Записки научных
семинаров ПОМИ
Том 271, 2000 г.

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BOUNDARY ESTIMATES FOR SOLUTIONS TO THE PARABOLIC FREE BOUNDARY PROBLEM

To the memory of A. I. Ivanov

ABSTRACT. Let u and Ω (an open set in $\mathbb{R}^{n+1}_+ = \{(x,t) : x \in \mathbb{R}^n, t \in \mathbb{R}^1, x_1 > 0\}, n \geqslant 2$) solve the following problem:

$$H(u) = \chi_\Omega, \quad u = |Du| = 0 \quad \text{in} \quad Q_1^+ \setminus \Omega, \quad u = 0 \quad \text{on} \quad \Pi \cap Q_1,$$

where $H = \Delta - \partial_t$ is the heat operator, χ_Ω denotes the characteristic function of Ω, Q_1 is the unit cylinder in $\mathbb{R}^{n+1}$, $Q_1^+ = Q_1 \cap \mathbb{R}^{n+1}_+$, $\Pi = \{(x,t) : x_1 = 0\}$, and the first equation is satisfied in the sense of distributions. We obtain the optimal regularity of the function u, i.e., we show that $u \in C_x^{1,1} \cap C_t^{0,1}$.

This paper deals with the regularity properties of solutions to a certain type of parabolic free boundary problems. Mathematically the problem is formulated as follows. Let Ω be an open set in $\mathbb{R}^{n+1}_+$ and suppose, locally, there exists a function u such that

$$\begin{gathered} H(u) = \chi_\Omega \quad \text{in} \quad Q_1^+, \\ u = |Du| = 0 \quad \text{in} \quad Q_1^+ \setminus \Omega, \quad u = 0 \quad \text{on} \quad \Pi \cap Q_1 \end{gathered} \tag{1.1}$$

and the first equation in (1.1) is understood in the weak (distributional) sence.

Results for an elliptic problem related to (1.1) can be found in [6]. It should be mentioned also a recent work [3] where the parabolic problem (1.1) was considered in the whole unit cylinder, without boundary restriction on Π. Results for an elliptic free boundary problem in the unit ball, without conditions on Π were obtained in [2]. In papers [6, 3, 2] the regularity properties of the free boundary $\partial\Omega$ were studied in addition to the optimal regularity of solutions.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 35R35; Secondary 35J85.

Key words and phrases. Free boundary problems, regularity.

D. E. Apushkinskaya and N. N. Uraltseva were partially supported by the Russian Foundation of Fundamental Research, grant no. 99-0100684. H. Shahgholian was partially supported by the Swedish Natural Sciences Research Council.

Notation. Throughout the paper we will use the following notations:
$z = (x, t)$ are points in $\mathbb{R}^{n+1}$, where $x = (x_1, x') = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $n \geqslant 2$ and $t \in \mathbb{R}^1$;
$\mathbb{R}^{n+1}_+ = \{(x,t) \in \mathbb{R}^{n+1} : x_1 > 0\}$;
$\Pi = \{(x,t) \in \mathbb{R}^{n+1} : x_1 = 0\}$;
χ_Ω denotes the characteristic function of the set Ω ($\Omega \subset \mathbb{R}^{n+1}$);
$B_r(x)$ is the open ball in $\mathbb{R}^n$ with center x and radius r;
$B_r = B_r(0)$;
$Q_r(z) = Q_r(x,t) = B_r(x,t) \times]t - r^2, t]$ denotes the open cylinder in $\mathbb{R}^{n+1}$;
$\partial' Q_r(x,t)$ is the parabolic boundary, i.e. the topological boundary minus the top of the cylinder;
$Q_r^+(z) = Q_r^+(x,t) = Q_r(x,t) \cap \mathbb{R}^{n+1}_+$;
$Q_r = Q_r(0,0)$;
$Q_r^+ = Q_r^+(0,0)$.

The indices i, j always vary from 1 to n, while the indices τ, μ vary from 2 to n. Repeated indices indicate summation, for example, $a^{ij} x_i x_j = \sum_{i,j=1}^n a^{ij} x_i x_j$.
D_i denotes the differential operator with respect to x_i; $\partial_t = \frac{\partial}{\partial t}$;
$D = (D_1, D') = (D_1, D_2, \dots, D_n)$ denotes the spatial gradient;
$H = \Delta - \partial_t$ is the heat operator;
$\Lambda(u) = \{(x,t) : u(x,t) = |Du(x,t)| = 0\}$ for any $C^1_x \cap C^0_t$- function u;
$\Omega(u) = \mathbb{R}^{n+1} \setminus \Lambda(u)$;
$\Gamma(u) = \partial\Omega(u) \cap \Lambda(u)$ is the free boundary;
$\|\cdot\|_{p,\Omega}$ denotes the norm in $L_p(\Omega)$, $1 < p \leqslant \infty$;
$W^{2,1}_p(\Omega)$ is the anisotropic Sobolev space with the norm

$$\|u\|_{W^{2,1}_p(\Omega)} = \|\partial_t u\|_{p,\Omega} + \|D(Du)\|_{p,\Omega} + \|u\|_{p,\Omega};$$

$e_1, \dots, e_n$ is the standard basis in $\mathbb{R}^n$;
$v^+ = \max\{v, 0\}, \quad v^- = \max\{-v, 0\}$;

We will use N, M and C with or without indices, to denote various constants. To indicate the quantities on which they depend we write $N(\dots)$.

In order to formulate our main result, let us define a local solution.

Definition. *We say a continuous function u (not identically zero) belongs to the class $P_r^+(M)$ if u satisfies:*

(1) $H(u) = \chi_\Omega$ *in* Q_r^+, *for some open set* $\Omega \subset \mathbb{R}^{n+1}_+$,
(2) $u = |Du| = 0$ *in* $Q_r^+ \setminus \Omega$,

(3) $\|u\|_{\infty,Q_r^+} \leqslant M$,
(4) $u = 0$ *on* $\Pi \cap Q_r$

and equation (1) *is understood in the sense of distributions.*

We also denote by $P_\infty^+(M)$ the class of "global solutions" with quadratic growth in x and linear growth in t, i.e., solutions in the entire half space $\mathbb{R}_+^{n+1}$ satisfying

$$|u(x,t)| \leqslant M(1 + |x|^2 + |t|). \tag{12}$$

Without loss of generality we assume that

$$M \geqslant 1. \tag{1.3}$$

Now we can state the main result of this paper.

Main Theorem. *There exists a universal constant* $C_0 = C_0(n)$ *such that, if* $u \in P_1^+(M)$*, then*

$$\sup_{Q_{1/8}^+ \cap \Omega} (|D_i D_j u(x,t)| + |\partial_t u(x,t)|) \leqslant C_0 M. \tag{1.4}$$

This paper is organized as follows. We collect all technical tools in Section 2. Most of these auxiliary results (Lemmas 2.1–2.5) are known, but probably not well-known in the context used in this paper. Whereas the other results (Lemmas 2.6–2.7) are completely new and crucial for our study. For the readers convenience, we have included the proofs of almost all technical lemmas. In Section 3 we carry out the proof for Main Theorem, using the modified monotonicity formulas (Lemmas 2.5 and 2.7) and the blow-up technique. We also obtain in §3 a simple corollary (Corollary 3.2) which states that global solutions are actually $C_x^{1,1} \cap C_t^{0,1}(\mathbb{R}_+^{n+1})$.

§2. Auxiliary results

In this section we will gather all basic tools used to prove Main Theorem.

Lemma 2.1 (Liouville type theorem). *Let* u *be caloric function in the whole space* $\mathbb{R}^{n+1}$ *and satisfy the inequality*

$$|u(x,t)| \leqslant C(1 + |x|^2 + |t|), \quad (x,t) \in \mathbb{R}^{n+1}.$$

Then there exist constants a^{ij}, a^i, a_0 such that

$$u(x,t) \equiv a^{ij} x_i x_j + 2\left(\sum_{i=1}^{n} a^{ii}\right) \cdot t + a^i x_i + a_0.$$

This result can be easily deduced from well-known estimates for derivatives of caloric functions.

Lemma 2.2. *For $u \in P_1^+(M)$ and $Q_\rho(z^0) \subset Q_1$ we have*

$$\|Du\|^2_{2,Q^+_{\rho/2}(z^0)} \leqslant C_1(n) \cdot \rho^n \cdot \{\rho^2 \cdot \|u\|_{\infty,Q^+_\rho(z^0)} + \|u\|^2_{\infty,Q^+_\rho(z^0)}\}. \quad (2.1)$$

Proof. Consider two cut-off functions:

$$\xi = \xi(x) \in C_0^\infty(\overline{B_\rho(x^0)}), \quad \xi(x) \equiv 1 \quad \text{in} \quad B_{\rho/2}(x^0), \quad |D\xi| \leqslant \frac{4}{\rho},$$

$$\eta = \eta(t) \in C_0^\infty([t^0 - \rho^2, t^0 + \rho^2]), \quad \eta(t) \equiv 1 \quad \text{in} \quad]t^0 - \frac{\rho^2}{4}, t^0], \quad \partial_t \eta \leqslant \frac{4}{\rho^2}.$$

Multiplying both sides of $H(u) = \chi_\Omega$ on $u\xi^2\eta^2$ and integrating by parts, we obtain

$$\int\limits_{Q^+_\rho(z^0)} (Du)^2 \xi^2 \eta^2 dxdt = -\frac{1}{2} \int\limits_{B^+_\rho(x^0)} (u|_{t=t^0})^2 \xi^2 dx + \int\limits_{Q^+_\rho(z^0)} u^2 \xi^2 \eta (\partial_t \eta) dxdt -$$
$$- \int\limits_{Q^+_\rho(z^0)} 2u\eta^2 \xi D_i u D_i \xi dxdt - \int\limits_{Q^+_\rho(z^0)} u\xi^2\eta^2 \chi_\Omega dxdt. \quad (2.2)$$

Using Young's inequality in the third integral of the right hand side of (2.2), we get (2.1). □

Remark. It follows from Lemma 2.2 and (1.3) that for any $\delta \in]0,1[$ we have

$$\|Du\|^2_{2,Q^+_{1-\delta}} \leqslant C_2(n,\delta) M^2, \quad (2.3)$$

$$\|D'u\|_{\infty,Q^+_{1-\delta}} \leqslant C_3(n,\delta) \|D'u\|_{2,Q^+_{1-\delta/2}} \leqslant C_4(n,\delta) M, \quad (2.4)$$

where the first inequality in (2.4) follows from the fact that $H(D'u) = \operatorname{div} f$ in Q_1^+, $f \in L_\infty(Q_1^+)$ and well-known local estimates [5, Theorem 8.1 Ch. III and Theorem 6.2 Ch. II].

Lemma 2.3. *Let v be a continuous function in $\overline{Q_1^+}$. Suppose also that v is smooth at points where it is positive and satisfies*

$$H(v) = f \quad \text{in} \quad Q_1^+ \cap \{(x,t) : v(x,t) > 0\},$$
$$v = 0 \quad \text{on} \quad \Pi \cap Q_1,$$

with $f \in L_\infty(Q_1^+)$.

There exists a positive constant C_5, depending on n only, such that

$$\frac{v(x,t)}{x_1} \leqslant C_5 \cdot \|v\|_{\infty, Q^+_{15/16}} + \|f^-\|_{\infty, Q^+_{15/16}}, \quad (x,t) \in Q^+_{1/2}. \tag{2.5}$$

Proof. Let $M_0 = \|v\|_{\infty, Q^+_{15/16}}$.

For arbitrary $(y', t^0) \in \Pi \cap Q_{1/2}$ we introduce the set

$$\mathcal{D}(y', t^0) = \Big\{ \ (x,t) : 0 < x_1 < \frac{1}{2}, \quad |x' - y'| < \frac{1}{4}, \quad t^0 - \frac{1}{16} < t < t^0, \ v(x,t) > 0 \ \Big\},$$

and consider in $\mathcal{D}(y', t^0)$ the barrier function

$$w(x,t) = M_0(16|x' - y'|^2 + N_1 x_1 - N_2 x_1^2) + 16 M_0 (t^0 - t),$$

where N_1 and N_2 are positive constants which will be choosen later.

Next, an elementary calculation gives that

$$H(v - w) \geqslant 0 \quad \text{in} \quad \mathcal{D}(y', t^0),$$
$$v - w|_{\partial' \mathcal{D}(y', t^0)} \leqslant 0,$$

if $N_2 = 16(n+1) + \frac{\|f^-\|_{\infty, Q_1^+}}{2M_0}$, $N_1 = \frac{1}{2} N_2 + 2$.

Applying the maximum principle to the difference $v - w$ in the set $\mathcal{D}(y', t^0)$ we get the estimate

$$v(x,t) \leqslant w(x,t) \quad \forall (x,t) \in \mathcal{D}(y', t^0).$$

In particular,

$$v(x_1, y', t^0) \leqslant w(x_1, y', t^0) = M_0(N_1 x_1 - N_2 x_1^2) \leqslant M_0 N_1 x_1, \quad 0 < x_1 < \frac{1}{2}. \tag{2.6}$$

Since $(y', t^0)$is an arbitrary point of $\Pi \cap Q_{1/2}$, we immediately obtain the desired inequality (2.5) from (2.6). $\square$

Now we formulate an extremely important tool used to prove Main Theorem. This is the following monotonicity formula due to L. A. Caffarelli and C. Kenig.

Lemma 2.4. *Let ψ be a time-independent cut-off function such that $\psi \equiv 1$ in $B_{1/8}$, it has support in $B_{1/4}$ and it is C^2 in $B_{1/2}$.*

Let h_1, h_2 be non-negative, sub-caloric and continuous functions in $Q_{1/2}$, satisfying

$$h_1(0,0) = h_2(0,0) = 0,$$

$$h_1(x,t) \cdot h_2(x,t) = 0 \quad in \quad Q_{1/2}.$$

Then, there exists $C_6 = C_6(n) > 0$ such that, for $0 < r \leqslant \frac{1}{4}$, we have:

$$\frac{1}{r^2} \int_{-r}^{0} \int_{\mathbb{R}^n} |D(h_1\psi)|^2 G(x,-s)\,dxds \int_{-r}^{0} \int_{\mathbb{R}^n} |D(h_2\psi)|^2 G(x,-s)\,dxds \leqslant$$
$$\leqslant C_6\{\|h_1\|^4_{2,Q_{1/2}} + \|h_2\|^4_{2,Q_{1/2}}\}, \quad (2.7)$$

where

$$G(x,s) = \frac{\exp(-|x|^2/4s)}{(4\pi s)^{n/2}} \quad for \quad s > 0 \quad and \quad G(x,s) = 0 \quad for \quad s \leqslant 0.$$

For a proof of this lemma we refer the reader to (the proof of) Theorem 2.3.1 in [1].

Remark. In some situations we need to use the following version of the monotonicity formula, which can be derived easily from (2.7) by rescaling:

$$\frac{1}{r^2} \int_{-r}^{0} \int_{\mathbb{R}^n} |D(h_1(x,s)\psi(x/\rho))|^2 G(x,-s)\,dxds \times$$
$$\times \int_{-r}^{0} \int_{\mathbb{R}^n} |D(h_2(x,s)\psi(x/\rho))|^2 G(x,-s)\,dxds \leqslant$$
$$\leqslant \frac{C_6}{\rho^{2n+8}}\{\|h_1\|^4_{2,Q_{\rho/2}} + \|h_2\|^4_{2,Q_{\rho/2}}\} \quad \forall \rho \leqslant 1. \quad (2.8)$$

Before stating the next results we need to extend $u \in P_1^+(M)$ across the plane Π to the whole unit cylinder Q_1 by the odd reflection, i.e., by setting it as $-u(-x_1, x_2, \ldots, x_n, t)$ for $(x,t) \in Q_1$, $x_1 < 0$. For simplicity of notation, we will use the same letter u for the extended function.

Lemma 2.5. *Let $u \in P_1^+(M)$, $z^0 = (x^0, t^0) \in \Gamma(u) \cap \overline{Q}^+_{1/4}$ and e be an arbitrary unit vector in $\mathbb{R}^n$, orthogonal to e_1 (x_1-axis).*

Suppose that we extend u to the whole cylinder Q_1 by the odd reflection. Then, for $v = D_e u$ and for all $r \leqslant 1/2$, we have

$$\Phi(r, v, \psi) := \frac{1}{r^4} \int\limits_{t^0 - r^2}^{t^0} \int\limits_{\mathbb{R}^n} |D(v^+(x,t)\psi(x - x^0))|^2 G(x - x^0, t^0 - t)\,dxdt \times$$
$$\times \int\limits_{t^0 - r^2}^{t^0} \int\limits_{\mathbb{R}^n} |D(v^-(x,t)\psi(x - x^0))|^2 G(x - x^0, t^0 - t)\,dxdt \leqslant$$
$$\leqslant C_7(n) M^4, \qquad (2.9)$$

where ψ is the same as in Lemma 2.4

Proof. Applying Lemma 2.4 to the functions

$$h_1(x,t) = v^+(x + x^0, t + t^0), \quad h_2(x,t) = v^-(x + x^0, t + t^0)$$

and making change of the variables, we get

$$\Phi(r, v, \psi) \leqslant 2C_6 \|Du\|^4_{2, Q_{1/2}(z^0)}.$$

Now, by inequalities (2.3) we immediately arrive at (2.9). □

Lemma 2.6. *Assume that we extend $u \in P_1^+(M)$ to the whole cylinder Q_1 by the odd reflection. Suppose also that $z = (x,t) \in \Lambda(u) \cap Q^+_{1/4}$, $R(z) = x_1 = \operatorname{dist}\{z, \Pi\}$ and*

$$\mathcal{M}(\rho, z, u) = \sup_{Q_\rho(z)} |u| \quad \text{for} \quad R(z) \leqslant \rho \leqslant \frac{1}{2}.$$

Then there exists a universal constant $C_8 = C_8(n)$ such that

$$\Psi(z, u) := \sup_{\rho \in [R(z), 1/2]} \frac{\mathcal{M}(\rho, z, u)}{\rho^2} \leqslant C_6 M. \qquad (2.10)$$

Remark. Since $\frac{\mathcal{M}(\rho,z,u)}{\rho^2}$ is continuous with respect to ρ it follows that the function $\frac{\mathcal{M}(\rho,z,u)}{\rho^2}$ attains its supremum for some $\overline{\rho} \in [R(z), 1/2]$.

Proof. We argue by contradiction. Suppose (2.10) fails. Then for every $j \in \mathbb{N}$ (natural numbers) there are $u_j \in P_1^+(M)$ and $z^j = (x^j, t^j) \in \Lambda(u_j) \cap Q_{1/4}^+$ such that

$$\Psi(z^j, u_j) > j \tag{2.11}$$

(here and in the sequel we regard u_j as functions extended by the odd reflection to the whole cylinder Q_1).

Let $R(z^j) = x_1^j$ and r_j be the value of $\rho \in [R(z^j), 1/2]$ such that

$$\Psi(z^j, u_j) = \frac{\mathcal{M}(r_j, z^j, u_j)}{(r_j)^2}.$$

It is evident that inequality (2.11) gives the estimate

$$r_j < \left(\frac{\mathcal{M}(r_j, z^j, u_j)}{j}\right)^{1/2} \leqslant \sqrt{\frac{M}{j}} \to 0 \quad \text{as} \quad j \to \infty. \tag{2.12}$$

Next, we define $\widetilde{u}_j$ as

$$\widetilde{u}_j(x,t) = \frac{u_j(r_j x + x^j, r_j^2 t + t^j)}{\mathcal{M}(r_j, z^j, u_j)}$$

for $(x,t) \in Q_{\frac{1}{2r_j}}$. Then $\widetilde{u}_j$ satisfies the following properties:

$$\begin{aligned}\|H(\widetilde{u}_j)\|_{\infty, Q_{\frac{1}{2r_j}}} \leqslant \frac{r_j^2}{\mathcal{M}(r_j, z^j, u_j)} &= \\ = \frac{1}{\Psi(z^j, u_j)} &< \frac{1}{j} \to 0 \quad \text{as} \quad j \to \infty, \end{aligned}\tag{2.13}$$

where the second inequality follows from (2.11),

$$\sup_{Q_1} |\widetilde{u}_j| = 1, \tag{2.14}$$

$$\widetilde{u}_j(0,0) = |D\widetilde{u}_j(0,0)| = 0, \tag{2.15}$$

$$\widetilde{u}_j\big|_{x_1 = -\frac{R(z^j)}{r_j}} = 0 \quad \text{and} \quad 2R(z^j) \leqslant \frac{R(z^j)}{r_j} \leqslant 1. \tag{2.16}$$

Further, for $R \in [1, \frac{1}{2r_j}]$ we have

$$\|\widetilde{u}_j\|_{\infty,Q_R} \leqslant \frac{\mathcal{M}(Rr_j, z^j, u_j)}{\mathcal{M}(r_j, z^j, u_j)} =$$
$$= \left(\frac{\mathcal{M}(Rr_j, z^j, u_j)}{R^2 r_j^2}\right) \cdot \left(\frac{r_j^2}{\mathcal{M}(r_j, z^j, u_j)}\right) \cdot R^2 \leqslant$$
$$\leqslant \Psi(z^j, u_j) \cdot \frac{1}{\Psi(z^j, u_j)} \cdot R^2 = R^2, \quad (2.17)$$

where the second inequality follows from the definitions of Ψ and r_j. Observe that, according to (2.12), the radius R in (2.17) can be chosen as large as desired, if j tends to infinity.

Now, by (2.13)–(2.17) we will have a subsequence of $\widetilde{u}_j$ weakly converging in $W^{2,1}_{q,\mathrm{loc}}(\mathbb{R}^{n+1})$, $q > n$, to a caloric function u_0, satisfying

$$|u_0(x,t)| \leqslant 1 + |x|^2 + |t|,$$
$$u_0(0,0) = |Du_0(0,0)| = 0, \quad (2.18)$$
$$\sup_{Q_1} |u_0| = 1, \quad (2.19)$$
$$u_0|_{x_1=-d_0} = 0, \quad (2.20)$$

where

$$d_0 = \lim_{j\to 0} \frac{R(z^j)}{r_j}, \quad d_0 \in [0,1].$$

According to Lemma 2.1, there exist constants a^{ij}, a^i, a_0 such that

$$u_0(x,t) = a^{ij}x_i x_j + 2\left(\sum_{i=1}^{n} a^{ii}\right)t + a^i x_i + a_0.$$

Obviously, conditions (2.18) and (2.20) imply

$$a_0 = a^i = a^{\tau\mu} = 0, \quad \text{for all indices} \quad i, \tau, \mu,$$

and

$$\sum_{j=1}^{n} a^{jj} = 0.$$

Consequently, $a^{11} = 0$ and the function u_0 takes the form

$$u_0(x,t) = 2a^{1\tau}x_1 x_\tau. \quad (2.21)$$

Applying Lemma 2.3 to the functions $v_j(x,t) = (D_\tau u_j(x,t))^{\pm}$ we obtain

$$\sup_{(x,t)\in Q^+_{r_j}(z^j)} |D_\tau u_j(x,t)| \leqslant 2C_5 r_j \|D_\tau u_j\|_{\infty, Q^+_{15/16}} \leqslant 2C_5 C_4 M r_j, \quad (2.22)$$

where the second inequality follows from (2.4)

By (2.22) and (2.11) it is evident that

$$\begin{aligned}\sup_{Q_1^+} |D_\tau \widetilde{u}_j| &\leqslant \frac{\sup_{Q_1^+} |D_\tau u_j(r_j x + x^j, r_j^2 t + t^j)|}{\mathcal{M}(r_j, z^j, u_j)} = \\ &= \frac{r_j}{\mathcal{M}(r_j, z^j, u_j)} \cdot \sup_{(\widetilde{x},\widetilde{t})\in Q^+_{r_j}(z^j)} |D_\tau u_j(\widetilde{x},\widetilde{t})| \leqslant \\ &\leqslant \frac{2C_5 C_4 M r_j^2}{\mathcal{M}(r_j, z^j, u_j)} < \frac{2C_5 C_4 M}{j} \to 0 \quad \text{as} \quad j \to 0.\end{aligned}$$

This gives

$$D_\tau u_0 \equiv 0 \quad \text{in} \quad Q_1^+. \quad (2.23)$$

Combining (2.21) and (2.23), we get $a^{1\tau} = 0$ and, hence $u_0 \equiv 0$ which contradicts (2.19). The proof is completed. □

Lemma 2.7. *Let $u \in P_1^+(M)$, $z^0 = (x^0, t^0) \in \Gamma(u) \cap Q^+_{1/4}$.*

Then, for $v = D_1 u$ and for all $r \leqslant R_0 = x_1^0 = \operatorname{dist}\{z^0, \Pi\}$, we have

$$\Phi(r, v, \psi_{R_0}) \leqslant C_9(n) M^4, \quad (2.24)$$

where $\psi_{R_0}(x) = \psi(\frac{x}{R_0})$ whereas ψ and Φ are the same as in Lemma 2.4 and Lemma 2.5, respectively.

Remark. Contrary to Lemma 2.5, we can apply the monotonicity formula to $\Phi(\rho, D_1 u, \widetilde{\psi})$ only if $\rho \leqslant R_0$ and $\operatorname{supp}\{\widetilde{\psi}\} \subset Q_1^+$ since, for u extended by the odd reflection to the whole unit cylinder, $(D_1 u)^-$ may not be sub-caloric in Q_1. As a result, case $v = D_1 u$ needs a more detailed anlysis.

Proof. Setting in (2.8) $\rho = R_0$,

$$h_1(x,t) = v^+(x + x^0, t + t^0), \quad h_2(x,t) = v^-(x + x^0, t + t^0)$$

and making change of the variables, we get

$$\Phi(r, v, \psi_{R_0}) \leqslant \frac{2C_6}{R_0^{2n+8}} \|Du\|^4_{2, Q_{\frac{R_0}{2}}(z^0)}. \quad (2.25)$$

Further, inequalities (2.1), (2.10) and (1.3) imply

$$\|Du\|^4_{2,Q_{R_0/2}(z^0)} \leqslant 2C_1^2 R_0^{2n}\{R_0^4\|u\|^2_{\infty,Q_{R_0}(z^0)} + \|u\|^4_{\infty,Q_{R_0}(z^0)}\} \leqslant \\ \leqslant 2C_1^2 C_8^2 M^2\{1+M^2C_8^2\}R_0^{2n+8}. \quad (2.26)$$

The proof is completed by combining (2.25) and (2.26). □

§3. Proof of Main Theorem

As in Section 2, we extend $u \in P_1^+(M)$ by the odd reflection to the whole cylinder Q_1 and preserve the notation u for the extended function.

Lemma 3.1. *There exists a universal constant $C_{10} = C_{10}(n) > 2$ such that, if $u \in P_1^+(M)$ and $z = (x,t) \in \Gamma(u) \cap \overline{Q}^+_{1/4}$, then*

$$\sup_{Q_r(z)} |u| \leqslant C_{10} M r^2 \quad \textit{for} \quad Q_r(z) \subset Q_1. \quad (3.1)$$

Proof. Define

$$M_k(z,u) = \sup_{Q_{2^{-k}}(z)} |u|, \quad k \in \mathbb{N}.$$

To obtain (3.1) it is sufficient to show that for some $C > 0$

$$4^{k+1}M_{k+1}(z,u) \leqslant \max\{4M_1(z,u), \dots, 4^k M_k(z,u), CM\} \\ \text{for all} \quad k \in \mathbb{N}. \quad (3.2)$$

Suppose (3.2) fails. Then for every $j \in \mathbb{N}$ there exist $u_j \in P_1^+(M)$, $z^j = (x^j, t^j) \in \Gamma(u_j) \cap \overline{Q}^+_{1/4}$ and $k_j \in \mathbb{N}$ such that

$$4^{k_j+1}M_{k_j+1}(z^j, u_j) > \max\{4M_1(z^j,u_j), \dots, 4^{k_j}M_{k_j}(z^j,u_j), j\}. \quad (3.3)$$

We regard u_j, in (3.3) and below, as functions extended to the whole Q_1 by the odd reflection.

Obviously, (3.3) and inequalities $M_{k_j}(z^j, u_j) \leqslant M$ imply

$$k_j \to \infty \quad \text{as} \quad j \to \infty. \quad (3.4)$$

Define now

$$\widetilde{u}_j(x,t) = \frac{u_j(2^{-k_j}x + x^j, 2^{-2k_j}t + t^j)}{M_{k_j+1}(z^j, u_j)} \quad \text{in} \quad Q_{2^{k_j-1}}.$$

Then it follows from the definition of $P_1^+(M)$ and (3.3) that

$$\widetilde{u}_j(0,0) = |D\widetilde{u}_j(0,0)| = 0, \tag{3.5}$$

$$\sup_{Q_{1/2}} |\widetilde{u}_j| = 1, \tag{3.6}$$

$$\|H(\widetilde{u}_j)\|_{\infty,Q_{2^{k_j-1}}} \leqslant \frac{1}{4^{k_j} M_{k_j+1}(z^j,u_j)} \leqslant \\ \leq \frac{4^{k_j+1}}{4^{k_j} j} \to 0 \quad \text{as} \quad j \to \infty. \tag{3.7}$$

Observe also that for integers m satisfying $1 \leqslant m \leqslant k_j - 1$ we have

$$\|\widetilde{u}_j\|_{\infty,Q_{2^m}} \leqslant \frac{M_{k_j-m}(z^j,u_j)}{M_{k_j+1}(z^j,u_j)} = \\ = 4^{m+1} \frac{4^{k_j-m} M_{k_j-m}(z^j,u_j)}{4^{k_j+1} M_{k_j+1}(z^j,u_j)} \leqslant 4 \cdot 2^{2m}, \tag{3.8}$$

where the second inequality follows from (3.3).

Now, by (3.4)–(3.8) we infer that a subsequence of $\widetilde{u}_j$ weakly converges in $W^{2,1}_{q,\mathrm{loc}}(\mathbb{R}^{n+1})$, $q > n$, to a caloric function u_0. Moreover, the limit function u_0 satisfies

$$u_0(0,0) = |Du_0(0,0)| = 0, \tag{3.9}$$

$$\sup_{Q_{1/2}} |u_0| = 1, \tag{3.10}$$

$$|u_0(x,t)| \leqslant 16(1 + |x|^2 + |t|).$$

Using Lemma 2.1 and equalities (3.9) we deduce that there are constants a^{ij} such that

$$u_0(x,t) = a^{ij} x_i x_j + 2\left(\sum_{i=1}^n a^{ii}\right) \cdot t. \tag{3.11}$$

We now proceed to show that u_0 is, in fact, one-space dimensional, i.e.,

$$u_0(x,t) = u_0(x_1,t) = a^{11}(x_1^2 + 2t). \tag{3.12}$$

To establish (3.12) we follow the steps from [3, Theorem 3.1].

For any fixed direction e such that $e \perp e_1$ define

$$v = D_e u_0, \quad v_j = D_e u_j, \quad \widetilde{v}_j = D_e \widetilde{u}_j.$$

Then, for a subsequence, $\widetilde{v}_j$ converges in $C_{\mathrm{loc}}(\mathbb{R}^{n+1})$ to v, where $H(v) = 0$.

Let ψ be the same cut-off function as in Lemma 2.4. Now according to Lemma 2.5, for all $j \in \mathbb{N}$ and for all $r \leqslant 1$, we have

$$\begin{aligned}
&\Phi(r, v_j, \psi) = \\
&= \left(\frac{4}{r}\right)^4 \int\limits_{t^j - r^2/16}^{t^j} \int\limits_{\mathbb{R}^n} |D(v_j^+(x,t)\psi(x - x^j))|^2 G(x - x^j, t^j - t)\,dxdt \times \\
&\quad \times \int\limits_{t^j - r^2/16}^{t^j} \int\limits_{\mathbb{R}^n} |D(v_j^-(x,t)\psi(x - x^j))|^2 G(x - x^j, t^j - t)\,dxdt \leqslant \\
&\leq C_7 M^4. \qquad (3.13)
\end{aligned}$$

Making change of the variables in (3.13) and letting $r = 2^{-k_j}$, $\psi_j(x) = \psi(2^{-k_j} x)$, we will obtain

$$\begin{aligned}
&\int\limits_{-1/16}^{0} \int\limits_{\mathbb{R}^n} |D(\widetilde{v}_j^+(x,t)\psi_j(x))|^2 G(x, -t)\,dxdt \times \\
&\quad \times \int\limits_{-1/16}^{0} \int\limits_{\mathbb{R}^n} |D(\widetilde{v}_j^-(x,t)\psi_j(x))|^2 G(x, -t)\,dxdt = \\
&= \left(\frac{2^{-2k_j}}{4M_{k_j+1}(z^j, u_j)}\right)^4 \Phi(2^{-k_j}, v_j, \psi) \leqslant C_7 \left(\frac{M}{4^{k_j+1} M_{k_j+1}(z^j, u_j)}\right)^4 \leqslant \\
&\leqslant C_7 \left(\frac{M}{j}\right)^4 \quad \text{for all} \quad j, \qquad (3.14)
\end{aligned}$$

where in the last inequality we have used (3.3).

Observe that $\psi_j \equiv 1$ in $B_{2^{k_j}/8}$ and, consequently,

$$|D(\widetilde{v}_j^\pm \psi_j)|^2 \geqslant |D\widetilde{v}_j^\pm|^2 \chi_{B_{1/8}} \quad \text{for} \quad -\frac{1}{16} \leqslant t < 0, \quad x \in B_{1/4}.$$

In addition, for $\varepsilon > 0$ (small and fixed) we have

$$G(x, -t) \geqslant N_3(n, \varepsilon) \quad \text{for} \quad -\frac{1}{16} \leqslant t < -\varepsilon, \quad x \in B_{1/8}.$$

Hence,

$$N_3(n,\varepsilon)\int\limits_{-1/16}^{-\varepsilon}\int\limits_{B_{1/8}}|D\widetilde{v}_j^{\pm}|^2dxdt\leqslant$$
$$\leqslant\int\limits_{-1/16}^{0}\int\limits_{\mathbb{R}^n}|D(\widetilde{v}_j^{\pm}\psi_j)|^2G(x,-t)dxdt. \quad (3.15)$$

Next, using (3.15) and invoking the Poincaré inequality we may reduce (3.14) to

$$\int\limits_{-1/16}^{-\varepsilon}\int\limits_{B_{1/8}}|\widetilde{v}_j^{+}-m_j^{+}(t)|^2dxdt\cdot\int\limits_{-1/16}^{-\varepsilon}\int\limits_{B_{1/8}}|\widetilde{v}_j^{-}-m_j^{-}(t)|^2dxdt\leqslant$$
$$\leqslant C_7N_3^{-2}\left(\frac{M}{j}\right)^4 \quad \text{for all} \quad j,$$

where $m_j^{\pm}(t)$ denotes the corresponding mean value of $\widetilde{v}_j^{\pm}$ on t-sections over $B_{1/8}$.

Letting j tend to infinity (and then ε tend to zero), we obtain

$$\int\limits_{Q_{1/8}}|v^{+}-m^{+}(t)|^2dxdt\cdot\int\limits_{Q_{1/8}}|v^{-}-m^{-}(t)|^2dxdt=0, \quad (3.16)$$

where $m^{\pm}(t)$ is the corresponding mean value of $v^{\pm}$ on t-sections over $B_{1/8}$.

Obviously (3.16) implies that either of $v^{\pm}$ is equivalent to $m^{\pm}(t)$ in $Q_{1/8}$, and thus independent of the spatial variables. Assume for the definiteness, that $v^{-}=m^{-}(t)$. Then $-\partial_t v^{-}=H(v^{-})=0$, i.e., m^{-} is constant in $Q_{1/8}$. Hence by the minimum principle

$$v\equiv 0 \quad \text{in} \quad Q_{1/8}. \quad (3.17)$$

Recall that $v=D_e u_0$, and e is arbitrary direction in $\mathbb{R}^n$ orthogonal to e_1. In such a manner, (3.17) in particular implies that u_0 depends only on x_1 and t, which together with (3.11) gives (3.12).

Now coming back to the sequence $z^j=(x^j,t^j)$ we see that two situations may arise:

(a) $x_1^j=0$, for all sufficiently large j;

(b) $x_1^j > 0$, at least for some subsequence of j (to shorten notation we continue to write j for a subsequence).

For case **(a)** we observe that

$$\widetilde{u}_j = 0 \quad \text{on} \quad \Pi \cap Q_{2^{k_j-1}} \quad \text{for all } j \quad \text{(sufficiently large)}$$

and, consequently,

$$u_0 = 0 \quad \text{on} \quad \Pi. \tag{3.18}$$

Equalities (3.12) and (3.18) imply $u_0 \equiv 0$ and we obtain the contradiction with (3.10).

For case **(b)** we need a more detailed analysis. Now one defines

$$v_{1,j} = D_1 u_j, \quad R_j = x_1^j, \quad \psi_{R_j}(x) = \psi(\frac{x}{R_j}).$$

By Lemma 2.7, for $k_j > -\log_2 R_j$, we have

$$2^{4k_j} \int\limits_{t^j-2^{-2k_j}}^{t^j} \int\limits_{\mathbb{R}^n} |D(v_{1,j}^+(x,t)\psi_{R_j}(x-x^j))|^2 G(x-x^j, t^j-t)dxdt \times$$
$$\times \int\limits_{t^j-2^{-2k_j}}^{t^j} \int\limits_{\mathbb{R}^n} |D(v_{1,j}^-(x,t)\psi_{R_j}(x-x^j))|^2 G(x-x^j, t^j-t)dxdt \leqslant$$
$$\leqslant \Phi(2^{-k_j}, v_{1,j}, \psi_{R_j}) \leqslant C_9 M^4. \tag{3.19}$$

Making change of the variables in the left hand side of (3.19) and repeating step by step all the arguments used to derive (3.12), we end up with $D_1 u_0 = 0$ in $Q_{1/8}$. The latter, combined with (3.12), gives $u_0 \equiv 0$ and we again obtain the contradiction with (3.10).

The proof is completed. □

Before starting to prove Main Theorem we recall the well-known fact.

Fact: Let $z = (x,t) \in \mathbb{R}^{n+1}_+$, $R > 0$, $H(v) = 1$ in $Q_R^+(z)$ and $v = 0$ on $Q_R(z) \cap \Pi$ (if the latter set is not empty).

Then

$$\sup_{Q_{R/2}^+(z)} \{|D_i D_j v| + |\partial_t v|\} \leqslant C_{11}(n)\{1 + R^{-2} \sup_{Q_R^+(z)} |v|\}. \tag{3.20}$$

Inequality (3.20) can be derived easily by rescaling from the standard parabolic estimates in the unit cylinder (see, Exercise 10.2.6, [4]).

Proof of Main Theorem. Let $z^0 = (x^0, t^0) \in \Gamma(u) \cap \overline{Q}^+_{1/4}$. By Lemma 3.1

$$|u(x,t)| \leqslant C_{10} M\{|x-x^0|^2 + (t-t^0)\} \quad \text{for} \quad (x,t) \in Q_1 \cap \{t \le t^0\}. \tag{3.21}$$

To obtain a similar estimate for $\{t > t^0\}$ we follow [3, Theorem 3.1]. Consider the barrier function

$$w(x,t) = C_{10} M\{|x-x^0|^2 + (2n+1)(t-t^0)\}.$$

One can easily verify that

$$|u(x,t)| \leqslant w(x,t) \quad \text{in} \quad Q_1 \cap \{t > t^0\}. \tag{3.22}$$

Let us introduce now the parabolic distance to $\Gamma(u)$

$$d(x,t) = \inf_{(y,\tau)\in\Gamma(u)} (|x-y|^2 + |t-\tau|)^{1/2}.$$

Then, combining (3.21) and (3.22) we have

$$|u(x,t)| \leqslant N_4(n) M d^2(x,t) \quad \text{for all} \quad (x,t) \in Q^+_{1/8}. \tag{3.23}$$

Now, we fix $(x,t) \in Q^+_{1/8} \cap \Omega$ and consider $Q^+_{d/2}(x,t)$ with $d = d(x,t)$. Since $H(u) = 1$ in $Q^+_{d/2}(x,t)$ and $u = 0$ on Π, we can apply (3.20) for $v = u$, $R = d/2$ and obtain

$$|D_i D_j u(x,t)| + |\partial_t u(x,t)| \leqslant N_5(n)(1 + d^{-2} \sup_{Q^+_{d/2}(x,t)} |u|). \tag{3.24}$$

Combining (3.23), (3.24) and (1.3) we get (1.4). □

Main Theorem has the following obvious implication.

Corollary 3.2. *Let $u \in P^+_\infty(M)$. Then*

$$|D_i D_j u(x,t)| + |\partial_t u(x,t)| \leqslant 3C_0 M \quad \textit{for all} \quad (x,t) \in \mathbb{R}^{n+1}_+.$$

In particular, if $(x^0, t^0) \in \Gamma(u)$, then

$$|u(rx + x^0, r^2 t + t^0)| \leqslant C_{12}(n) M r^2$$
$$\forall (x,t) \in Q_1 \cap \{x_1 > -\frac{x^0_1}{r}\} \quad \textit{and} \quad r > 0.$$

Proof. The proof of this statement is completely analogous to (that of) Corollary 3.2 [3]. For the reader's convenience and for the completeness, we provide it here.

Define

$$u_r(x,t) = \frac{u(rx, r^2 t)}{r^2} \quad \text{in} \quad Q_1^+, \quad r > 1.$$

Condition $u \in P_\infty^+(M)$ implies $\|u_r\|_{\infty, Q_1^+} \leqslant 3M$. Hence by Main Theorem we will have

$$\sup_{Q_{1/8}^+} (|D_i D_j u_r(x,t)| + |\partial_t u_r(x,t)|) \leqslant 3C_0 M,$$

i.e.,

$$\sup_{Q_{r/8}^+} (|D_i D_j u(x,t)| + |\partial_t u(x,t)|) \leqslant 3C_0 M.$$

Letting r tend to infinity we will have the first statement of the corollary.

The second part follows by using the first part and that

$$|u(rx + x^0, r^2 t + t^0)| \leqslant$$
$$\leqslant r^2 \left(\sup_{Q_r^+(x^0,t^0)} (|D_i D_j u(x,t)| + |\partial_t u(x,t)|) \right) \leqslant C_{12}(n) M r^2.$$

□

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October 16, 2000