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CANONICAL FORMS OF SKEW POLYNOMIAL RINGS

Abstract. We consider special relations which lead to skew polynomial rings with the following property: every commutation relation between the elements of the basis of the ring and the coefficients can be calculated. Such relations are called canonical forms of the skew polynomial ring. For example, the Weyl relation will be a canonical form for the Weyl algebra. Obviously, such skew polynomial rings can be applied to the representation theory and to mathematical physics.

1. Introduction

1. Let $\mathcal{A}$ be a ring with unit $\mathbf{1}$. Let $\mathcal{K}$ be a subring of $\mathcal{A}$, and τ an element of $\mathcal{A}$.

Definition 1. *The ring $\mathcal{A}$ will be called a (left) skew polynomial ring with respect to τ over the ring $\mathcal{K}$ if every element $a \in \mathcal{A}$ can be uniquely represented in the form*

$$a = \alpha_0 + \alpha_1\tau + \cdots + \alpha_{m_0}\tau^{m_0}, \tag{1}$$

where $\alpha_i \in \mathcal{K}$, $m_0 = m_0(\alpha)$.

The definition of a right skew polynomial ring is analogous. For every $\alpha \in \mathcal{K}$ and $\tau \in \mathcal{A}$, the element $\tau\alpha \in A$ is defined. Thus by (1) we have a unique expression

$$\tau\alpha = \alpha_0^{(1)} + \alpha_1^{(1)}\tau + \cdots + \alpha_{m_1}^{(1)}\tau^{m_1}, \tag{2}$$

where $\alpha_j^{(1)} \in \mathcal{K}$, $m_1 = m_1(\alpha)$.

Obviously, the mappings $\psi_j^{(1)}$: $\psi_j^{(1)}(\alpha) = \alpha_j^{(1)}$ are linear mappings of $\mathcal{K}$ into itself.

Thus for each $\alpha \in \mathcal{K}$, (2) can be represented in the form

$$\tau\alpha = \psi_0^1(\alpha) + \psi_1^1(\alpha)\tau + \cdots + \psi_{m_1}^1(\alpha)\tau^{m_1}, \quad m_1 = m_1(\alpha). \tag{3}$$

Usually, one denotes $\Psi = (\psi_0^1, \psi_1^1, \dots)$, $\mathcal{A} = \mathcal{K}[\tau, \Psi]$. The set $\Psi = (\psi_0^1, \psi_1^1, \dots)$ is usually called the general derivation of the ring $\mathcal{K}$, see [4].

The following two possibilities appear:

1. For any positive integer m, there exists $\alpha \in \mathcal{K}$ such that $\psi_m^1(\alpha) \neq 0$.
2. There exists the smallest integer s such that for $m > s$ and for each $\alpha \in \mathcal{K}$ we have $\psi_m^1(\alpha) = 0$.

In the first case the ring $\mathcal{K}[\tau, \Psi]$ will be called a skew polynomial ring of infinite order. In the second case $\mathcal{K}[\tau, \Psi]$ is of order s, see [9].

If $s = 1$ the right-hand side of (3) is usually represented in the form

$$\tau\alpha = \varphi(\alpha)\tau + \psi(\alpha). \tag{4}$$

Relation (4) defines Ore's skew polynomial ring [7].

The general case of skew polynomial rings for $\psi_0^1 = 0$ was considered in [8]. In this case one can define the ring of skew series, i.e., the right-hand side of (2) is of the form $\sum_{i=1}^{\infty} \alpha_i^{(1)} \tau^i$. Skew series rings are very well studied, see [1–4].

The case $\psi_0^1 \neq 0$ is very important for applications. For example, it appears in quadratic algebras when they can be considered as skew polynomial rings, [5, 10].

Here we will consider the general case of skew polynomial rings when the operator ψ_0^1 is not zero.

2. Recall some well-known properties of skew polynomial rings.

We obtain from (3) infinitely many equations

$$\tau^m \alpha = \psi_0^m(\alpha) + \psi_1^m(\alpha)\tau + \dots, \tag{5}$$

where $m = 1, 2, \dots, \alpha \in \mathcal{K}$, and ψ_i^m are linear mappings of the ring $\mathcal{K}$ into itself. Obviously, the sum (5) is finite for every m and for every $\alpha \in \mathcal{K}$.

The operators ψ_i^m can be expressed in terms of $\psi_0^1, \psi_1^1, \dots$, see [2]. If $\psi(t) = \psi_0^1 + \psi_1^1 t + \psi_2^1 t^2 + \dots$ is the generating function for ψ_m^1 in a commuting variable t then we have

$$\psi(t)^n = \psi_0^n + \psi_1^n t + \psi_2^n t^2 + \dots. \tag{6}$$

On the other hand, from the equations $\tau^m(\alpha\beta) = (\tau^m\alpha)\beta$, $\forall \alpha, \beta \in \mathcal{K}$,

$m = 1, 2, \ldots$, and the uniqueness of the representation (3) we obtain

$$\begin{aligned} \psi_0^m(\alpha\beta) &= \psi_0^m(\alpha)\beta + \psi_1^m(\alpha)\psi_0^1(\beta) + \psi_2^m(\alpha)\psi_0^2(\beta) + \ldots \\ \psi_1^m(\alpha\beta) &= \psi_1^m(\alpha)\psi_1^1(\beta) + \psi_2^m(\alpha)\psi_1^2(\beta) + \ldots \\ \psi_2^m(\alpha\beta) &= \psi_1^m(\alpha)\psi_2^1(\beta) + \psi_2^m(\alpha)\psi_2^2(\beta) + \ldots \\ &\vdots \end{aligned} \tag{7}$$

If we consider the infinite matrix

$$\Psi(\alpha) = \begin{pmatrix} \alpha & 0 & 0 & \ldots \\ \psi_0^1(\alpha) & \psi_1^1(\alpha) & \psi_2^1(\alpha) & \ldots \\ \psi_0^2(\alpha) & \psi_1^2(\alpha) & \psi_2^2(\alpha) & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{8}$$

then the equations (7) are equivalent to

$$\Psi(\alpha\beta) = \Psi(\alpha)\Psi(\beta), \quad \alpha, \beta \in \mathcal{K}. \tag{9}$$

Also, if

$$\Phi(\alpha) = \begin{pmatrix} \psi_1^1(\alpha) & \psi_2^1(\alpha) & \ldots \\ \psi_1^2(\alpha) & \psi_2^2(\alpha) & \ldots \\ \vdots & \vdots & \ddots \end{pmatrix} \tag{10}$$

then for every α, $\beta \in \mathcal{K}$ we have

$$\Phi(\alpha\beta) = \Phi(\alpha)\Phi(\beta). \tag{11}$$

In the case $\psi_0^1 = \mathbf{0}$ this representation was indicated in [8].

Proposition 1 (Existence Theorem). *Let ψ_0^m, $\psi_1^m, \ldots$, $m = 1, 2, \ldots$, be linear mappings of a ring $\mathcal{K}$ into itself which satisfy* (6) *and* (7) *for $m = 1$. Then the skew polynomial ring $\mathcal{K}[\tau, \Psi]$ exists.*

The proof is analogous to the proof of Theorem 8.1 in [3] (the case of Ore's skew polynomial ring) and will be given below.

3. We would like to emphasize different roles of the operators ψ_0^1 and ψ_1^1 (in the case $\psi_0^1 = 0$).

If $\psi_0^1 = 0$ then $\psi_1^n = 0$ for $n > 1$, by (6). Then (7) implies

$$\psi_1^1(\alpha\beta) = \psi_1^1(\alpha)\psi_1^1(\beta), \quad \alpha, \beta \in \mathcal{K}. \tag{12}$$

Let us consider elements $\alpha \in \mathcal{K}$ as the operators of left multiplication by α in the ring $\mathcal{K}$. Then (12) is equivalent to the operator equation

$$\psi_1^1 \cdot \alpha = \psi_1^1(\alpha)\psi_1^1. \tag{13}$$

Hence if $\mathcal{A}[\psi_1^1, \mathcal{K}]$ is the ring generated by the operator ψ_1^1 and all $\alpha \in \mathcal{K}$ then $\mathcal{A}[\psi_1^1, \mathcal{K}]$ is a homomorphic image of Ore's skew polynomial ring. In the case $\psi_0^1 \neq 0$ we define the ring $\mathcal{A}[\psi_0^1, \mathcal{K}]$ analogously.

Proposition 2. *The ring $\mathcal{A}[\psi_0^1, \mathcal{K}]$ is a homomorphic image of the ring $\mathcal{K}[\tau, \Psi]$.*

Proof. By (6) we have $[\psi_0^1]^k = \psi_0^k$. Then the first equation of (7) for $m = 1$ can be represented by the equivalent operator equation

$$\psi_0^1 \cdot \alpha = \psi_0^1(\alpha) + \psi_1^1(\alpha)\psi_0^1 + \psi_2^1(\alpha)[\psi_0^1]^2 + \dots. \tag{14}$$

In view of (3), the right-hand side of (14) is finite for every $\alpha \in \mathcal{K}$, thus it makes sense.

Define a mapping $\sigma : \mathcal{K}[\tau, \Psi] \to \mathcal{A}[\psi_0^1, \mathcal{K}]$ by

$$\sigma\Big(\sum \alpha_k \tau^k = \sum \alpha_k [\psi_0^1]\Big)^k, \quad \sigma(\alpha) = \alpha. \tag{15}$$

Obviously, the mapping σ is linear and unique. It follows from (15) that $\sigma(\alpha\beta) = \sigma(\alpha)\sigma(\beta)$ and $\sigma(\alpha\tau^k) = \sigma(\alpha)\tau^k$ for all $\alpha \in \mathcal{K}$. Then (3) and (14) imply $\sigma(\tau\alpha) = \sigma\Big(\sum \psi_k^1(\alpha)\tau^k\Big) = \sum \psi_k^1(\alpha)[\psi_0^1]^k = [\psi_0^1]\alpha = \sigma(\tau)\sigma(\alpha)$. $\square$

It is clear that the mapping σ is an isomorphism if and only if the operators $\psi_0^k = [\psi_0^1]^k$, $k = 1, 2, \dots$, are left linearly independent over $\mathcal{K}$.

2. Canonical forms

Definition 2. *Let $\mathcal{A} = \mathcal{K}[\tau, \Psi]$ be a ring of skew polynomials over $\mathcal{K}$. If for each $\alpha \in \mathcal{K}$ we have*

$$\begin{aligned} \tau\alpha &= \psi_0^1(\alpha) + \psi_1^1(\alpha)\tau + \dots + \psi_s^1(\alpha)\tau^s \\ &\vdots \\ \tau^s\alpha &= \psi_0^s(\alpha) + \psi_1^s(\alpha)\tau + \dots + \psi_s^s(\alpha)\tau^s \end{aligned} \tag{16}$$

then we say that the skew polynomial ring $\mathcal{K}[\tau, \Psi]$ has canonical forms of order s.

If g_m, $m = 1, 2, \ldots$, are some generators of the ring $\mathcal{K}$ then the relations

$$\tau^i \alpha = \psi_0^i(g_m) + \psi_1^i(g_m)\tau + \cdots + \psi_s^i(g_m)\tau^s,$$

$i = 1, 2, \ldots, s$; $m = 1, 2, \ldots$, will be called the canonical forms of the skew polynomial ring $\mathcal{K}[\tau, \Psi]$.

The ring $\mathcal{A}$ is a special case of a skew polynomial ring. For example, Ore's skew polynomial ring (4) has a canonical form of order 1. It is also easy to see from (4) that Ore's ring has canonical forms of arbitrary order s, $s > 1$.

Canonical forms are very useful. For example, they allow us to construct efficiently a skew polynomial ring over the ring of ordinary polynomials $\mathcal{F}[x_1, \ldots, x_n]$.

It also follows from (7), (8), and (9) that the matrix (16) defines two finite-dimensional representations of the ring $\mathcal{K}$ as special cases of (8) and (10).

Moreover, in this case the Existence Theorem (Proposition 1) can be represented in the following form.

Proposition 3. *Given $\alpha \in \mathcal{K}$, let*

$$\Psi(\alpha) = \begin{pmatrix} \alpha & 0 & \ldots & 0 \\ \psi_0^1(\alpha) & \psi_1^1(\alpha) & \ldots & \psi_s^1(\alpha) \\ \ldots & \ldots & \ldots & \ldots \\ \psi_0^s(\alpha) & \psi_1^s(\alpha) & \ldots & \psi_s^s(\alpha) \end{pmatrix} \tag{17}$$

be a matrix representation of the ring $\mathcal{K}$, where ψ_j^i are linear mappings of $\mathcal{K}$ into itself. If for every k, $1 \leq k \leq s$, we have

$$\psi_0^k + \psi_1^k t + \cdots + \psi_s^k t^s = \left(\psi_0^1 + \psi_1^1 t + \cdots + \psi_s^1 t^s\right)^k, \tag{18}$$

then the ring $\mathcal{K}[\tau, \Psi]$ exists and has canonical forms of order s.

Proof. It is easy to see that if $\psi_m^n = 0$ for any $n \leq s$, $m > s$, and $\alpha, \beta \in \mathcal{K}$, then equations (7) for $m = 1$ reduce to $(s+1)$ equations

$$\begin{aligned} &\psi_0^1(\alpha\beta) = \psi_0^1(\alpha)\beta + \psi_1^1(\alpha)\psi_0^1(\beta) + \cdots + \psi_s^1(\alpha)\psi_0^s(\beta) \\ &\psi_1^1(\alpha\beta) = \psi_1^1(\alpha)\psi)1^1(\beta) + \cdots + \psi_s^1(\alpha)\psi_1^s(\beta) \\ &\vdots \\ &\psi_s^1(\alpha\beta) = \psi_1^1(\alpha)\psi_s^1(\beta) + \cdots + \psi_s^1(\alpha)\psi_s^s(\beta). \end{aligned} \tag{19}$$

Thus consider a set of new operators $\widetilde{\psi}_m^{(n)}$, $n = 1, 2, \ldots; m = 0, 1, 2, \ldots$. For $i, j \le s$ put $\widetilde{\psi}_j^{(i)} = \psi_j^i$, where ψ_j^i are defined by (17). For $n \le s$, $m > s$, put $\psi_m^1 = \mathbf{0}$, and for $n > s$, let the operators $\widetilde{\psi}_m^{(n)}$ be defined by the equations

$$\widetilde{\psi}_0^{(n)} + \widetilde{\psi}_1^{(n)} t + \cdots + \widetilde{\psi}_{sn}^{(n)} t^{sn} = \left(\psi_0^1 + \psi_1^1 t + \cdots + \psi_s^1 t^s\right)^n.$$

Since $\Psi(\alpha)$ is a representation of $\mathcal{K}$, the operators $\widetilde{\Psi}_m^{(n)}$ satisfy (19). Therefore, such system of operators $\widetilde{\psi}_m^{(m)}$ satisfies the conditions of Proposition 1. □

2. In various applications there are very interesting examples of skew polynomial rings $\mathcal{K}[\tau, \Psi]$ of order s, $s \ge 2$, when relations (16) follow from the unique relation

$$\tau\alpha = \psi_0^1(\alpha) + \psi_1^1(\alpha)\tau + \cdots + \psi_s^1(\alpha)\tau^s.$$

The following two propositions can be used for constructing skew polynomial rings defined only by the first relation of (16) over the rings of usual polynomials $\mathcal{F}[x_1, \ldots, x_n]$.

Proposition 4. *Let $\mathcal{A} = \mathcal{K}[\tau, \Psi]$ be a ring of skew polynomials of order s over $\mathcal{K}$, i.e., $\Psi = \left(\psi_0^1, \psi_1^1, \ldots, \psi_s^1, \mathbf{0}, \mathbf{0}, \ldots\right)$. If the ring $\mathcal{K}$ is commutative and the operators $\psi_1^1, \ldots, \psi_s^1$ are linearly independent over $\mathcal{K}$ then the ring $\mathcal{K}[\tau, \Psi]$ has canonical forms of order s.*

Proof. As we have observed in the above proof, it suffices to show that $\psi_m^n = \mathbf{0}$ for any $n \le s$ and $m > s$.

By definition of a skew polynomial ring $\mathcal{A} = \mathcal{K}[\tau, \Psi]$ of order s, for each $\alpha \in \mathcal{K}$ we have

$$\tau\alpha = \psi_0^1(\alpha) + \psi_1^1(\alpha)\tau + \cdots + \psi_s^1(\alpha)\tau^s, \quad \psi_s^1 \ne \mathbf{0}. \tag{20}$$

Therefore, for any α, $\beta \in \mathcal{K}$ and $k > s$, equations (7) and (20) yield

$$0 = \psi_k^1(\alpha\beta) = \psi_1^1(\alpha)\psi_k^1(\beta) + \psi_2^1(\alpha)\psi_k^2(\beta) + \cdots + \psi_s^1(\alpha)\psi_k^s(\beta). \tag{21}$$

By assumption, the ring $\mathcal{K}$ is commutative, so equations (21) are equivalent to the operator equations

$$\psi_k^1(\beta)\psi_1^1 + \psi_k^2(\beta)\psi_2^1 + \cdots + \psi_k^s(\beta)\psi_s^1 = \mathbf{0}, \quad k > s. \tag{22}$$

By assumption, the operators ψ_i^1, $1 \leq i \leq s$, are linearly independent. Therefore, for every $\beta \in \mathcal{K}$ we have

$$\psi_k^1(\beta) = \psi_k^2(\beta) = \cdots = \psi_k^s(\beta) = 0,$$

and hence $\psi_k^1 = \psi_k^2 = \cdots = \psi_k^s = \mathbf{0}$ for all $k > s$. □

3. A ring of skew polynomials of order 2 is the first step of generating Ore's polynomials, and hence it can be very interesting for applications. In this case we will use the special notation:

$$\psi_0^1 = \psi, \quad \psi_1^1 = \varphi, \quad \psi_1^2 = \theta, \quad \theta \neq \mathbf{0}.$$

Then we obtain from (6)

$$\psi_0^2 = \psi^2, \quad \psi_1^2 = \psi\varphi + \varphi\psi, \quad \psi_2^2 = \theta\psi + \psi\theta + \varphi^2. \tag{23}$$

Proposition 5. *Let $\mathcal{K}$ be a domain. Then a ring of skew polynomials of order 2 over $\mathcal{K}$ has canonical forms of order 2.*

Proof. For any skew polynomial ring of order 2 we have

$$\tau\alpha = \psi(\alpha) + \varphi(\alpha)\tau + \theta(\alpha)\tau^2, \quad \forall \alpha \in \mathcal{K}. \tag{24}$$

Then from (6), (7), and (23) we obtain

$$\begin{aligned}
&1) \quad \psi(\alpha\beta) = \psi(\alpha)\beta + \varphi(\alpha)\psi(\beta) + \theta(\alpha)\psi^2(\beta)\\
&2) \quad \varphi(\alpha\beta) = \varphi(\alpha)\varphi(\beta) + \theta(\alpha)[\varphi(\psi(\beta)) + \psi(\varphi(\beta))]\\
&3) \quad \theta(\alpha\beta) = \varphi(\alpha)\theta(\beta) + \theta(\alpha)[\theta(\psi(\beta)) + \psi(\theta(\beta)) + \varphi(\varphi(\beta))]\\
&4) \quad \theta(\alpha)[\theta(\varphi(\beta)) + \varphi(\theta(\beta))] = 0\\
&5) \quad \theta(\theta(\alpha))[\theta(\theta(\beta))] = 0, \quad \forall \alpha,\ \beta \in \mathcal{K}.
\end{aligned} \tag{25}$$

Since $\theta \neq \mathbf{0}$, we have $\theta(\alpha) \neq 0$ for some $\alpha \in \mathcal{K}$. By assumption, the ring $\mathcal{K}$ is a domain. Then from 4), 5) in (25) we obtain

$$\theta(\varphi(\beta)) + \varphi(\theta(\beta)) = 0, \quad \theta(\theta(\beta)) = 0, \quad \forall \beta \in \mathcal{K}.$$

Obviously, these equations are equivalent to the operator equations

$$\theta\varphi + \varphi\theta = \mathbf{0} \ \text{ or } \ \theta\varphi = -\varphi\theta \tag{26}$$

and

$$\theta^2 = \mathbf{0}. \tag{27}$$

Then simple calculations and (26), (27) yield

$$\begin{aligned}\tau^2\alpha &= \tau(\tau\alpha) = \tau(\psi(\alpha)) + (\tau(\varphi(\alpha)))\tau + (\tau\theta(\alpha))\tau^2 \\ &= \psi^2(\alpha) + [\psi(\varphi(\alpha)) + \varphi(\psi(\alpha))]\tau + [\theta(\psi(\alpha)) + \psi(\theta(\alpha)) + \varphi(\varphi(\alpha))]\tau^2 \\ &= \psi_0^2(\alpha) + \psi_1^2(\alpha)\tau + \psi_2^2(\alpha)\tau^2.\end{aligned}$$

□

3. Canonical generating systems. Compatibility conditions

1. Assume that a skew polynomial ring $\mathcal{K}[\tau, \Psi]$ has canonical forms (16) of order s. Let $\mathcal{G} = \{g_m\}$, $m = 1, 2, \dots$, be a generating set of the ring $\mathcal{K}$. Then, in view of (7), the values of the operators ψ_j^i, $i = 1, 2, \dots, s$; $j = 0, 1, \dots, s$, are uniquely determines by the values $\psi_j^i(g_m) = c_{mj}^i$, $g_m \in \mathcal{G}$.

Definition 3. *A set of relations*

$$\tau^i g_m = c_{m0}^i + c_{m1}^i\tau + \dots + c_{ms}^i\tau^s, \tag{28}$$

where $c_{mj}^i \in \mathcal{K}$, $i = 1, 2, \dots, s$; $j = 0, 1, \dots, s$; $m = 1, 2, \dots$, will be called a canonical generating system if they generate a skew polynomial ring in the variable τ over $\mathcal{K}$.

It is clear that this skew polynomial ring has canonical forms of order s, and relations (28) are its canonical forms.

If (28) is a canonical generating system then the mapping Ψ, see (8), defined on $\mathcal{G}$ by

$$\Psi(g_m) = \begin{pmatrix} g_m & 0 & \dots & 0 \\ c_{m0}^1 & c_{m1}^1 & \dots & c_{ms}^1 \\ \dots & \dots & \dots & \dots \\ c_{m0}^s & c_{m1}^s & \dots & c_{ms}^s \end{pmatrix}, \quad g_m \in \mathcal{G}, \tag{29}$$

can be extended to a homomorphism of the ring $\mathcal{K}$.

Definition 4. *Suppose that for a given set $\mathbf{K} = \{c_{mj}^i\} \subset \mathcal{K}$, $i = 1, 2, \dots, s$; $j = 0, 1, \dots, s$; $m = 1, 2, \dots$, and for given generators $\mathcal{G} = \{g_m\}$, $m = 1, 2, \dots$, of the ring $\mathcal{K}$, the mapping (29) can be extended to a homomorphism of the ring $\mathcal{K}$. Then we say that the set $\mathbf{K}$ satisfies the correctness condition with respect to the generators $\mathcal{G}$.*

Then, by the correctness condition, equations (29) define linear mappings ψ_j^i of the ring $\mathcal{K}$ into itself, and for all $\alpha \in \mathcal{K}$ we have

$$\Psi(\alpha) = \begin{pmatrix} \alpha & 0 & \dots & 0 \\ \psi_0^1(\alpha) & \psi_1^1(\alpha) & \dots & \psi_s^1(\alpha) \\ \dots & \dots & \dots & \dots \\ \psi_0^s(\alpha) & \psi_1^s(\alpha) & \dots & \psi_s^s(\alpha) \end{pmatrix}. \tag{30}$$

Notice that in general the operators ψ_j^i are not related by (6). However, by the above definition, the mapping $\Psi(\alpha)$ is a representation of $\mathcal{K}$, hence we have the operator equation

$$\Psi \cdot \alpha = \Psi(\alpha)\Psi, \tag{31}$$

where the elements α and $\psi_j^i(\alpha)$ are regarded as the operators of left multiplication by α and $\psi_j^i(\alpha)$ in the ring $\mathcal{K}$.

2. Assume that a ring $\mathcal{K}[\tau, \Psi]$ has canonical forms (16) of order s. Then for $i = 1, 2, \dots, s$; $j = 1, 2, \dots, s$, we have

$$\tau^i(\tau^j \alpha) = \sum_{u=0}^{s} \Big(\tau^i \psi_u^j(\alpha)\Big)\tau^u = \sum_{w=0}^{s}\left[\sum_{u=0}^{w} \psi_u^i\Big(\psi_{w-u}^j(\alpha)\Big)\right]\tau^w + \\ + \sum_{k=1}^{s}\left[\sum_{u=0}^{s-k} \psi_{s-u}^i\Big(\psi_{u+k}^j(\alpha)\Big)\right]\tau^{s+k}. \tag{32}$$

Let

$$S_w^{ij}(\alpha) = \sum_{u=0}^{w} \psi_u^i\Big(\psi_{w-u}^j(\alpha)\Big), \tag{33}$$

$$R_k^{i,j}(\alpha) = \sum_{u=0}^{s-k} \psi_{s-u}^i\Big(\psi_{u+k}^j(\alpha)\Big), \quad k = 1, 2, \dots, s. \tag{34}$$

Then it is clear that for every $\alpha \in \mathcal{K}$, $i + j = h + l$, $w \le s$, we have $R_k^{i,j}(\alpha) = R_k^{h,l}(\alpha)$, $S_w^{i,j}(\alpha) = S_w^{h,l}(\alpha)$, and if $i + j \le s$ then $R_k^{i,j}(\alpha) = \mathbf{0}$.

Definition 5. *We say that a set* $\mathbf{K} = \{c_{mj}^i\} \subset \mathcal{K}$, $i = 1, 2, \dots, s$; $j = 0, 1, \dots, s$; $m = 1, 2, \dots,$ *and generators* $\mathcal{G} = \{g_m\}$, $m = 1, 2, \dots,$ *of a ring* $\mathcal{K}$ *satisfy the compatibility conditions if*

1. *The set* $\mathbf{K}$ *and the generators* $\mathcal{G}$ *satisfy the correctness condition. Hence, the operators* ψ_j^i, $i = 1, 2, \dots, s$; $j = 0, 1, \dots, s$, *are defined by* (30).

2. *For every* $g_m \in \mathcal{G}$ *and* $i,\ j,\ h,\ l = 1,2,\dots,s;\ i+j = h+l$, *the equations*

$$S_w^{i,j}(g_m) = S_w^{h,l}(g_m), \quad R_k^{i,j}(g_m) = R_k^{h,l}(g_m), \quad w = 1,2,\dots,s, \quad (35)$$

hold, where $S_w^{i,j}(g_m)$ *and* $R_k^{i,j}(g_m)$ *are expressed in* $\Psi(g_m)$, $\psi_j^i(g_m) = c_{mj}^i$, (29).

3. *For every* $g_m \in \mathcal{G}$ *and* $i, j = 1,2,\dots,s$ *with* $i+j \le s$, *the following equations hold:*

$$S_w^{i,j}(g_m) = \psi_w^{i+j}(g_m), \quad w = 1,2,\dots,s, \quad (36)$$

$$R_k^{i,j}(g_m) = 0, \quad k = 1,2,\dots,s, \quad (37)$$

where $S_w^{i,j}(g_m)$ *and* $R_k^{i,j}(g_m)$ *are expressed in terms of* $\Psi(g_m)$, *see* (29).

For brevity we will sometimes represent conditions (35), (36), and (37) in the form $\tau^i(\tau^j g_m) = \tau^h(\tau^l g_m)$ for $i+j = h+l$, and $\tau^i(\tau^j g_m) = \tau^{i+j} g_m$ for $i+j \le s$, $g_m \in \mathcal{G}$.

3. Theorem 1. *Assume that a set* $\mathbf{K} = \{c_{mj}^i\} \subset \mathcal{K}$, $i = 1,2,\dots,s;$ $j = 0,1,\dots,s;$ $m = 1,2,\dots,$ *and generators* $\mathcal{G} = \{g_m\}$, $m = 1,2,\dots,$ *of a ring* $\mathcal{K}$, *satisfy the compatibility conditions. Then relations* (28)

$$\tau^i g_m = c_{m0}^i + c_{m1}^i \tau + \cdots + c_{ms}^i \tau^s, \quad m = 1,2,\dots, \quad (38)$$

form a canonical generating system for a certain skew polynomial ring $\mathcal{K}[\tau,\Psi]$.

Theorem 1 has many applications. We consider the case when $\mathcal{K} = \mathcal{F}[x]$ is the usual polynomial ring over a field $\mathcal{F}$ of characteristic 0.

Let we are given algebraic relations

$$\begin{aligned} \tau x &= c_0^1 + c_1^1\tau + \cdots + c_s^1\tau^s \\ &\vdots \\ \tau^s x &= c_0^s + c_1^s\tau + \cdots + c_s^s\tau^s, \end{aligned} \quad (39)$$

where the coefficients $c_j^i = c_j^i(x)$ are polynomials from $\mathcal{F}[x]$.

Then we can define linear mappings ψ_j^i, $i = 1,\dots,s;\ j = 0,1,\dots,s$, of the ring $\mathcal{F}[x]$ into itself by determining the values $\psi_j^i(x^n)$ from equations

(40) for $n = 0, 1, \ldots$.

$$c^n(x) = \begin{pmatrix} x & 0 & \ldots & 0 \\ c_0^1(x) & c_1^1(x) & \ldots & c_s^1(x) \\ \multicolumn{4}{c}{\dotfill} \\ c_0^s(x) & c_1^s(x) & \ldots & c_s^s(x) \end{pmatrix}^n = \begin{pmatrix} x^n & 0 & \ldots & 0 \\ \psi_0^1(x^n) & \psi_1^1(x^n) & \ldots & \psi_s^1(x^n) \\ \multicolumn{4}{c}{\dotfill} \\ \psi_0^s(x^n) & \psi_1^s(x^n) & \ldots & \psi_s^s(x^n) \end{pmatrix}. \tag{40}$$

Then Theorem 1 implies

Theorem 2. *The algebra over $\mathcal{F}[x]$ generated by relations* (39) *has a basis of the form $\{x^p\tau^q\}$, $p, q = 0, 1, 2, \ldots$ ($x^0 = \tau^0 = \mathbf{1}$) (classical Poincaré–Birkhoff–Witt basis) if and only if the operators ψ_j^i defined by* (39) *and* (40) *satisfy the following equations:*

$$\begin{aligned} &1) \quad \sum_{u=0}^{w} \psi_u^i(c_{w-u}^j(x)) = \sum_{u=0}^{w} \psi_u^h(c_{w-u}^l(x)), \quad i, j, h, l = 1, 2, \ldots, s, \\ &2) \quad \sum_{u=0}^{w} \psi_u^i(c_{w-u}^j(x)) = c_w^{i+j}(x)), \quad i + j \le s, \\ &3) \quad \sum_{u=0}^{s-k} \psi_{s-u}^i(c_{u+k}^j(x)) = 0, \quad k = 1, 2, \ldots, s. \end{aligned} \tag{41}$$

Remark 1. Let n_0 be the highest power of the polynomials $\{c_j^i(x)\}$, and m_0 be the highest power of the polynomials $\{\psi_q^{(p)}(c_j^i(x))\}$, $i, p = 1, 2, \ldots, s$; j, $q = 0, 1, \ldots, s$. Then to verify (41) it suffices to know the matrices $c(x)$, $c^2(x), \ldots, c^{\max(n_0, m_0)}(x)$ defined by (40).

3. In the case $s = 2$ equations (41) reduce to $\tau(\tau x) = \tau^2 x$ and $\tau(\tau^2 x) = \tau^2(\tau x)$. If we represent (39) in the form

$$\tau x = a_0(x) + a_1(x)\tau + a_1(x)\tau^2, \quad \tau^2 x = b_0(x) + b_1(x)\tau + b_1(x)\tau^2,$$

then we will determine the values of linear operators ψ, φ, and θ on x^n, $n = 1, 2, \ldots$, from the equations

$$\begin{pmatrix} x & 0 & 0 \\ a_0(x) & a_1(x) & a_2(x) \\ b_0(x) & b_1(x) & b_2(x) \end{pmatrix}^n = \begin{pmatrix} x^n & 0 & 0 \\ \psi(x^n) & \varphi(x^n) & \theta(x^n) \\ \psi_1(x^n) & \varphi_1(x^n) & \theta_1(x^n) \end{pmatrix}.$$

Then the condition $\tau(\tau x) = \tau^2 x$ yields

$$\theta(\theta(x)) = 0, \quad \theta(\varphi(x)) = -\varphi(\theta(x)), \quad \psi_1(x) = \psi(\psi(x)),$$
$$\varphi_1(x) = \psi(\varphi(x)) + \varphi(\psi(x)), \quad \theta_1(x) = \theta(\psi(x)) + \psi(\theta(x)) + \varphi(\varphi(x)).$$

The condition $\tau^2(\tau x) = \tau(\tau^2 x)$ yields

$$\psi_1(\psi(x)) = \psi(\psi_1(x)), \quad \varphi_1(\psi(x)) + \psi_1(\varphi(x)) = \varphi(\psi_1(x)) + \psi(\varphi_1(x)),$$
$$\theta_1(\varphi(x)) + \varphi(\theta_1(x)) = \theta(\varphi_1(x)) + \varphi_1(\theta(x)), \quad \theta_1(\theta(x)) = \theta(\theta_1(x)).$$

We see that the operators ψ_1, φ_1, and θ_1 are related with ψ, φ, and θ by (23).

4. Proofs of the main theorems

1. *Proof of Theorem 1.* To prove Theorem 1, it suffices to show that the operators ψ_j^i defined by (29) satisfy the conditions of Proposition 3 provided that the compatibility conditions are satisfied.

The correctness condition coincides with the first condition of Proposition 3. Thus it suffices to verify (18) for the operators ψ_j^i defined by (29). To this end, we define operators U_i, $i = 1, 2, \dots, s$, acting on the ring $\mathcal{K}[t]$ of usual polynomials in a variable t over $\mathcal{K}$.

By definition, for $k = 1, 2, \dots, s$ we put

$$U_k = \psi_0^k + \psi_1^k t + \cdots + \psi_s^k t^s, \tag{42}$$

where the variable t commutes with the operators ψ_j^i, and the powers $\{t^n\}$ are linearly independent over the ring generated by the operators $\{\psi_j^i\}$. For $\alpha \in \mathcal{K}$ we set

$$U_k(\alpha t^n) = \psi_0^k(\alpha) t^n + \psi_1^k(\alpha) t^{n+1} + \cdots + \psi_s^k(\alpha) t^{n+s}, \tag{43}$$

i.e., the operators U_k are uniquely determined by their values on $\mathcal{K}$.

Lemma 1. *Equations* (18) *are equivalent to* $U_k = U_1^k$, $k = 1, 2, \dots, s$.

The proof is straightforward.

Hence, to proof Theorem 1, instead of demonstrating (18) it suffices to show that

$$U_k = U_1^k, \quad k = 1, 2, \dots, s \tag{44}$$

or $U_k(\alpha) = U_1(U_{k-1}(\alpha)) = \cdots = U_1(U_1(\dots(U_1(\alpha))\dots))$ for every $\alpha \in \mathcal{K}$.

Lemma 2. *For all $\alpha \in \mathcal{K}$ the following equations hold:*

$$\begin{aligned} U_1\alpha &= \psi_0^1(\alpha) + \psi_1^1(\alpha)U_1 + \cdots + \psi_s^1(\alpha)U_s \\ &\vdots \\ U_s\alpha &= \psi_0^s(\alpha) + \psi_1^s(\alpha)U_1 + \cdots + \psi_s^s(\alpha)U_s. \end{aligned} \tag{45}$$

Proof. Let Ψ_k denote the kth column of the matrix Ψ from (31), $k = 0, 1, \dots, s$. Then (31) yields

$$\Psi_k \cdot \alpha = \Psi(\alpha) \cdot \Psi_k.$$

Multiplying this equation by t^k and summing over all k yields

$$\begin{pmatrix} id_{\mathcal{K}} \\ U_1 \\ \vdots \\ U_s \end{pmatrix} \cdot \alpha = \Psi(\alpha) \begin{pmatrix} id_{\mathcal{K}} \\ U_1 \\ \vdots \\ U_s \end{pmatrix},$$

which is equivalent to (45). □

2. First we will prove (44) when the ring $\mathcal{K}$ is the ring of usual polynomials over $\mathcal{F}$, i.e., $\mathcal{K} = \mathcal{F}[x]$, where $\mathcal{F}$ is a field of characteristic zero.

In this case we have one generator x. Obviously, the correctness condition is satisfied automatically.

The canonical generating system is given by (39), and by the definition of the operators ψ_j^i in (40) we have

$$\psi_j^i(x) = c_j^i(x). \tag{46}$$

Denote by X the operator of multiplication by x acting on the ring $\mathcal{K} = \mathcal{F}[x]$. Thus in this case for every $\alpha(x) \in \mathcal{F}[x]$ equations (45) can be represented in the form

$$U_k\alpha(x) = \psi_0^k(\alpha(x)) + \psi_1^k(\alpha(x))U_1 + \cdots + \psi_s^k(\alpha(x))U_s, \tag{47}$$

$k = 1, 2, \dots, s$. Then for $\alpha(X) = X$ we have

$$U_kX = \psi_0^k(X) + \psi_1^k(X)U_1 + \cdots + \psi_s^k(X)U_s, \quad k = 1, 2, \dots, s. \tag{48}$$

The operators ψ_j^i are defined on the elements x^n, $n \geq 0$ by (40). But we should define ψ_j^i on $\mathbf{1} = x^0$ in a correct way. Since $\tau^i\mathbf{1} = \tau^i$, we must put $\psi_i^i(\mathbf{1}) = \mathbf{1}$, $i = 1, 2, \dots, s$, and $\psi_j^i(\mathbf{1}) = 0$ for $i \neq j$, which coincides with the definition of $\psi_j^i(\mathbf{1})$ in (40).

Then from (42) we have

$$U_1(\mathbf{1}) = t, \quad U_2(\mathbf{1}) = t^2 \quad , \dots, \quad U_s(\mathbf{1}) = t^s. \tag{49}$$

Lemma 3. *Equations* (48) *with initial conditions* (49) *uniquely define operators* U_i, $i = 1, 2, \dots, s$.

Proof. Acting by both sides of (48) on $\mathbf{1}$ we obtain

$$U_i(x) = \psi_0^1(x) + \psi_1^1(x)t + \cdots + \psi_s^1(x)t^s,$$

which coincides with (42) applied to x. Then the values $U_i(x^n)$, $n = 2, 3, \dots$, can be uniquely defined from (48) by trivial induction. $\square$

For this reason we will prove (44) for the solutions $U_1, \dots, U_s$ of (48) with initial conditions (49).

3. Lemma 4. *Let* $i + j = h + l = d$; $i, j, h, l = 1, 2, \dots, s$. *Then* $U_iU_j = U_hU_l$.

Proof. By (49) we have $U_iU_j(\mathbf{1}) = U_i(t^j) = t^{i+j} = U_{i+j}(\mathbf{1})$; hence we can proceed by induction. Assume that we have

$$U_iU_j(x^k) = U_hU_lx^k = f_{d,k}(x) \tag{50}$$

for $k = 0, 1, \dots, n$; $i + j = h + l = d$.

Multiplying the left-hand side of the jth equation of (48) by U_i we obtain

$$U_iU_jX = U_i\psi_0^j(X) + U_i\psi_1^j(X)U_1 + \cdots + U_i\psi_s^j(X)U_s.$$

Then replacing $U_i\psi_k^j(X)$ by (47) for $\alpha(X) = \psi_k^j(X)$ yields

$$\begin{aligned} U_iU_jX = &\sum_{w=0}^{s}\sum_{u=0}^{w}\psi_u^i(\psi_{w-u}^j(X))U_uU_{w-u} \\ &+ \sum_{k=1}^{s}\sum_{u=0}^{s-k}\psi_{s-u}^i(\psi_{u+k}^j(X))U_{s-u}U_{u+k}. \end{aligned} \tag{51}$$

Acting by both sides of (51) on x^n, using the induction hypothesis (50), and (33), (34), we obtain

$$\begin{aligned} U_iU_j(x^{n+1}) &= \sum_{w=0}^{s} S_w^{i,j}(x)f_{w,n}(x) + \sum_{k=1}^{s} R_k^{i,j}(x)f_{s+k,n}(x) \\ U_hU_l(x^{n+1}) &= \sum_{w=0}^{s} S_w^{h,l}(x)f_{w,n}(x) + \sum_{k=1}^{s} R_k^{h,l}(x)f_{s+k,n}(x). \end{aligned} \tag{52}$$

Therefore, if $i + j = h + l$; $i, j, h, l = 1, 2 \dots, s$, we obtain from (35) that $U_iU_j(x^{n+1}) = U_hU_l(x^{n+1})$. Thus $U_iU_j = U_hU_l$. $\square$

3. Proof of equation (44). Let us consider (51) with $i+j=d\le s$. Acting on $\mathbf{1}$ by both sides of (51) and using (49), (35), (36), and (37) in the compatibility conditions, we obtain

$$U_iU_j(x)=\sum_{w=0}^{s} S_w^{i,j}(x)U_l(\mathbf{1})=\sum_{w=0}^{s}\psi_w^d(x)U_w(\mathbf{1}),\quad i+j=d\le s. \tag{53}$$

On the other hand, acting by both sides of the dth equation of (48) and using (49) we obtain

$$U_d(x)=\sum_{w=0}^{s}\psi_w^d(x)U_w(\mathbf{1}). \tag{54}$$

Hence the right-hand sides of (53) and (54) coincide. Thus we have

$$U_iU_j(x)=U_d(x),\quad i+j=d\le s.$$

Now, suppose by induction that for $i+j=d\le s$ and for all k with $0\le k\le n$ we have

$$U_iU_j(x^k)=U_d(x^k),\quad i+j=d\le s.$$

Then acting on x^n by both sides of operator equations (51) and using Lemma 4, (35), and (36) we obtain

$$U_iU_j(x^{n+1})=\sum_{w=0}^{s} S_w^{i,j}(x)f_{w,n}(x)=\sum_{w=0}^{s}\psi_w^d(x)f_{w,n}(x). \tag{55}$$

By the induction hypothesis

$$f_{w,n}(x)=U_uU_{w-u}(x^n)=U_w(x^n).$$

Hence (55) can be written as

$$U_iU_j(x^{n+1})=\sum_{w=0}^{s}\psi_w^d(x)U_w(x^n),\quad i+j=d,\ U_0=\mathbf{1}. \tag{56}$$

On the other hand, acting on x^n by both sides of the dth equation of (48) we obtain

$$U_d(x^{n+1})=\psi_0^d(x)x^n+\psi_1^d(x)U_1(x^n)+\cdots+\psi_s^d(x)U_s(x^n). \tag{57}$$

Comparing the right-hand sides of (56) and (57) shows that

$$U_iU_j(x^{n+1})=U_d(x^{n+1}),\quad i+j=d\le s.$$

Thus (44) holds. □

5. Thus Theorem 1 is proved in the case $\mathcal{K} = \mathcal{F}[x]$. Obviously, the same lines of the proof can be used for a ring $\mathcal{K}$ with generators $\mathcal{G}$. To apply the above arguments, it suffices to consider the induction on the length of the monomials described by the generators $\mathcal{G}$. □

6. *Proof of the Existence Theorem.* Let us consider a left $\mathcal{K}$-module $\mathcal{L}[\mathcal{K}]$ with basis $\{e_i\}$, $i = 0, 1, 2, \ldots$. Then each element $\alpha \in \mathcal{K}$ can be regarded as a linear operator on $\mathcal{L}[\mathcal{K}]$ defined as the left multiplication by α. Moreover, let us introduce a linear mapping h of the module $\mathcal{L}[\mathcal{K}]$ by

$$h(\alpha e_i) = \psi_0^{(1)}(\alpha)e_i + \psi_1^{(1)}(\alpha)e_{i+1} + \cdots = (h\alpha)e_i. \tag{58}$$

In the particular case $\alpha = \mathbf{1}$ (see (49)), we have $he_i = e_{i+1}$, $i = 0, 1, \ldots$. It follows from (58) by simple calculations that

$$h^k(\alpha e_i) = \psi_0^{(k)}(\alpha)e_i + \psi_1^{(k)}(\alpha)e_{i+1} + \cdots = (h^k\alpha)e_i. \tag{59}$$

Let $\mathcal{A}(h, \mathcal{K})$ be the ring of all linear mappings of $\mathcal{L}(\mathcal{K})$ generated by h and all $\alpha \in \mathcal{K}$ regarded as the mappings generated by the left multiplication by α.

We will prove that $\mathcal{A}(h, \mathcal{K})$ is a skew polynomial ring over $\mathcal{K}$ with respect to h regarded as a variable and $\Psi = (\psi_0^{(1)}, \psi_1^{(1)}, \cdots)$ regarded as the general derivation of the ring $\mathcal{K}$, see (2). First we will prove that the powers h^k are left linearly independent over $\mathcal{K}$. Suppose they are left linearly dependent. Then we have

$$\alpha_0 + \alpha_1 h + \cdots + \alpha_n h^n = 0 \tag{60}$$

for some $\alpha_i \in \mathcal{K}$, where not all α_i vanish. Acting on the element e_0 by both sides of (60) and using the equation $he_i = e_{i+1}$, $i = 0, 1, \cdots$, we obtain $\alpha_0 e_0 + \alpha_1 e_1 + \cdots + \alpha_n e_n = 0$. But this equation contradicts to the left linear independence of the base elements e_i. Hence the powers h^n are left linearly independent over $\mathcal{K}$. So it remains to show that the operators h and every $\alpha \in \mathcal{K}$ are related by (3). Obviously, (3) is equivalent to

$$(h\alpha)\beta e_i = [\psi_0^{(1)}(\alpha) + \psi_1^{(1)}(\alpha)h + \cdots]\beta e_i, \tag{61}$$

$\forall \alpha, \beta \in \mathcal{K}$, $i = 0, 1, \ldots$. We will prove these equations. By the definition of the operators h and α,

$$(h\alpha)\beta e_i = h(\alpha\beta)e_i = \psi_0^{(1)}(\alpha\beta)e_i + \sum_{k\geq 1} \psi_k^{(1)}(\alpha\beta)e_{i+k} + \cdots. \tag{62}$$

On the other hand, we obtain from (59) that the right-hand side of (61) equals

$$\begin{aligned}&\left(\sum_{k\geq 0}\psi_k^{(1)}(\alpha)h^k\right)(\beta e_i)=\sum_{k\geq 0}\psi_k^{(1)}(\alpha)(h^k(\beta e_i)=\\&\left\{\psi_0^{(1)}(\alpha)\beta+\psi_1^{(1)}(\alpha)\psi_0^{(1)}(\beta)+\psi_2^{(!)}(\alpha)\psi_0^{(2)}(\beta)+\cdots\right\}e_i+\\&\sum_{k\geq 1}\left\{\sum_{d\geq 1}\psi_d^{(1)}(\alpha)\psi_k^{(d)}(\beta)\right\}e_{i+k}.\end{aligned}\tag{63}$$

By (7) with $m=1$, the coefficients of e_{i+k}, $k=0,1,\dots$, in the right-hand sides of (62) and (63) coincide. Hence (61) holds, and $(h\alpha)=\psi_0^{(1)}(\alpha)+\psi_1^{(1)}(\alpha)h+\cdots$. □

5. Examples

1. The algebra defined by the relations

$$\begin{aligned}&\tau x=\gamma_2 x^2\tau^2+\cdots+\gamma_s x^s\tau^s\\&\tau^2 x=0,\dots,\tau^s x=0,\quad \gamma_s\neq 0,\quad \gamma_i\in\mathcal{F}\end{aligned}\tag{64}$$

gives a simple example of a nondegenerate ring of skew polynomials over $\mathcal{F}[x]$ with canonical forms of order s.

That fact can be derived from Proposition 1. It also follows directly from Theorem 1. Indeed, by Theorem 1 it suffices to verify the equations $\tau^i(\tau^j x)=\tau^h(\tau^l x)$, $i+j=h+l$; $i,j,h,l=1,2,\dots,s$, and $\tau^i(\tau^j x)=\tau^{i+j}x$ if $i+j\leq s$. But in view of (64) both sides of these relations are zero.

For this reason the algebra defined by (64) will be called the null skew polynomial ring.

2. The algebra defined over a field $\mathcal{F}$ by the quadratic relation

$$\tau x=\alpha x^2-x\tau+\gamma\tau^2,\quad \alpha,\gamma\in\mathcal{F},\quad \alpha\gamma\neq 1\tag{65}$$

is an interesting example of a skew polynomial ring with canonical forms of order 2, see [5].

Proposition 6. *Relation* (65) *generates a skew polynomial ring over* $\mathcal{F}[x]$, *and for all* $p(x)\in\mathcal{F}[x]$ *the following equations hold:*

$$\begin{aligned}&1)\quad \tau p(x)=\psi(p(x))+\varphi(p(x))\tau+\theta(p(x))\tau^2,\\&2)\quad \tau^2 p(x)=p(x)\tau^2,\end{aligned}\tag{66}$$

where

$$\begin{aligned}
&\text{i)}\quad \psi(x^{2n}) = 0, \quad \psi(x^{2n+1}) = \alpha x^{2n+2},\\
&\text{ii)}\quad \varphi(x^n) = (-x)^n, \\
&\text{iii)}\quad \theta(x^{2n}) = 0, \quad \theta(x^{2n+1}) = \gamma x^{2n}, \quad n = 0, 1, 2, \dots.
\end{aligned} \tag{67}$$

Obviously, (67) *implies*

$$\begin{aligned}
&\psi^2 = \mathbf{0}, \quad \varphi^2 = \mathbf{1}, \quad \psi\varphi = -\varphi\psi, \quad \theta\varphi = -\varphi\theta,\\
&\theta^2 = \mathbf{0}, \quad \theta\psi = \psi\theta = \mathbf{0}.
\end{aligned} \tag{68}$$

Proof. We will show that the equation $\tau^2 p(x) = p(x)\tau^2$ follows from (65). Then it suffices to show that

$$\tau x^2 = x^2\tau, \quad \tau^2 x = x\tau^2. \tag{69}$$

Indeed, multiplying (65) from the left by τ yields

$$\begin{aligned}
\tau^2 x &= \alpha\tau x^2 - (\tau x)\tau + \gamma\tau^3\\
&= \alpha\tau x^2 - (\alpha x^2 - x\tau + \gamma\tau^2)\tau + \gamma\tau^3 = \alpha\tau x^2 - \alpha x^2\tau + x\tau^2.
\end{aligned} \tag{70}$$

Analogously, multiplying (65) from the right by x yields

$$\begin{aligned}
\tau x^2 &= \alpha x^3 - x(\tau x) + \gamma\tau^2 x = \\
&= \alpha x^3 - x(\alpha x^2 - x\tau + \gamma\tau^2) + \gamma\tau^2 x = x^2\tau - \gamma x\tau^2 + \gamma\tau^2 x.
\end{aligned} \tag{71}$$

Multiplying (71) by α and substituting the right-hand part of the obtained expression for $\alpha\tau x^2$ in (70) yields, after trivial reductions under assumption $\alpha\gamma \neq 1$, that $\tau^2 x = x\tau^2$. Substituting this in (71) we find that $\tau x^2 = x^2\tau$. Hence (69) holds.

Then from (65) and (69) we obtain

$$\begin{aligned}
&\tau x^{2n} = x^{2n}\tau\\
&\tau x^{2n+1} = x^{2n}(\tau x) = \alpha x^{2n+2} - x^{2n+1}\tau + \gamma x^{2n}\tau^2.
\end{aligned} \tag{72}$$

If we suppose that the algebra over $\mathcal{F}$ defined by (65) has a basis of the form $\{x^m\tau^n\}$, $m, n = 0, 1, 2, \dots$, then the operators ψ, φ, and θ uniquely determined by (72) coincide with the operators defined by (67).

Therefore, in view of (66) and (68), we can apply Proposition 3 for $s = 2$. In our case it suffices to verify (18). By (68)

$$\psi_0^2 = \mathbf{0} = \psi^2, \quad \psi_1^2 = \mathbf{0} = \psi\varphi + \varphi\psi, \quad \psi_2^2 = \mathbf{1} = \theta\psi + \psi\theta + \varphi^2.$$

Then (23) shows that (18) holds for the operators ψ, φ, and θ. Hence the algebra generated by (66) has the basis $\{x^m\tau^n\}$, $m, n = 0, 1, 2, \dots$, over $\mathcal{F}$. Finally, observe that the algebras defined by (65) and (66) coincide.

Indeed, relations (66) with $p(x) = x$ reduce to equation (65) and $\tau^2 x = x\tau^2$. But these equations determine a canonical generating system for the algebra defined by (66). □

Remark 2. A direct and simple proof of the fact that the elements $\{x^m\tau^n\}$, $m, n = 0, 1, 2, \dots$, form a basis of the algebra defined by (65) can be obtained by Proposition 4 from [5].

3. At the end we consider one example of a skew polynomial ring of order 2 defined by the relations

$$\tau x = a_0 + a_1 x\tau + a_2 x^2\tau^2 \quad \text{and} \quad \tau^2 x = b_1\tau + b_2 x\tau^2, \tag{73}$$

where x and τ are generators, a_i, $b_j \in \mathcal{F}$, $\mathcal{F}$ is a field of characteristic zero, $a_2 \neq 0$.

In the case $a_2 = 0$ the skew polynomial ring defined by (73) is the a_1-deformation of Weyl's algebra defined by the relation $\tau x = a_0 + a_1 x\tau$.

It is not difficult to find the compatibility conditions for the system (73). Applying the equations $\tau(\tau x) = \tau^2 x$ and $\tau^2(\tau x) = \tau(\tau^2 x)$ to (73) like in Sect. 3.4, we obtain, for $a_2 \neq 0$,

$$a_1 + b_1 = 0, \quad b_0 = a_0(1 + a_1), \quad a_1^2 + a_0 a_2 + a_0 a_1 a_2 = -a_1, \tag{74}$$

If $a_0 \neq 0$, $a_1 \neq -1$, then $a_2 = -\frac{a_1}{a_0}$.

Hence under conditions (74) the algebra defined by (73) is a skew polynomial ring over $\mathcal{F}[x]$, by Theorem 2. Thus there exist unique operators ψ, φ, and θ on $\mathcal{F}[x]$ such that for all $p(x) \in \mathcal{F}[x]$

$$\tau p(x) = \psi(p(x)) + \varphi(p(x))\tau + \theta(p(x))\tau^2. \tag{75}$$

In addition, from $\tau \cdot \mathbf{1} = \tau$, $x^0 = \mathbf{1}$ and the first equation of (73) we can obtain the initial conditions

$$\begin{aligned} &\psi(\mathbf{1}) = 0, \quad \varphi(\mathbf{1}) = \mathbf{1}, \quad \theta(\mathbf{1}) = 0 \\ &\psi(x) = a_0, \quad \varphi(x) = a_1 x, \quad \theta(x) = a_2 x^2. \end{aligned} \tag{76}$$

Denote by X the operator of multiplication by x in the ring $\mathcal{F}[x]$, like in Sect. 4.2 above. Then we obtain

Proposition 7. *The system of operator equations*

$$\begin{aligned} \psi_0 X &= X\psi_0 + a_0\varphi_0 \\ \varphi_0 X &= a_1 X\varphi_0 + a_0(1+a_1)\theta_0 \\ \theta_0 X &= -a_1 X\theta_0 + a_2 X^2\varphi_0 \end{aligned} \tag{77}$$

with a_0, a_1, $a_2 \in \mathcal{F}$ *has a unique operator solution* ψ_0, φ_0, θ_0 *for arbitrary initial conditions* $\psi_0(\mathbf{1}) = h_0(x)$, $\varphi_0(\mathbf{1}) = h_1(x)$, *and* $\theta_0(\mathbf{1}) = h_2(x)$.

If the constants a_0, a_1, *and* a_2 *satisfy the equation* $a_1^2 + a_0a_2 + a_0a_1a_2 = -a_1$, *and* $\psi_0(\mathbf{1}) = 0$, $\varphi_0(\mathbf{1}) = 1$, $\theta_0(\mathbf{1}) = 0$, *then the operators* ψ_0, φ_0, *and* θ_0 *coincide with* ψ, φ, *and* θ.

Proof. The first part is obvious, since if the values $\psi_0(x^n)$, $\varphi_0(x^n)$, and $\theta_0(x^n)$ are known then we have from (77) that

$$\begin{aligned} \psi_0(x^{n+1}) &= x\psi_0(x^n) + a_0\varphi_0(x^n) \\ \varphi_0(x^{n+1}) &= a_1 x\varphi_0(x^n) + a_0(1+a_1)\theta_0(x^{n+1}) \\ \theta_0(x^{n+1}) &= -a_1 x\theta_0(x^n) + a_2 x^2\varphi_0(x^n). \end{aligned}$$

To prove the second part, it suffices to recall that the operators ψ, φ, and θ satisfy (25). In our special case $\beta = X$ and α is an arbitrary polynomial. Then the first three equations of (25) are equivalent to the following operator equations:

$$\begin{aligned} \psi X &= X\psi + \psi(X)\varphi + \psi(\psi(X))\theta \\ \varphi X &= \varphi(X)\varphi + [\varphi(\psi(X)) + \psi(\varphi(X))]\theta \\ \theta X &= \theta(X)\varphi + [\theta(\psi(X)) + \psi(\theta(X)) + \varphi(\varphi(X))]\theta. \end{aligned} \tag{78}$$

The values $\psi(\varphi(X))$, $\varphi(\psi(X))$, $\theta(\psi(X))$, $\psi(\psi(X))$, and $\varphi(\varphi(X))$ can be found from the initial conditions (76). We have

$$\begin{aligned} \psi(\varphi(x)) &= \psi(a_1 x) = a_1\psi(x) = a_0a_1 \\ \varphi(\psi(x)) &= \varphi(a_0) = a_0\varphi(\mathbf{1}) = a_0 \\ \theta(\psi(x)) &= \theta(a_0) = a_0\theta(\mathbf{1}) = \mathbf{0} \\ \psi(\psi(x)) &= \psi(a_0) = a_0\psi(\mathbf{1}) = \mathbf{0} \\ \varphi(\varphi(x)) &= \varphi(a_1 x) = a_1\varphi(x) = a_1^2. \end{aligned} \tag{79}$$

In order to compute $\psi(\theta(x)) = \psi(a_2x^2) = a_2\psi(x^2)$, it suffices to find the expansion of τx^2 using (73). Multiplying the first equation of (73) from the right by x and using (73) we obtain $\tau x^2 = a_0(1+a_1)x + (a_1^2 + a_2 b_0)x^2\tau + a_2(a_1+b_1)x^3\tau^2$. Hence

$$\psi(x^2) = a_0(1+a_1)x. \tag{80}$$

Substituting the values (79) and (80) into (78) yields (77). Hence $\psi_0 = \psi$, $\varphi_0 = \varphi$, and $\theta_0 = \theta$. □

Proposition 8. *Let $a_2 \neq 0$ and assume that a_0, a_1, and a_2 satisfy (74). Then the operators ψ, φ, and θ defined by (75) satisfy*

$$\psi(x^n) = \psi_n x^{n-1}, \quad \varphi(x^n) = \varphi_n x^n, \quad \theta(x^n) = \theta_n x^{n+1}, \tag{81}$$

where

$$\begin{aligned} &\psi_0 = 0, \quad \psi_{2n+1} = a_0, \quad \psi_{2n+2} = a_0[1 + (-1)^n a_1^{n+1}] \\ &\varphi_{2n} = (-a_1)^n, \quad \varphi_{2n+1} = (-1)^n a_1^{n+1} \\ &\theta_{2n} = 0, \quad \theta_{2n+1} = (-a_1)^n a_2, \quad fn = 0, 1, 2, \ldots . \end{aligned} \tag{82}$$

Proof. Elementary calculations show that the operators ψ, φ, and θ defined by (81) and (82) satisfy equations (76) and (77). Therefore we can apply Proposition 7. □

4. The a_1-deformation of Weyl's algebra defined by the equation $\tau x = a_0 + a_1 x\tau$ gives an example of a homogeneous relation of order 1. Relation (73) is an example of a homogeneous relation of order 2.

The general case of homogeneous relations of order s can be defined as follows:

$$\begin{aligned} \tau x &= a_0 + a_1 x\tau + \cdots + a_s x^s \tau^s \\ \tau^2 x &= \quad b_0\tau + \cdots + b_{s-1}x^{s-1}\tau^s \\ &\vdots \\ \tau^s x &= \qquad\qquad d_0\tau^{s-1} + d_1 x\tau^s, \end{aligned} \tag{83}$$

where $a_i, b_j, \ldots, d_k \in \mathcal{F}$, and $\mathcal{F}$ is a field of characteristic zero. For $s = 3$ and $s = 4$ the compatibility conditions were found. For $s = 3$ we obtain

$$\tau x = a_2 x^2\tau^2 + a_3 x^3\tau^3, \quad \tau^2 x = \tau^3 x = 0,$$

and for $s = 4$,

$$\tau x = a_2 x^2\tau^2 + a_3 x^3\tau^3 + a_4 x^4\tau^4, \quad \tau^2 x = \tau^3 x = \tau^4 x = 0.$$

Thus in the cases $s = 3$ and $s = 4$ relations (83) give null skew polynomial rings (see Example 1). It seems very probable that the following conjecture is true.

Null conjecture. Suppose that for some $s \geq 3$ the algebra generated by (83) is a skew polynomial ring over $\mathcal{F}[x]$. Then it is a null skew polynomial ring.

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