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PARTIAL REGULARITY OF WEAK SOLUTIONS OF THE STATIONARY 3D- BOUSSINESQ SYSTEM

Abstract. In this paper we study smoothness of weak solutions of the three-dimensional stationary Boussinesq system describing steady state motion of viscous heat-convergent Newtonian fluid whose viscosity may depend on the temperature of the fluid. The principal feature of the system under consideration is the dissipative term involved into the energy balance equation. This term is equal to the product of the stress and the strain velocity tensors and has the quadratic growth with respect to the gradient of the velocity field. We prove the partial regularity of solutions to this system and give an estimate of the Hausdorff measure of the singular set.

Dedicated to Olga Alexandrovna Ladyzhenskaya

1. Introduction and Main Results

Let Ω be a domain in $\mathbb{R}^3$. We consider the system describing the steady-state flow of a heat-convergent Newtonian liquid whose viscosity μ can depend on the temperature. This system (known as the Boussinesq system) has the following form, see, for instance, [12]:

$$\left.\begin{aligned} -\operatorname{div}(\mu(\theta)\varepsilon(v)) + (\nabla v)v = f - \nabla p \\ \operatorname{div} v = 0 \end{aligned}\right\} \quad \text{in} \quad \Omega, \tag{1.1}$$

$$-\Delta\theta + \nabla\theta\cdot v = \mu(\theta)|\varepsilon(v)|^2 \quad \text{in} \quad \Omega. \tag{1.2}$$

Here unknown functions are the velocity field $v : \Omega \to \mathbb{R}^3$, pressure $p : \Omega \to \mathbb{R}$ and temperature $\theta : \Omega \to \mathbb{R}$. We denote by $\varepsilon(v) = \frac{1}{2}\big(\nabla v + (\nabla v)^T\big)$ the strain velocity tensor, $f : \Omega \to \mathbb{R}^3$ is a given external force and $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is a given viscosity function.

We assume that the function μ is uniformly continuous on $\mathbb{R}_+$ and satisfies the following restrictions: there exist positive constants μ_0 and μ_1 such that

$$0 < \mu_0 \leqslant \mu(s) \leqslant \mu_1. \tag{1.3}$$

Existence of weak solutions of some boundary value problem for the system (1.1), (1.2) (as well as some more general systems in both stationary and non-stationary cases) was studied by many authors, see [13, 14, 1, 2], and references there.

In this paper we investigate solutions of system (1.1), (1.2) belonging, roughly speaking, to the classes $v \in W^{1,2}(\Omega;\mathbb{R}^3)$, $p \in L^2(\Omega)$, and $\theta \in W^{2,1}(\Omega)$. We show (see Theorem 1.1 below) that really any such solution possesses additional smoothness, namely, there exists $q > 1$ such that

$$v \in W^{1,2q}(\Omega;\mathbb{R}^3), \quad p \in L^{2q}(\Omega), \quad \theta \in W^{2,q}(\Omega). \tag{1.4}$$

Existence of at least one solution (v, p, θ) belonging to (1.4) in the case of Dirichlet boundary data ($v|_{\partial\Omega} = 0$, $\theta|_{\partial\Omega} = 0$) was proved in [3]. The proof of existence is based on derivation of reverse Hölder inequality for ∇v from which the higher integrability of ∇v follows. Namely, the following theorem is true:

Theorem 1.1. *Assume condition* (1.3) *holds and let*

$$f \in L^{\frac{6q_0}{5}}(\Omega;\mathbb{R}^3) \quad \textit{for some} \quad q_0 > 1. \tag{1.5}$$

Then there is $q \in (1, \frac{5}{4})$ depending only on q_0, μ_0, and μ_1 such that for any measurable function θ and any $v \in W^{1,2}(\Omega;\mathbb{R}^3)$ satisfying (1.1) *in the sense of distributions we have the inclusion*

$$v \in W^{1,2q}_{\mathrm{loc}}(\Omega;\mathbb{R}^3).$$

Moreover, if $v \in \overset{\circ}{W}{}^{1,2}_0(\Omega;\mathbb{R}^3)$ and $\partial\Omega$ is smooth enough then $v \in W^{1,2q}(\Omega;\mathbb{R}^3)$ and the following estimate holds:

$$\|v\|_{W^{1,2q}(\Omega)} \leqslant C\|f - (\nabla v)v\|_{\frac{6q}{5},\Omega}. \tag{1.6}$$

We remind here that $(\nabla v)v \in L^{3/2}(\Omega;\mathbb{R}^3)$ providing $v \in W^{1,2}(\Omega;\mathbb{R}^3)$ and hence for $q \leqslant 5/4$ (which means $\frac{6q}{5} \leqslant \frac{3}{2}$) the right-hand side of (1.6) is finite.

Theorem 1.1 was proved in [3]. In this paper we give a sketch of the proof, see Lemma 2.2 below. Existence of a solution (v, p, θ) in the class (1.4) follows form Theorem 1.1 by Schauder fixed point theorem. Namely, in view of (1.6) nondecreasing functions $\mathcal{F}_1$ and $\mathcal{F}_2$ exist such that

$$\|v\|_{W^{1,2q}(\Omega)} \leqslant \mathcal{F}_1(\|f\|_{\frac{6q}{5},\Omega}), \quad \|\theta\|_{W^{2,q}(\Omega)} \leqslant \mathcal{F}_2(\|v\|_{W^{1,2q}(\Omega)}).$$

These a priori estimates allow us to apply Schauder theorem.

In this paper we study smoothness of solutions to Boussinesq system (1.1), (1.2). Let us explain the difficulties we have to overcome. Assuming μ be smooth we can introduce a new unknown vector function $U = \binom{cv}{\theta}$ and get for it (after projecting on the subspace of the solenoidal fields), roughly speaking the system of type

$$-\Delta U = b(U, \nabla U)$$

where function b has quadratic growth with respect to ∇U. It is well known that systems of such type (in contrast to the scalar equations) in general can have nonsmooth solutions, see, for instance, [4]. Nevetheless in [3] we show that in 2D- case solutions of (1.1), (1.2) are always smooth.

The main result of the present paper is the following local condition for Hölder continuity of (v, θ) at the neighborhood of a given point:

Theorem 1.2. *Assume condition* (1.3) *holds and suppose that*

$$f \in L^{\frac{3q}{3-q}}(\Omega; \mathbb{R}^3), \tag{1.7}$$

where q is the exponent from Theorem 1.1. *Let functions (v, p, θ) belong to* (1.4) *and satisfy* (1.1), (1.2) *in the sense of distributions. Suppose that for some point $x_0 \in \Omega$ there is a sequence $R_m \to 0$ such that*

$$\Phi(x_0, R_m) \to 0 \text{as} \quad m \to \infty,$$

where

$$\Phi(x_0, R) := \Big(\fint_{B(x_0,R)} |v - (v)_{x_0,R}|^2 \, dx \Big)^{1/2} + R\Big(\fint_{B(x_0,R)} |\nabla \theta|^q \, dx \Big)^{1/q}.$$

Then for any $\alpha \in (0, q_1)$ there is $r = r(x_0, \alpha) > 0$ such that

$$v \in C^\alpha(\bar{B}(x_0, r); \mathbb{R}^3), \quad \theta \in C^\alpha(\bar{B}(x_0, r)).$$

Here $q_1 = \min\{1, \frac{3}{q}(q-1)\}$.

A direct consequence of Theorem 1.2 is the following result on partial regularity of (v, θ):

Theorem 1.3. *Assume all conditions of Theorem* 1.2 *be satisfied. Then an open set $\Omega_* \subset \Omega$ exists such that for any $x_0 \in \Omega_*$ and any $\alpha \in (0, q_1)$ functions (v, θ) are Hölder continuous with the exponent α at some neighborhood of x_0. Moreover,*

$$\mathcal{H}^{\frac{3(3-q)}{3-2q}}(\Omega \setminus \Omega_*) = 0,$$

where $\mathcal{H}^\beta$ is the β- dimensional Hausdorff measure.

Roughly speaking, Theorem 1.2 says that Hausdorff dimension of the singular set of (v,θ) is strictly less than $3/2$. We shall refer to the points of Ω_* as *regular points* of (v,θ). The following theorem explains this definition:

Theorem 1.4. *Assume all conditions of Theorem* 1.2 *hold and let the point x_0 be regular. Then functions (v,θ) are smooth at some neighborhood of x_0 providing μ and f are smooth.*

In this paper we shall use the standard notations $L^q(\omega;\mathbb{R}^k)$, $W^{l,q}(\omega;\mathbb{R}^k)$, and $W^{l,q}_0(\omega;\mathbb{R}^k)$ for the usual Lebegue and Sobolev spaces of functions with values in $\mathbb{R}^k$. Sometimes we omit the target space in notations. $\|\cdot\|_{q,\omega}$ is the norm in $L^q(\omega)$. For a, $b\in\mathbb{R}^3$, $A\in\mathbb{M}^{3\times3}$ we denote by $a\cdot b=a_ib_i$, $a\otimes b=(a_ib_j)\in\mathbb{M}^{3\times3}$, $a\odot b=\frac{1}{2}(a\otimes b+b\otimes a)$, $Ab=(A_{ij}b_j)\in\mathbb{R}^3$ and repeated indexes stands for the summation from 1 to 3. For any $v:\Omega\to\mathbb{R}^3$ $v_{,j}\equiv\frac{\partial v}{\partial x_j}$, ∇v is (3×3)- matrix $(v_{i,j})$ and hence the convective term $(\nabla v)v$ is a vector with components $(v_jv_{i,j})$. By C we denote constants (whose values can be different from line to line) depending possibly only on μ_0, μ_1 unless otherwise declared. For any measurable $\omega\subset\mathbb{R}^3$ and $g\in L^1(\omega)$ we denote by $|\omega|$ three-dimensional Lebegue measure of ω and $(g)_\omega:=\fint_\omega g\,dx\equiv\frac{1}{|\omega|}\int_\omega g\,dx$. In case of $\omega=B(x_0,R)$ (which is an open ball in $\mathbb{R}^3$ of radius R centered at x_0) we use the notation $(g)_{x_0,R}$ instead of $(g)_{B(x_0,R)}$.

2. Auxiliary results and proof of Theorem 1.1

Our proof of Theorem 1.1 is based on the following lemma:

Lemma 2.1 (Caccioppoli inequality)**.** *Assume conditions* (1.3) *hold and $f\in L^{6/5}(\Omega;\mathbb{R}^3)$. Let θ be any measurable functions and $v\in W^{1,2}(\Omega;\mathbb{R}^3)$ satisfy* (1.1) *in the sense of distributions. Then the estimate*

$$\begin{gathered}\int\limits_{B(x_0,\frac{4}{5}R)}\left(|\nabla v|^2+|p-(p)_{x_0,\frac{4}{5}R}|^2\right)\,dx\leqslant\\ \leqslant\frac{C}{R^2}\int\limits_{B(x_0,R)}|v-a|^2\,dx+C\Big(\int\limits_{B(x_0,R)}\left(|f-(\nabla v)v|^{6/5}\right)\,dx\Big)^{5/3}\end{gathered}\tag{2.8}$$

holds for any $B(x_0,R)\Subset\Omega$ and $a\in\mathbb{R}^3$.

Proof. Denote by $\widetilde{f} = f - (\nabla v)v$. Then we have

$$\int\limits_{B(x_0,R)} \mu(\theta)\varepsilon(v) : \varepsilon(w)\ dx = \int\limits_{B(x_0,R)} (\widetilde{f}\cdot w + (p-b)\operatorname{div} w)\ dx, \qquad \text{for any} \quad w \in W_0^{1,2}(\Omega;\mathbb{R}^3). \tag{2.9}$$

where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ are arbitrary.

Let $\varphi \in C_0^\infty(B(x_0,R))$ and take in (II) $w = \varphi^2(v-a)$. Then

$$\varepsilon(w) = \varphi^2\varepsilon(v) + (v-a)\odot\nabla\varphi^2, \quad \operatorname{div} w = (v-a)\cdot\nabla\varphi^2.$$

The term containing $\widetilde{f}$ can be estimated with the help of the imbedding theorem:

$$\Big|\int\limits_{\Omega} \varphi^2\widetilde{f}\cdot(v-a)\ dx\Big| \leqslant \|\varphi\widetilde{f}\|_{6/5,\Omega}\|\varphi(v-a)\|_{6,\Omega} \leqslant$$
$$\leqslant \|\widetilde{f}\|_{6/5,B(x_0,R)}\|\nabla(\varphi(v-a))\|_{2,\Omega}.$$

By Korn's inequality we get

$$\int\limits_{B(x_0,R)} \varphi^2|\nabla v|^2 dx \leqslant C \int\limits_{B(x_0,R)} \varphi^2|\varepsilon(v)|^2 dx + C\int\limits_{B(x_0,R)} |\nabla\varphi|^2|v-a|^2 dx.$$

Hence for any $\sigma > 0$ the following estimate holds:

$$\int\limits_{B(x_0,R)} \varphi^2|\nabla v|^2\ dx \leqslant C_\sigma \int\limits_{B(x_0,R)} |\nabla\varphi|^2|v-a|^2\ dx +$$
$$+\sigma\int\limits_{B(x_0,R)} \varphi^2|p-b|^2\ dx + C\Big(\int\limits_{B(x_0,R)} |\widetilde{f}|^{6/5}\ dx\Big)^{5/3},$$

where $C_\sigma = \frac{C}{\sigma}$. Let $\frac{4}{5}R \leqslant r < s \leqslant R$ and $\varphi \equiv 1$ on $B(x_0,r)$, $|\nabla\varphi| \leqslant \frac{C}{s-r}$. Denote by A the integral

$$A(t) := \int\limits_{B(x_0,t)} |\nabla v|^2\ dx.$$

Then

$$A(r) \leqslant \frac{C_\sigma}{(s-r)^2}\int\limits_{B(x_0,s)} |v-a|^2\ dx + \sigma\int\limits_{B(x_0,s)} |p-b|^2\ dx + \|\widetilde{f}\|^2_{6/5,B(x_0,R)},$$

where $\sigma > 0$ can be taken arbitrary small. Let $b = (p)_{x_0,s}$ and take in the relation

$$\int\limits_{B(x_0,s)} (p-(p)_{x_0,s}) \operatorname{div} w\, dx = \int\limits_{B(x_0,s)} \mu(\theta)\varepsilon(v) : \varepsilon(w) dx - \int\limits_{B(x_0,s)} \widetilde{f} \cdot w\, dx,$$

the test function $w \in W_0^{1,2}(B(x_0,s);\mathbb{R}^3)$ satisfying conditions

$$\operatorname{div} w = p - (p)_{x_0,s} \quad \text{in} \quad B(x_0,s),$$

$$\|\nabla w\|_{2,B(x_0,s)} \leqslant C\|p-(p)_{x_0,s}\|_{2,B(x_0,s)}$$

(existence of such a function is proved in [11]). Then we obtain

$$\int\limits_{B(x_0,s)} |p-(p)_{x_0,s}|^2\, dx \leqslant C_0 A(s) + \|\widetilde{f}\|^2_{6/5,B(x_0,R)}. \tag{2.10}$$

Hence for all $\frac{4}{5}R \leqslant r < s \leqslant R$ we have the estimate

$$A(r) \leqslant C_0\sigma A(s) + \frac{C_\sigma}{(s-r)^2} \int\limits_{B(x_0,s)} |v-a|^2\, dx + \|\widetilde{f}\|^2_{6/5,B(x_0,R)}.$$

Choosing σ so that $C_0\sigma = \frac{1}{2}$ and applying well-known lemma from [6] we obtain the inequality

$$A(\frac{4}{5}R) \leqslant \frac{C}{R^2} \int\limits_{B(x_0,R)} |v-a|^2\, dx + \|\widetilde{f}\|^2_{6/5,B(x_0,R)}.$$

Taking into account (2.10) with $s = \frac{4}{5}R$ we obtain the statement of the lemma. Lemma 2.1 is proved.

Lemma 2.2 (Higher integrability)**.** *Suppose all conditions of Lemma* 2.1 *be satisfied and moreover condition* (1.5) *holds. Then there is a number* $q \in (1, \frac{5}{4})$ *depending only on* q_0, μ_0, *and* μ_1 *such that for all* $B(x_0,R) \Subset \Omega$

$$\begin{aligned}\Big(\ \fint\limits_{B(x_0,\frac{3}{5}R)} |\nabla v|^{2q}\, dx\Big)^{1/2q} &\leqslant C\Big(\ \fint\limits_{B(x_0,\frac{4}{5}R)} |\nabla v|^2\, dx\Big)^{1/2} + \\ &+ CR\Big(\ \fint\limits_{B(x_0,R)} |f-(\nabla v)v|^{\frac{6q}{5}}\, dx\Big)^{\frac{5}{6q}}.\end{aligned} \tag{2.11}$$

The proof of this lemma essentially follows to the proof of the Lemma 3.2 of [3]. For the readers' convenience we remind it hear briefly.

Denote as before $\widetilde{f} = f - (\nabla v)v$. Take in (2.8) $a = (v)_{x_0,R}$. With the help of the estimate

$$\int\limits_{B(x_0,R)} |v - (v)_{x_0,R}|^2\, dx \leqslant C\Big(\int\limits_{B(x_0,R)} |\nabla v|^{6/5}\, dx\Big)^{5/3}$$

we get the inequality

$$\Big(\fint\limits_{B(x_0,\frac{4}{5}R)} |\nabla v|^2\, dx\Big)^{1/2} \leqslant$$

$$\leqslant C\Big(\fint\limits_{B(x_0,R)} |\nabla v|^{6/5}\, dx\Big)^{5/6} + CR\Big(\fint\limits_{B(x_0,R)} |\widetilde{f}|^{\frac{6}{5}}\, dx\Big)^{\frac{5}{6}}. \quad (2.12)$$

As the function $\widetilde{f}$ is integrable with the power greater than 6/5 the estimate (2.12) with some $q > 1$ follows now from some generalization of Gering's lemma, see [5] and [6, p. 122]. Lemma 2.2 is proved.

Remark 2.1. If $\partial\Omega$ is smooth enough and one assumes some boundary conditions for v, for example, Dirichlet boundary data $v|_{\partial\Omega} = 0$ then it is possible to show that (2.12) holds up to the boundary, i.e., for any $x_0 \in \bar{\Omega}$

$$\Big(\fint\limits_{\Omega\cap B(x_0,\frac{3}{5}R)} |\nabla v|^{2q}\, dx\Big)^{1/2q} \leqslant$$

$$\leqslant C\Big(\fint\limits_{\Omega\cap B(x_0,\frac{4}{5}R)} |\nabla v|^2\, dx\Big)^{1/2} + CR\Big(\fint\limits_{\Omega\cap B(x_0,R)} |\widetilde{f}|^{\frac{6q}{5}}\, dx\Big)^{\frac{5}{6q}}. \quad (2.13)$$

For the proof of (2.13) in case of $v|_{\partial\Omega} = 0$ see [3].

Remark 2.2. From Lemma 2.2 by Hölder inequality we get relations

$$(\nabla v)v \in L^{\frac{3q}{3-q}}(\Omega;\mathbb{R}^3), \quad \|(\nabla v)v\|_{\frac{3q}{3-q},\Omega} \leqslant C\|v\|^2_{W^{1,2q}(\Omega)}, \quad (2.14)$$

and hence for all $B(x_0,R) \Subset \Omega$

$$\|\widetilde{f}\|_{\frac{6q}{5},B(x_0,R)} \leqslant C\|\widetilde{f}\|_{\frac{3q}{3-q},\Omega} R^{1-\frac{1}{2q}} \quad (2.15)$$

providing (2.15) is valid. Here $\widetilde{f} = f - (\nabla v)v$.

Lemma 2.3. *Let $v \in W^{1,2q}(\Omega;\mathbb{R}^3)$ and $\theta \in W^{2,q}(\Omega)$ satisfy (1.2). Then for all $B(x_0,R) \Subset \Omega$ the following estimate holds*

$$\|\nabla^2\theta\|_{q,B(x_0,\frac{2}{5}R)} \leqslant$$
$$\leqslant C\|\varepsilon(v)\|^2_{2q,B(x_0,\frac{3}{5}R)} + \frac{C}{R}(1+R\|v\|^2_{6,\Omega})\|\nabla\theta\|_{q,B(x_0,\frac{3}{5}R)}. \quad (2.16)$$

Proof. Let $\varphi \in C_0^\infty(B(x_0,\frac{3}{5}R))$ be a cut-off function satisfying conditions

$$\varphi \equiv 1 \text{ on } B(x_0,\frac{2}{5}R), \quad |\nabla^k\varphi| \leqslant \frac{C}{R^k},\ k=1,2.$$

Then from (1.2) we get for $\bar\theta = \theta - (\theta)_{x_0,R}$

$$-\Delta(\varphi\bar\theta) = \varphi\mu(\theta)|\varepsilon(v)|^2 + 2\nabla\theta\cdot\nabla\varphi + \bar\theta\Delta\varphi - \varphi v\cdot\nabla\theta.$$

Hence

$$\|\varphi\nabla^2\theta\|_{q,B(x_0,\frac{3}{5}R)} \leqslant C\|\varepsilon(v)\|^2_{2q,B(x_0,\frac{3}{5}R)} + \frac{C}{R}\|\nabla\theta\|_{q,B(x_0,\frac{3}{5}R)} +$$
$$+\frac{C}{R^2}\|\theta-(\theta)_{x_0,R}\|_{q,B(x_0,\frac{3}{5}R)} + \|\varphi v\cdot\nabla\theta\|_{q,B(x_0,\frac{3}{5}R)}.$$

By Hölder inequality we obtain

$$\|\varphi v\cdot\nabla\theta\|_{q,B(x_0,\frac{3}{5}R)} \leqslant \|v\|_{6,B(x_0,\Omega)}\|\varphi\nabla\theta\|_{\frac{6q}{6-q},B(x_0,\frac{3}{5}R)}.$$

Choose $\lambda = 1/2$. Then $\frac{6-q}{6q} = \frac{1}{q}(1-\lambda) + \frac{3-q}{3q}\lambda$ and hence the interpolation inequality provides the estimate

$$\|\varphi\nabla\theta\|_{\frac{6q}{6-q},B(x_0,\frac{3}{5}R)} \leqslant C\|\varphi\nabla\theta\|^{1/2}_{q,B(x_0,\frac{3}{5}R)}\|\varphi\nabla\theta\|^{1/2}_{\frac{3q}{3-q},B(x_0,\frac{3}{5}R)} \leqslant$$
$$\leqslant C\|\varphi\nabla\theta\|^{1/2}_{q,B(x_0,\frac{3}{5}R)}(\|\varphi\nabla^2\theta\|_{q,B(x_0,\frac{3}{5}R)} + \frac{1}{R}\|\nabla\theta\|_{q,B(x_0,\frac{3}{5}R)})^{1/2}$$

from which (2.16) follows. Lemma 2.3 is proved.

In conclusion of this section we remind the readers one well-known statement we shall use below.

Lemma 2.4. *Let $v^0 \in W^{1,2}(B;\mathbb{R}^3)$ and $p^0 \in L^2(B)$ be a solution of the homogeneous Stokes system*

$$-\nu\Delta v^0 + \nabla p^0 = 0, \quad \nu > 0,$$
$$\operatorname{div} v^0 = 0.$$

Then $v^0 \in C^\infty_{\rm loc}(B;\mathbb{R}^3)$, $p^0 \in C^\infty_{\rm loc}(B)$ *and hence for all* $\tau \in (0,1)$

$$\fint\limits_{B(\tau)} |v^0 - (v^0)_{B(\tau)}|^2\, dx \leqslant C_1\tau^2 \int\limits_B |v^0|^2\, dx. \tag{2.17}$$

For the proof of smoothness of v^0, p^0 see, for instance, [10]. Estimate (2.17) is an obvious consequence of the estimate $\|v^0\|_{W^{k,2}(B(1/2))} \leqslant C(k)\|v^0\|_{2,B}$ which follows from Lemma 2.1 and holds for arbitrary k.

3. Proof of Theorem 1.2

Theorem 3.1 (Decay Estimate)**.** *Assume all conditions of Theorem* 1.2 *hold and suppose* $\alpha \in (0, q_1)$, *where* $q_1 = \min\{1, \frac{3}{q}(q-1)\}$. *Then for all* $\tau \in (0, \frac{2}{5})$ *there are positive numbers* $\gamma_0 = \gamma_0(\tau)$ *and* $R_0 = R_0(\tau)$ *such that for all* $x_0 \in \Omega$ *and all* $R < R_0$ *if* $B(x_0,R) \Subset \Omega$ *and*

$$\Phi(x_0,R) + R^\alpha < \gamma_0 \quad \text{then} \quad \Phi(x_0,\tau R) \leqslant C_*\tau(\Phi(x_0,R) + R^\alpha).$$

Here $C_* > C_0$ *(and* $C_0 = C_0(\mu_0,\mu_1)$ *is an absolute constant) and excess* Φ *is defined as in Theorem* 1.2.

Proof. By contradiction assume that there is a number $\tau \in (0,1)$ and sequences $x_h \in \Omega$, $\gamma_h \to 0$, $R_h \to 0$ such that

$$\Phi(x_h,R_h) + R_h^\alpha = \gamma_h, \quad \Phi(x_h,\tau R_h) \geqslant C_*\tau\gamma_h. \tag{3.18}$$

As before we introduce the new external force $\widetilde{f} = f - (\nabla v)v$ and denote also

$$a^h = (v)_{x_h,R_h}, \quad b^h = (\theta)_{x_h,R_h}.$$

By Lemma 2.1 we get the estimate

$$\Big(\fint\limits_{B(x_h,\frac{4}{5}R_h)} |\nabla v|^2\, dx\Big)^{1/2} \leqslant$$

$$\leqslant \frac{C}{R_h}\Phi(x_h,R_h) + CR_h\Big(\fint\limits_{B(x_h,R_h)} |\widetilde{f}|^{6/5}\, dx\Big)^{5/6}. \tag{3.19}$$

Lemma 2.2 provides the inequality

$$\Big(\fint\limits_{B(x_h,\frac{3}{5}R_h)} |\nabla v|^{2q}\, dx\Big)^{1/2q} \leqslant$$

$$\leqslant C\Big(\fint\limits_{B(x_h,\frac{4}{5}R_h)} |\nabla v|^2\, dx\Big)^{1/2} + CR_h\Big(\fint\limits_{B(x_h,R_h)} |\widetilde{f}|^{6q/5}\, dx\Big)^{5/6q} \tag{3.20}$$

for some $q \in (1, \frac{5}{4})$. Finally, by Lemma 2.3 we get

$$\begin{aligned}\Big(\fint_{B(x_h, \frac{2}{5}R_h)} |\nabla^2 \theta|^q \, dx\Big)^{1/q} \leqslant \\ \leqslant C\Big(\fint_{B(x_h, \frac{3}{5}R_h)} |\varepsilon(v)|^{2q} \, dx\Big)^{1/q} + \frac{C}{R_h^2}(1 + R_h\|v\|^2_{6,\Omega})\Phi(x_h, R_h).\end{aligned} \tag{3.21}$$

Note also that for $\Omega \subset \mathbb{R}^3$, $v \in W^{1,2}(\Omega; \mathbb{R}^3) \subset L^6(\Omega; \mathbb{R}^3)$

$$R_h|a^h| \equiv R_h|(v)_{x_h, R_h}| \leqslant CR_h^{1/2}\|v\|_{6,\Omega} \to 0 \quad \text{as} \quad h \to \infty. \tag{3.22}$$

For $y \in B \equiv B(0,1)$ we introduce rescaled functions v^h, p^h, θ^h, and $\widetilde{f}^h$ given by

$$v^h(y) := \frac{1}{\gamma_h}(v(x_h + R_h y) - (v)_{x_h, R_h}),$$

$$p^h(y) := \frac{R_h}{\gamma_h}(p(x_h + R_h y) - (p)_{x_h, R_h}),$$

$$\theta^h(y) := \frac{1}{\gamma_h}(\theta(x_h + R_h y) - (\theta)_{x_h, R_h}),$$

$$\widetilde{f}^h(y) := \frac{R_h^2}{\gamma_h} f(x_h + R_h y).$$

Let us also introduce a notation

$$V^h(\tau) := \Big(\fint_{B(\tau)} |v^h - (v^h)_{B(\tau)}|^2 \, dy\Big)^{1/2} + \tau\Big(\fint_{B(\tau)} |\nabla \theta^h|^q \, dy\Big)^{1/q}.$$

Rescaling (3.18) we obtain

$$V^h(1) + \frac{R_h^\alpha}{\gamma_h} = 1, \qquad V^h(\tau) \geqslant C_* \tau, \tag{3.23}$$

Taking into account (2.15) it is easy to see that

$$\|\widetilde{f}^h\|_{\frac{6q}{5}, B} \leqslant C\frac{R_h^2}{\gamma_h} R_h^{-\frac{5}{2q}} \|\widetilde{f}\|_{\frac{6q}{5}, B(x_h, R_h)} \leqslant C\frac{R_h^{\frac{3}{q}(q-1)}}{\gamma_h} \|\widetilde{f}\|_{\frac{3q}{3-q}, \Omega}.$$

Since $L^{\frac{3q}{3-q}}$- norm of $\widetilde{f}$ is finite (see (1.7), (2.14)), $\alpha < \frac{3}{q}(q-1)$, and $\frac{R_h^\alpha}{\gamma_h} \leqslant 1$ we get the convergence

$$\widetilde{f}^h \to 0 \quad \text{in} \quad L^{\frac{6q}{5}}(B; \mathbb{R}^3). \tag{3.24}$$

Changing variables in (3.19)–(3.21) we get relations

$$\|\nabla v^h\|_{2,B(4/5)} + \|p^h\|_{2,B(4/5)} \leqslant C(V^h(1) + \|\widetilde{f}^h\|_{6/5,B}), \leqslant C$$

$$\|\nabla v^h\|_{2q,B(3/5)} \leqslant C(\|\nabla v^h\|_{2,B(4/5)} + \|\widetilde{f}^h\|_{6q/5,B(4/5)}) \leqslant C, \qquad (3.25)$$

and

$$\|\nabla^2 \theta^h\|_{q,B(2/5)} \leqslant C\gamma_h \|\varepsilon(v^h)\|^2_{2q,B(3/5)} + C(1 + \|v\|^2_{6,\Omega})V^h(1) \leqslant C,$$

where all constants C do not depend on h.

Rescaling equations (1.1), (1.2) we get

$$\begin{gathered} -\operatorname{div}\Big(\mu(b^h + \gamma_h\theta^h)\varepsilon(v^h)\Big) + \nabla p^h = \widetilde{f}^h \quad \text{in} \quad W_2^{-1}(B;\mathbb{R}^3), \\ \operatorname{div} v^h = 0 \text{ a.e. in } \ B, \end{gathered} \qquad (3.26)$$

$$-\Delta\theta^h = \gamma_h \mu(b^h + \gamma_h\theta^h)|\varepsilon(v^h)|^2 + R_h(a^h + \gamma_h v^h)\cdot\nabla\theta^h \quad \text{a.e. in} \quad B. \quad (3.27)$$

Estimates (3.23)–(3.25) allow us to extract a subsequence $\{v^h\}$ such that

$$v^h \rightharpoonup v^0 \quad \text{in} \quad L^2(B;\mathbb{R}^3), \qquad \theta^h \rightharpoonup \theta^0 \quad \text{in} \quad W^{1,q}(B), \qquad (3.28)$$

$$v^h \to v^0 \quad \text{in} \quad L^2(B(2/5);\mathbb{R}^3), \qquad \theta^h \to \theta^0 \quad \text{in} \quad W^{1,q}(B(2/5)). \qquad (3.29)$$

From the strong convergence $\gamma_h\theta^h \to 0$ in $L^q(B)$ we also get $\gamma_h\theta^h \to 0$ a.e. in B. Taking into account uniform continuity of μ on $\mathbb{R}_+$ from (1.3) we conclude that a number $\widetilde{\mu} \in [\mu_0, \mu_1]$ exists such that

$$\mu(b^h + \gamma_h\theta^h(y)) \to \widetilde{\mu} \quad \text{for almost all} \quad y \in B.$$

Thanks to (3.22)

$$R_h a^h \to 0 \quad \text{as} \quad h \to \infty. \qquad (3.30)$$

Denote by

$$V^0(\tau) := \Big(\fint_{B(\tau)} |v^0 - (v^0)_{B(\tau)}|^2\, dy\Big)^{1/2} + \tau\Big(\fint_{B(\tau)} |\nabla\theta^0|^q\, dy\Big)^{1/q}.$$

Convergence (3.28) imply that

$$V^0(1) \leqslant \liminf_{h\to\infty} V^h(1) \leqslant 1.$$

From (3.29) we get for $\tau \in (0, 2/5)$

$$V^0(\tau) = \lim_{h\to\infty} V^h(\tau) \geqslant C_*\tau. \qquad (3.31)$$

Taking into account (3.24), (3.28), (3.29), (3.30), and passing to the limit in (3.26) and (3.27) we obtain the homogeneous Stokes system for v^0

$$\left.\begin{aligned} -\frac{\widetilde{\mu}}{2}\Delta v^0 + \nabla p^0 = 0 \\ \operatorname{div} v^0 = 0 \end{aligned}\right\} \quad \text{in} \quad B(2/5), \tag{3.32}$$

and also the Laplace equation for θ^0:

$$-\Delta\theta^0 = 0 \quad \text{in} \quad B(2/5). \tag{3.33}$$

As v^0 is a solution of the Stokes system (3.32) by Lemma 2.4 we obtain

$$\Big(\fint_{B(\tau)} |v^0 - (v^0)_{B(\tau)}|^2 dy\Big)^{1/2} \leqslant C_1\tau\Big(\fint_B |v^0|^2 dy\Big)^{1/2}.$$

On the other hand, θ^0 is a harmonic function (see (3.33)). Hence (see [8])

$$\Big(\fint_{B(\tau)} |\nabla\theta^0|^2 dy\Big)^{1/2} \leqslant \sup_{B(2/5)} |\nabla\theta^0| \leqslant C_2\Big(\int_B |\nabla\theta^0|^q dy\Big)^{1/q}.$$

Then

$$V^0(\tau) \leqslant C_3\tau V^0(1), \quad \text{where} \quad C_3 = C_1 + C_2.$$

This contradicts to (3.31) if $C_* > C_3$. Theorem 3.1 is proved.

Now we turn to the proof of Theorem 1.2. It follows from Theorem 3.1 by routine arguments. We remind them to the reader.

Define $\Omega_* \subset \Omega$ by

$$\Omega_* := \Big\{x_0 \in \Omega : \liminf_{R\to 0} \Phi(x_0, R) = 0\Big\}.$$

Let us prove that v, θ are Hölder continuous in some neighborhood of any point of Ω_*. Suppose $\alpha \in (0, q_1)$ and take $\alpha_1 \in (\alpha, q_1)$. Let us choose $\tau \in (0, \frac{2}{5})$ in such a way that $\tau^{\alpha_1} = C_*\tau$, and, moreover, $2\tau^{\alpha_1} \leqslant 1$ (it is possible if C_* is large enough). Define γ_0 and R_0 as in Theorem 3.1.

Let $x_0 \in \Omega_*$. Then there is $R_1 \leqslant R_0$ such that $\Phi(x_0, R_1) + R_1^{\alpha_1} < \gamma_0$. As the function $x \mapsto \Phi(x, R_1)$ is continuous there is $r = r(x_0) > 0$ such that $\Phi(x, R_1) + R_1^{\alpha_1} < \gamma_0$ for all $x \in \bar{B}(x_0, r)$. We claim at the estimates

$$\Phi(x, \tau^{l-1}R_1) + (\tau^{l-1}R_1)^{\alpha_1} < \gamma_0 \quad \text{and} \quad \Phi(x, \tau^l R_1) \leqslant l\tau^{l\alpha_1}\gamma_0 \tag{3.34}$$

to be valid for each $l \in \mathbb{N}$. Indeed, for $l = 1$ (3.34) were established in Theorem 3.1. Assume (3.24) hold for $l = k$. Then for $l = k + 1$ by the same theorem we get

$$\Phi(x, \tau^k R_1) + (\tau^k R_1)^{\alpha_1} \leqslant \tau^{\alpha_1} \Phi(x, \tau^{k-1} R_1) + 2(\tau^k R_1)^{\alpha_1}.$$

As $2\tau^{\alpha_1} \leqslant 1$ we have by induction assumption

$$\Phi(x, \tau^k R_1) + (\tau^k R_1)^{\alpha_1} \leqslant \Phi(x, \tau^{k-1} R_1) + (\tau^{k-1} R_1)^{\alpha_1} < \gamma_0$$

Thus the first inequality in (3.34) holds for $l = k + 1$ and hence we can apply Theorem 3.1 again:

$$\Phi(x, \tau^{k+1} R_1) \leqslant \tau^{\alpha_1} (\Phi(x, \tau^k R_1) + (\tau^k R_1)^{\alpha_1}).$$

Using induction assumption one more time we conclude

$$\Phi(x, \tau^{k+1} R_1) \leqslant (k + 1) \tau^{(k+1)\alpha_1} \gamma_0.$$

So the second estimate in (3.34) is also established.

Now we take an advantage of $\alpha < \alpha_1$ and get from (Iterations) the estimate

$$\Phi(x, \tau^k R_1) \leqslant C(\tau, \gamma_0, \alpha, \alpha_1) \tau^{k\alpha} \quad \text{for all} \quad k \in \mathbb{N}, \ x \in \bar{B}(x_0, r).$$

This estimate implies Hölder continuity of v and θ on $\bar{B}(x_0, r)$ with the exponent α.

4. Proof of Theorems 1.3 and 1.4

To estimate Hausdorff measure of $\Sigma := \Omega \setminus \Omega_*$ we decompose

$$\Sigma = \Sigma_1 \cup \Sigma_2,$$

where

$$\Sigma_1 = \{x \in \Omega : \liminf_{R \to 0} \Phi_1(x, R) > 0\},$$

$$\Phi_1(x, R) := \Big(\fint_{B(x,R)} |v - (v)_{x,R}|^2 dx \Big)^{1/2},$$

$$\Sigma_2 = \{x \in \Omega : \liminf_{R \to 0} \Phi_2(x, R) > 0\},$$

$$\Phi_2(x, R) := R \Big(\fint_{B(x,R)} |\nabla \theta|^q dx \Big)^{1/q}.$$

As $v \in W^{1,2q}_{\rm loc}(\Omega;\mathbb{R}^3)$ by Poincaré inequality we obtain

$$\Phi_1(x,R) \leqslant R\Big(\fint\limits_{B(x,R)} |\nabla v|^{2q} dx\Big)^{1/2q}$$

and hence by well-known result (see, for example, Theorem 6.2 in [7]) we get $\mathcal{H}^{3-2q}(\Sigma_1) = 0$. Unfortunately, the estimate of Σ_2 we can obtain is much worse. By imbedding theorem we have $\nabla\theta \in W^{1,q}(\Omega;\mathbb{R}^3) \subset L^{\frac{3q}{3-q}}(\Omega;\mathbb{R}^3)$ and similarly to above we get $\mathcal{H}^{\frac{3(3-2q)}{3-q}}(\Sigma_2) = 0$. Theorem 1.2 is proved.

We are left to prove Theorem 1.4. Assuming (v,θ) be Hölder continuous from the general regularity theory for linear systems we get Hölder continuity of ∇v (the proof is essentially follows to the proof of Theorem 3.5 of [7] or Theorem 3.13 of [9] with some trivial modifications). Now smoothness of (v,θ) follows by regular boot-strap arguments. Theorem 1.4 is proved.

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