Записки научных
семинаров ПОМИ
Том 267, 2000 г.

V. O. Manturov

CHORD DIAGRAMS, D-DIAGRAMS, AND KNOTS

Abstract. A method of representing knots, links, and singular knots and links by means of words in a finite alphabet and so-called d-diagrams is given. Also a representation of the chord diagram algebra in terms of words in a finite alphabet is considered. This method, unlike coding by Gauss diagrams, allows to consider knots and links simultaneously. An algorithm for recognition diagrams of classical knots in the terms of d-diagrams is given.

§1. Introduction

1.1. d-Diagrams. In the present work we describe a new way for encoding knots[1] by words in the alphabet of four letters [,], (, and). The main idea behind such an isomorphism is the geometrical construction of correspondence between knots and the so-called atoms and d-diagrams (see [3, 4]).

This construction allows us to give a simple combinatorial coding of knots. Using this encoding, we can construct a simple algebraic description of the semigroup of knots[1] elements of the semigroup are represented by means of equivalence classes of some words in the alphabet of four brackets. Now let us describe this encoding and the semigroup.

Geometrically, the knot semigroup K can be described as follows:

(i) Elements of K are isotopy classes of links with chosen component, an orientation of this component, and a point on it.

(ii) The unit of K is the trivial knot isotopy class.

(iii) For two elements $A, B \in K$, the product AB is defined as follows. Let A' and B' be representatives of the classes A and B respectively. Let A_1, A_2 be points on A' in a small neighborhood of the fixed point such that A_1 is located after A_2 with respect to the orientation of the component. Analogously, choose B_1, B_2 on the knot B'. Place A' and B' in space in such a way that the initial points of A' and B' are near each other. Besides, let us require that the intersection of the knots A' and

[1]By a "knot" we mean a link with a fixed point on an oriented component, for example, an oriented knot with a fixed point

B' with a small neighborhood U of corresponding fixed point represent two arcs. Finally, let us require the existence of a line l on the projection plane intersecting neither A' nor B' such that projections of A' and B' lie on the different sides of l.

Delete the arc A_1A_2 from A' and the arc B_1B_2 from B' and connect A_1 with B_2 and B_1 with A_2 by a straight interval. The class of this link where the fixed point is $A_1 = B_2$ and the orientation in the neighborhood of this point coincides with that for A and B, is called AB, see Fig. 1.

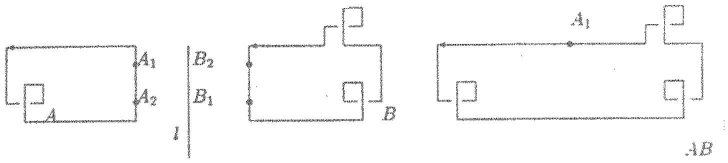


Fig. 1. Product of knots AB.

Remark. It is easy to see that the product does not depend on the fixed points of two knots, but only on the selected components.

Definition 1.1.1. *A word A in the four-bracket alphabet (,), [,] is a proper double-bracket structure if the words $A_{[]}$ and $A_{()}$ obtained from A by means of deleting round (resp. square) brackets are proper bracket structures in the ordinary sense.*

Now we define the semigroup G of proper double-bracket structures as follows.

(i) The elements of the semigroup G are equivalence classes of proper double-bracket structures by relations (1)–(7), enumerated below. In the sequel, we call them the *global relations*.

Each of the relations (2)–(6) means that some word I is a proper double-bracket structure and one of its subwords has the type described in the right (left) part of the global relation. Then the word obtained by replacing I by the subword J described in the other part of the global relation is said to be equivalent to the initial word. Global relations can be used only if the initial word has a proper double-bracket structure. In this case the obtained word is a proper double-bracket structure too.

(1) $() = [] = 1$.

(2) $([\,)A_1A_2] = ([\,)(A_2A_1)']$; here one of the words A_i, $i = 1, 2$ has a proper (,)-bracket structure; the prime means that if some bracket [from

A_1 was closed by] from A_2, then in $A_2{A_1}'$ the corresponding bracket is opening in A_2 and the bracket in A_1 is closing.

(3) $([\,)(\,]A) = A$; here A has a proper (,)-structure.

(4) $([(A))B] = [([A)B]]$; here the words A and B have proper (,)-structures; the word AB has a proper [,]-structure.

(5)

$$A[((\,]B)C)$$

$$= A((B[[\,)]](C)[[\,)]]$$

here the words A, B, C have proper (,)-structures; the word ABC has proper [,]-structure.

(6) $ABCD = ([\,)A[(\,]D'\,]C')B'$; here AD has a proper [,]-structure and the word DC has a proper (,)-structure. Here B', C', D' means that if an opening bracket from one word is closed by a closing bracket from another word then in the right part we write the closing bracket instead of opening and vice versa. For example, if in the left part a bracket [from B is closed by] from C, then in the right part the opening bracket from C is closed by] from B.

(7) $A = A^*$, where A is a proper double-bracket structure and A^* is a proper double-bracket structure, obtained from A by replacing round brackets by square brackets and vice versa.

(8) $(A)B = (B')A'; A(B) = B'(A')$.

(ii) The unit of G is the empty word.

(iii) The semigroup operation of G is concatenation.

The main result of this work is the construction of a canonical isomorphism between the algebraic semigroup G and the geometrically-defined semigroup of knots K (proof in 2.4).

An important step in this construction is the representation of semigroup K in terms of so-called d-diagrams.

There is an important problem — to find a semigroup H isomorphic to G whose elements are equivalence classes of some words in a finite alphabet by *local* relations, i.e., relations in the proper sense.

In §3, we consider virtual knots (see [2]) and try to apply our geometrical constructions to them. The basic notions used in this paper are Reidemeister moves (cf. [6]), vertical atoms [3], and chord diagrams (as a combinatorial object, unrelated to singular knots) [3].

1.2. The Vasil'ev knot and link invariants. Now we give definitions of the Vasil'ev link and knot invariants.

Definition 1.2.1. *A singular link is a smooth immersion of a finite set of circumferences to $\mathbb{R}^3$ such that there is only finite numbers of points in $\mathbb{R}^3$ having more than one preimage and each point having more than one preimage is the image of two points lying on transversally intersecting arcs. A singular knot is a knot of order K if the number of double points equals K.*

If the number of components equal one, we say that a singular knot is given, i.e., a singular knot is a one-component singular link.

A *link invariant* is a function on links, not changing under isotopies in $\mathbb{R}^3$.

Given a link invariant L. We define its derivative[1] on singular links of the order 1 as follows:

The relation above means that the value of derivative on a singular link can be defined by the values on links distinguishing from the initial link only at one point.

Analogously, one can define an n-derivative on singular links of the order n by using $n-1$-derivative on singular links of the order $n-1$.

Definition 1.2.2. *A link invariant L said to be the Vasil'ev invariant of the order n if the value of its $n+1$-derivative equals zero on all links.*

We denote the space of the Vasil'ev invariants of the order n by V_n.

§2. Constructing the semigroup G

2.1. Basic notions. Following [1, 3, 5], we give the definitions of an atom, a vertical atom, and an f-graph that will be necessary for constructing G.

Definition 2.1.1. *An atom (introduced by Fomenko [1]) is a pair $(f^{-1}(c-\varepsilon, c+\varepsilon), f^{-1}(c))$, where f is a Morse function on a closed 2-manifold, c is its critical level, ε is a small number such that there are no other critical points of f in $f^{-1}(c-\varepsilon, c+\varepsilon)$ except for those lying on the critical level.*

Remark. In fact, for the definition of an atom, we must know only the behavior of f only on the first component of this atom, but not on M.

[1] For the sake of simplicity, we use the same letter L for the derivative

The second component of an atom (critical level) is called the *frame of the atom.* The frame is a graph; all its vertices have valency 4.

In the sequel, the set of circumferences $f^{-1}(c+\varepsilon, c+\varepsilon)$ is called *supercritical level of the atom.*

An atom is called *orientable* if its first component is orientable as a manifold with boundary.

Atoms are usually considered up to a natural equivalency that is called isomorphism. Two atoms are called *isomorphic* if there exists a diffeomorphism of their first components, mapping the second component of one atom to that of the other atom and preserving the direction of the function growth.

Definition 2.1.2 [3]**.** *A vertical atom is an atom isomorphic to an atom given by the third projection function on a 2-manifold in* $\mathbb{R}^3$.

Important remark. Each vertical atom is orientable.

Considering atoms up to isomorphism, one can obtain it by means of its frame with a special structure.

Definition 2.1.3. *A graph of valency* 4 *(atom's frame) is said to have a B-structure if four semiedges, outcoming from each vertex are divided into two pairs, called opposite and at each vertex two pairs of adjacent (non-opposite) edges are chosen.*

A frame with B-structure on it uniquely generates an atom up to the isomorphism. Here B-structure of the frame is also called the B-structure of the atom.

Definition 2.1.4. *A chord diagram is called d-diagram if its chords can be divided into two sets of chords such that chords of the same set do not intersect each other.*

2.2. Constructing the correspondence between knots and d-diagrams. We need some definitions and results obtained in [5, 3, 4].

It is easy to see that an atom is vertical if and only if its frame is embeddable in plane in such a way that the initial and the induced orders of opposite edges coincide [3].

The correspondence between vertical atoms and knots can be defined as follows [3]. Given a vertical atom V. Embed its frame to plane in such a way that the order of opposite edges (A-structure) stays the same. Four semiedges outcome from each vertex of the embedded frame. We transform this frame to a diagram of a knot. This knot will correspond to our atom. To do it, we show, which pair of opposite edges generates undercrossing, and which one – overcrossing. This can be defined by the rule: moving from the edge of "undercrossing" to the edge of the adjacent "overcrossing" is clockwise.

The theorem on atoms and knots in [3] states that the isotopy class of knot, obtained from an atom, does not depend on an embedding with preserving the order of opposite edges.

Thus an isotopy class of knot, corresponding to an atom, is well-defined.

Definition 2.2.1. *The obtained knot is called the knot corresponding to vertical atom V. All vertical atoms, to which K corresponds, are called atoms generated by knot K.*

The back operation is also defined: for each planar knot diagram one can construct a vertical atom.

It turns out [5, 4] that there exists a one-to-one correspondence between orientable atoms and the so-called polychord diagrams. A polychord diagram is a diagram consisting of some oriented circumferences and chords connecting points of these circumferences (the number of these circles is finite, each point of a circumference corresponds to no more than one circle), see Fig. 2. Analogously, diagrams with labels on nonoriented edges correspond to nonorientable atoms (cf. the notion of f-graph in [5]).

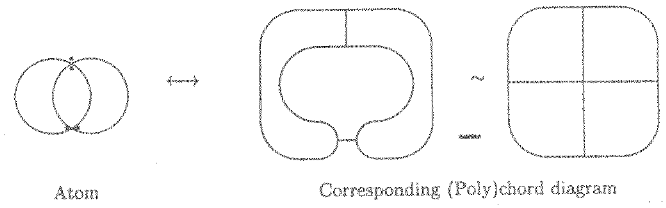


Fig. 2.

Given an orientable atom V, the polychord diagram D corresponding to V can be constructed as follows: consider the supercritical circumferences of V, i.e., the connected components of the supercritical level and orient them compatibly. In the neighborhood of each vertex of V we put a chord, connecting arcs of the supercritical circumferences. The obtained graph, consisting of circumferences and chords is called *polychord diagram, corresponding to the atom V*.

Remark. The link corresponding to the atom shown in Fig. 2 consists of two unlinked circles with two intersection points on its projection.

Theorem 2.2.1. *For each homotopy class of a non-oriented classical knot (or link) K without fixed points there exists a vertical atom V, generated by a knot from this class such that the polychord diagram, corresponding to V, is a diagram with one circumference, i.e., diagram in the proper sense.*

Proof. Consider some atom generated by K and its polychord diagram D. Performing second Reidemeister move, we do not change the isotopy type of the knot, but this move can decrease the total number of circumferences of the corresponding polychord diagram until this quantity equals one. $\bullet$

Definition 2.2.2. *A chord diagram D is called d-diagram if chords of D can be divided in two sets of non-intersecting chords.*

Theorem 2.2.2 [4]. *If an atom V corresponds to a classical chord diagram C then (the atom V is vertical) $\Longleftrightarrow$ (the diagram C is a d-diagram).*

Now we construct the semigroup G. First we construct the semigroup D isomorphic to G, in the terms of d-diagrams, Then we reconstruct D to G in the terms of proper double-bracket structures.

Given a knot K with a fixed point. The set of d-diagrams with fixed point whose atoms generate knot K with its point, are called *d-diagrams, corresponding to K*.

The d-diagram with the empty set of chords corresponds to the trivial knot.

Thus we obtain that for each knot with a fixed point there is a corresponding d-diagram (with a fixed point); besides, we can recover a knot by each d-diagram corresponding to it.

Definition 2.2.3. *Two d-diagrams are called equivalent, if they correspond to isotopic knots.*

Definition 2.2.4. *Let D be the semigroup of all equivalence classes of d-diagrams with fixed point; semigroup operation in D is defined as follows. Given two d-diagrams D_1 and D_2, described as sets of point pairs on the circle. These pairs are connected by chords. They are given by angles between Ox and radius-vectors to them. Thus $x_{1,i}$, $x_{2,i}$ are coordinates of* *chords from D_1 and $y_{1,i}$, $y_{2,i}$ are coordinates of chords from D_2. Consider the diagram whose chords connect points with coordinates $x_{1,i}/2$, $x_{i,2}/2$ and also $y_{1,i}/2+\pi$, $y_{i,2}/2+\pi$. This diagram defines the product D_1D_2, see Fig. 3.*

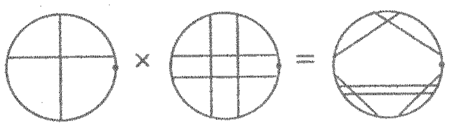


Fig. 3. Product of d-diagrams.

Thus we obtain the following theorem.

Theorem 2.2.3. *The semigroup K is isomorphic to D.* •

2.3. Interpretation of d-diagrams in the terms of proper bracket structures. Consider a d-diagram D and an arbitrary division of its chords set into two subsets of non-intersecting chords. We start to walk from the fixed point of D along the circumference counterclockwise. For each chord *the first* vertex is that we meat earlier.

Passing through a first chord vertex, we write an opening bracket and meeting a second chord vertex, we write a closing vertex. Here we use round brackets for the chords of the first set and square brackets for that of the second set. Returning to the initial point, we obtain a word $A(D)$. It is obvious that $A(D)$ is a proper double-bracket structure.

Definition 2.3.1. *All proper double-bracket structures $A(D)$, obtained as $A(D)$ above are called structures, corresponding to the chord diagram D with fixed point.*

Example. See Fig. 4.

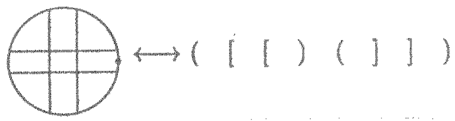


Fig. 4. Correspondence between d-diagram and a proper double-bracket structure.

It is easy to see that the back statement holds: each proper double-bracket structure generates a d-diagram with a fixed point.

Consider a knot K with a fixed point.

Definition 2.3.2. *Let $\{D\}$ be the set of all d-diagrams with a fixed point, corresponding to knot K. Each proper double-bracket structure, corresponding to a d-diagram from $\{D\}$, is called a proper double-bracket structure, corresponding to K.*

Thus we obtain the following:

1) All isotopy types of knots can be encoded by proper double-bracket structures.

2) For each proper double-bracket structure there exists a knot, corresponding to it.

3) The trivial knot isotopy class corresponds to the empty word.

4) Semigroup operation in K corresponds to the concatenation in the semigroup of proper double-bracket structures.

Definition 2.3.3. *Two proper double-bracket structure are called equivalent if they generate isotopic knots.*

Examples. In G the right trefoil knot is generated by ([[[[ )]]]], see Fig. 5, the left trefoil knot is generated by (([[ ))]].

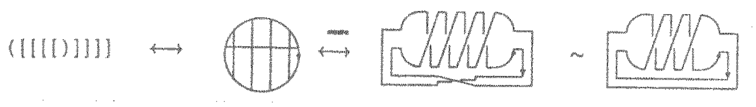


Fig. 5. Right trefoil knot.

Remark 2.3.1. Each classical knot can be obtained from a rectangular loop inside the first coordinate quadrant of Oxy. Here we consider only loops outcoming from $(0,0)$ and coming to $(0,0)$ such that all its links are horizontal or vertical unit intervals. Besides, each such loop generates a knot.

Really, each proper double-bracket structure can be treated as such a loop: (means one step forward,) is backward, [is one step to the right, and] is a step to the left. •

2.4. Constructing the isomorphism between G and K. Thus we know that there exists a bijection between isotopy equivalence classes of knots and equivalence classes of proper double-bracket structures; here semigroup operation in G corresponds to that in K. We must only prove that two proper double-bracket structures are equivalent if and only if one of them can be transformed to the other by means of a finite sequence of the global relations (1)–(8), described in the introduction.

It is obvious that two proper double-bracket structures generate the same d-diagram if and only if one can be transformed to the other by means of move (7). Besides, by using the move (8) we can choose the initial point of the knot everywhere.

Thus it is left to prove just the following fact:

Statement 2.4.1. *Two d-diagrams A and B generate equivalence class of the same knot if and only if there exist corresponding proper double-bracket structures a and b and a sequence of the global moves* (1)–(6), *transforming a to b.*

For the proof, see [7].

Given a proper double-bracket structure D. Consider the following set of brackets of D: the first opening bracket A_1, the bracket A_2, which is following[2] the bracket, closing A_1, the bracket A_3, preceding the bracket corresponding to A_2 etc. The bracket A_n follows the bracket, corresponding to A_{n-1} for odd n and precedes bracket corresponding to A_{n-1} for even n. By a_i we mean the pair of adjacent brackets: A_{i+1} and bracket, corresponding to A_i.

Definition 2.4.2. *A proper double-bracket structure D is called good if there is no repetitions among a_i until all pairs of adjacent brackets are among a_i.*

Example. Proper double-bracket structure $[a_8[a_5(a_2(a_7]a_4]a_1{=}a_9)a_6)a_3$ is good and $(a_4[[a_2[)a_1 = a_5]]a_3]$ is not good.

It is easy to see that good proper double-bracket structures encode all knots, i.e., links with one component (a_i shows the circuit of the knot). Herewith the relations (1)–(8), described in the introduction, preserve the number of the links component (since they are isotopy equivalences). Thus we obtain the following corollary.

[2] The bracket after the last is the first

Corollary 2.4.4.1. *The subsemigroup $G' \subset G$ of equivalence classes of good proper double-bracket structures factorized by global relations* (1)–(8), *is isomorphic to the semigroup $K' \subset K$ of all knots (one-component links) with the trivial knot class as the unity and with product, induced from K.*

§3. Virtual case

3.1. Basic definitions. In 1996, L. Kauffman introduced the notion of virtual knot[3], a natural generalization of the notion of knot, based on the notion of knot diagram.

Definition 3.1.1. *A virtual knot diagram is a planar graph each whose vertex of which is incident to four edges. Each vertex can be marked with undrecrossings, overcrossings, or with so-called virtual crossings; this is usually shown symbolically as in Fig. 6.*

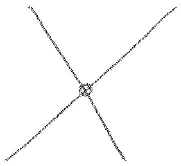


Fig. 6. Virtual crossing.

Two virtual knot diagrams are called equivalent if there exists a sequence of so-called virtual Reidemeister moves transforming one diagram to the other.

Definition 3.1.2. *Following* [2], *we give the list of virtual Reidemeister moves.*

1. *Classical Reidemeister moves relating classical vertices.*
2. *Shadowed versions of classical Reidemeister moves relating virtual crossings, see Fig. 7.*
3. *A version of the third "triangle" Reidemeister move relating two virtual crossings and one classical crossing, see Fig. 8.*

Definition 3.1.3. *A virtual knot is an equivalence class of virtual knot diagrams by virtual Reidemeister moves.*

[3] as above, under "knot" we mean knot or link

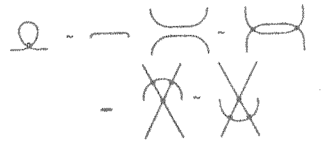


Fig. 7.

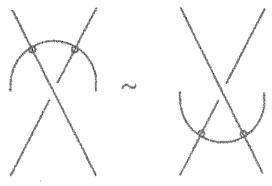


Fig. 8.

Definition 3.1.4. *Given an orientable atom V with a fixed compatible orientation of supercritical circumferences. A planar realization of an atom V is a smooth immersion of its frame to $\mathbb{R}^2$ such that:*

(i) *Each point on the plane having more than one preimage is a transversal intersection point of two images of different lines;*

(ii) *The A-structure (see the definition in 2.1) of V at each vertex is preserved under this immersion. Besides, the image of each oriented supercritical circumference goes locally counterclockwise, see Fig. 9.*

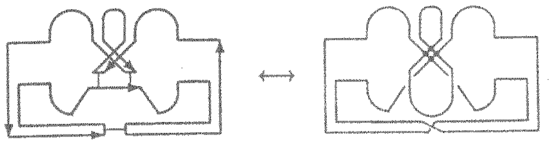


Fig. 9. The planar realization of an orientable atom on the left from the virtual knot diagram.

3.2. Orientable atoms and "orientable" virtual knots. Given an orientable atom V and its planar realization, we define a virtual knot diagram corresponding to it. At each image of the atom's vertex we put over- or undercrossing information like that for the case of classical knot; at intersection points of images of two edges we put a virtual crossing.

Theorem 3.2.1 (Generalized theorem on atoms and knots). *Given an* *orientable atom V, then virtual knot diagrams, obtained from its different planar realizations, represent the same virtual knot.*

Proof. Follows from the fact that transformation from one realization to another can be performed by means of moves, corresponding to the virtual Reidemeister moves. •

Definition 3.2.1. *Given an orientable atom V, virtual knot constructed by a planar realization of V is called virtual knot corresponding to V.*

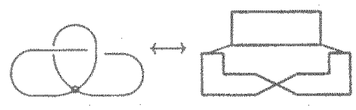
Definition 3.2.2. *All virtual knots, corresponding to at least one orientable atom, are called orientable virtual knots.*

Now given a virtual knot K. Consider its different diagrams and construct "corresponding" atoms (at each classical vertex we put an atom vertex with correspondent B-structure, and at each virtual vertex we put nothing: it is just the intersection of two atom's edges). Note, that the obtained might not be orientable.

Theorem 3.2.2. *The parity of the correspondent atom Euler characteristic stays the same under virtual Reidemeister moves, i.e., it is a virtual knot invariant.*

Proof. Can be easily obtained from an accurate consideration of all virtual Reidemeister moves. •

Example of a non-orientable knot. Consider the virtual knot, shown in Fig. 10.



V=2, E=4, S=3,

V-E+S=1.

Fig. 10.

An atom, corresponding to this diagram of this knot, is lying on $\mathbb{R}P^2$, i.e., its Euler characteristic equals one. Thus this virtual knot is non-orientable (all 2-manifolds with odd Euler characteristic are non-orientable).

3.3. Orientable virtual knots and chord diagrams. We recall that each orientable atom has a polychord diagram (f-graph with coordinately oriented circles, without labels at the edges).

Theorem 3.3.1. *For each orientable virtual knot K there exists a corresponding orientable atom V such that the polychord diagram, corresponding to V, is a diagram with one circumference, i.e., monochord diagram.*

Proof. (Analogous to the proof of Theorem 2.2.2) can be obtained by multiple modification of this diagram with several circles by means of classical second Reidemeister moves. Each of them decreases the number of circumferences by two and finally it becomes equal to one, but the virtual knot stays the same. •

Thus the set of all chord diagrams encodes the set of virtual knots. Here one can construct a system of relations on chord diagrams, encoding the same virtual knot. Chord diagrams, equivalent to d-diagrams, encode classical knots.

However, we have found no way to encode virtual knots (and even orientable virtual knots) in terms of words in finite alphabet.

§4. d-Diagrams and singular knots

4.1. Basic definitions and theorems. In this section, we generalize the theory of d-diagrams to the case of virtual knots.

Definition 4.1.1. *A d-diagram is said to be d-diagram of type n if n (hard) chords in it are selected; these chords are called hard and are shown by thick lines; the other chords are shown by thinner lines.*

Definition 4.1.2. *A d-diagram is called oriented, if each arc of it has an orientation such that at each chord $(ab), a, b \in [0, 2\pi]$, the pairs of arcs $((a-\varepsilon, a)$ and $(b-\varepsilon, b))$ and $((a, a+\varepsilon)$ and $(b, b+\varepsilon))$ have opposite orientations.*

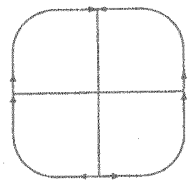


Fig. 11.

Following [7], we construct a correspondence between oriented d-diagrams of type n (possibly, $n = 0$) and oriented singular knots.

Consider a d-diagram D in the plane $\mathbb{R}^2$. Let us modify it as shown in the picture (Note that an overcrossing corresponds to a thin chord, and a double point corresponds to a hard chord).

First, this d-diagram is embeddable to $\mathbb{R}^2$ (first set of chords can be interpreted as inside chords, and others are outside chords). Let us embed D to $\mathbb{R}^2$. In a neighborhood of each edge we set two arcs of our knot. At each singular edge we set a crossing (singular point), and at a non-singular edge we set under-, and overcrossing as it is shown in the figure below:

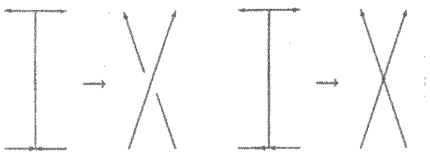


Fig. 12.

Then we obtain a knot diagram of some (singular) knot K.

Theorem 4.1.1.[4] *Different embeddings of the same d-diagram D generate the same isotopy class of the corresponding knot diagram.*

(For the proof, see 4.2.)

Definition 4.1.3. *This (singular, oriented) knot is called (singular, oriented) knot, corresponding to D.*

Definition 4.1.4. *A word B in the alphabet of $($, $)$, $[$, $]$, $\Big($, $\Big)$, $\Big[$, and $\Big]$ is called a proper double-bracket structure with hard brackets*[5], *if the following words are proper bracket structures (in the proper sense):*

(i) *Each of four words, containing only one of four types of brackets: $[$, $]$, and $\Big($, $\Big)$, and $($, $)$, and $\Big[$, $\Big]$.*

[4] This is the generalization of theorem on atoms and knots for the case of singular knots

[5] In the sequel, we use hard brackets for singular chords

(ii) *Each of two words, containing only square brackets, or only round brackets.*

All these words can be obtained from B by deleting brackets of other types, e.g. the order of brackets in these words is the same that their order in B.

It was shown in [7] (cf. [3, 4]) that for each knot isotopy class there exists a d-diagram (without hard chords), corresponding to this knot (see Sec. 2).

Theorem 4.1.2. *For each isotopy class of a singular knot there exists a corresponding d-diagram.*

(For the proof, see 4.2)

4.2. Proofs of basic theorems. First, let us give proofs of Theorems 4.1.1 and 4.1.2.

Proof of Theorem 4.1.1. Consider a d-diagram D. We divide chords of it into equivalence classes according to the following rule: two chords a, b are called equivalent, if there exists a finite set of chords $a = a_1, \dots, a_n = b$, such that for each $i = 1, \dots, n$ a_i intersects a_{i+1}.

Consider two different embeddings $A : D \to \mathbb{R}^2$, $B : D \to \mathbb{R}^2$. Without loss of generality, assume $A(x) = B(x)$ for all points lying on the circumference.

It is easy to see that A can be obtained from B by means of the following trick: all outside chords of S become inside chords, and all inside chords become outside chords, where $S = \cup S_i, S_i$ are equivalence classes.

It obvious that such a trick preserves the isotopy class of the knot (proof of this fact is left for the reader). •

Proof of Theorem 4.1.2. Given a singular knot planar diagram S, consider an edge of it and start to walk along this edge. Meeting a vertex, we change the circuit direction: if the vertex is singular, we may choose any of two direcions (to the right or to the left). If not, we choose the direction as it is shown below:

Returning to the initial point and edge, we obtain some loop on the diagram.

If we have not walked around all edges, we choose any other edge and perform the same operation.

Thus we obtain several loops. If their number equals 1, we obtain the desired d-diagram. Really, walking around this loop we meat each crossing

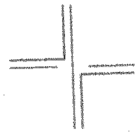


Fig. 13

two times. We connect corresponding points of the circle by chord. If this crossing is singular, we use hard chord. It is easy to see that we obtain a d-diagram, corresponding to the knot K.

If the number of loops is greater than one, then there exists a crossing, incident to two different loops. If this crossing is singular, we just change the directions as it is shown below:

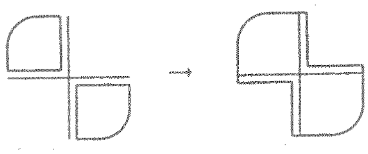


Fig. 14.

If the crossing is not singular, we performing the second Reidemeister move, as it is shown below.

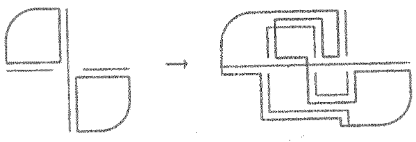


Fig. 15

Thus we preserve the isotopy class of the singular knot and decrease the number of loops by one until it equals one. Thus we obtain the desired d-diagram. $\bullet$

To simplify the method of encoding singular knots by means of d-diagrams let us allow also *negative* non-hard chord (non-hard chords existing before are called *positive*). The way for encoding knot by means of such diagrams is just this:

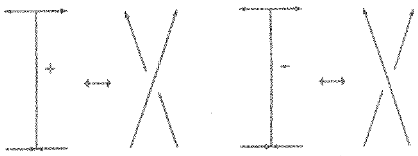


Fig. 16

Definition 4.2.1. *Such a diagram is called d-diagram with negative chords.*

It is easy to prove the following lemma:

Lemma 4.2.1. *For each singular knot diagram there exists a corresponding d-diagram with negative chords.*

This lemma can be proved by means of a good choice of positive and negative chords at each vertex. •

Thus for each singular knot having a planar diagram on n vertices there exists a d-diagram with negative chords having n vertices.

This means that one can simplify the type of diagram (d-diagrams with hard chords are simpler than Gauss diagrams because they can be encoded by means of words in the finite alphabet) without increasing the number of chords.

4.3. Chord diagrams and d-diagrams. In this subsection, we set the correspondence between chord diagrams of (in a proper sense) singular knots and d-diagrams with hard chords.

We can encode knots (in the proper sense, not links) by means of Gauss diagrams or d-diagrams. It is well known, that not each Gauss diagram encodes some classical knot[6].

We give a simple algorithm to recognize Gauss diagrams of classical knots.

Using the definitions described above we obtain the following algorithm.

Let us give an algorithm for d-diagrams without types of crossing, corresponding to chord diagrams for embedding knots to $\mathbb{R}^3$ and then for Gauss diagrams with arrows and local writhe numbers.

Consider a d-diagram D of a singular knot (not link). Divide chords of D into two sets A_1 and A_2 of non-intersecting chords. In the sequel, chords from A_1 will be outer chords, and chords from A_2 – inner chords.

[6] Some diagrams encode virtual knots, see, e.g., [9].

Then the Gauss diagram G, encoding corresponding planar diagram can be obtained as follows:

We start to walk along the circumference at a starting point O counterclockwise. Meeting a chord ab at the point a, we pass to b and change the direction of walking. We choose a parametrization of our walk continuous on circumference arcs such that $t(a) = t(b)$. This time is called time of passing ab. Besides, we return to O at the moment 2π.

Since we have a knot d-diagram, returning to O we have already passed through all chords twice.

Chords of the diagram G connect points $t_1(ab)$, $t_2(ab)$ – two moments of passing the same chord.

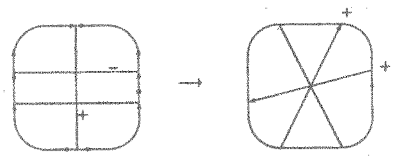
Thus we obtain a chord diagram GG without writhe numbers and arrows.

Definition 4.3.1. *We say that GG is a chord diagram, corresponding to the d-diagram D.*

It is easy to see that for each d-diagram D of knot corresponding chord diagram is a chord diagram for embedding knot in $\mathbb{R}^3$.

Thus for the d-diagram D we have a chord diagram GG. Now we put orientations of edges and writh numbers to obtain G as follows:

If ab is a hard chord of D then corresponding chord of G is without arrow. Otherwise one can easily obtain the correspondence between the local write numbers of the Gauss diagram and the labels on chords of the d-diagram.



Definition 4.3.2. *We say that a chord diagram is good if all its chords have odd lengths, i.e., for each chord ab arc (ab) contains an even number of chord vertices.*

Definition 4.3.3. *Consider a chord diagram C. A trip along its arcs is called good if meeting a chord ab at the point a, we pass to b. Then we can rither change or preserve the direction of walking.*

It is obvious that a chord diagram for embedding a knot to $\mathbb{R}^3$ is good.

The back algorithm is just the following:

Consider a Gauss diagram G of a singular classical knot (not link).

Since G is a Gauss diagram of a classical knot, it is good.

Then the d-diagram D, encoding the corresponding planar diagram can be obtained as follows:

Choose a good trip starting from the point O.

Every time we choose such directions of walking that returning to O we pass all arcs of circumference.

We choose a parametrization of our walk continuous on circumference arcs such that $t(a) = t(b)$. This time is called time of passing ab. Besides, we return to O at the moment 2π.

Chords of the diagram D (maybe, hard chords) connect points $t_1(ab)$, $t_2(ab)$ which are two moments of passing the same chord.

Thus we obtain a D-diagram with hard chords, but without information about orientation and types of non-hard chords (hard chords correspond to chords without arrows).

Now we give the rest of this algorithm, defining the orientation and types of chords.

Now we divide chords of obtained d-diagram into two sets. If the d-diagram is "connected", there are just two ways to do it.

If ab is a chord of G without arrows, then the corresponding chord of D is a hard chord.

If not, the rule for the sign of chord is the following.

Consider three boolean numbers.

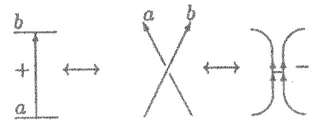
$c_1 = \langle \textit{we do not change direction} \rangle$[7],

$c_2 = \langle \textit{arrow is from } a \textit{ to } b \textit{ in } G \rangle$,

$c_3 = \langle \textit{the sign of } ab \textit{ at } G \textit{ is } + \rangle$,

$c_4 = \langle ab \textit{ is an inner chord y of } d-\textit{diagram} \rangle$. *Then the sign for ab at D is positive if and only if* $((c_1 + c_2 + c_4) \mod 2) \equiv 0$

Example. $c_1 = c_2 = c_4 = 1$, ab is negative.



[7]i.e., passing from a to b, the direction remains the same.

The second algorithm is just the back algorithm for the first one. Thus starting with a Gauss diagram of a classical knot, we obtain a d-diagram.

So, the label $\pm$ at the Gauss diagram of a knot can be uniquely restored if we choose orientation of corresponding d-diagram and the division of its chords in two sets. To do it, we perform the second algorithm (there we do not use labels $\pm$ at chords) and then the first algorithm.

For each good diagram G denote diagram, obtained from G by means of the second algorithm, by E_G.

We say, that a chord diagram G' with starting point and orientation of chords but without writh numbers is *classical*, if there exist writh numbers for all chords, such that obtained Gauss diagram G is Gauss diagram of a classical knot.

Definition 4.3.4. *Classical Gauss diagram G is proper, if there exists such orientation and choice of inner and outer chords of E_G such that labels of diagram obtained from E_G by means of the first algorithm correspond to those of G.*

Lemma 4.3.1. *Consider a chord diagram G' with starting point and orientation of chords but without writh numbers. Then G' is classical if and only if G' is good and E_G is a d-diagram.*

Proof. The necessity is obvious.

Since E_G is good, there exists a good trip, such that at each vertex A two arcs incident to A have opposite orientation.

Then the diagram obtained from E_G by means of the second algorithm is G. Thus E_G is classical. •

Remark. If one diagram obtained from a good chord diagram G is a d-diagram, then all such diagrams are d-diagrams.

Example. Consider a good chord diagram G with corresponding d-diagram E_G. Set some orientations for chords of G.

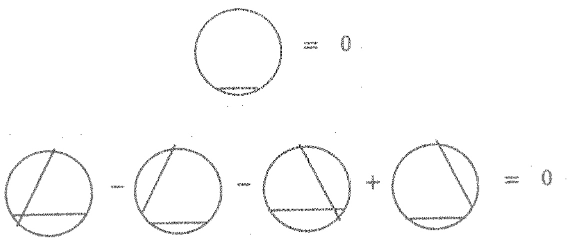
Thus we obtain the following theorem.

Theorem 4.3.1 (A criterion of whether Gauss diagram is a diagram of a classical knot). *Consider a Gauss diagram G, such that diagram G', obtained from G by means of deleting writh numbers at all chords is good. Then G is a diagram of a classical (singular) knot if and only if G is proper.*

Analogously, one can expand this method for the case of diagrams with orientation on edges and writhe numbers.

4.4. A theorem on the algebra of chord diagrams. In this section, we describe the algebra of chord diagrams factored by all 1-term and 4-term relations (see, for example, [8]) in terms of d-diagrams. Here we use the notion of d-diagram, described below.

It was shown (see, e.g., [8]) that the Vasil'ev knot invariants generate an algebra, more precisely, the product of two invariants of orders p and q is an invariant of order $p+q$. Besides, according to the Kontsevich theorem, the space V_n/V_{n-1} is isomorphic to the algebra A^c of chord diagrams described below.



1T-relation i 4T-relation.

Here the first relation means that a chord diagram with an isolated chord equals zero;

The second relation means that if four chord diagrams differ from each other outside small neighborhoods of three points, then the four-term relation holds.

The main idea of our d-diagram construction is the following:

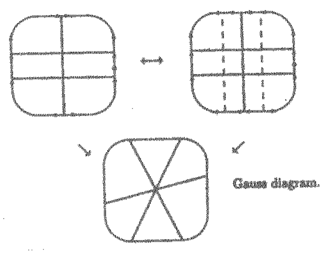
All singular knots can be encoded by means of proper double-bracket structures with hard brackets. Thus we may consider chord diagram as an equivalency class of the singular knot with fixed order of singular points. All singular knots can be encoded by d-diagrams or double-bracket structures with singular brackets. Then all chord diagrams can be represented by means of classes of equivalent d-diagrams, or double-bracket structures. All chord diagrams from A^c are encoded by means of D-diagrams from DD with hard chords.

In the sequel, we consider only d-diagrams and proper double-bracket structures, encoding knots.

Definition 4.1. *Two oriented proper double-bracket structures (resp. two d-diagrams) are called C-equivalent, if they encode two knots with the same order of singular vertices.*

Thus we may not use signs $\pm$ at chords, because it does not act on order of singular vertices. Starting point of an oriented d-diagram does not act on corresponding Gauss diagram.

Remark. In the sequel, we use dotted line for non-hard chords. Besides, we do not put starting point to d-diagrams.



Gauss diagram.

Gauss diagram.

Consider the following algebra DD.

(i) Elements of this algebra are linear combinations of C-equivalency of knot d-diagrams with hard chords.

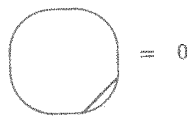
(ii) The unit is chord C-equivalency class of d-diagram without hard chords.

(iii) Product on d-diagrams is defined as follows.

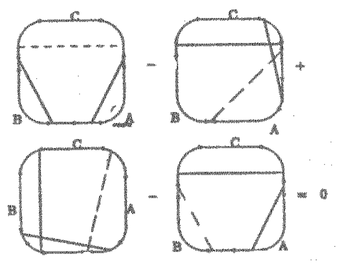
Given two d-diagrams D_1 and D_2, described as sets of point pairs on the circle. These pairs are connected by chords. They are given by angles between Ox and radius-vectors to them. Thus $x_{1,i}$, $x_{2,i}$ are coordinates of chords from D_1 and $y_{1,i}$, $y_{2,i}$ are coordinates of chords from D_2. Consider the diagram whose chords connect points with coordinates $x_{1,i}/2$, $x_{i,2}/2$ and also $y_{1,i}/2+\pi$, $y_{i,2}/2+\pi$. This diagram defines the product D_1D_2.

Besides C-equivalency, there exist the following relations.

(iv) ("1T-relation").



.



(v) ("4T-relation").

Denote the algebra of chord diagrams factored by 1T and 4T-relations by A^c.

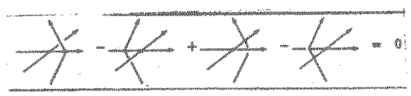
Theorem 4.4.1. *The algebra DD is isomorphic to A^c.*

Proof. Each element of DD generates some element of A^c.

Besides, the algebra DD' without properties (iv), (v) is isomorphic to algebra of all chord diagrams.

Relation (iv) means that in DD d-diagram with a separate hard chords equal zero. Thus in algebra, corresponding to DD, each diagram, obtained from such d-diagram equals zero. It is easy to see that each chord diagram from D with a separate chord can be obtained form a D-diagram with a separate hard chord. Really, consider a chord diagram E that is divided by a chord a into two parts A and B. Consider two D-diagrams with hard chords A_d and B_d, corresponding to A and B. Then d-diagram E_d consisting from A and B, divided by a separate hard chord corresponds to E.

It is easy to see that relation (v) for DD generates just 4T-relation for A^c. More precisely, if for some d-diagrams A_d, B_d, C_d, D_d $A_d - B_d + C_d - D_d = 0$ is relation (v), then for diagrams $A, B, C, D \in A^c$ corresponding to A_d, B_d, C_d, D_d $A - B + C - D = 0$ is a $4T$-relation (see the figure below).



For each chord diagram there exists a lot of d-diagrams, corresponding to it. Without loss of generality, assume that arcs of the d-diagram circles are connected as it is shown in the picture above. Thus using the equivalence relation we obtain the reverse statement. Namely, if for some diagrams $A, B, C, D \in A^c$, corresponding to A_d, B_d, C_d, D_d, $A - B + C - D = 0$ is a 4T-relation, then there exist corresponding d-diagrams A_d, B_d, C_d, D_d, such that $A_d - B_d + C_d - D_d = 0$ is relation (v).

That completes the proof. $\bullet$

Thus we obtained a method of representing all weight systems in the terms of words in our finite alphabet.

The author expresses thanks to professors Vasil'ev, Fomenko, and Sossinsky for constant attention to this work.

References

1. A. T. Fomenko, *The Theory of Multidimensional Integrable Hamiltonian Systems (with Arbitrary Many Degrees of Freedom). Molecular Table of All Integrable Systems with Two Degrees of Freedom.* — Adv. Soviet Math. **6** (1991), Am. Math. Soc..
2. L. H. Kauffman, *Introduction to the Virtual Knot Theory.* private communication.
3. V. O. Manturov, *Bifurcations, Atoms, and Knots.* — Vestnik of the MSU, ser. math. no. 1 (2000), 1–7.
4. V. O. Manturov, *Atoms, Vertical Atoms, Chord diagrams, and Knots. Enumeration of Atoms of Low Complexity Using Mathematica 3.0.* Topological Methods in Hamiltonian Systems Theory. Moscow, "Factorial", 1998, pp. 203–212.
5. A. A. Oshemkov, *Morse functions on two-dimensional surfaces. Coding singularities.* — Works of mathematical institute of V. A. Steklov **205 131–141**. (Russian)
6. V. V. Prasolov, A. B. Sossinskii, *Knots, Links, Braids, and 3-Manifolds.* MCCME. (Russian)
7. V. O. Manturov, *Bracket Semigroup of Knots.* — Mat. Zametki **67** (2000), 549–562.
8. D. Bar-Natan, *On the Vasil'ev Knot Invariants.* — Topology **34** (1995), 423–475.
9. M. Polyak, M. Gussarov, and O. Viro, *Finite-type invariants of virtual knots.* — Topology (1999) (to appear).

Московский государственный
университет им. М. В. Ломоносова
email manturov@matov.msk.ru

Поступило 31 октября 1999 г.