde{a}_3$ is expressed as follows:

$$\widetilde{a}_3 = \widetilde{b}'_3 + \frac{\sigma\xi^2}{s}\widetilde{v}_3\bigg|_{x_3=0} = \widetilde{b}'_3 + \frac{\sigma\xi^2}{s}\omega_3. \tag{2.8}$$

Problem (2.1), (2.7) is a diffraction problem with a system parabolic in the sense of Petrovskii and with boundary conditions for which the complementary condition in the sense of Lopatinskii holds. Hence, it will be uniquely classical solvable and for its solution the coercive estimate will take place if we prove that the norm of a_3 is evaluated in a corresponding Hölder class. We do it after the inverse Fourier–Laplace transformation of (2.8) by estimating convolutions.

Now we prove some auxiliary propositions, assuming that the dynamical viscosities and the densities satisfy the restrictions

$$\begin{gathered}\frac{\mu^-}{2} \leqslant \mu^+ \leqslant 2\mu^-, \quad 0 < \beta^\pm \equiv \frac{\lambda^\pm}{\mu^\pm} \leqslant 1,\\ \frac{\rho^-}{2} \leqslant \rho^+ \leqslant 2\rho^-.\end{gathered} \tag{2.9}$$

Lemma 2.1. *If $s \in \mathbb{C}$ is such that* $\operatorname{Re} s > 0$, $\operatorname{Im} s \geqslant 0$ *then*

$$\arg r^\pm, \quad \arg r_1^\pm \in \Big(0, \frac{\pi}{4}\Big); \tag{2.10}$$

$$\arg\Big(\frac{r^\pm}{r_1^\pm}\Big), \quad \arg\Big(\frac{r^\pm}{r^\pm + r_1^\pm}\Big) \in \Big(0, \frac{\pi}{12}\Big); \tag{2.11}$$

$$\arg\Big(\frac{r_1^\pm}{r_1^\mp}\Big), \quad \arg\Big(\frac{r^\pm}{r_1^\mp}\Big) \in \Big(-\frac{\pi}{6}, \frac{\pi}{6}\Big); \tag{2.12}$$

$$\arg\frac{r^\pm r_1^\pm - \xi^2}{s} \in \Big(-\frac{\pi}{12}, 0\Big); \tag{2.13}$$

$$\arg q \in \Big(-\frac{\pi}{12}, \frac{\pi}{4}\Big). \tag{2.14}$$

Remark 2.1. We suppose here and in the next that $\operatorname{Im} s \geqslant 0$. It does not restrict the generality because in the case $\operatorname{Im} s \leqslant 0$, we have a symmetric picture with respect to the axis $\operatorname{Im} s = 0$.

Proof. Relation (2.10) is evident.

To prove (2.11), we consider the function

$$\varphi_1(|\xi|) = 2\arg\Big(\frac{r^+}{r_1^+}\Big) = \arg\Big(\frac{{r^+}^2}{{r^+_1}^2}\Big) = \arg\Big(\frac{s}{\nu^+} + \xi^2\Big) - \arg\Big(\frac{s}{2\nu^+ + \kappa^+} + \xi^2\Big)$$

$$= \operatorname{arcctg}\Big(\frac{\xi^2 + \operatorname{Re} s/\nu^+}{\operatorname{Im} s/\nu^+}\Big) - \operatorname{arcctg}\Big(\frac{\xi^2 + \operatorname{Re}(s/(2\nu^+ + \kappa^+))}{\operatorname{Im} s/(2\nu^+ + \kappa^+)}\Big) \geqslant 0.$$

We find the extrema of this function. The derivative

$$\varphi_1'(|\xi|) = -\frac{2|\xi|}{1 + (\frac{\xi^2 + \operatorname{Re} s/\nu^+}{\operatorname{Im} s/\nu^+})^2} \cdot \frac{1}{\operatorname{Im} s/\nu^+} + \frac{2|\xi|}{1 + (\frac{\xi^2 + \operatorname{Re} s/(2\nu^+ + \kappa^+)}{\operatorname{Im} s/(2\nu^+ + \kappa^+)})^2} \frac{1}{\operatorname{Im} s/(2\nu^+ + \kappa^+)}$$

equals zero at

$$|\xi_*| = 0 \quad \text{and} \quad |\xi_{**}| = \sqrt{|s|/\sqrt{\nu^+(2\nu^+ + \kappa^+)}}.$$

So, we have

$$\varphi_{1\,\min} = \varphi_1(0) = 0$$

and

$$\varphi_{1\,\max} = \varphi_1(|\xi_{**}|) = \operatorname{arcctg}\Big(\frac{|s|/\sqrt{\nu^+(2\nu^+ + \kappa^+)} + \operatorname{Re} s/\nu^+}{\operatorname{Im} s/\nu^+}\Big) - \operatorname{arcctg}\Big(\frac{|s|/\sqrt{\nu^+(2\nu^+ + \kappa^+)} + \operatorname{Re} s/(2\nu^+ + \kappa^+)}{\operatorname{Im} s/(2\nu^+ + \kappa^+)}\Big).$$

It is obvious that the maximal angle between r^+ and r_1^+ will be in the case when $\kappa^+ = \nu^+$. Hence,

$$\begin{aligned}
\varphi_{1\,\max} &\leqslant \operatorname{arcctg}\Big(\frac{|s|/\sqrt{3} + \operatorname{Re} s}{\operatorname{Im} s}\Big) - \operatorname{arcctg}\Big(\frac{\sqrt{3}\,|s| + \operatorname{Re} s}{\operatorname{Im} s}\Big) \\
&= \operatorname{arctg}\frac{\operatorname{Im} s}{|s|/\sqrt{3} + \operatorname{Re} s} - \operatorname{arctg}\frac{\operatorname{Im} s}{\sqrt{3}\,|s| + \operatorname{Re} s} \\
&= \operatorname{arctg}\left\{\frac{\dfrac{\operatorname{Im} s}{|s|/\sqrt{3} + \operatorname{Re} s} - \dfrac{\operatorname{Im} s}{\sqrt{3}\,|s| + \operatorname{Re} s}}{1 + \dfrac{(\operatorname{Im} s)^2}{(|s|/\sqrt{3} + \operatorname{Re} s)(\sqrt{3}\,|s| + \operatorname{Re} s)}}\right\} \\
&= \operatorname{arctg}\Big\{\frac{\operatorname{Im} s}{\sqrt{3}\,|s| + 2\operatorname{Re} s}\Big\} = \operatorname{arcctg}\Big\{\frac{\sqrt{3}}{\sin(\arg s)} + 2\operatorname{ctg}(\arg s)\Big\}.
\end{aligned}$$

Since $\arg s \in (0, \pi/2)$,

$$\varphi_{1\,\max} \leqslant \operatorname{arcctg}\Big(\frac{\sqrt{3}}{\sin(\arg s)}\Big) \leqslant \operatorname{arcctg}\sqrt{3} = \frac{\pi}{6}.$$

So, $\arg\big(\dfrac{r^+}{r_1^+}\big)$, $\arg\big(\dfrac{r^-}{r_1^-}\big) \leqslant \dfrac{\pi}{12}$, and we have proved (2.11) because $0 \leqslant \arg\big(\dfrac{r^\pm}{r^\pm + r_1^\pm}\big) \leqslant \arg\big(\dfrac{r^\pm}{r_1^\pm}\big)$.

Next, since

$$\frac{r^\pm r_1^\pm - \xi^2}{s} = \frac{1}{\nu^\pm(2+\beta^\pm)}\Big(1 + \frac{(1+\beta^\pm)r_1^\pm}{r^\pm + r_1^\pm}\Big) \tag{2.14$'$}$$

and

$$0 \geqslant \arg \frac{r_1^\pm}{r^\pm + r_1^\pm} \geqslant \arg \frac{r_1^+}{r^\pm},$$

we obtain (2.13).

Adding up the arguments of the multipliers in q, we arrive at (2.14).

It remains to verify (2.12). Let us consider for definiteness that the angle between r_1^+ and r_1^- is positive. Then the function

$$\varphi_2(|\xi|) = 2\arg\Big(\frac{r^+}{r_1^-}\Big)$$

takes its maximum value at the point $|\xi_0| = \sqrt{\dfrac{|s|}{\sqrt{\nu^+(2\nu^- + \kappa^-)}}}$. It is obvious that

$$\begin{aligned}\varphi_{2\max}(|\xi_0|) &\leqslant \operatorname{arcctg}\Big(\frac{|s|/\sqrt{3\nu^+\nu^-} + \operatorname{Re} s/\nu^+}{\operatorname{Im} s/\nu^+}\Big)\\ &\quad - \operatorname{arcctg}\Big(\frac{|s|/\sqrt{3\nu^+\nu^-} + \operatorname{Re} s/\nu^-}{\operatorname{Im} s/\nu^-}\Big)\\ &\leqslant \operatorname{arcctg}\Big(\frac{|s|/\sqrt{12} + \operatorname{Re} s}{\operatorname{Im} s}\Big) - \operatorname{arcctg}\Big(\frac{\sqrt{12}\, s + \operatorname{Re} s}{\operatorname{Im} s}\Big)\end{aligned}$$

because $\nu^- \leqslant 4\nu^+$. By applying the above arguments, we have

$$\varphi_{2\max} \leqslant \operatorname{arcctg}\Big(\frac{2\sqrt{12}}{11\sin(\arg s)}\Big) \leqslant \operatorname{arcctg}\Big(\frac{\sqrt{3}}{3}\Big) = \frac{\pi}{3}.$$

Hence,

$$\arg\frac{r_1^+}{r_1^-} \leqslant \arg\frac{r^+}{r_1^-} \leqslant \pi/6. \qquad \square$$

Lemma 2.2. *Let $s \in \mathbb{C}$ be such that*

$$\operatorname{Re} s + \varkappa |\operatorname{Im} s| \geqslant -\delta\xi^2,$$

where $\varkappa$ and δ are small positive numbers. In the case $\operatorname{Im} s \geqslant 0$, *we have*

$$\arg r^{\pm}, \quad \arg r_{-}^{\pm} \in \Big(0, \frac{k}{4}\Big), \tag{2.15}$$

$$\arg q \in \Big(-\frac{\pi}{12}k, \frac{\pi}{4}k\Big), \tag{2.16}$$

where $k = \dfrac{\pi/2+\gamma}{\pi/2}$, $\gamma = \operatorname{arcctg} \varkappa$. *In addition,*

$$c_1^{\pm}(|s|+\xi^2)^{1/2} \leqslant \operatorname{Re} r^{\pm}(\xi, s) \leqslant |r^{\pm}(\xi, s)| \leqslant c_2^{\pm}(|s|+\xi^2)^{1/2}, \tag{2.17}$$

$$c_3^{\pm}(|s|+\xi^2)^{1/2} \leqslant \operatorname{Re} r_1^{\pm}(\xi, s) \leqslant |r_1^{\pm}(\xi, s)| \leqslant c_4^{\pm}(|s|+\xi^2)^{1/2}, \tag{2.18}$$

$$c_5(|s|+\xi^2)^{1/2} \leqslant |q(\xi, s)| \leqslant c_6(|s|+\xi^2)^{1/2}. \tag{2.19}$$

Proof. Relation (2.15) follows from the fact that $\arg r^{\pm 2}$, $\arg r_1^{\pm 2} \in (0, \pi/2+\gamma)$. Taking into account (2.14), we arrive at (2.16).

To prove (2.17), we consider two cases.

1. If $\operatorname{Re} s \geqslant 0$, (2.17) is obvious.

2. In the case when $\operatorname{Re} s < 0$, we have $|\operatorname{Im} s| \geqslant \dfrac{|s|-\delta\xi^2}{\varkappa+1}$.

Let us find a lower bound of the expression $\operatorname{Re} r^{+2} + \varkappa_1 |\operatorname{Im} r^{+2}|$ with some $\varkappa_1 > 0$. So,

$$\operatorname{Re} r^{+2} + \varkappa_1 |\operatorname{Im} r^{+2}| \geqslant c\Big(1-\frac{\delta}{\nu^+}\Big)\xi^2 + \Big(\varkappa_1 - \frac{\varkappa}{\nu^+}\Big)|\operatorname{Im} s|$$

$$\geqslant c\Big\{\Big(1-\frac{\delta}{\nu^+} - \Big(\varkappa_1 - \frac{\varkappa}{\nu^+}\Big)\frac{\delta}{\varkappa+1}\Big)\xi^2 + \frac{\varkappa_1 - \varkappa/\nu^+}{\varkappa+1}|s|\Big\}.$$

Hence, if we take a number $\varkappa_1 > \varkappa/\nu^+$ and sufficiently small δ, we will obtain

$$|r^+|^2 \geqslant \frac{1}{2}\max(1, \varkappa_1)\Big(\operatorname{Re} r^{+2} + |\operatorname{Im} r^{+2}|\Big)$$
$$\geqslant \operatorname{Re} r^{+2} + \varkappa_1 |\operatorname{Im} r^{+2}| \geqslant c(|s|+\xi^2).$$

And due to (2.15), $\operatorname{Re} r^+ \geqslant |r^+| \cos\left(\frac{\pi k}{4}\right) \geqslant c|r^+|$.

Thus, we have found the lower bound of $\operatorname{Re} r^+$ and $|r^+|$. The upper bound is evident.

Inequalities (2.18) are proved in the same way. It remains to verify (2.19). Taking into account (2.14′), we have

$$|q| \leqslant c(|r_1^+| + |r_1^-|) \leqslant c_6(|s| + \xi^2)^{1/2}.$$

On the other hand, $\arg(q/r_1^+) \in \Bigl(\frac{\pi k}{3}, \frac{\pi k}{4}\Bigr)$. Hence, $\operatorname{Re}(q/r_1^+) > 0$ if we choose $k < 3/2$. Therefore,

$$\begin{aligned} |q| &\geqslant |r_1^+| \operatorname{Re}\Bigl\{\rho_0^+ \frac{r^- r_1^- - \xi^2}{s} + \rho_0^- r_1^- c^+ \Bigl(\frac{1+\beta^+}{r^+ + r_1^+} + \frac{1}{r_1^+}\Bigr)\Bigr\} \\ &\geqslant c|r_1^+| \Bigl\{1 + \frac{|r_1^-|}{|r^- + r_1^-|} \cos\frac{\pi k}{12} + \frac{|r_1^-|}{|r^+ + r_1^+|} \cos\frac{\pi k}{4} + \frac{|r_1^-|}{|r_1^+|} \cos\frac{\pi k}{6}\Bigr\} \\ &\geqslant c_5(|s| + \xi^2)^{1/2}. \qquad \square \end{aligned}$$

Lemma 2.3. *If $s \in \mathbb{C}$: $\operatorname{Re} s + \varkappa|\operatorname{Im} s| \geqslant -\delta\xi^2$, where $\varkappa$ and δ are such as in Lemma 2.2, then for the expression $Q = \sum_{j=2}^5 I_j/s$, the following relations*

$$\arg(Q/q) \in \Bigl(-\frac{5\pi k}{12}, \frac{\pi k}{4}\Bigr), \quad k \leqslant \frac{6}{5}; \tag{2.20}$$

$$\operatorname{Re}\frac{Q}{q} - \varkappa_1\Bigl|\operatorname{Im}\frac{Q}{q}\Bigr| \geqslant c_6\Bigl(\cos\frac{5\pi k}{12} - \varkappa_1 \sin\frac{5\pi k}{12}\Bigr)\frac{\xi^2}{|r_1^+|} > 0 \tag{2.21}$$

hold with some positive $\varkappa_1$.

Proof. First, we will study the arguments of the terms $I_j/(sq)$, $j = 2, \dots, 5$. Next, we will obtain the estimate (2.21) for each $I_j/(sq)$.

Let us start with $\dfrac{I_2}{sq} = \dfrac{\rho_0^+\rho_0^-\xi^2}{q}\Bigl\{\dfrac{r_1^+}{r_1^-} + \dfrac{r_1^-}{r_1^+}\Bigr\}$. In virtue of Lemmas 2.1 and 2.2,

$$\begin{aligned} \arg I_2/(sq) &\leqslant \arg\frac{r_1^\pm}{r_1^\mp} - \arg q \in \Bigl(-\Bigl(\frac{\pi}{6} + \frac{\pi}{4}\Bigr)k, \Bigl(\frac{\pi}{6} + \frac{\pi}{12}\Bigr)k\Bigr) \\ &= \Bigl(-\frac{5\pi}{12}k, \frac{\pi k}{4}\Bigr) \in \Bigl(-\frac{\pi}{2}, \frac{3\pi}{10}\Bigr) \quad \text{for} \quad k \leqslant \frac{6}{5}. \end{aligned}$$

Hence,

$$\operatorname{Re}\frac{I_2}{sq} - \varkappa_1\Bigl|\operatorname{Im}\frac{I_2}{sq}\Bigr| \geqslant \Bigl(\cos\frac{5\pi}{12}k - \varkappa_1\sin\frac{5\pi k}{12}\Bigr)\Bigl|\frac{I_2}{sq}\Bigr|$$

$$\geqslant c\Big(\cos\frac{5\pi}{12}k-\varkappa_1\sin\frac{5\pi k}{12}\Big)\frac{\xi^2}{|q|}\operatorname{Re}\Big\{\frac{r_1^+}{r_1^-}+\frac{r_1^-}{r_1^+}\Big\}. \tag{2.22}$$

Next, we study $\arg\left(\frac{sq}{I_3}\right)$. We have

$$\frac{sq}{I_3}=\frac{c\rho_0^+r_1^+(r^++r_1^+)}{\xi^2r^+}+\frac{c\rho_0^-r_1^-(r^++r_1^+)(r^+r_1^+-\xi^2)}{\xi^2r^+(r^-r_1^--\xi^2)}.$$

The argument of the first term belongs to $\left(-\frac{\pi k}{12},\frac{\pi k}{4}\right)$, the argument of the second term does to $\left(\frac{\pi k}{12}\cdot 2,\frac{\pi k}{12}+\frac{\pi k}{4}\right)=\left(-\frac{\pi k}{6},\frac{\pi k}{3}\right)$.

Consequently,

$$\arg\Big(\frac{I_3}{sq}\Big)\in\Big(-\frac{\pi k}{3},\frac{\pi k}{6}\Big),$$

and

$$\begin{aligned}\operatorname{Re}\frac{I_3}{sq}-\varkappa_1\Big|\operatorname{Im}\frac{I_3}{sq}\Big| &\geqslant \Big(\cos\frac{\pi k}{3}-\varkappa_1\sin\frac{\pi k}{3}\Big)\Big|\frac{I_3}{sq}\Big|\\ &\geqslant c\Big(\cos\frac{\pi k}{3}-\varkappa_1\sin\frac{\pi k}{3}\Big)\frac{\xi^2}{|q|}.\end{aligned} \tag{2.23}$$

In the same way, we evaluate I_4. Finally, the arguments of the first and the second terms in $I_5/(sq)$ belong to $\left(-\frac{\pi k}{4},2\frac{\pi k}{12}\right)$. The argument of the third one equals $-\arg(q)$. As for the forth term in $I_5/(sq)$, we conclude that its argument belongs to the interval

$$\Big(-\frac{\pi k}{4}-2\frac{\pi k}{12},2\frac{\pi k}{12}\Big)=\Big(-\frac{5\pi k}{12},\frac{\pi k}{6}\Big).$$

Therefore,

$$\operatorname{Re}\frac{I_5}{sq}-\varkappa_1\Big|\operatorname{Im}\frac{I_5}{sq}\Big|\geqslant\Big(\cos\frac{5\pi k}{12}-\varkappa_1\sin\frac{5\pi k}{12}\Big)\Big|\frac{I_5}{sq}\Big|. \tag{2.24}$$

Since $\operatorname{Re}\left(\frac{I_5}{s}\right)$ is positive, so is the real part of each term of it , we can write

$$\begin{aligned}\Big|\frac{I_5}{qs}\Big|\geqslant\frac{c\xi^2}{|q|}\Big\{\sum_{i=+,-}\Big\{\operatorname{Re}\frac{r^i}{r^i+r_1^i}+\operatorname{Re}\frac{r^ir_1^{-i}}{(r^++r_1^+)(r^-+r_1^-)}\Big\}\\ +1+\operatorname{Re}\frac{r_1^+r_1^-}{(r^++r_1^+)(r^-+r_1^-)}\Big\}\geqslant\frac{c\xi^2}{|q|}.\end{aligned} \tag{2.25}$$

By joining the intervals corresponding to $\frac{I_j}{sq}$, $j=2,\dots,5$, we find an interval to which the argument of Q/q belongs. It is just one as in (2.20). Choosing $k<\frac{6}{5}$, we see that $\arg(Q/q)\in(-\pi/2, 3\pi/10)$ in this case. Let be $k=11/10$.

Adding the inequalities (2.22)–(2.24) and granting (2.25), we deduce (2.21). It remains to choose $\varkappa_1>0$ in such a way that $\cos\frac{5\pi k}{12}-\varkappa_1\sin\frac{5\pi k}{12}>0$. $\square$

Lemma 2.4. *If $s\in\mathbb{C}$ is such that*

$$\operatorname{Re}s+\varkappa|\operatorname{Im}s|\geqslant-\delta+\xi^2$$

with some small positive $\varkappa$ and δ, then the expression

$$M=s\mathcal{D}+\frac{Q}{q},\quad \mathcal{D}=\Big(\frac{\rho_0^+}{r_1^+}+\frac{\rho_0^-}{r_1^-}\Big),$$

satisfies the following inequalities

$$c_8(|s|+\xi^2)^{1/2}\leqslant|M(\xi,s)|\leqslant c_9(|s|+\xi^2)^{1/2},\quad \xi_j\in\mathbb{R},\quad j=1,2. \tag{2.26}$$

The same estimates hold for $M(\zeta,s)$, where $\zeta_j=\xi_j+\mathrm{i}\,\eta_j$, ξ_j, $\eta_j\in\mathbb{R}$, $j=1,2$,

$$|\eta|\leqslant\delta_1|\xi|$$

with a sufficiently small positive number δ_1.

Proof. In order to prove (2.26), we find a lower bound of the expression $\operatorname{Re}M(\xi,s)+\varkappa_1\,|\operatorname{Im}M(\xi,s)|$ with some $\varkappa_1>0$. So,

$$\begin{aligned}\operatorname{Re}M+\varkappa_1|\operatorname{Im}M|&=\operatorname{Re}\mathcal{D}\operatorname{Re}s-\operatorname{Im}\mathcal{D}\operatorname{Im}s+\operatorname{Re}(Q/q)\\&+\varkappa_1|\operatorname{Im}(\mathcal{D}s)+\operatorname{Im}(Q/q)|\geqslant\operatorname{Re}\mathcal{D}(\operatorname{Re}s+\varkappa_1|\operatorname{Im}s|)\\&-\varkappa_1|\operatorname{Re}s|\,|\operatorname{Im}\mathcal{D}|+\operatorname{Re}(Q/q)-\varkappa_1|\operatorname{Im}(Q/q)|\end{aligned}\tag{2.27}$$

since $\operatorname{Re}\mathcal{D}=\rho_0^+\operatorname{Re}\frac{\overline{r_1^+}}{|r_1^+|^2}+\rho_0^-\operatorname{Re}\frac{\overline{r_1^-}}{|r_1^-|^2}>0$ because of (2.15). Note: $|\operatorname{Im}\mathcal{D}|\leqslant|\mathcal{D}|\leqslant\sum_{i=\pm}\frac{\rho_0^i}{|r_1^i|}$.

We consider two cases.

1°. Let be $\operatorname{Re}s\geqslant0$. Then $\arg(r_1^i)\in(0,\pi/4)$ and $\operatorname{Re}r_1^i\geqslant|r_1^i|/\sqrt{2}$, $i=\pm$. Hence, in view of (2.21), we have

$$\operatorname{Re}M+\varkappa_1|\operatorname{Im}M|\geqslant\sum_{i=\pm}\frac{\rho_0^i}{|r_1^i|}\Big\{\Big(\frac{1}{\sqrt{2}}-\varkappa_1\Big)\operatorname{Re}s+\frac{1}{\sqrt{2}}\varkappa_1|\operatorname{Im}s|\Big\}$$

$$+\operatorname{Re}(Q/q)-\varkappa_1|\operatorname{Im}(Q/q)|\geqslant\sum_{i=\pm}\frac{\rho_0^i}{|r_1^i|}\min\Big\{\frac{1}{\sqrt{2}}-\varkappa_1,\frac{\varkappa_1}{\sqrt{2}}\Big\}\frac{|s|}{2}$$

$$+c_6\Big(\cos\frac{5\pi k}{12}-\varkappa_1\sin\frac{5\pi k}{12}\Big)\frac{\xi^2}{r_1^+}\geqslant c_7\frac{|s|+\xi^2}{|r_1^+|}\geqslant c_{10}(|s|+\xi^2)^{1/2}$$

if $\varkappa_1$ is small enough, $k<6/5$.

2°. If $\operatorname{Re}s<0$ then $|\operatorname{Im}s|\geqslant\frac{|s|-\delta\xi^2}{1+\varkappa}$. Now, $\operatorname{Re}\mathcal{D}\geqslant\cos\frac{\pi k}{4}\sum_{i=\pm}\frac{\rho_0^i}{|r_1^i|}>0$. Consequently, we can continue (2.27) as follows:

$$\operatorname{Re}M+\varkappa_1|\operatorname{Im}M|\geqslant\operatorname{Re}\mathcal{D}\Big(-\delta\xi^2+(\varkappa_1-\varkappa)\frac{|s|-\delta\xi^2}{1+\varkappa}\Big)$$

$$-|\operatorname{Im}\mathcal{D}|(\varkappa_1\delta\xi^2+\varkappa_1\varkappa|\operatorname{Im}s|)+\operatorname{Re}(Q/q)-\varkappa_1|\operatorname{Im}(Q/q)|$$

$$\geqslant\frac{c\xi^2}{|r_1^+|}\Big(\cos\frac{5\pi k}{12}-\varkappa_1\sin\frac{5\pi k}{12}\Big)-\cos\frac{\pi k}{4}\Big(\delta+\frac{\delta(\varkappa_1-\varkappa)}{1+\varkappa}+\varkappa_1\delta\Big)\Big\}$$

$$+\sum_{i=\pm}\frac{\rho_0^i|s|}{|r_1^i|}\Big\{\frac{\varkappa_1-\varkappa}{1+\varkappa}-\varkappa_1\varkappa\Big\}\geqslant c\frac{(\xi^2+|s|)}{|r_1^+|}\geqslant c_{11}(|s|+\xi^2)^{1/2}.$$

Here we have taken $\varkappa_1$ the same as in 1°, and $\varkappa<\varkappa_1/2$, δ sufficiently small. Thus, in the both cases, we obtain

$$\operatorname{Re}M+\varkappa_1|\operatorname{Im}M|\geqslant c(|s|+\xi^2)^{1/2}$$

which gives the lower bound of $|M|$ in (2.26).

The upper bound for $|M(\xi,s)|$ is evident enough. The estimates for $M(\zeta,s)$, where ξ^2 is replaced by $\langle\zeta\rangle^2$, $\langle\zeta\rangle=\sqrt{\zeta_1^2+\zeta_2^2}$, follow from (2.26) and from the inequalities

$$|M(\zeta,s)-M(\xi,s)|\leqslant c|\eta|\,|\xi|\leqslant c_{12}\delta_1\xi^2$$

if δ_1 is small enough. □

Lemma 2.5. *For $\forall s\in\mathbb{C}$ such that*

$$\operatorname{Re}s+\varkappa|\operatorname{Im}s|\geqslant a\gg\tau^2$$

and for $\forall\zeta_\beta=\xi_\beta+\mathrm{i}\,\eta_\beta$, ξ_β, $\eta_\beta\in\mathbb{R}$, $\beta=1,2$, such that $|\eta|\leqslant\delta_1|\xi|$ the expression

$$P^{(\tau)}(\zeta,s)=sM(\zeta,s)q(\zeta,s)+\tau\sigma q(\zeta,s)\langle\zeta\rangle^2$$

satisfies the inequalities

$$c_{13}(|s|(|s|+\xi^2)+\tau|r_1^+|\xi^2)\leqslant|P^{(\tau)}(\zeta,s)|\leqslant c_{14}(|s|(|s|+\xi^2)+\tau|r_1^+|\xi^2), \tag{2.28}$$

$$\begin{aligned}c_{15}(|s|(|s|+\xi^2)+\tau|r_1^+|\xi^2)&\leqslant|P^{(\tau)}(\zeta,s-\delta_2\langle\zeta\rangle)|\\&\leqslant c_{16}(|s|(|s|+\xi^2)+\tau|r_1^+|\xi^2)\end{aligned} \tag{2.29}$$

provided that $\varkappa$, δ^2/τ *and* $\tau/\sqrt{a}$ *are sufficiently small. The constants are inependent of* τ.

Proof. First, we examine the absolute value of the ratio

$$\frac{P^{(\tau)}(\xi,s)}{M(\zeta,s)q(\xi,s)}=s+\frac{\tau\sigma\xi^2}{M}=s+\frac{\tau\sigma\xi^2}{|M|^2}\Big(\overline{s\mathcal{D}}+\Big(\frac{\overline{Q}}{q}\Big)\Big).$$

To this end, we find a lower found of the sum

$$\begin{aligned}N^{(\tau)}&\equiv\operatorname{Re}\frac{P^{(\tau)}}{Mq}+\varkappa_1\Big|\operatorname{Im}\frac{P^{(\tau)}}{Mq}\Big|\\&=\operatorname{Re}s+\frac{\tau\sigma\xi^2}{|M|^2}\operatorname{Re}M+\varkappa_1\Big|\operatorname{Im}s-\frac{\tau\sigma\xi^2}{|M|^2}\operatorname{Im}M\Big|\\&\geqslant\operatorname{Re}s(1+R)+\varkappa_1|\operatorname{Im}s|(1-R)+\frac{\tau\sigma\xi^2}{|M|^2}\Big\{\operatorname{Re}M-\varkappa_1|\operatorname{Im}M|\Big\}.\end{aligned}$$

We introduce the quantity

$$R=\frac{\tau\sigma\xi^2\operatorname{Re}\mathcal{D}}{|M|^2}.$$

Since $\operatorname{Re}\mathcal{D}>0$, we have

$$\begin{aligned}N^{(\tau)}&\geqslant(1+R)\operatorname{Re}s+\varkappa_1(1-R)|\operatorname{Im}s|\\&+\frac{\tau\sigma\xi^2}{|M|^2}\{-\operatorname{Im}\mathcal{D}\operatorname{Im}s-\varkappa_1\operatorname{Re}s|\,|\operatorname{Im}\mathcal{D}|+\operatorname{Re}(Q/q)-\varkappa_1|\operatorname{Im}(Q/q)|\}.\end{aligned}$$

Here we can omit the term $-\operatorname{Im}\mathcal{D}\operatorname{Im}s$ because $\operatorname{Im}\mathcal{D}$ and $\operatorname{Im}s$ have inverse signs to each other. Moreover, $\arg\mathcal{D}\in\Big(-\dfrac{\pi k}{4},0\Big)$ and, hence,

$$|\operatorname{Im}\mathcal{D}|\leqslant\operatorname{tg}\Big(\frac{\pi k}{4}\Big)\operatorname{Re}\mathcal{D}\leqslant\operatorname{tg}\frac{\pi}{3}\operatorname{Re}\mathcal{D}=\sqrt{3}\operatorname{Re}\mathcal{D}.$$

In addition, it was shown in [1] that

$$\operatorname{Re} s + \varkappa_1 |\operatorname{Im} s| \geqslant \frac{\varkappa_1 - \varkappa}{\sqrt{1+\varkappa^2}}\, |s|$$

under the lemma hypotheses. Therefore, by virtue of (2.21),

$$N^{(\tau)} \geqslant (1+R)\frac{\varkappa_1 - \kappa}{\sqrt{1+\varkappa^2}}\, |s| - 2\varkappa_1 R |\operatorname{Im} s| - \sqrt{3}\, \varkappa_1 R |\operatorname{Re} s|$$

$$+\frac{c_6}{c_{17}}\Big(\cos\frac{5\pi k}{12} - \varkappa_1 \sin\frac{5\pi k}{12}\Big)\xi^2 R. \tag{2.30}$$

We have taken here into account also the fact that $\operatorname{Re}\mathcal{D} \leqslant \dfrac{c_{17}}{|r_1^+|}$. Now, we fixed $\varkappa_1$ so small that the last term in (2.30) would be positive if $k = 1.1$. We take then $\varkappa \leqslant \varkappa_1/2 \leqslant 1/2$ and such that $k = 1.1$. Thus, we have

$$\begin{aligned} N^{(\tau)} &\geqslant \frac{\varkappa_1 |s|}{2\sqrt{1+\varkappa^2}}\,[1 + R - 2(2+\sqrt{3})R\sqrt{1+\varkappa^2}] + c_{18} R\xi^2 \\ &\geqslant \frac{\varkappa_1(1-8R)}{2\sqrt{1+\varkappa^2}}\, |s| + c_{18} R\xi^2. \end{aligned} \tag{2.31}$$

We note that $\min |s| = \dfrac{a}{\sqrt{1+\varkappa^2}}$ under the lemma hypotheses. Therefore

$$\begin{aligned} R &\leqslant \frac{\tau\sigma\sqrt{|s|}\,\xi^2 \operatorname{Re}\mathcal{D}}{c_8^2 \min\sqrt{|s|}\,(|s|+\xi^2)} \\ &\leqslant c_{19}\frac{\tau\sigma(1+\varkappa^2)^{1/4}}{\sqrt{a}}\ \frac{\sqrt{|s|}\,\xi^2}{(|s|^{1/2}+|\xi|)^3} \leqslant \frac{1}{10} \end{aligned}$$

provided that $\dfrac{\tau}{\sqrt{a}} \leqslant \dfrac{1}{10 c_{19}\sigma(1+\varkappa^2)^{1/4}}$.

Hence, we can continue (2.31) as follows:

$$N^{(\tau)} \geqslant \frac{\varkappa_1}{10\sqrt{1+\varkappa^2}}\, |s| + \frac{c_{18}(\varkappa_1)\tau\sigma\xi^2}{|M|}.$$

Consequently,

$$\Big|\frac{P^{(\tau)}}{Mq}\Big| \geqslant c_{19}(\varkappa_1)\Big(|s| + \frac{\tau\sigma\xi^2}{|M|}\Big)$$

and, finally,

$$|P^{(\tau)}(\xi,s)|\geqslant c_{19}(\varkappa_1)\Big(|s|\,|M|\,|q|+\tau\sigma\xi^2|q|\Big)$$
$$\geqslant c_{20}\Big(|s|(|s|+\xi^2)+\tau\xi^2|r_1^+|\Big).$$

The upper bound for $|P^{(\tau)}(\xi,s)|$ is clear. So, we have (2.28) for $|P^{(\tau)}(\xi,s)|$ which implies similar estimates for $|P^{(\tau)}(\zeta,s)|$, $\zeta=\xi+\mathrm{i}\,\eta$, $|\eta|\leqslant\delta_1|\xi|$. Indeed,

$$|P^{(\tau)}(\zeta,s)-P^{(\tau)}(\xi,s)|\leqslant c|\eta|\Big(|s|\;|\xi|+\tau|r_1^+|\;|\xi|+\tau\xi^2\Big)$$
$$\leqslant c\delta_1\Big(|s|\xi^2+\tau|r_1^+|\xi^2\Big)$$

whence it follows (2.28) if δ_1 is small enough.

In order to prove (2.29), we consider the difference

$$P^{(\tau)}(\zeta,s-\delta_2\langle\,\zeta\,\rangle)-P^{(\tau)}(\zeta,s)=-\delta_2\langle\,\zeta\,\rangle\int\limits_0^1\frac{\partial}{\partial s}P^{(\tau)}(\zeta,s-\delta_2\lambda\langle\,\zeta\,\rangle)\mathrm{d}\lambda$$

$$=-\delta_2\langle\,\zeta\,\rangle\int\limits_0^1\Big[M(\zeta,s-\delta_2\lambda\langle\,\zeta\,\rangle)q(\zeta,s-\delta_2\lambda\langle\,\zeta\,\rangle)+(s-\delta_2\lambda\langle\,\zeta\,\rangle)\frac{\partial(Mq)}{\partial s}\Big]\mathrm{d}\lambda$$

$$-\delta_2\tau\sigma\langle\,\zeta\,\rangle^3\int\limits_0^1\frac{\partial q(\zeta,s-\delta_2\lambda\langle\,\zeta\,\rangle)}{\partial s}\mathrm{d}\lambda.$$

Since

$$\mathrm{Re}(s-\delta_2\lambda\langle\,\zeta\,\rangle)+\varkappa|\,\mathrm{Im}(s-\delta_2\lambda\langle\,\zeta\,\rangle)|\geqslant a-c_{21}\delta_2|\xi|$$
$$\geqslant-\delta\xi^2+a-\frac{c_{21}^2\delta_2^2}{4\delta}\geqslant-\delta\xi^2$$

for $\delta_2\leqslant\dfrac{2\sqrt{a\delta}}{c_{21}}$, the functions $r^{\pm}(\zeta,s-\delta_2\lambda\langle\,\zeta\,\rangle)$, $r_1^{\pm}(\zeta,s-\delta_2\lambda\langle\,\zeta\,\rangle)$, $q(\zeta,s-\delta_2\lambda\langle\,\zeta\,\rangle)$ and $M(\zeta,s-\delta_2\lambda\langle\,\zeta\,\rangle)$ satisfy the inequalities of Lemmas 2.2 and 2.4. Therefore we have

$$\Big|P^{(\tau)}(\zeta,s-\delta_2\langle\,\zeta\,\rangle)-P^{(\tau)}(\zeta,s)\Big|\leqslant c_{22}\delta_2|\xi|\Big\{|s|+\xi^2+\delta_2|\xi|+\tau\frac{\xi^2}{|r_1^+|}\Big\}$$

$$\leqslant c_{22}\Big\{\delta_2\Big(|s|^2+\frac{|s|\xi^2}{\min|s|}\Big)+\delta_2|r_1^+|\xi^2+\frac{\delta_2^2|s|\xi^2}{\min|s|}+\frac{\delta_2\tau|r_1^+|\xi^2}{\min|r_1^+|}\Big\}$$

$$\leqslant 2c_{22}\max\Big\{\delta_2,\frac{\delta_2}{a},\frac{\delta_2^2}{a},\frac{\delta_2}{\tau},\frac{\sqrt{3\nu^+}\,\delta_2}{\sqrt{a}}\Big\}\Big(|s|(|s|+\xi^2)+\tau|r_1^+|\xi^2\Big) \qquad (2.32)$$

because $|\xi|\leqslant|r_1^+|$. The left-hand inequality in (2.29) follows from (2.28) and from (2.32) in the case when $\max\{\dots\}$ is a small value.

The right-hand inequality on (2.29) is obvious. □

Now, our purpose is to estimate the Hölder norm of the function a_3 in terms of the known functions.

First of all, we substitute ω_3 in relation (2.8). Note that ω_3, the third components of the velocity $\widetilde{\mathbf{v}}$ on the boundary, can be expressed on the basis of (2.6) as follows:

$$\omega_3=\frac{A}{P}\Big\{\rho_0^+(r^-r_1^- -\xi^2)-\rho_0^-(r^+r_1^+-\xi^2)$$
$$-2(\mu^+-\mu^-)\frac{(r^+r_1^+-\xi^2)(r^-r_1^- -\xi^2)}{s}\Big\}-\frac{\widetilde{b}_3'}{P}\,qs.$$

We remind that $A=\sum_{\beta=1}^2 \mathrm{i}\,\xi_\beta\widetilde{b}_\beta$, $\widetilde{b}_3'=\widetilde{b}_3+\frac{\sigma\widetilde{B}}{s}$. In this way, we obtain

$$\widetilde{a}_3=\widetilde{b}_3'+\frac{\sigma\xi^2}{s}\omega_3=\frac{\sigma\xi^2A}{P}\Big\{\rho_0^+\frac{r^-r_1^- -\xi^2}{s}-\rho_0^-\frac{r^+r_1^+-\xi^2}{s}$$
$$-2(\mu^+-\mu^-)\frac{r^+r_1^+-\xi^2}{s}\;\frac{r^-r_1^- -\xi^2}{s}\Big\}+\widetilde{b}_3$$
$$+\sum_{\beta=1}^2\frac{\sigma\mathrm{i}\,\zeta_\beta q}{P}\mathrm{i}\,\xi_\beta\widetilde{b}_3+\frac{\sigma Mq}{P}\widetilde{B}. \qquad (2.33)$$

Let us take the inverse Fourier–Laplace transformation by the formula

$$(FL)^{-1}\widetilde{u}\equiv u(x',x_3,t)$$
$$=\frac{1}{(2\pi)^3\mathrm{i}}\int\limits_{\mathbb{R}^2}\mathrm{e}^{\mathrm{i}\,x'\xi'}\int\limits_{\mathrm{Re}\,s=a}\mathrm{e}^{st}\widetilde{u}(\xi,x_3,s)\mathrm{d}s\mathrm{d}\xi,\quad a>0, \qquad (2.34)$$

in (2.33). Then we have

$$a_3(x',t)=b_3(x',t)+\sum_{\beta=1}^2\int\limits_0^t\int\limits_{\mathbb{R}^2}W_\beta(x'-y',t-\tau)\frac{\partial b_3(y',\tau)}{\partial y_\beta}\mathrm{d}y'\,\mathrm{d}\tau$$

$$+\sum_{\beta=1}^{2}\int_0^t\int_{\mathbb{R}^2} W_3(x'-y',t-\tau)\frac{\partial b_\beta(y',\tau)}{\partial y_\beta}\mathrm{d}y'\,\mathrm{d}\tau \tag{2.35}$$

$$+\int_0^t\int_{\mathbb{R}^2} W_4(x'-y',t-\tau)\sigma B(y',\tau)\mathrm{d}y'\,\mathrm{d}\tau \equiv \sum_{i=1}^{4} A_i(x',t),$$

where

$$W_\beta = (FL)^{-1}\frac{\sigma \mathrm{i}\,\xi_\beta q}{P}, \quad \beta=1,2,$$

$$W_3 = (FL)^{-1}\frac{\sigma\xi^2}{P}\Big\{\rho_0^+\frac{r^-r_1^- - \xi^2}{s} - \rho_0^-\frac{r^+r_1^+ - \xi^2}{s}$$

$$-2(\mu^+-\mu^-)\frac{r^+r_1^+ - \xi^2}{s}\,\frac{r^-r_1^- - \xi^2}{s}\Big\},$$

$$W_4 = (FL)^{-1}\frac{Wq}{P}.$$

To evaluate the kernels W_j and their derivatives, we use the following lemma.

Lemma 2.6. *The functions*

$$V_{1,k}(x',t) = (FL)^{-1}\frac{r_1^\pm(\xi,s)\varphi(r^\pm,r_1^\pm)p_{k-1}(\xi)}{P(\xi,s)}, \quad k\geqslant 2,$$

$$V_{2,k}(x',t) = (FL)^{-1}\frac{\varphi(r^\pm,r_1^\pm)p_k(\xi)}{P(\xi,s)}, \quad k\geqslant 1,$$

$$V_{3,k}(x',t) = (FL)^{-1}\frac{r^\pm M(\xi,s)}{P(\xi,s)}\varphi(r^\pm,r_1^\pm)p_{k-2}(\xi), \quad k\geqslant 2,$$

($p_k(\xi)$ is a homogeneous polynomial of order k with respect to ξ_j, $|\xi|$, and $\varphi(r^\pm,r_1^\pm)$ is a homogeneous function of zero order with respect to $r^\pm$, $r_1^\pm$) and their derivatives satisfy the inequalities

$$|\mathcal{D}_x^{\mathbf{j}} V_{\beta,k}(x',t)| \leqslant \frac{c_{23}(t)t^{\theta/2}}{(|x'|+t)^{k+\theta+|\mathbf{j}|}}, \quad \beta=1,2, \quad \theta\in[0,1], \tag{2.36}$$

$$|\mathcal{D}_x^{\mathbf{j}} V_{3,k}(x',t)| \leqslant c_{24}(t)(|x'|+t)^{-(k+|\mathbf{j}|}, \tag{2.37}$$

$$|\mathcal{D}_t V_{\beta,k}(x',t)| \leqslant \frac{c_{25}(t)}{t^{1-\theta/2}(|x'|+t)^{k+\theta}}, \quad \beta=1,2, \quad \theta\in[0,1],$$

for $t>0$. Here c_{23}, c_{24}, c_{25} are incresing functions of t. Each function $V_{i,k}$ equals zero when $t<0$.

This lemma is proved in just the same way as Lemma 2.3 from [1].

Corollary 2.1. *For the functions $V_{j,2}(x',t)$, $j=1,2,3$, the estimates*

$$\int\limits_0^t \int\limits_{\mathbb{R}^2} |\Delta_\beta^m(h) V_{j,2}(x',\tau)\mathrm{d}x'\mathrm{d}\tau \leqslant c(t)h, \quad m>1, \tag{2.38}$$

hold. Here

$$\Delta_\beta^m(h)V_{j,2}(x',\tau) = \sum_{k=0}^m (-1)^{m-k} C_m^k V_{j,2}(x'+e_\beta kh,\tau)$$

is a finite difference of $V_{j,2}$ with respect to x_β of order m, $\beta=1,2$, $e_1=(1,0)$, $e_2=(0,1)$.

It was shown in [5] that inequalities (2.37), (2.36) with $\theta=0$ imply (2.3).

Let us formulate now the existence theorem for the homogeneous system (2.1), (2.2).

Theorem 2.1. *Let $\alpha\in(0,1)$ and $T>0$. For arbitrary functions $b_j\in \overset{\circ}{C}{}^{1+\alpha,\frac{1+\alpha}{2}}(\mathbb{R}^2_T)$, $j=1,2,3$, and $B\in \overset{\circ}{C}{}^{\alpha,\alpha/2}(\mathbb{R}^2_T)$ decreasing as power functions together with their derivatives as $|x'|\to\infty$, system (2.1), (2.2) is solvable in $C^{2+\alpha,1+\alpha/2}(D_T)$ and the estimate for the solution norm*

$$\|\mathbf{v}\|_{D_T}^{(2+\alpha,1+\alpha/2)} \leqslant C(T)\Big\{\sum_{j=1}^3 \langle\, b_j\,\rangle_{\mathbb{R}^2_T}^{(1+\alpha,\frac{1+\alpha}{2})} + \sigma\langle\, B\,\rangle_{\mathbb{R}^2_T}^{(\alpha,\alpha/2)}\Big\} \tag{2.39}$$

holds. Here $C(T)$ is a nondecreasing function of T, $D_T\equiv\mathbb{R}^-_T\cup\mathbb{R}^+_T$, $\mathbb{R}^\pm_T=\mathbb{R}^3_\pm\times(0,T)$, $\mathbb{R}^2_T=\mathbb{R}^2\times(0,T)$.

Proof. We note ones more that the vector-valued function (2.5) is a solution of the problem (2.1), (2.7) with parabolic in the sense of Petrovskii system and with the boundary conditions satisfying the complementary Lopatinskii condition. Therefore this diffraction problem can be treated in the same way as other classical initial-boundary value problems [6] and for its solution the estimate

$$\|\mathbf{v}\|_{D_T}^{(2+\alpha,1+\alpha/2)} \leqslant c\Big\{\sum_{\beta=1}^{2}\|b_\beta\|_{\mathbb{R}_T^2}^{(1+\alpha,\frac{1+\alpha}{2})} + \|a_3\|_{\mathbb{R}_T^2}^{(1+\alpha,\frac{1+\alpha}{2})}\Big\} \tag{2.40}$$

holds.

Since the function a_3 belongs to $\overset{\circ}{C}{}^{1+\alpha,\frac{1+\alpha}{2}}(\mathbb{R}_T^2)$, the complete norm of it is equivalent to its leading term. Hence, we need to estimate only $\langle\, a_3\,\rangle_{\mathbb{R}_T^2}^{(1+\alpha,\frac{1+\alpha}{2})}$ which is equivalent to the sum of seminorms $\langle\,\nabla' a_3\,\rangle_{x',\mathbb{R}_T^2}^{(\alpha)} + \langle\, a_3\,\rangle_{t,\mathbb{R}_T^2}^{(\frac{1+\alpha}{2})}$, $\nabla' = (\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2})$ (see [7, Lemma 2]).

The function a_3 is expressed by (2.35) where each integral is convergent due to the decay of $\nabla' b_j$ and B at infinity and to the fact that the kenels W_j satisfy (2.36) with $k=2$.

Using Corollary 2.1, we can write

$$\sup_{h>0} h^{-(1+\alpha)} \sup_{x'\in\mathbb{R}^2}\Big|\Delta_\beta^{m+i}(h)a_3(x',t)\Big|$$
$$\leqslant c\Big\{\sup_{h>0} h^{-(1+\alpha)} \sup_{x'\in\mathbb{R}^2}\Big|\Delta_\beta^{m+i}(h)b_3(x',t)\Big|$$
$$+\sum_{j=1}^{3}\sup_{h>0} h^{-\alpha}\sup_{x'\in\mathbb{R}^2}\Big|\Delta_\beta^{i}(h)\nabla' b_j(x',t)\Big| + \sup_{h>0} h^{-\alpha}\sup_{x'\in\mathbb{R}^2}\Big|\Delta_\beta^{i}(h)B(x',t)\Big|$$

for integers $m>1$ and $i>\alpha$. This implies in view of the equivalence of the seminorms $\langle\, u\,\rangle_{\mathbb{R}^2}^{(l)}$ and $\sum_{j=1}^{2}\sup_{h>0} h^{-l}\sup_{x'\in\mathbb{R}^2}\big|\Delta_\beta^i(h)u(x')\big|$, $i>l$, (see [8]) that

$$\langle\,\nabla' a_3\,\rangle_{x',\mathbb{R}_T^2}^{(\alpha)} \leqslant c\Big\{\sum_{j=1}^{3}\langle\,\nabla' b_j\,\rangle_{x',\mathbb{R}_T^2}^{(\alpha)} + \sigma\langle\, B\,\rangle_{x',\mathbb{R}_T^2}^{(\alpha)}\Big\}. \tag{2.41}$$

Next, let us estimate $\langle\, a_3\,\rangle_{t,\mathbb{R}_T^2}^{(\frac{1+\alpha}{2})}$. We note that the kernels W_β, $\beta=1,2$, are derivatives with respect to x_β of some functions, therefore

$$\int\limits_{\mathbb{R}^2} W_\beta(x'-y',t-\tau)\mathrm{d}y' = 0,$$

and we have

$$A_2(x',t+h) - A_2(x',t)$$

$$= \sum_\beta \Big\{ \int\limits_{t-h}^{t+h} \int\limits_{\mathbb{R}^2} W_\beta(x'-y', t+h-\tau)\Big(\frac{\partial b_3(y',\tau)}{\partial y_\beta} - \frac{\partial b_3(x',\tau)}{\partial x_\beta}\Big)\mathrm{d}y'\,\mathrm{d}\tau$$

$$- \int\limits_{t-h}^{t} \int\limits_{\mathbb{R}^2} W_\beta(x'-y', t-\tau)\Big(\frac{\partial b_3}{\partial y_\beta} - \frac{\partial b_3}{\partial x_\beta}\Big)\mathrm{d}y'\,\mathrm{d}\tau$$

$$+ \int\limits_0^h \int\limits_0^{t-h} \int\limits_{\mathbb{R}^2} \frac{\partial W_\beta(x'-y', t+\lambda-\tau)}{\partial t}\Big(\frac{\partial b_3}{\partial y_\beta} - \frac{\partial b_3}{\partial x_\beta}\Big)\mathrm{d}y'\,\mathrm{d}\tau\mathrm{d}\lambda\Big\}.$$

If $h < t$, $t+h < T$, and $\theta \in (\alpha, 1)$, one can conclude that

$$\big|A_2(x', t+h) - A_2(x', t)\big| \leqslant c \sup_{\tau < t+h} \langle \nabla' b_3(\,\cdot\,, \tau) \rangle^{(\alpha)}_{\mathbb{R}^2} h^{1+\alpha-\theta/2}$$
$$\leqslant C(T)\langle \nabla' b_3 \rangle^{(\alpha)}_{x', \mathbb{R}^2_T} h^{\frac{1+\alpha}{2}}.$$

In the case when $h > t$, we obtain

$$\big|A_2(x', t+h) - A_2(x', t)\big| \leqslant \big|A_2(x', t+h) + A_2(x', t)\big|$$

$$\leqslant c\langle \nabla' b_3 \rangle^{(\alpha)}_{x', \mathbb{R}^2_T} (t+h)^{1+\alpha-\theta/2} \leqslant C(T)\langle \nabla' b_3 \rangle^{(\alpha)}_{x', \mathbb{R}^2_T} h^{\frac{1+\alpha}{2}}.$$

We can estimate A_3 in the same way if we take into account that $\xi^2 = -\sum_{\beta=1}^2 (\mathrm{i}\,\xi_\beta)^2$. So,

$$\big|A_3(x', t+h) - A_3(x', t)\big| \leqslant C(T) \sum_{\beta=1}^2 \langle \nabla' b_\beta \rangle^{(\alpha)}_{x', \mathbb{R}^2_T} h^{\frac{1+\alpha}{2}}.$$

Finally, by virtue of the equality

$$\frac{Mq}{P} = \frac{1}{s}\Big(1 - \frac{\sigma\xi^2 q}{P}\Big),$$

we have

$$A_4(x', t+h) - A_4(x', t)$$

$$= \int\limits_0^h \frac{\partial}{\partial t} \int\limits_{\mathbb{R}^2}^{t+\rho} \sigma W_4(x'-y', t+\rho-\tau) B(y', \tau)\mathrm{d}y'\,\mathrm{d}\tau\mathrm{d}\rho = \int\limits_0^h \sigma B(x', t+\rho)\mathrm{d}\rho$$

$$+\sum_{\beta=1}^{2}\int\limits_0^h\int\limits_0^{t+\rho}\int\limits_{\mathbb{R}^2}\frac{\partial W_\beta}{\partial x_\beta}(x'-y',t-\tau)\sigma\big[B(y',\tau)-B(x',\tau)\big]\mathrm{d}y'\mathrm{d}\tau\mathrm{d}\rho.$$

Hence,

$$\big|A_4(x',t+h)-A_4(x',t)\big|\leqslant\sigma h\sup_{\tau<T}|B(x',\tau)|$$

$$+c(t)\sigma\langle\, B\,\rangle^{(\alpha)}_{x',\mathbb{R}^2_T}\int\limits_0^h\int\limits_0^{t+\rho}\int\limits_{\mathbb{R}^2}\frac{|x'-y'|^\alpha\mathrm{d}y'}{(|x'-y'|+t+\rho-\tau)^3}\mathrm{d}\tau\mathrm{d}\rho$$

$$\leqslant C(T)\big\|B\big\|^{(\alpha)}_{x',\mathbb{R}^2_T}h^{\frac{1+\alpha}{2}}\leqslant C(T)\big(T^{\alpha/2}+1\big)\langle\, B\,\rangle^{(\alpha,\alpha/2)}_{\mathbb{R}^2_T}h^{\frac{1+\alpha}{2}}.$$

Since the estimate of A_1 is trivial, we deduce from the above estimates for A_i:

$$\langle\, a_3\,\rangle^{(\frac{1+\alpha}{2})}_{t,\mathbb{R}^2_T}\leqslant C(T)\Big\{\langle\, b_3\,\rangle^{(1+\alpha,\frac{1+\alpha}{2})}_{t,\mathbb{R}^2_T}+\sum_{\beta=1}^{2}\langle\,\nabla' b_\beta\,\rangle^{(\alpha)}_{x',\mathbb{R}^2_T}+\sigma\langle\, B\,\rangle^{(\alpha,\alpha/2)}_{\mathbb{R}^2_T}\Big\}$$

which together with (2.40), (2.41) yields the inequality (2.39). □

3. The existence theorem for the nonhomogeneous model problem

Now we will study the nonhomogeneous linear problem

$$\mathcal{D}_t\mathbf{w}-\nu^\pm\nabla^2\mathbf{w}-(\nu^\pm+\kappa^\pm)\nabla(\nabla\cdot\mathbf{w})=\mathbf{f}\quad\text{in}\quad D_T=(\mathbb{R}^3_+\cup\mathbb{R}^3_-)\times(0,T),$$

$$\mathbf{w}\Big|_{t=0}=\mathbf{w}_0,\quad \mathbf{w}\underset{|x|\to\infty}{\longrightarrow}0, \tag{3.1}$$

$$[\mathbf{w}]\Big|_{x_3=0}=0,\quad -\Big[\mu^\pm\Big(\frac{\partial w_\beta}{\partial x_3}+\frac{\partial w_3}{\partial x_\beta}\Big)\Big]\Big|_{x_3=0}=d_\beta(x',t),\quad \beta=1,2,$$

$$\Big[\lambda^\pm\nabla\cdot\mathbf{w}+2\mu^\pm\frac{\partial w_3}{\partial x_3}\Big]\Big|_{x_3=0}+\sigma\Delta'\int\limits_0^t w_3\mathrm{d}\tau=d_3+\sigma\int\limits_0^t D\mathrm{d}\tau$$

$$\text{on}\quad \mathbb{R}^2_T=\mathbb{R}^2\times(0,T).$$

The following theorem takes place.

Theorem 3.1. *Let $\alpha \in (0,1)$, $\gamma > 0$ and $T \in (0,\infty)$. Assume that* $\mathbf{f} \in C_\gamma^{\alpha,\alpha/2}(D_T)$, $\mathbf{w}_0 \in C_\gamma^{2+\alpha}(\mathbb{R}^3_- \cup \mathbb{R}^3_+)$, $d_j \in C_\gamma^{1+\alpha,\frac{1+\alpha}{2}}(\mathbb{R}^2_T)$, $j=1,2,3$, $D \in C_\gamma^{\alpha,\alpha/2}(\mathbb{R}^2_T)$. *Moreover, we suppose also that the compatibility conditions*

$$[\mathbf{w}]\Big|_{x_3=0} = 0, \quad \big[(\nu^\pm + \kappa^\pm)\nabla(\nabla\cdot\mathbf{w}_0) + \nu^\pm\nabla^2\mathbf{w}_0 + \mathbf{f}(x,0)\big]\Big|_{x_3=0} = 0, \tag{3.2}$$

$$-\Big[\mu^\pm\Big(\frac{\partial w_{03}}{\partial x_\beta} + \frac{\partial w_{0\beta}}{\partial x_3}\Big)\Big]\Big|_{x_3=0} = d_\beta(x',0), \quad \beta = 1,2, \tag{3.3}$$
$$\Big[\lambda^\pm\nabla\cdot\mathbf{w}_0 + 2\mu^\pm\frac{\partial w_{03}}{\partial x_3}\Big]\Big|_{x_3=0} = d_3(x',0), \quad x' \in \mathbb{R}^2,$$

hold.

Then there exists a unique solution $\mathbf{w} \in C_\gamma^{2+\alpha,1+\alpha/2}(D_T)$ *of problem* (3.1)*. This solution satisfies the estimate*

$$\|\mathbf{w}\|^{(2+\alpha,1+\alpha/2)}_{\gamma,D_T} \leqslant C(T)\Big\{\|\mathbf{f}\|^{(\alpha,\alpha/2)}_{\gamma,D_T} + \|\mathbf{w}_0\|^{(2+\alpha)}_{\gamma,\mathbb{R}^3_-\cup\mathbb{R}^3_+}$$
$$+\sum_{j=1}^{2}\|d_j\|^{(1+\alpha,1+\alpha/2)}_{\gamma,\mathbb{R}^2_T} + \sigma\|D\|^{(\alpha,\alpha/2)}_{\gamma,\mathbb{R}^2_T}\Big\} \tag{3.4}$$

with nondecreasing function $C(T)$*.*

Proof. We find the solution $\mathbf{w}$ in the form:

$$\mathbf{w} = \mathbf{w}^+ + \mathbf{w}^- + \mathbf{w}',$$

where

$$\text{i)} \qquad \mathbf{w}^+ = \begin{cases} \mathbf{w}_1 & \text{in } \mathbb{R}^+_T, \\ 0 & \text{in } \mathbb{R}^-_T, \end{cases}$$

$\mathbf{w}_1$ is a solution of the Cauchy problem

$$\begin{aligned} &\mathcal{D}_t\mathbf{w}_1 - (\nu^+ + \kappa^+)\nabla(\nabla\cdot\mathbf{w}_1) - \nu^+\Delta\mathbf{w}_1 = \mathbf{f}^+, \\ &\mathbf{w}_1\big|_{t=0} = \mathbf{w}_0^+, \end{aligned} \tag{3.5}$$

(here $\mathbf{f}^+$, $\mathbf{w}_0^+$ are extentions of the constrictions $\mathbf{f}\big|_{\mathbb{R}^+_T}$ and $\mathbf{w}_0\big|_{\mathbb{R}^3_+}$ into $\mathbb{R}^3_T$ and $\mathbb{R}^3$, respectively, with preservation of class and decay at infinity);

$$\text{ii)} \qquad \mathbf{w}^- = \begin{cases} 0 & \text{in } \mathbb{R}^+_T, \\ \mathbf{w}_2 & \text{in } \mathbb{R}^-_T, \end{cases}$$

$\mathbf{w}_2$ is a solution to the Dirichlet problem

$$\begin{aligned} &\mathcal{D}_t\mathbf{w}_2-(\nu^-+\kappa^-)\nabla(\nabla\cdot\mathbf{w}_2)-\nu^-\Delta\mathbf{w}_2=\mathbf{f}^-,\\ &\mathbf{w}_2\big|_{t=0}=\mathbf{w}_0^-,\quad \mathbf{w}_2\big|_{x_3=0}=\mathbf{w}_1\big|_{x_3=0}, \end{aligned} \tag{3.6}$$

($\mathbf{f}^-\equiv\mathbf{f}\big|_{\mathbb{R}_T^-}$ and $\mathbf{w}_0^-\equiv\mathbf{w}_0\big|_{\mathbb{R}^3_-}$);

iii) finally, $\mathbf{w}'$ is a solution to the homogeneous problem (2.1) with the following functions on the right-hand sides in (2.2):

$$\begin{aligned} &b_\beta=d_\beta+\Big[\mu^\pm\Big(\frac{\partial w_\beta^\pm}{\partial x_3}+\frac{\partial w_3^\pm}{\partial x_\beta}\Big)\Big]\Big|_{x_3=0},\quad \beta=1,2,\\ &b_3=d_3-\Big[\lambda^\pm\nabla\cdot\mathbf{w}^\pm-2\mu^\pm\frac{\partial w_3^\pm}{\partial x_3}\Big]\Big|_{x_3=0},\\ &B=D-\Delta' w_3^+\big|_{x_3=0}. \end{aligned} \tag{3.7}$$

According to Theorems 4.9 and 4.10 from [6] and due to (3.2), the problems (3.5) and (3.6) are uniquely solvable in $C^{2+\alpha,1+\alpha/2}(\mathbb{R}^3_T)$ and the norms of $\mathbf{w}^\pm$ satisfy the inequalities

$$\|\mathbf{w}^\pm\|_{D_T}^{(2+\alpha,1+\alpha/2)}\leqslant c^\pm\Big(\|\mathbf{f}\|_{\mathcal{D}_t}^{(\alpha,\alpha/2)}+\|\mathbf{w}_0\|_{\mathbb{R}^3_-\cup\mathbb{R}^3_+}^{(2+\alpha)}\Big). \tag{3.8}$$

Next, we could apply Theorem 2.1 to problem (2.1), (2.2), (3.7), but we need to verify that the functions in (3.7) take zero values when $t=0$. This follows from (3.3) for b_j, $j=1,2,3$, but $B\big|_{t=0}$ is not equal to zero in general. Therefore, we replace it by $B'=B+\mathcal{D}_tA$, where $A\in C^{(2+\alpha,1+\alpha/2)}(\mathbb{R}^2_T)$, $A\Big|_{t=0}=0$, $\mathcal{D}_tA\Big|_{t=0}=-B\Big|_{t=0}\equiv\Delta' w_{03}-D\Big|_{t=0}$ and

$$\|A\|_{\mathbb{R}^2_T}^{(2+\alpha,1+\alpha/2)}\leqslant c\Big(\|D\|_{\mathbb{R}^2_T}^{(\alpha,\alpha/2)}+\|\mathbf{w}_0\|_{\mathbb{R}^3_-\cup\mathbb{R}^3_+}^{(2+\alpha)}\Big).$$

In this connection, b_3 is to be transformed into $b_3''=b_3-\sigma A\in\overset{\circ}{C}{}^{1+\alpha,1+\alpha/2}(\mathbb{R}^2_T)$. So, the hypotheses of Theorem 2.1 are satisfied and we conclude that $\mathbf{w}'$ exists and

$$\begin{aligned} &\|\mathbf{w}'\|_{D_T}^{(2+\alpha,\frac{1+\alpha}{2})}\\ &\quad\leqslant C(T)\Big\{\sum_{\beta=1}^{2}\langle\, b_\beta\,\rangle_{\mathbb{R}^2_T}^{(1+\alpha,\frac{1+\alpha}{2})}+\langle\, b_3''\,\rangle_{\mathbb{R}^2_T}^{(1+\alpha,\frac{1+\alpha}{2})}+\sigma\langle\, B'\,\rangle_{\mathbb{R}^2_T}^{(\alpha,\alpha/2)}\Big\} \end{aligned}$$

$$\leqslant C'(T)\Big\{\sum_{j=1}^{3}\|d_j\|_{\mathbb{R}^2_T}^{(1+\alpha,\frac{1+\alpha}{2})}+\sigma\|D\|_{\mathbb{R}^2_T}^{(\alpha,\alpha/2)}$$

$$+\|\mathbf{w}_0\|_{\mathbb{R}^3_-\cup\mathbb{R}^3_+}^{(2+\alpha)}+\sum_{i=+,-}\|\mathbf{w}^i\|_{D_T}^{(2+\alpha,1+\alpha/2)}\Big\}.$$

This together with (3.8) implies (3.4) in ordinary Hölder spaces, i.e., with $\gamma=0$.

Thus, the existence of a solution of (3.1) is proved, and it remains only to verify the estimate (3.4) for it in the weighted spaces.

To this end, we consider a point $x_0\in\mathbb{R}^3\setminus B_4$, $B_\rho\equiv\{|x|<\rho\}$, and a smooth cut-off function $\eta(y)$:

$$\eta(y)=\begin{bmatrix}1 & \text{for} \;\; |x_0-y|\leqslant\dfrac{|x_0|}{4}\equiv d,\\ 0 & \text{for} \;\; |x_0-y|\geqslant\dfrac{|x_0|}{2}.\end{bmatrix}$$

Let for η the inequalities

$$\|\nabla^k\eta\|_{\mathbb{R}^3}\leqslant cd^{-k},\quad k\in\mathbb{N},$$

$$\langle\,\eta\,\rangle_{\mathbb{R}^3}^{(\beta)}\leqslant cd^{-\beta},\quad \beta\in(0,1),$$

hold.

We introduce the new unknown function $\mathbf{u}=\mathbf{w}\eta$ which satisfies the problem

$$\mathcal{D}_t\mathbf{u}-\nu^\pm\nabla^2\mathbf{u}-(\nu^\pm+\kappa^\pm)\nabla(\nabla\cdot\mathbf{u})=\mathbf{f}^{(1)}\quad\text{in}\;\;D_T,$$

$$\mathbf{u}\big|_{t=0}=\mathbf{w}_0\eta,\quad \mathbf{u}\underset{|x|\to\infty}{\longrightarrow}0, \tag{3.9}$$

$$[\mathbf{u}]\Big|_{x_3=0}=0,\quad -\Big[\mu^\pm\Big(\frac{\partial u_\beta}{\partial x_3}+\frac{\partial u_3}{\partial x_\beta}\Big)\Big]\Big|_{x_3=0}=d_\beta^{(1)}(x',t),\quad \beta=1,2,$$

$$\Big[\lambda^\pm\nabla\cdot\mathbf{u}+2\mu^\pm\frac{\partial u_3}{\partial x_3}\Big]\Big|_{x_3=0}+\sigma\Delta'\int_0^t w_3\mathrm{d}\tau=d_3^{(1)}+\sigma\int_0^t D^{(1)}\mathrm{d}\tau\quad\text{on}\quad\mathbb{R}^2_T,$$

where

$$\mathbf{f}^{(1)}=\mathbf{f}\eta-\nu^\pm\big(\mathbf{w}\nabla^2\eta+2(\nabla\eta\cdot\nabla)\mathbf{w}\big)-(\nu^\pm+\kappa^\pm)\Big\{\nabla(\mathbf{w}\cdot\nabla\eta)+(\nabla\cdot\mathbf{w})\nabla\eta\Big\},$$

$$d_\beta^{(1)} = d_\beta \eta - \Big[\mu^\pm \Big(w_\beta \frac{\partial \eta}{\partial x_3} + w_3 \frac{\partial \eta}{\partial x_\beta}\Big)\Big]\Big|_{x_3=0}, \quad \beta = 1,2,$$

$$d_3^{(1)} = d_3 \eta + \Big[\lambda^\pm \mathbf{w}\cdot\nabla\eta + 2\mu^\pm w_3 \frac{\partial \eta}{\partial x_3}\Big]\Big|_{x_3=0},$$

$$D^{(1)} = D\eta + w_3 {\nabla'}^2 \eta + 2(\nabla'\eta\cdot\nabla')w_3, \quad \nabla' = \Big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\Big).$$

The compatibility conditions are fulfilled for (3.9) and we can use inequality (3.4) with $\gamma = 0$. As a result, we obtain for $\mathbf{u}$ the following estimates:

$$\|\mathbf{w}\|^{(2+\alpha,1+\alpha/2)}_{\bigcup\limits_{i=\pm} B^i_{d,T}} \leqslant \|\mathbf{u}\|^{(2+\alpha,1+\alpha/2)}_{\bigcup\limits_{i=\pm} B^i_{2d,T}} \leqslant$$

$$\leqslant c\Big\{\|\mathbf{f}^{(1)}\|^{(\alpha,\alpha/2)}_{\bigcup\limits_{i=\pm} B^i_{2d,T}} + \|\mathbf{w}_0\eta\|^{(2+\alpha)}_{\bigcup\limits_{i=\pm} B^i_{2d}(x_0)} +$$

$$+\sum_{j=1}^{3} \|d_j^{(1)}\|^{(1+\alpha,\frac{1+\alpha}{2})}_{K_{2d,T}} + \sigma\|D^{(1)}\|^{(\alpha,\alpha/2)}_{K_{2d,T}}\Big\} \leqslant$$

$$\leqslant c\Big\{\|\mathbf{f}\|^{(\alpha,\alpha/2)}_{\bigcup\limits_{i=\pm} B^i_{2d,T}} + \|\mathbf{w}_0\|^{(2+\alpha)}_{\bigcup\limits_{i=\pm} B^i_{2d}(x_0)} +$$

$$+\sum_{j=1}^{3} \|d_j\|^{(1+\alpha,\frac{1+\alpha}{2})}_{K_{2d,T}} + \sigma\|D\|^{(\alpha,\alpha/2)}_{K_{2d,T}} +$$

$$+d^{-2}\|\mathbf{w}\|^{(\alpha,\alpha/2)}_{\bigcup\limits_{i=\pm} B^i_{2d,T}} + d^{-1}\|\nabla\mathbf{w}\|^{(\alpha,\alpha/2)}_{\bigcup\limits_{i=\pm} B^i_{2d,T}}\Big\}. \tag{3.10}$$

Here we have used the notation:

$$\bigcup_{i=\pm} B^i_{\rho,T} = B^-_{\rho,T} \cup B^+_{\rho,T}, \quad B^\pm_{\rho,T} = B^\pm_\rho(x_0)\times(0,T),$$

$$B^\pm_\rho(x_0) = \big\{\mathbb{R}^3_\pm \,\big|\, |x-x_0| < \rho\big\}, \quad K_{\rho,T} = K_\rho(x_0)\times(0,T),$$

$$K_\rho(x_0) = B_\rho(x_0)\cap\{x_3 = 0\}, \quad B_\rho(x_0) = \{|x-x_0| < \rho\}.$$

Now, we multiply (3.10) by $(1+d)^\gamma$ and take into account that $|x| \leqslant 5d$ for $\forall x \in B_d(x_0)$, and $|x| \geqslant 2d > 2$ for $\forall x \in B_{2d}(x_0)$. It is easily seen that

$$\frac{1}{5^\gamma}\|\mathbf{w}\|^{(2+\alpha,1+\alpha/2)}_{\gamma,\bigcup\limits_{i} B^i_{d,T}} \leqslant c\Big\{\|\mathbf{f}\|^{(\alpha,\alpha/2)}_{\gamma,\bigcup\limits_{i} B^i_{2d,T}} + \|\mathbf{w}_0\|^{(2+\alpha)}_{\gamma,\bigcup\limits_{i} B^i_{2d,T}}$$

$$+\sum_{i=1}^{3}\|d_i\|^{(1+\alpha,(1+\alpha)/2)}_{\gamma,K_{2d,T}}+\sigma\|D\|^{(\alpha,\alpha/2)}_{\gamma,K_{2d,T}}+2^{\gamma}\|\mathbf{w}\|^{(2+\alpha,1+\alpha/2)}_{\bigcup\limits_i B^i_{2d,T}}\Big\}.$$

We note that the norms in $C^{2+\alpha,1+\alpha/2}_{\gamma}(B_5)$ and $C^{2+\alpha,1+\alpha/2}(B_5)$ are equivalent. Using again (3.4) with $\gamma=0$ and taking supremum over all $x_0\in\mathbb{R}^3\setminus B_4$, we arrive at the estimate (3.4) with $\gamma\in(0,1]$. If $\gamma>1$ then we will have on the right-hand side of (3.4) the additional term $2\|\mathbf{w}\|^{(2+\alpha,1+\alpha/2)}_{\gamma-1,D_T}$ which we can take off by means of the Young inequality

$$(1+|x|)^{\gamma-1}|u(x)|^{p^*}|u(x)|^{q^*}\leqslant\varepsilon(1+|x|)^{\gamma}|u(x)|+c(\varepsilon)|u(x)|,$$

where $p^*=\frac{\gamma-1}{\gamma}$, $q^*=\frac{1}{\gamma}$.

So, we have proved (3.4) for all $\gamma>0$, which implies the uniqueness of the solution $\mathbf{w}$. □

4. Problem (1.7)

Theorem 1.1 for problem (1.7) is proved in just the same way as a similar theorem for the case of a single compressible fluid (see [2, §3]). The authors used there the Banach theorem to establish the solvability of a nonlinear problem which was considered as a perturbation of a linear one. In our case, the role of this linear system is played by the following problem

$$\begin{gathered}
\mathcal{D}_t\mathbf{w}-\rho_0^{-1}(x)\nabla\cdot\mathbb{T}'(\mathbf{w})=\mathbf{f}\quad\text{in}\quad D_T=Q^-_T\cup Q^+_T,\\
\mathbf{w}\big|_{t=0}=\mathbf{w}_0\quad\text{in}\quad\Omega^-_0\cup\Omega^+_0,\quad\mathbf{w}\xrightarrow[|x|\to\infty]{}0,\\
[\mathbf{w}]\big|_{G_T}=0,\quad[\mu^{\pm}\Pi_0\mathbb{S}(\mathbf{w})\mathbf{n}_0]\big|_{G_T}=\Pi_0\mathbf{a},\\
[\mathbf{n}_0\cdot\mathbb{T}'(\mathbf{w})\mathbf{n}]\big|_{\Gamma}-\sigma\mathbf{n}_0\cdot\Delta_\Gamma\int_0^t\mathbf{w}\big|_\Gamma d\tau=a+\sigma\int_0^t A\,d\tau,\quad t\in(0,T),
\end{gathered}\tag{4.1}$$

where Δ_Γ is the Beltrami–Laplace operator on Γ;

$$\mathbb{T}'(\mathbf{v})=\begin{cases}(\lambda^+\nabla\cdot\mathbf{v})\mathbb{I}+\mu^+\mathbb{S}(\mathbf{v})&\text{in}\quad Q^+_T,\\(\lambda^-\nabla\cdot\mathbf{v})\mathbb{I}+\mu^-\mathbb{S}(\mathbf{v})&\text{in}\quad Q^-_T;\end{cases}$$

ρ_0, $\mathbf{f}$, $\mathbf{w}_0$, $\mathbf{a}$, a, and A are given functions, σ is a nonnegative constant.

For problem (4.1), the following theorem takes place.

Theorem 4.1. *Let* $\alpha \in (0,1)$, γ, $T > 0$. *We assume that* $\Gamma \in C^{2+\alpha}$, $\rho_0 \in C^{l+\alpha}(\Omega_0^- \cup \Omega_0^+)$, $R_1 \geqslant \rho_0(\xi) \geqslant R_0 > 0$, $\mathbf{f} \in \mathbf{C}_\gamma^{\alpha,\frac{\alpha}{2}}(D_T)$, $\mathbf{w}_0 \in \mathbf{C}_\gamma^{2+\alpha}(\Omega_0^- \cup \Omega_0^+)$, $\mathbf{a} \in \mathbf{C}^{1+\alpha,\frac{1+\alpha}{2}}(G_T)$, $a \in C^{1+\alpha,\frac{1+\alpha}{2}}(G_T)$, $A \in C^{\alpha,\frac{\alpha}{2}}(G_T)$. *In addition, let the dynamical viscosities satisfy the inequalities* (1.8) *and the compatibility conditions*

$$[\mathbf{w}_0]|_\Gamma = 0, \quad \left[\rho_0^{-1}(x)(\nabla \cdot \mathbb{T}'(\mathbf{w}_0) + \mathbf{f}(x,0)\right]\Big|_\Gamma = 0,$$

$$\left[\mu^\pm \Pi_0 \mathbb{S}(\mathbf{w}_0)\mathbf{n}_0\right]\Big|_\Gamma = \Pi_0 \mathbf{a}\Big|_{t=0}, \quad \left[\mathbf{n}_0 \cdot \mathbb{T}'(\mathbf{w}_0)\mathbf{n}_0\right]\Big|_\Gamma = a\Big|_{t=0}$$

hold.

Then the problem (4.1) *has a unique solution* $\mathbf{w} \in \mathbf{C}_\gamma^{2+\alpha,1+\alpha/2}(D_T)$ *and this solution satisfies the estimate*

$$\|\mathbf{w}\|_{\gamma,D_T}^{(2+\alpha,1+\alpha/2)} \leqslant c_1(T)\big\{\|\mathbf{f}\|_{\gamma,D_T}^{(\alpha,\alpha/2)} + \|\mathbf{w}_0\|_{\gamma,\Omega_0^-\cup\Omega_0^+}^{(2+\alpha)}$$

$$+\|\mathbf{a}\|_{G_T}^{(1+\alpha,(1+\alpha)/2)} + \|a\|_{G_T}^{1+\alpha,(1+\alpha)/2} + \sigma\|A\|_{G_T}^{(\alpha,\alpha/2)}\big\}, \qquad (4.2)$$

where $c_1(T)$ *is nondecreasing function of* T.

This theorem can be established on the basis of Theorem 3.1 by following §2 from [2]. First, estimate (4.2) is proved by the Schauder method as a priori one. Next, we construct a regularizer using the solvability of problems like (3.1). This operator solves a system close to (4.1). Finally, we show that the difference between the latter one and the problem (4.1) is small and the operator corresponding to it in sum with the identity operator is invertible. The uniqueness of a solution follows from a priori estimate (4.2).

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