Записки научных
семинаров ПОМИ
Том 256, 2000 г.

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CONVEX HULLS OF INTEGRAL POINTS

ABSTRACT. The convex hull of all integral points contained in a compact polyhedron C is obviously a compact polyhedron. When C is not compact, the convex hull K of its integral points need not be a closed set. However under some natural assumptions, K is a closed set and a generalized polyhedron.

1. INTRODUCTION

Let C be a compact polyhedron. The convex hull K of $C \cap \mathbb{Z}^d$ is a polyhedron, but if C is not compact, this is no longer the case, K need not be a polyhedron or a generalized polyhedron, or even a closed set.

The purpose of this paper is to give sufficient conditions on C so that the convex hull of $C \cap \mathbb{Z}^d$ is a generalized polyhedron. The corresponding theorems are generalization of results contained in [7] and [6].

The motivation for such questions arises from the study of sails and Klein polyhedra. One can find a general presentation of this topic in [6].

We shall first recall the definitions of these notions together with some basic facts. Then we shall present some example, formulate the theorems and give the complete proofs.

2. INTEGRAL SPACES

We shall always consider $\mathbb{R}^d$ with its standard Euclidean structure and make the natural identification between $\mathbb{R}^d$ and its dual. The scalar product will be noted $\langle x, y\rangle$. If $h \in \mathbb{R}^d$, $\ker h$ will denote the hyperplane $\{x \in \mathbb{R}^d \text{ s.t. } \langle h, x\rangle = 0\}$. This last set will sometimes be noted $\{h = 0\}$. Similarly the set $\{x \in \mathbb{R}^d \text{ s.t. } \langle h, x\rangle \geqslant c\}$ will be noted $\{h \geqslant c\}$. The coordinates functions in $\mathbb{R}^d$ will be noted $\mathtt{x}_1, \dots, \mathtt{x}_d$.

Let A be a subset of $\mathbb{R}^d$. The affine space spanned by A is the set of affine combinations of points of A. It will be noted (A). The vector space spanned by A will be noted $\langle A\rangle$. Therefore if $u \in \mathbb{R}^d$, $\langle u\rangle$ will denote the line spanned by u. The half-line $\{tu \text{ with } t \geqslant 0\}$, will be noted $\langle u\rangle^+$.

We say that elements of $\mathbb{Z}^d$ are integral points or integral vectors, according to the context. Let us first recall the definition of two important groups.

Definition 1. *The group* $\mathrm{SL}(d,\mathbb{Z})$ *is the group of* $d \times d$ *matrices with integer coefficients that have determinant* 1.

The group $\mathrm{GL}(d,\mathbb{Z})$ *is the group of* $d \times d$ *matrices with integer coefficients that have determinant* ± 1.

Remark that $\mathrm{GL}(d,\mathbb{Z})$ may also be defined as the group of invertible elements of $\mathcal{M}(d,\mathbb{Z})$, and that $\mathrm{SL}(d,\mathbb{Z})$ is the group of invertible elements of $\mathcal{M}(d,\mathbb{Z})$ that preserve the orientation.

Since we always use standard coordinates in $\mathbb{R}^d$, there is a natural identification between $\mathrm{GL}(d,\mathbb{Z})$ and the group of linear transformations of $\mathbb{R}^d$ that map $\mathbb{Z}^d$ to and onto $\mathbb{Z}^d$.

Similarly, elements of $\mathrm{SL}(d,\mathbb{Z})$ are identified with the linear transformations of $\mathbb{R}^d$ that map $\mathbb{Z}^d$ to and onto $\mathbb{Z}^d$, and that preserve orientation.

Definition 2. *A vector space* $L \subset \mathbb{R}^d$ *is said to be integral if it is spanned by* $\dim L$ *vectors of* $\mathbb{Z}^d$.

Proposition 1. *Let* H *be an integral hyperplane of* $\mathbb{R}^d$. *There exists* $h \in \mathbb{Z}^d$ *such that* $H = \ker h$. *Conversely, if* $h \in \mathbb{Z}^d$, *then* $\ker h$ *is an integral hyperplane.*

Proof. Let H be an integral hyperplane, and let $h \in \mathbb{R}^d$ be such that $H = \ker h$. Suppose that $H = \langle e_1, \dots, e_{d-1}\rangle$ with $e_1, \dots, e_{d-1} \in \mathbb{Z}^d$. Choose $e_d \in \mathbb{Z}^d \setminus H$, and let $(w_1, \dots, w_d)$ be the dual base of $(e_1, \dots, e_d)$. The coordinates of w_d are rational numbers, hence for $n \in \mathbb{N}^*$ conveniently chosen, $nw_d \in \mathbb{Z}^d$, and $H = \ker w_d$.

Conversely, if $h \in \mathbb{Z}^d$, let $h_1, \dots, h_{d-1} \in \mathbb{Z}^d$ be such that h, h_1, $\dots$, h_{d-1} are independent over $\mathbb{R}$. Let e_i denote the solution of

$$\begin{aligned}\langle h, e_i\rangle &= 0\\ \langle h_1, e_i\rangle &= 0\\ &\vdots\\ \langle h_i, e_i\rangle &= 1\\ &\vdots\\ \langle h_{d-1}, e_i\rangle &= 0\end{aligned}$$

Each e_i belongs to $\mathbb{Q}^d$, thus for n_i suitably chosen in $\mathbb{N}^*$, $n_i e_i \in \mathbb{Z}^d$. Hence H contains $d-1$ linearly independent integral points. $\square$

Proposition 2. *Let L be a vector subspace of $\mathbb{R}^d$.*

$$L \text{ is integral} \iff \exists w_1, \ldots, w_{d-l} \in \mathbb{Z}^d \text{ s.t. } L = \bigcap_{i=1}^{d-l} \ker w_i.$$

Proof. If L is integral, let $e_1, \ldots, e_l$ be independent integral vectors spanning L. One can find $e_{l+1}, \ldots, e_d \in \mathbb{Z}^d$ such that $(e_1, \ldots, e_d)$ is a basis of $\mathbb{R}^d$. Let $w_1, \ldots, w_d$ be the vectors of the dual basis. One checks easily that these vectors have rational coordinates, thus for n_i suitably chosen in $\mathbb{N}^*$, $n_i w_i \in \mathbb{Z}^d$, and $L = \cap_{i=l+1}^d \ker w_i$.

Conversely, if $w_{l+1}, \ldots, w_d \in \mathbb{Z}^d$ are $d-l$ independent elements in $\mathbb{Z}^d$, and if $L = \cap_{i=l+1}^d \ker w_i$, one can find $w_1, \ldots, w_l \in \mathbb{Z}^d$ such that $(w_1, \ldots, w_d)$ is a basis of $\mathbb{R}^d$. Let then $(e_1, \ldots, e_d)$ be the dual basis. For conveniently chosen integers $n_1, \ldots, n_d$, each $n_i e_i \in \mathbb{Z}^d$, and hence $L = \langle e_1, \ldots, e_l \rangle$ is integral. $\square$

Proposition 3. *Intersection of integral vector subspaces is an integral vector subspace.*

Proof. The proof is straightforward. Suppose first that L is the intersection of finitely many integral vector subspaces $L_1, \ldots, L_k$, and let

$$\begin{aligned} L_1 &= \cap \ker w_i^1 \\ &\vdots \\ L_k &= \cap \ker w_i^k \end{aligned}$$

We choose $\alpha_1, \ldots, \alpha_p$ among the w_i^j satisfying $\langle w_i^j \rangle = \langle \alpha_1, \ldots, \alpha_p \rangle$. Then $L = \cap \ker \alpha_i$.

The previous reasoning applies with natural changes to the case of a general intersection. $\square$

Proposition 4. *Let L be an integral vector subspace of $\mathbb{R}^d$ endowed with its ordinary Euclidean structure, then $L^\perp$ is integral.*

Proof. Let $u_1, \ldots, u_k$ be an integral basis of L. We add to this basis $d-k$ integral vectors $u_{k+1}, \ldots, u_d$ to get an integral basis of $\mathbb{R}^d$. Let $h_1, \ldots, h_d$ be the corresponding dual base, then $h_{k+1}, \ldots, h_d$ is a basis of $L^\perp$. Since these vectors have rational coordinates, we get an integral basis of $L^\perp$ if we multiply them with conveniently chosen integers. $\square$

Theorem 1. *Let L be an integral vector space of dimension l in $\mathbb{R}^d$. The lattice $L \cap \mathbb{Z}^d$ has a basis with l elements: $a_1, \dots, a_l$. Moreover, one can find $a_{l+1}, \dots, a_d \in \mathbb{Z}^d$ such that $\det(a_1, \dots, a_d) = 1$.*

Proof. The set $L \cap \mathbb{Z}^d$ is an Abelian free subgroup of $\mathbb{Z}^d$. It has a basis: $a_1, \dots, a_k$. Since these vectors are independent and span L, $k = l$.

Let $u \in \mathbb{Z}^d \setminus L$. The lattice $G = (L \oplus \langle u \rangle) \cap \mathbb{Z}^d$ is a free Abelian group that contains $H = L \cap \mathbb{Z}^d$. The quotient group G/H is a free Abelian group with one generator α_{l+1}. One of the corresponding elements of G: a_{l+1} gives the first element of the desired basis. The set $a_1, \dots, a_d$ is a basis of the lattice $\mathbb{Z}^d$, it must therefore have determinant ± 1. After rearranging the a_i, one gets the desired basis. $\square$

The previous theorem may be rephrased in the following way.

Theorem 2. *Let L be an integral vector space in $\mathbb{R}^d$ of dimension l. There exists $g \in \mathrm{GL}(d, \mathbb{Z})$ that maps L to the plane*

$$\bigcap_{i=l+1}^{d} \ker \mathbf{x}_i,$$

where $\mathbf{x}_i$ denotes the standard coordinates in $\mathbb{R}^d$.

Proof. Let $a_1, \dots, a_d$ be a set of elements of $\mathbb{Z}^d$ that match the requirements of Theorem 1. Let A be the matrix of $\mathrm{SL}(d, \mathbb{Z})$ whose columns contain the coordinates of a_i in the standard basis of $\mathbb{R}^d$. Its inverse $A^{-1} \in \mathrm{SL}(d, \mathbb{Z})$ maps a_1 to $(1, 0, \dots, 0)$, a_2 to $(0, 2, \dots, 0)$, $\dots$ and a_l to $(0, 0, \dots, 1, \dots, 0)$, where the 1 occurs in the lth position. The integral vector space L is therefore mapped to the integral vector space with equation $\cap_{i=l+1}^{d} \ker \mathbf{x}_i$. $\square$

Remark. There is a constructive way of proving Theorem 2. It amounts to build a matrix that contains the coordinates of the vectors that span L and add or subtract lines together so that $d - l - 1$ rows contain zeros. Then permute lines so that the $d - l - 1$ last rows are filled with zeros. The addition and subtraction of lines correspond to the action of $g_1 \in \mathrm{SL}(d, \mathbb{Z})$ on the left-hand side, whereas the permutation of lines corresponds to the action of $g_2 \in \mathrm{GL}(d, \mathbb{Z})$. Composing these operations we get $g = g_2 g_1 \in \mathrm{GL}(d, \mathbb{Z})$, and g^{-1} maps the standard simplex of

$$\bigcap_{i=d-l+1}^{d} \ker \mathbf{x}_i$$

to a basis of L. This process is standard and corresponds to the use of Hermite normal forms (see [10]).

Definition 3. *An integral affine subspace of* $\mathbb{R}^d$ *is an affine space* L *that contains* $\dim L + 1$ *independent integral points.*

Definition 4. *The group* $\mathrm{Aff}(d, \mathbb{Z})$ *is the group of affine transformations of* $\mathbb{R}^d$ *that map* $\mathbb{Z}^d$ *to and onto* $\mathbb{Z}^d$*.*

Proposition 5. *Let* H *be an integral hyperplane of* $\mathbb{R}^d$ *containing the origin. There exists* $h \in \mathbb{Z}^d$ *such that:*

(1) *each affine integral hyperplane parallel to* H *has an equation* $\langle h, x\rangle = k$, $k \in \mathbb{Z}$.
(2) *for each* $k \in \mathbb{Z}$ *the equation* $\langle h, x\rangle = k$ *defines an integral affine hyperplane.*

Proof. We showed in Proposition 1 that there exists $h \in \mathbb{Z}^d$ such that $H = \ker h$. We may assume that the greatest common divisor of the coordinates of h is 1 (if not we divide every coordinate of h by this number).

If H' is an affine integral hyperplane parallel to H, and $x_0 \in H' \cap \mathbb{Z}^d$, then $\langle h, x\rangle = \langle h, x_0\rangle$ defines H', and since $\langle h, x_0\rangle \in \mathbb{Z}$, the first statement is proved.

Since the greatest common divisor of the coordinates of h is 1, Bezout's theorem implies the existence of $x_0 \in \mathbb{Z}^d$ satisfying $\langle h, x_0\rangle = 1$.

The hyperplane with equation $\langle h, x\rangle = 1$ is thus parallel to H and contains an integral point. It is therefore an integral hyperplane.

If $k \in \mathbb{Z}$, kx_0 is an integral point in the hyperplane with equation $\langle h, x\rangle = k$, which is therefore integral. $\square$

Proposition 6. *Let* L *be an affine integral subspace of* $\mathbb{R}^d$*. There exist* $h_1, \dots h_k \in \mathbb{Z}^d$*, and* $n_1, \dots n_k \in \mathbb{Z}$ *such that* L *is defined by the equations*

$$\begin{aligned} \langle h_1, x\rangle &= n_1 \\ &\vdots \\ \langle h_k, x\rangle &= n_k \end{aligned}$$

Proof. Let L' denote the integral vector space parallel to L. We use Proposition 2 to get $h_1, \dots, h_k \in \mathbb{Z}^d$ such that $L' = \cap_{i=1}^k \ker h_i$. There is at least one integral point in L. Let x_0 be such a point. Since $\langle h_i, x_0\rangle \in \mathbb{Z}$, the equations

$$\begin{aligned} \langle h_1, x\rangle &= \langle h_1, x_0\rangle \\ &\vdots \\ \langle h_k, x\rangle &= \langle h_k, x_0\rangle \end{aligned}$$

define L and satisfy our requirements. $\square$

One can adapt Theorems 1 and 2 to affine integral spaces.

Theorem 3. *Let L be an integral affine space of dimension l in $\mathbb{R}^d$. Let $a_0 \in L \cap \mathbb{Z}^d$. There exist $a_1, \dots, a_d \in \mathbb{Z}^d$ such that*

(1) *$(a_1, \dots, a_l)$ is a basis of the lattice $(L \cap \mathbb{Z}^d) - a_0$. In other words, elements of $L \cap \mathbb{Z}^d$ may be written as*

$$a_0 + \sum_{i=1}^{l} n_i a_i \quad \text{with} \quad n_1, \dots n_l \in \mathbb{Z}.$$

(2) $\det(a_1, \dots, a_d) = 1$.

The set $(a_0, \dots, a_l)$ is said to be an affine integral basis of $L \cap \mathbb{Z}^d$.

Theorem 4. *Let L be an integral affine space of dimension l. There exists $g \in \mathrm{Aff}(d, \mathbb{Z})$ that maps L to the integral affine space*

$$\bigcap_{i=l+1}^{d} \ker \mathbf{x}_i,$$

where $\mathbf{x}_i$ denotes the standard coordinates in $\mathbb{R}^d$.

3. Kronecker's theorem

Let $u \in \mathbb{R}^d$. Kronecker's theorem is about the structure of the set $\langle u\rangle + \mathbb{Z}^d$. Consider for instance in $\mathbb{R}^3$, $u = (1, \sqrt{2}, 0)$, and suppose that $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ denote the standard coordinates.

The horizontal plane $\mathbf{x}_3 = 0$ is densely filled with the lines

$$\langle u\rangle + \begin{pmatrix} n_1 \\ n_2 \\ 0 \end{pmatrix} \quad \text{with} \quad n_1, n_2 \in \mathbb{Z},$$

and one sees that the closure of $\langle u\rangle + \mathbb{Z}^3$ is the set of affine planes with equation $\mathbf{x}_3 = n_3$, $n_3 \in \mathbb{Z}$. This is general. We give a precise statement in the following. But Kronecker's theorem may be seen differently.

Consider in the previous example the tube

$$B(0,\varepsilon) + \langle u\rangle \quad \text{with} \quad \varepsilon > 0,$$

where $B(x,\varepsilon)$ denotes the open ball of center x and radius ε.

This set contains infinitely many points, which are all placed (for small ε) in the plane $\mathbf{x}_3 = 0$. Informally this means that the rational points that approximate the half-line u all belong to the same vector space, which will be called the rational approximation space of u and will be noted R_u.

Definition 5. *Let $u \in \mathbb{R}^d$. We call integral approximation space of u, the intersection of every integral subspace of $\mathbb{R}^d$ that contains u. It will be noted R_u.*

Since intersection of integral vector spaces is an integral vector space, R_u is an integral vector space for any $u \in \mathbb{R}^d$. The space R_u may also be thought (and defined) as the smallest integral space containing u. We refer to [6] for the proof of the following result.

Theorem 5 (Kronecker). *Let $u \in \mathbb{R}^d$.*

(1) *Let $L = \mathrm{R}_u \cap \mathbb{Z}^d$, then $L + \langle u\rangle$ is dense in R_u.*
(2) *For any $x \in \mathrm{R}_u$, and $\varepsilon > 0$, the set $B(x,\varepsilon) + \langle u\rangle$ contains infinitely many integral points.*
(3) *For any $x \in \mathrm{R}_u$, and $\varepsilon > 0$, the set $B(x,\varepsilon) + \langle u\rangle^+$ contains infinitely many integral points.*
(4) *$\overline{\langle u\rangle + \mathbb{Z}^d} = \mathrm{R}_u \oplus T$, where T is a lattice of rank $d - \dim \mathrm{R}_u$.*

Proposition 7. *Let $u \in \mathbb{R}^d$, $H = \ker h$, a hyperplane containing u, and suppose that $\mathrm{R}_u \not\subset H$.*

(1) *There are infinitely many integral points x inside $B(o,\varepsilon) + \langle u\rangle$ verifying $\langle h, x\rangle > 0$, and infinitely many integral points x inside $B(o,\varepsilon) + \langle u\rangle$ verifying $\langle h, x\rangle < 0$.*
(2) *There are infinitely many integral points x inside $B(o,\varepsilon) + \langle u\rangle^+$ verifying $\langle h, x\rangle > 0$, and infinitely many integral points x inside $B(o,\varepsilon) + \langle u\rangle^+$ verifying $\langle h, x\rangle < 0$.*

Proof. Let $v \in \mathrm{R}_u \setminus H$, with $\langle h, v\rangle > 0$. Then $B(\varepsilon v/(2\langle h, v\rangle), \varepsilon/2) + \langle u\rangle$ contains infinitely many integral points, and is contained in $\{y \in \mathbb{R}^d \text{ s.t. } \langle h, y\rangle > 0\}$. Since $B(\varepsilon v/(2\langle h, v\rangle), \varepsilon/2) \subset B(o,\varepsilon)$, one half of the first assertion is proved. We get the second half considering $v \in \mathrm{R}_u \setminus H$, with $\langle h, v\rangle < 0$. The proof of the second proposition is similar. $\square$

We get from the previous proposition the following immediate consequence.

Proposition 8. *Let x be an integral point, $u \in \mathbb{R}^d$, $H_o = \ker h$ a hyperplane containing u and $H_x = \{y \in \mathbb{R}^d \text{ s.t. } \langle h, y\rangle = \langle h, x\rangle\}$. We suppose that $\mathrm{R}_u \not\subset H_o$.*

(1) *There are infinitely many integral points y inside $B(x, \varepsilon) + \langle u\rangle$ verifying $\langle h, y\rangle > \langle h, x\rangle$, and infinitely many integral points y inside $B(x, \varepsilon) + \langle u\rangle$ verifying $\langle h, y\rangle < \langle h, x\rangle$.*
(2) *There are infinitely many integral points y inside $B(x, \varepsilon) + \langle u\rangle^+$ verifying $\langle h, y\rangle > \langle h, x\rangle$, and infinitely many integral points y inside $B(x, \varepsilon) + \langle u\rangle^+$ verifying $\langle h, y\rangle < \langle h, x\rangle$.*

4. Convex sets

We review here some general results about convex sets and convex polyhedral sets. We borrowed presentation, notation and results from a set of lecture notes written by A. Pajor [8]. One can also consult [2] or [10].

Definition 6. *Let A be a subset of $\mathbb{R}^d$. The convex hull of A is the smallest convex set that contains A. It will be noted $[A]$.*

If $x_0, \ldots, x_d$ are $d+1$ points in $\mathbb{R}^d$, the simplex with vertices $x_0, x_1, \ldots, x_d$ will be noted $[x_0, \ldots, x_d]$.

Definition 7. *Let C be a convex set, $C \subset \mathbb{R}^d$. A point $x \in C$ is said to be extremal if C contains no open segment containing x. The set of extremal points of C is noted $\operatorname{ext}(C)$.*

Definition 8. *Let C be a convex subset of $\mathbb{R}^d$, and x a point of C. We define*

$$F(x, C) = \{y \in C \text{ s.t. } (x, y) \cap C \text{ contains an open subset of } (x, y)\}$$

and call it the face of x in C. Notice that $x \in F(x, C)$. The dimension of $F(x, C)$ is the dimension of the affine space spanned by $F(x, C)$. Notice that

$$F(x, C) = C \cap (F(x, C)).$$

Definition 9. *Let C be a convex subset of $\mathbb{R}^d$, and $E \subset C$. We define*

$$F(E,C) = \cup_{e \in E} F(e,C)$$

and call it the face of E in C.

Proposition 9. *If C is a convex subset of $\mathbb{R}^d$, and $E \subset C$. The face of E in C is convex.*

Definition 10. *A cone $C \subset \mathbb{R}^d$ is a set which satisfies*

$$x \in C \text{ and } \lambda \geqslant 0 \Rightarrow \lambda x \in C.$$

The cone spanned by U is the set of linear positive combination of elements of U.

$$\text{cone} = \{\sum_{i=1}^{k} \lambda_i u_i, \text{ where } \lambda_i \geqslant 0 \text{ and } u_i \in U\}.$$

The faces of a convex cone always contain the origin, hence the affine space spanned by a face of a convex cone is actually a vector space. Notice also that the definition we chose for cones implies that convex cones are semi-groups.

Definition 11. *The asymptotic cone of a closed convex set $C \subset \mathbb{R}^d$, is the convex cone*

$$C_{\text{as}}(C) = \{u \in \mathbb{R}^d \text{ s.t. } x + \lambda u \in C \quad \text{for any} \quad x \in C \quad \text{and} \quad \lambda \geqslant 0\}.$$

We define

$$N(C) = \{u \in \mathbb{R}^d \text{ s.t. } \langle u \rangle \subset C_{\text{as}}(C)\}.$$

It is a vector space, and we say that C contains no line when its dimension is 0.

Remark. It follows immediately from definition that if C is a closed convex set then $C_{\text{as}}(C)$ is closed.

Remark. One checks easily that the asymptotic cone of C may also be defined as

$$C_{\text{as}}(C) = \{u \in \mathbb{R}^d \text{ s.t. } x + \lambda u \in C \quad \text{and} \quad \lambda \geqslant 0\}$$

for some $x \in C$.

Definition 12. *Let C be a cone in $\mathbb{R}^d$. An element $u \in C$ is said to be extremal if the decomposition $u = u_1 + u_2$, $u_1, u_2 \in C$, is only possible with $u_1, u_2 \in \langle u \rangle$. The set of extremal directions of a cone C will be noted $D_{\text{ext}}(C)$.*

We now state the separation theorems, and representation theorems for convex compact sets, convex cones that contain no lines and closed convex sets that contain no line.

Theorem 6. *Let C be a non-empty convex set in $\mathbb{R}^d$ and $x \notin C$. There exists a linear form h such that*

$$\langle h, x \rangle \leqslant \inf_{c \in C} \langle h, c \rangle.$$

Theorem 7. *Let A and B be two disjoint non-empty convex subsets of $\mathbb{R}^d$. There exists a linear form h such that*

$$\sup_{a \in A} \langle h, a \rangle \leqslant \inf_{b \in B} \langle h, b \rangle.$$

Theorem 8 (Caratheodory). *Let A be a subset of $\mathbb{R}^d$.*

(1) *If $x \in [A]$, there are $d+1$ points of A, $a_0, \ldots, a_d$, and $d+1$ real numbers, $\lambda_0, \ldots, \lambda_d \geqslant 0$ such that*

$$x = \sum_{i=0}^{d} \lambda_i a_i \quad \text{and} \quad \sum_{i=0}^{d} \lambda_i = 1.$$

(2) *If $u \in$ cone, there are d points of A, $a_1, \ldots, a_d$, and d positive real numbers, $\lambda_1, \ldots, \lambda_d$ such that*

$$u = \sum_{i=1}^{d} \lambda_i a_i.$$

Theorem 9. *Let C be a convex compact set of $\mathbb{R}^d$. For any $x \in C$, there are $k \leqslant d$ affine independent extremal points $x_0, \ldots, x_k$, and $\lambda_0, \ldots, \lambda_k > 0$ with $\sum \lambda_i = 1$ such that*

$$x = \sum_{i=0}^{k} \lambda_i x_i.$$

In other words,

$$C = [\text{ext}(C)].$$

Theorem 10. *Let C be a closed cone in $\mathbb{R}^d$ that contains no line. If $u \in C$, there are $k \leqslant d$ independent extremal directions of C, $u_1, \dots,$ u_k, and $\lambda_1, \dots \lambda_k > 0$ such that*

$$u = \sum_{i=1}^{k} \lambda_i u_i .$$

In other words,

$$C = \text{cone}\,.$$

Proposition 10. *Let C be a closed cone in $\mathbb{R}^d$ that contains no lines.*

$$u \in D_{\text{ext}}(C) \iff \begin{cases} \textit{if} \quad u = \sum_{i=1}^{k} \lambda_i u_i \quad \textit{with} \quad \lambda_i > 0 \\ \textit{and} \quad u_i \quad \textit{independent}, \\ \textit{then} \quad k = 1. \end{cases}$$

Proof. Suppose that u is extremal and

$$u = \sum_{i=1}^{k} \lambda_i u_i \quad \text{with} \quad \lambda_i > 0 \quad \text{and} \quad u_i \quad \text{independent}.$$

If $k \geqslant 2$, then

$$u = \lambda_1 u_1 + \sum_{i=2}^{k} \lambda_i u_i$$

and since u is extremal, $u_1 \in \langle u \rangle$. If $u \neq u_1$, $u_1 \in \langle u_2, \dots, u_k \rangle$ which is impossible. Hence $u = u_1$ and

$$\sum_{i=2}^{k} \lambda_i u_i = 0.$$

Since the u_i are independent we get a contradiction, and $k = 1$. The converse is obvious. □

Theorem 11. *Let C be a closed convex subset of $\mathbb{R}^d$ that contains no lines*

$$C = [\text{ext}(C)] + C_{\text{as}}(C).$$

Since C contains no lines, $C_{\text{as}}(C)$ is spanned by its extremal directions, and the previous relation may be rewritten as

$$C = [\text{ext}(C)] + \text{cone}\,.$$

5. Polyhedra

Definition 13. *A set $P \subset \mathbb{R}^d$ is said to be a polyhedron if there are h_1, ..., $h_d \in \mathbb{R}^d$ and $c_1, \dots, c_d \in \mathbb{R}$ such that*

$$P = \{x \in \mathbb{R}^d \text{ s.t. } \langle h_i, x\rangle \geqslant c_i \text{ for } i = 1, \dots n\}.$$

One checks easily that P is a closed convex set, and that

$$C_{\mathrm{as}}(P) = \{u \in \mathbb{R}^d \text{ s.t. } \langle h_i, u\rangle \geqslant 0\},$$
$$N(P) = \{u \in \mathbb{R}^d \text{ s.t. } \langle h_i, u\rangle = 0\}.$$

Proposition 11. *Let P be a polyhedron in $\mathbb{R}^d$ with non-empty interior,*

$$P = \{x \in \mathbb{R}^d \text{ s.t. } \langle h_i, x\rangle \geqslant c_i \text{ for } i = 1, \dots n\}.$$

One can choose the linear forms h_1, ..., h_n such that for all $i = 1, \dots, n$, there exist $x_i \in P$ satisfying

$$\langle h_i, x_i\rangle = c_i \quad \text{and} \quad \langle h_j, x_j\rangle > c_j \quad \text{for} \quad j \neq i.$$

This representation of P is said to be minimal.

This representation allows us to say that P has precisely n $(d-1)$-dimensional faces.

Proposition 12. *Let P be a polyhedron in $\mathbb{R}^d$ with minimal representation*

$$P = \{x \in \mathbb{R}^d \text{ s.t. } \langle h_i, x\rangle \geqslant c_i \text{ for } i = 1, \dots n\}.$$

Then

$$\operatorname{int} P = \{x \in \mathbb{R}^d \text{ s.t. } \langle h_i, x\rangle > c_i \text{ for } i = 1, \dots n\}.$$

Theorem 12. *A convex polyhedral set has finitely many extremal points. Its asymptotic cone has finitely many extremal directions.*

Theorem 13. *Let P be a convex polyhedral set. If P contains no lines, then*

$$P = [e_1, \dots, e_q] + \text{cone}\,,$$

where e_1, ..., e_q are the extremal points of P, and u_1, ..., u_r are the extremal directions of $C_{\mathrm{as}}(C)$.

In addition to the general representation theorems we have listed, we shall need some results.

Lemma 1. *Let C be a closed convex subset of $\mathbb{R}^d$. If $(x_n)_{n\geqslant 0}$ is an unbounded sequence of C, one can extract from $(x_n)_{n\geqslant 0}$ a subsequence $(y_n)_{n\geqslant 0}$ such that the sequence $(y_n/\|y_n\|)_{n\geqslant 0}$ converges to $u \in C_{\mathrm{as}}(C)$.*

Proof. Let $(x_n)_{n\geqslant 0}$ be a sequence of points of C. Since $x_n/\|x_n\|$ belongs to the sphere, one can extract from $(x_n/\|x_n\|)_{n\geqslant 0}$ a sequence that converges to $u \in \mathbb{R}^d$.

If $u \notin C_{\mathrm{as}}(C)$, there exists $\varepsilon > 0$ such that $B(u,\varepsilon)\cap C_{\mathrm{as}}(C) \neq 0$. Choose $x \in C$ and define for $v \in B(u,\varepsilon)$

$$\varphi(v) = \inf_{\substack{t\geqslant 0\\ x_1+tv\notin C}} t.$$

Let $v \in B(u,\varepsilon)$ such that $\varphi(v) < t_0$ and take $t \in]\varphi(v), t_0[$. Then $x_1 + tv \notin C$, and $B(x_1+tv,\alpha)\cap C = \emptyset$ for some $\alpha > 0$. Thus, for any $w \in B(v,\alpha)\cap B(u,\varepsilon)$, $x_1 + tw \notin C$, therefore $\varphi(w) \leqslant t < t_0$. This shows that the set

$$\{v \in B(u,\varepsilon) \ \text{ s.t. } \ \varphi(v) < t_0\}$$

is open.

The covering of $\overline{B(u,\varepsilon/2)}$

$$\overline{B(u,\varepsilon/2)} \subset \bigcup_{k\in\mathbb{N}^*} \{v \in B(u,\varepsilon) \ \text{ s.t. } \ \varphi(v) < k\}$$

is therefore an open covering, and since $\overline{B(u,\varepsilon/2)}$ is compact,

$$\overline{B(u,\varepsilon/2)} \subset \{v \in B(u,\varepsilon) \ \text{ s.t. } \ \varphi(v) < k_0\}$$

for some $k_0 \in \mathbb{N}^*$.

There exists $N \in \mathbb{N}$ such that $n \geqslant N$ implies

$$\|x_n\| \geqslant k_0 \quad \text{and} \quad \frac{x_n - x_1}{\|x_n\|} \in \overline{B(u,\varepsilon/2)}.$$

But this implies that $x_n \notin C$, thus $u \in C_{\mathrm{as}}(C)$. $\square$

Remark. An easy consequence of this lemma is that the asymptotic cone of a closed convex set is reduced to $\{0\}$ if and only if it is compact.

Proposition 13. *Let K be a closed convex set, and F a face of K. The set $K \setminus F$ is convex.*

Proof. Let $x \in K$ be such that $F = F(x,K)$ and suppose that $y, z \in K \setminus F$. If $w = ty + (1-t)z \in F$, with $t \in]0,1[$ (t cannot be 0 or 1), then there exists $u \in K$ such that the segment $[w, u]$ contains an open segment that contains x. Hence x belongs to int $[y, z, u]$, which shows that the line (y, x) contains an open segment that contains x, and $y \in F$. Hence $w \notin F$. $\square$

Proposition 14. *Let K be a convex subset of $\mathbb{R}^d$, and F a face of K. If $y \in F$, then $F(y, K) \subset F$.*

Proof. The proof of Proposition 13 applies with minor changes. Suppose that $F = F(x, K)$. If $y = x$, there is nothing to do. Similarly, if y is extremal, $F(y, K) = \{y\}$ and we are done.

Else, let $z \in F(y, K)$, $z \neq y$. There exists $z' \in K$ such that $y \in]z, z'[$, and there exists $y' \neq x$, $y' \in K$ such that $x \in]y, y'[$. One checks easily that x lies in the interior of the triangle $[z, z', y']$, which shows that $z \in F$. $\square$

Proposition 15. *Let K be a convex subset of $\mathbb{R}^d$, and F a face of K.*

(1) *If $x_1, x_2 \in K$ and $t \in]0, 1[$ are such that $tx_1 + (1-t)x_2 \in F$, then $x_1, x_2 \in F$.*
(2) $\operatorname{ext}(F) = \operatorname{ext}(K) \cap F$.

Proof. Let $y = tx_1 + (1-t)x_2$. If $x_1 = x_2$, then $x_1 = x_2 = y \in F$, and if $x_1 \neq x_2$, then x_1 and x_2 belong to $F(y, K)$ which is included in F according to Proposition 14.

Obviously $\operatorname{ext}(K) \cap F \subset \operatorname{ext}(F)$. Suppose that $x \in \operatorname{ext}(F)$. If $]x_1, x_2[$ is an open segment of K that contains x, then $x_1, x_2 \in F(x, K) \subset F$. Since x is extremal in F we get a contradiction, and $x \in \operatorname{ext}(K)$. $\square$

Proposition 16. *Let C be a convex cone of $\mathbb{R}^d$, and F a face of C.*

(1) *If $u_1, u_2 \in C$ and $\lambda_1, \lambda_2 \geqslant 0$, $\lambda_1 + \lambda_2 \neq 0$, are such that $\lambda_1 u_1 + \lambda_2 u_2 \in F$, then $u_1, u_2 \in F$.*
(2) $D_{\text{ext}}(F) = D_{\text{ext}}(C) \cap F$.

Proof. The proof of Proposition 15 applies with minor changes. $\square$

Definition 14. *A set $K \subset \mathbb{R}^d$, is said to be a generalized polyhedron if its intersection with any compact polyhedron is a polyhedron.*

Remark. Generalized polyhedra are obviously closed sets.

Definition 15. *A polytope σ in $\mathbb{R}^d$ is the convex hull of finitely many points. We usually write $\sigma = [x_1, \ldots, x_n]$, with each x_i extremal.*

A polytope is obviously a convex compact set. Remark that a polytope is a polyhedron, but the converse is not true since polyhedra may fail to be compact.

6. Examples

We may now consider the question we asked in the introduction. Suppose that e_1, e_2 and e_3 are three vectors in $\mathbb{R}^3$ with non-zero determinant. We also assume that

$$\text{cone} \cap \mathbb{Z}^3 = \{na, n \in \mathbb{N}\} \quad \text{with} \quad a \in \mathbb{Z}^3,$$
$$\text{cone} \cap \mathbb{Z}^3 = \text{cone} \cap \mathbb{Z}^3 = \{0\}.$$

Let now w_1, w_2, w_3 denote the dual base of e_1, e_2, e_3, ($\langle w_i, e_j \rangle = \delta_{ij}$, where δ_{ij} denotes Kronecker's symbol) and let

$$C = \{w_1 \geqslant 0\} \cap \{w_2 \geqslant 0\} \cap \{w_3 \geqslant 0\} \cap \{w_1 + w_2 + w_3 \geqslant 1\}.$$

This set is represented at Fig. 1. The convex hull of all integral points contained in C is noted K.

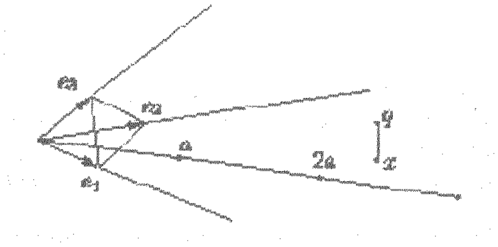

Fig. 1. The convex hull of $C \cap \mathbb{Z}^3$ is not a closed set

One checks easily in this case that the convex hull of integral points contained in C is not a closed set. To see this consider a point x lying in cone but not on the half-line spanned by a: $x = a + \lambda_1 e_1 + \lambda_2 e_2$. Let now q be a point with rational coordinates lying in the interior of C and close to a. If q is chosen close enough, the half-line $a + \langle q - a \rangle^+$ is contained in C, and since $q - a$ has rational coordinates, there is a strictly positive integer n such that $b = a + n(q - a) \in C \cap \mathbb{Z}^3$ and $q \in [a, b]$. Hence x belongs to the closure of K. But since x does not belong to $\langle a \rangle^+$, x does not belong to K, and hence, K is not closed.

If cone $\cap \mathbb{Z}^3$ had been reduced to $\{0\}$, the above construction would not have been possible. Similarly, if the plane $\langle e_1, e_2 \rangle$ had been integral, it would have contained a two-dimensional lattice and, once again, the above construction would not have been possible. This suggests that the convex hull is not closed if and only if

$$F \cap \mathbb{Z}^d = \emptyset \quad \text{or} \quad C_{\text{as}}(F) \subset (F \cap \mathbb{Z}^d) \tag{1}$$

for every face F of C.

This is indeed the case if C is a polyhedral set that contains no lines and if int $C_{\text{as}}(C) \neq \emptyset$. Before we state the main theorems let us give another example. Consider in the plane the polyhedron

$$C = \{(x, y) \in \mathbb{R}^2 \ \text{ s.t. } \ -1/2 + x\sqrt{2} \leqslant y \leqslant 1/2 + x\sqrt{2}\}.$$

Obviously, the border of C contains no integral points and matches (1), but $\left[C \cap \mathbb{Z}^2\right]$ is not closed, which shows that condition (1) alone is not enough.

7. A-HULLS AND A-POLYHEDRA

Definition 16. *We say that a set $C \subset \mathbb{R}^d$ is an A-polyhedron if*

(1) *C contains no lines,*
(2) *int $C_{\text{as}}(C)$ is not empty,*
(3) *$F \cap \mathbb{Z}^d = \emptyset$ or $C_{\text{as}}(F) \subset (F \cap \mathbb{Z}^d)$ for every face F of C.*

An A-hull is the convex hull of the integral points that lie inside an A-polyhedron.

We may now state the two main theorems.

Theorem 14. *A-hulls are closed sets.*

Theorem 15. *A-hulls are generalized polyhedra.*

Let C be a polyhedron. We first check that if $C_{\text{as}}(C)$ has non-empty interior, then condition (1) is necessary for $\left[C \cap \mathbb{Z}^d\right]$ to be closed. The following construction is directly inspired from the first example given. Let F be a face of C, and suppose that

$$F \cap \mathbb{Z}^d \neq \emptyset \quad \text{and} \quad \exists u \in C_{\text{as}}(F) \setminus (F \cap \mathbb{Z}^d).$$

We choose $a \in F \cap \mathbb{Z}^d$. Since $u \in C_{\text{as}}(C)$, and since $C_{\text{as}}(C)$ is a polyhedral cone with non-empty interior,

$$\forall \varepsilon > 0, \quad B(u, \varepsilon) \cap C_{\text{as}}(C) \cap \mathbb{Q}^d \neq \emptyset.$$

We choose v in this last set. We have $a+\langle v\rangle^{+}\subset C$, and for $n\in\mathbb{N}$, $n\geqslant 2$ conveniently chosen, $b=a+nv\in C\cap\mathbb{Z}^{d}$ and $a+v\in\left[C\cap\mathbb{Z}^{d}\right]$. Hence $x\in\overline{[C\cap\mathbb{Z}^{d}]}$.

One checks easily (we give the proof of a similar statement in the sequel) that

$$\left[F\cap\mathbb{Z}^{d}\right]=\left[C\cap\mathbb{Z}^{d}\right]\cap F.$$

Since $x\notin\left[F\cap\mathbb{Z}^{d}\right]$ and $x\in F$, $x\notin\left[C\cap\mathbb{Z}^{d}\right]$. This shows that $\left[C\cap\mathbb{Z}^{d}\right]$ is not a closed set.

We now give the proofs for the two theorems. The first step is to show that an A-polyhedron C can be transformed slightly so that ∂C contains no integral points.

Lemma 2. *Let C be an A-polyhedron. There exists C', A-polyhedron such that*

(1) $C\cap\mathbb{Z}^{d}=C'\cap\mathbb{Z}^{d}$,
(2) $\partial C'\cap\mathbb{Z}^{d}=\emptyset$.

Proof. Suppose that C has the following minimal representation

$$C=\cap_{i=1}^{l}c_{i}\geqslant w_{i}$$

with $w_{i}\in\mathbb{R}^{d}$ and $c_{i}\in\mathbb{R}^{d}$ for $i=1,\ldots,l$. Let $F_{1}=\{w_{1}=c_{1}\}$ be a face of C. If F_{1} contains an integral point, then $C_{\mathrm{as}}(F)$ is contained in an integral space. Let then

$$D_{\alpha}=\{c_{1}-\alpha\leqslant w_{1}\leqslant c_{1}+\alpha\}\cap C.$$

The asymptotic cone of D_{α} is obviously contained in an integral space, and for $\alpha>0$ small enough,

$$C\cap\mathbb{Z}^{d}=\{c_{1}-\alpha\leqslant w_{1}\}\cap\{c_{2}\leqslant w_{2}\}\cap\cdots\cap\{c_{l}\leqslant w_{l}\}.$$

Repeating this procedure for $i=2,\ldots,l$ we get the desired polyhedron C'. $\square$

Proposition 17. *If A and B are two closed convex subsets of $\mathbb{R}^{d}$ satisfying*

$$\begin{cases}\forall a\in A\quad\forall\beta\in C_{\mathrm{as}}(B)\quad\exists t\in\mathbb{R}^{+}\ \ s.t.\ \ a+t\beta\in A\cup B\\ \forall b\in B\quad\forall\alpha\in C_{\mathrm{as}}(A)\quad\exists t\in\mathbb{R}^{+}\ \ s.t.\ \ b+t\alpha\in A\cup B\\ C_{\mathrm{as}}(A)\cup C_{\mathrm{as}}(B)\ \ \text{contains no lines}\end{cases}$$

then $[A \cup B]$ *is closed.*

Proof. Let $(x_n)_{n \geqslant 0}$ be a sequence of $[A \cup B]$ with $\lim_{n \to \infty} x_n = x$. We may write for any n

$$x_n = t_n a_n + (1 - t_n) b_n \qquad \text{with} \quad t_n \in [0,1], \quad a_n \in A, \quad b_n \in B.$$

If the sequences $(b_n)_{n \geqslant 0}$ and $(a_n)_{n \geqslant 0}$ are not bounded, then after extraction

$$\begin{aligned} \lim_{n \to +\infty} a_n / \|a_n\| &= \alpha \in C_{\mathrm{as}}(A) \\ \lim_{n \to +\infty} b_n / \|b_n\| &= \beta \in C_{\mathrm{as}}(B) \\ \lim_{n \to +\infty} t_n &= t \in [0,1]. \end{aligned}$$

Suppose first that $t = 0$. Then $t_n = 1$ for finitely many n. Similarly, since

$$\lim_{n \to +\infty} b_n / \|b_n\| = \beta \neq 0,$$

$b_n = 0$ for finitely many n. Consequently, up to extraction we may assume that $t_n \neq 1$ and $b_n \neq 0$ for $n \in \mathbb{N}$. We get

$$\frac{x_n}{(1 - t_n)\|b_n\|} = \frac{a_n}{\|a_n\|} \frac{t_n \|a_n\|}{(1 - t_n)\|b_n\|} + \frac{b_n}{\|b_n\|}.$$

The limit of the left-hand side is 0, and since $\alpha \neq 0$, $\beta \neq 0$, the positive sequence

$$\left(\frac{t_n \|a_n\|}{(1 - t_n)\|b_n\|} \right)_{n \geqslant 0}$$

has a strictly positive limit ρ, and

$$0 = \rho\alpha + \beta.$$

We get that $C_{\mathrm{as}}(A) \cup C_{\mathrm{as}}(B)$ contains a line which is impossible, hence $t \neq 0$. Similarly $t \neq 1$.

Hence $0 < t < 1$. One checks easily that $\lim_{n \to \infty} \|b_n\| / \|a_n\|$ exists and is strictly positive. We note it ρ, and we get that

$$\rho\beta = \frac{-t}{1 - t} : \alpha.$$

This is a contradiction with the assumption that $C_{\mathrm{as}}(A) \cup C_{\mathrm{as}}(B)$ contains no lines.

We may therefore assume that the sequence $(a_n)_{n\geqslant 0}$ is bounded, and hence (after extraction) that it converges toward $a \in A$.

If the sequence $(b_n)_{n\geqslant 0}$ is also bounded, then we may assume that it converges toward $b \in B$, we get

$$x = ta + (1-t)b \quad \text{with} \quad t \in [0,1], \quad a \in A, \quad b \in B,$$

which shows that $x \in [A \cup B]$.

If $(b_n)_{n\geqslant 0}$ is not bounded, then we may assume that $b_n/\|b_n\|$ converges toward $\beta \in C_{\text{as}}(B)$. One checks easily that $t = 1$ and that $(1-t_n)\|b_n\|$ has a positive limit ρ. We get

$$x = a + \rho\beta \quad \text{with} \quad a \in A, \quad \rho \geqslant 0, \quad \beta \in C_{\text{as}}(B).$$

Let $s \geqslant 0$ be such that $b = a + s\beta \in B$, then $x \in [a,b] \subset [A \cup B]$. $\square$

Notice that in the previous proposition we cannot weaken the assumptions. One may consider simple examples of A and B closed convex sets in $\mathbb{R}^2$ and check whether the convex hull of their union is closed.

If A is a point and B a half-line that does not contain A, then $[A \cup B]$ is not closed – the first condition is not satisfied.

If

$$\begin{aligned} A &= \{(x,y) \in \mathbb{R}^2 \ \text{ s.t. } \ 0 \leqslant x \quad 0 \leqslant y \leqslant 1 \quad 1 \leqslant xy\}, \\ B &= \{(x,y) \in \mathbb{R}^2 \ \text{ s.t. } \ x \leqslant 0 \quad 0 \leqslant y \leqslant 1 \quad xy \leqslant -1\}, \end{aligned}$$

then $[A \cup B]$ is no closed – the third condition is not fulfilled.

Proposition 18. *Let C be an A-polyhedron, K its A-hull and $x \in \operatorname{ext}(K)$. Then $x \in \mathbb{Z}^d$.*

Proof. Because of Caratheodory's theorem, x may be written as

$$\begin{aligned} x &= \sum_{i=0}^{k} \lambda_i m_i \quad \text{with} \quad \lambda_0, \ldots, \lambda_k > 0, \\ &\sum_{i=0}^{k} \lambda_i = 1 \quad \text{and} \quad m_0, \ldots, m_k \in C \cap \mathbb{Z}^d. \end{aligned}$$

If $k \geqslant 1$, x cannot be extremal, therefore $k = 0$, and $x \in C \cap \mathbb{Z}^d$. $\square$

Proposition 19. *Let C be a polyhedron in $\mathbb{R}^d$ that contains no lines. If $C_{\mathrm{as}}(C)$ is contained in an integral vector space L, there are finitely many integral affine spaces parallel to L that intersect C.*

Proof. Since C is a polyhedron that contains no lines, C coincides with the convex hull of its extremal points and directions. More precisely,

$$C = [x_1, \ldots, x_p] + \text{cone}\,,$$

where x_i are the extremal points of C, and u_i the extremal directions of $C_{\mathrm{as}}(C)$.

Let $L = \ker h_1 \cap \cdots \cap \ker h_n$, with $h_1, \ldots, h_n \in \mathbb{Z}^d$. Since

$$C_{\mathrm{as}}(C) = \text{cone} \subset L,$$

$\langle h_i, u_j \rangle = 0$ for $i = 1, \ldots, n$, $j = 1, \ldots, q$. And since $[x_1, \ldots, x_p]$ is compact every h_i is bounded on C. Thus h_i takes only finitely many integral values on C.

Each integral affine hyperplane parallel to L is defined by equations of the following form

$$\begin{aligned} \langle h_1, x \rangle &= k_1 \\ &\vdots \\ \langle h_n, x \rangle &= k_n \end{aligned}$$

with $k_1, \ldots, k_n \in \mathbb{Z}$, the number of such affine spaces that have with C a non-empty intersection is therefore finite. $\square$

Lemma 3. *Let C be a polyhedron that contains no lines. Let $C_{\mathrm{as}}(C) =$ cone , and $x_0 \in C$. Let h be a linear form, and H_{x_0} the plane with equation $\langle h, x \rangle = \langle h, x_0 \rangle$.*

$$H_x \cap C \text{ is not compact} \iff \exists i, j\ i \neq j \text{ s.t. } \langle h, u_i \rangle > 0 \text{ and } \langle h, u_j \rangle < 0.$$

Proof. Suppose that $\langle h, u_1 \rangle > 0$ and $\langle h, u_2 \rangle < 0$. Let

$$y_t = x_0 + t u_1 - t \frac{\langle h, u_1 \rangle}{\langle h, u_2 \rangle} u_2$$

for $t > 0$. We have $y_t \in C \cap H_{x_0}$. And since $\|y_t\| \to \infty$ with t, $H_{x_0} \cap C$ is not bounded, and therefore not compact.

Suppose now that $H_{x_0} \cap C$ is not compact, since it is closed, it must be unbounded. Therefore one can find $(y_n)_{n \geqslant 0}$, an unbounded sequence of

$H_{x_0} \cap C$. We may assume that $\|y_n\| \neq 0$ and extract from $(y_n/\|y_n\|)_{n\geqslant 0}$ a sequence that converges toward $u \neq 0$, $u \in C_{\text{as}}(C)$. Since $y_n \in H_{x_0}$,

$$\langle h, u\rangle = \lim_{n\to+\infty} \langle h, y_n\rangle/\|y_n\| = \lim_{n\to+\infty} \langle h, x_0\rangle/\|y_n\| = 0.$$

Therefore, we get from the decomposition

$$u = \lambda_1 u_1 + \cdots + \lambda_k u_k \quad \text{with} \quad \lambda_1, \ldots, \lambda_k > 0$$

that $\langle h, \lambda_1 u_1 + \cdots + \lambda_k u_k\rangle = 0$, and $\langle h, u_1\rangle, \ldots, \langle h, u_k\rangle$ cannot be all positive or all negative. $\square$

Lemma 4. *Let* $U = \text{cone} \subset \mathbb{R}^d$.

$$U \textit{ contains no lines} \quad \Longleftrightarrow o \notin [u_1, \ldots, u_q].$$

Proof. If $o \in [u_1, \ldots, u_q]$, then one can find $\lambda_1, \ldots, \lambda_q \geqslant 0$, $\sum_{i=1}^q \lambda_i = 1$, that satisfy

$$o = \sum_{i=1}^q \lambda_i u_i.$$

We may for instance assume that $\lambda_1 \neq 0$. Then $\lambda_1 u_1 \in U$, $-\lambda_1 u_1 \in U$, and U contains a line.

Conversely, suppose that U contains the line $\langle u\rangle$. Decompose

$$u = \sum_{i=1}^q \lambda_i u_i \quad \text{and} \quad -u = \sum_{i=1}^q \mu_i u_i$$

with $\lambda_i, \mu_i \geqslant 0$ for $i = 1, \ldots, q$. Since $u \neq 0$,

$$\sum_{i=1}^q \lambda_i > 0 \quad \text{and} \quad \sum_{i=1}^q \mu_i > 0.$$

Therefore

$$v = \frac{u}{\sum_{i=1}^q \lambda_i} \in [u_1, \ldots, u_q],$$
$$w = \frac{-u}{\sum_{i=1}^q \mu_i} \in [u_1, \ldots, u_q]$$

and $o \in [v, w] \subset [u_1, \ldots, u_q]$. $\square$

Lemma 5. *Let C be a polyhedron in $\mathbb{R}^d$ that contains no lines. Suppose that*

$$C_{\mathrm{as}}(C) = \mathrm{cone}\,.$$

There exists $h \in \mathbb{R}^d$ such that $\langle h, u_i \rangle > 0$ for $i = 1, \dots, q$.

Proof. Since $C_{\mathrm{as}}(C)$ contains no lines, $o \notin [u_1, \dots, u_q]$. We use strict separation theorem to get $h \in \mathbb{R}^d$ satisfying our requirement. □

Lemma 6. *Let C, a polyhedron that contains no lines, be such that $C_{\mathrm{as}}(C)$ has not empty interior, $C_{\mathrm{as}}(C) = \mathrm{cone}$. Let $H = \{h = h_0\}$, with $h \in \mathbb{R}^d$, $h_0 \in \mathbb{R}$, be an affine hyperplane, and suppose that integral points of C lie on one side of H. Then, up to a change of sign of h, $\langle h, u_i \rangle \geqslant 0$ for $i = 1, \dots, q$.*

Proof. Suppose that $\langle h, u_1 \rangle > 0$ and $\langle h, u_2 \rangle < 0$, and let $x \in C$. If $\langle h, x \rangle > 0$, then

$$x_0 = x + \frac{\langle h, x \rangle}{\langle h, u_1 \rangle} u_1 \in C \cap H.$$

This shows that $C \cap H \neq \emptyset$.

For $\lambda_1, \dots, \lambda_q > 0$ suitably chosen,

$$u = \lambda_1 u_1 + \dots + \lambda_q u_q \in \mathrm{int}\ C_{\mathrm{as}}(C) \quad \text{and} \quad \langle h, u \rangle = 0.$$

One can therefore find $\mu_1, \mu_2 > 0$ and $\varepsilon_1, \varepsilon_2 > 0$ such that

$$B(u + \mu_1 u_1, \varepsilon_1) \subset \{h > 0\} \cap C_{\mathrm{as}}(C)$$

and

$$B(u + \mu_2 u_2, \varepsilon_2) \subset \{h < 0\} \cap C_{\mathrm{as}}(C).$$

Let now $x_0 \in C \cap H$,

$$C_1 = x_0 + \mathrm{cone} \quad \text{and} \quad C_2 = x_0 + \mathrm{cone}\,.$$

These cones are included in C and contain infinitely many integral points. Since their intersection with H is reduced to x_0, we get a contradiction. All $\langle h, u_i \rangle$ must have the same sign. □

Lemma 7. *Let C, a polyhedron that contains no lines, be such that $C_{\mathrm{as}}(C)$ has not empty interior. Let K denote the convex hull of $C \cap \mathbb{Z}^d$, let $x_0 \in C \setminus K$. If H is an affine hyperplane separating x_0 from K, and* *if* $H \cap C \cap \mathbb{Z}^d = \emptyset$*, then there exists* G*, affine hyperplane separating* x_0 *and* K*, having with* C *compact intersection.*

Proof. Using Lemma 6 we suppose for instance that $\langle h, u_i \rangle \geqslant 0$ for $i = 1, \dots, q$. Since C contains no lines, one can find a linear form k that satisfies $\langle k, u_i \rangle > 0$ for $i = 1, \dots, q$. Any affine hyperplane parallel to $\ker k$ will have compact or empty intersection with C.

The last step of the proof is to show that it is possible to modify slightly H with the help of k so that the intersection of H with C becomes compact.

Let $g_\eta = h + \eta k$ and $G_\eta = \{g_\eta = \langle g_\eta, x_0 \rangle\}$ for $\eta > 0$. Since $\langle g_\eta, u_i \rangle > 0$ for $i = 1, \dots, q$, $G_\eta \cap C$ is compact. The number of integral points a of C such that $\langle g_\eta, a \rangle \leqslant \langle g_\eta, x_0 \rangle$ is finite because $\{g_\eta \leqslant \langle g_\eta, x_0 \rangle\}$ is compact.

Let $y_1, \dots, y_n$ denote these points. Since $H \cap C \cap \mathbb{Z}^d = \emptyset$, $\langle h, y_1 \rangle > 0$, $\dots$, $\langle h, y_n \rangle > 0$, and one can find $\eta_1, \dots, \eta_n > 0$ such that $\langle g_{\eta_i}, y_i \rangle \geqslant \langle g_{\eta_i}, x_0 \rangle$ for $i = 1, \dots, n$.

If $\alpha = \min(\eta, \eta_1, \dots, \eta_n)$, then

$$\forall y \in C \cap \mathbb{Z}^d \quad \langle g_\alpha, y \rangle \geqslant \langle g_\alpha, x_0 \rangle.$$

The affine hyperplane $G_\alpha = \{g_\alpha = \langle g_\alpha, x_0 \rangle\}$ has compact intersection with C and separates x_0 from K. $\square$

Lemma 8. *Let* C*, a polyhedron that contains no lines, be such that* $C_{\mathrm{as}}(C)$ *has non-empty interior. Let* $K = \left[C \cap \mathbb{Z}^d\right]$ *and* $x_0 \in C \setminus K$*. If* H *is an affine hyperplane that separates* x_0 *from* K*, and if* $H \cap C$ *is compact, then one can find* $\varepsilon > 0$ *such that* $B(x_0, \varepsilon) \cap K = \emptyset$*.*

Proof. Since $H \cap C$ is compact it contains finitely many points, $\left[H \cap C \cap \mathbb{Z}^d\right]$ is compact, and hence closed.

If x_0 and K are strictly separated, there is nothing to do, and we may assume that $H = \{h = \langle h, x_0 \rangle\}$, with $h \in \mathbb{R}^d$, and that

$$\forall x \in C \cap \mathbb{Z}^d \ \langle h, x \rangle \geqslant \langle h, x_0 \rangle.$$

Consider the set

$$\{\langle h, x_0 \rangle \leqslant \langle h, x \rangle \leqslant \langle h, x_0 \rangle + \alpha\} \cap C \cap \mathbb{Z}^d \quad \text{for} \quad \alpha > 0.$$

It contains finitely many points because it is compact. Hence

$$\inf_{x \in C \cap \mathbb{Z}^d \setminus H} \langle h, x \rangle > \langle h, x_0 \rangle,$$

and there exists $\eta > 0$ such that

$$C \cap \mathbb{Z}^d = C \cap \mathbb{Z}^d \cap H \quad \cup \quad C \cap \mathbb{Z}^d \cap H_\eta,$$

where $H_\eta = \{h \geqslant \langle h, x_0\rangle + \eta\}$. Therefore

$$K \subset [K \cap H, C \cap H_\eta].$$

One checks easily that $K \cap H$ and $C \cap H_\eta$ match the conditions of Proposition 17, and therefore that $[K \cap H, C \cap H_\eta]$ is closed. It is enough to remark

$$x_0 \notin [K \cap H, C \cap H_\eta],$$

to see that one can find $\varepsilon > 0$ matching our requirements. □

Lemma 9. *Let C be a polyhedron that contains no lines, and let H be an affine hyperplane such that integral points of C lie on one side of H. Then*

$$\left[H \cap C \cap \mathbb{Z}^d\right] = H \cap \left[C \cap \mathbb{Z}^d\right].$$

Proof. Let $K = \left[C \cap \mathbb{Z}^d\right]$. We assume for instance that $H = \{h = h_0\}$ with $h \in \mathbb{R}^d$, $h_0 \in \mathbb{R}$ and that

$$\forall x \in C \cap \mathbb{Z}^d,\ \langle h, x\rangle \geqslant h_0.$$

Since

$$H \cap C \cap \mathbb{Z}^d \subset H$$

and

$$H \cap C \cap \mathbb{Z}^d \subset K$$

we get

$$\left[H \cap C \cap \mathbb{Z}^d\right] \subset K \cap H.$$

And if $x \in K \cap H$, then $x \in K$ and

$$x = \sum_{i=0}^{k} \lambda_i x_i \quad \text{with} \quad \lambda_0, \dots, \lambda_k > 0 \quad \text{and} \quad x_0, \dots, x_k \in C \cap \mathbb{Z}^d.$$

But since $\langle h, x_0\rangle \geqslant h_0, \ \dots\ , \langle h, x_k\rangle \geqslant h_0$,

$$\langle h, x_i\rangle = h_0 \quad \text{for} \quad i = 0, \dots k,$$

and $x \in \left[H \cap C \cap \mathbb{Z}^d\right]$. □

Lemma 10. *Let C be a polyhedron of $\mathbb{R}^d$ that contains no lines. Suppose that* $\operatorname{int} C_{\mathrm{as}}(C)$ *is not empty,* $C_{\mathrm{as}}(C) = \text{cone}$. *Let H be an affine hyperplane such that integral points of C lie on one side of H. If $H \cap \operatorname{int} C$ contains one integral point, and $\langle h, u_i\rangle = 0$, then $\mathrm{R}_{u_i} \subset \ker h$.*

Proof. We saw in Lemma 6 that all $\langle h, u_i\rangle$ have the same sign. We assume for instance that $\langle h, u_i\rangle \geqslant 0$ for $i = 1, \ldots, q$.

Suppose for instance that $\langle h, u_1\rangle = 0$. Let $x_0 \in \operatorname{int} C \cap H \cap \mathbb{Z}^d$. For $\varepsilon > 0$ small enough $B(x_0, \varepsilon) \in C$. The half-tube

$$T_\varepsilon = B(x_0, \varepsilon) + \langle u\rangle^+$$

is contained in C.

If $\mathrm{R}_{u_1} \not\subset \ker h$, we use Proposition 8, and get integral points in T_ε strictly above and strictly under H. This is a contradiction with the assumption that all integral points of C lie on one side of H. $\square$

Lemma 11. *Let C be a convex polyhedron in $\mathbb{R}^d$ that contains no lines, $K = \left[C \cap \mathbb{Z}^d\right]$ and $x_0 \in C \setminus K$. If $H = \{h > \langle h, x_0\rangle\}$ is a hyperplane that separates x_0 from K, then*

$$\inf_{y \in C \cap \mathbb{Z}^d \setminus H} \langle h, y\rangle > \langle h, x_0\rangle.$$

Proof. We write $C_{\mathrm{as}}(C) = \text{cone}$. Lemma 6 implies that $\langle h, u_1\rangle, \ldots, \langle h, u_q\rangle \geqslant 0$, and we assume that $\langle h, u_1\rangle, \ldots, \langle h, u_i\rangle = 0$ while $\langle h, u_{i+1}\rangle, \ldots, \langle h, u_q\rangle > 0$.

We have just seen (Lemma 10) that $\mathrm{R}_{u_1}, \ldots, \mathrm{R}_{u_i} \subset \ker h$. Let L denote the direct sum of these spaces. We have

$$\langle u_1, \ldots, u_i\rangle \subset L \subset H.$$

Consider now for some $\alpha > 0$

$$D = C \cap \{\langle h, x_0\rangle \leqslant h \leqslant \langle h, x_0\rangle + \alpha\}.$$

One checks easily that $C_{\mathrm{as}}(D) = \text{cone}$, which is therefore included in L.

This shows that there are only finitely many affine integral spaces parallel to L that have non-empty intersection with D. Since $L \subset \ker h$, h is constant on these spaces, hence

$$\inf_{y \in D \cap \mathbb{Z}^d \setminus H} \langle h, y\rangle > \langle h, x_0\rangle$$

and

$$\inf_{y \in C \cap \mathbb{Z}^d \setminus H} \langle h, y\rangle > \langle h, x_0\rangle.$$

$\square$

Theorem 16. *Let C be a polyhedron of $\mathbb{R}^d$ that contains no lines and with non-empty interior. Suppose that ∂C contains no integral points. Let $K = \left[C \cap \mathbb{Z}^d\right]$ and $x \in C \setminus K$. There exists $\varepsilon > 0$ such that $B(x, \varepsilon) \cap K = \emptyset$. In other words K is closed.*

Proof. Before we go into details, we present an overview of the proof. We first construct a hyperplane H separating x from K. If the separation is strict, there is nothing to do. If the intersection between H and C is compact, we only have to apply Lemma 8. We therefore assume that the intersection of H and C is not compact, and that $x \in H \cap C$, and we consider two cases.

(1) There are no integral points in $H \cap C$.
(2) $H \cap C$ contains at least one integral point.

The first case is easy and we prove it immediately: we use Lemma 7 to get a hyperplane separating x from K that has with C a compact intersection, and we apply Lemma 8 to get the conclusion.

The second case requires some care. Suppose that $H = \{h = \langle h, x\rangle\}$. The integral points of $H \cap C$ must all belong to int C, and $H \cap \text{int } C \cap \mathbb{Z}^d \neq \emptyset$. We can apply Lemma 11, and deduce from it that K is included in

$$\left[\overline{[H \cap C \cap \mathbb{Z}^d]}, \{\langle h, x\rangle + \alpha \leqslant h\} \cap C\right]$$

for some $\alpha > 0$. We use Proposition 17 and induction to show that it is closed and that it does not contain x. This gives the conclusion.

So, let $H = \{h = \langle h, x\rangle\}$ be a hyperplane separating x from K. Let cone denote the asymptotic cone of C. We assume (see Lemma 6) that $\langle h, u_1\rangle, \ldots, \langle h, u_i\rangle = 0$, and that $\langle h, u_{i+1}\rangle, \ldots, \langle h, u_q\rangle > 0$. Since $H \cap$ int C contains an integral point, we may apply Lemma 11. There exists $\alpha > 0$ such that

$$\langle h, x\rangle + \alpha \leqslant \inf_{y \in C \cap \mathbb{Z}^d \setminus H} \langle h, y\rangle.$$

The integral points of C belong to $H \cap C$ or to $\{\langle h, x\rangle + \alpha \leqslant h\} \cap C$, therefore

$$K \subset \left[\left[H \cap C \cap \mathbb{Z}^d\right], \{\langle h, x\rangle + \alpha \leqslant h\} \cap C\right].$$

Remark that this last set does not contain x. If it did, then x would belong to

$$\left[H \cap C \cap \mathbb{Z}^d\right] = K \cap H \text{ (see Lemma 9)},$$

and x would belong to K.

Let

$$A = \overline{[H \cap C \cap \mathbb{Z}^d]} \quad \text{and}$$
$$B = \{\langle h, x\rangle + \alpha \leqslant h\} \cap C.$$

If $a \in A$ and $u \in C_{\mathrm{as}}(B)$, then

$$u = \sum_{j=1}^{k} \lambda_j u_j \quad \text{with} \quad \lambda_1, \dots, \lambda_k > 0,$$

with $k > i$, $\langle h, u\rangle > 0$ and $\langle h, a + tu\rangle \geqslant \langle h, x\rangle + \alpha$ for some $t \geqslant 0$. We also see that $C_{\mathrm{as}}(A) \cup C_{\mathrm{as}}(B)$ contains no lines and we may apply Proposition 17 and conclude that $[A \cup B]$ is closed.

We apply now induction to get that

$$\left[H \cap C \cap \mathbb{Z}^d\right] = \overline{[H \cap C \cap \mathbb{Z}^d]}.$$

And since

$$x \notin \left[\left[H \cap C \cap \mathbb{Z}^d\right], \{\langle h, x\rangle + \alpha \leqslant h\} \cap C\right],$$

x is out of a closed set that contains K. This shows that there exists $\varepsilon > 0$ such that

$$B(x, \varepsilon) \cap K = \emptyset.$$

It also shows that K is closed. $\square$

Proof of Theorem 14. Let C be an A-polyhedron. We may assume that the border of C contains no integral points. We may therefore apply Theorem 16. $\square$

Proposition 20. *Let C be an A-polyhedron, and K the corresponding A-hull, then*

$$C_{\mathrm{as}}(K) = C_{\mathrm{as}}(C).$$

Proof. Let $x \in K$ and $u \notin C_{\mathrm{as}}(C)$. Since $x \in C$, $x + tu \notin C$ for t suitably chosen. Since $K \subset C$, $x + tu \notin K$ and $u \notin C_{\mathrm{as}}(K)$.

Conversely, let $u \in C_{\mathrm{as}}(C)$ and $x \in \operatorname{int} C \cap \mathbb{Z}^d$. For $\varepsilon > 0$ small enough, the tube $B(x, \varepsilon) + \langle u\rangle^+$ is contained in C, and one checks easily that for any $t \geqslant 0$, $x + tu$ belongs to the convex hull of integral points lying in $B(x, \varepsilon) + \langle u\rangle^+$. This shows that $u \in C_{\mathrm{as}}(K)$, and proves the proposition. $\square$

Lemma 12. *Let C be a polyhedron in $\mathbb{R}^d$, with minimal representation*

$$C = \cap_{i=1}^{q} \{h_i \geqslant c_i\} \quad \textit{with} \quad h_1, \dots, h_q \in \mathbb{R}^d \quad \textit{and} \quad c_1, \dots, c_q \in \mathbb{R}.$$

Let $x \in \operatorname{int} C$, and $(z_n)_{n\geqslant 0}$ an unbounded sequence in C.

There exists $(y_n)_{n\geqslant 0}$ subsequence of $(z_n)_{n\geqslant 0}$ such that all y_n are different and different from x, and

$$\langle h_i, x\rangle + \langle h_i, y_n - y_0\rangle \geqslant c_i \quad \textit{for} \quad i = 1, \dots, q.$$

Proof. Remark first that x extremal implies $x \in \mathbb{Z}^d$ and $x \in \operatorname{int} C$. The minimal representation of C implies that (see Proposition 12)

$$\langle h_i, x\rangle > c_i \quad \text{for} \quad i = 1, \dots, q.$$

Since $z_n \in C$,

$$\langle h_i, z_n\rangle \geqslant c_i \quad \text{for} \quad i = 1, \dots q.$$

If for some i the sequence $(\langle h_i, z_n\rangle)_{n\geqslant 0}$ is not bounded, we may extract from $(z_n)_{n\geqslant 0}$ a subsequence that we shall still note $(z_n)_{n\geqslant 0}$ such that $(\langle h_i, z_n\rangle)_{n\geqslant 0}$ is increasing and

$$\lim_{n\to\infty} \langle h_i, z_n\rangle = +\infty.$$

Hence, since $\langle h_i, x\rangle > c_i$, for some $n_i \in \mathbb{N}$

$$n \geqslant n_i \Rightarrow \langle h_i, x\rangle + \langle h_i, z_n - z_{n_i}\rangle \geqslant c_i.$$

Suppose now that for some other i, $(\langle h_i, z_n\rangle)_{n\geqslant 0}$ is bounded. We may assume (after extraction) that it has a limit, and that the sequence $(\langle h_i, z_n\rangle)_{n\geqslant 0}$ is monotonous. If it is increasing, once again, for some $n_i \in \mathbb{N}$

$$n \geqslant n_i \Rightarrow \langle h_i, x\rangle + \langle h_i, z_n - z_{n_i}\rangle \geqslant c_i.$$

If the sequence $(\langle h_i, z_n\rangle)_{n\geqslant 0}$ is decreasing, there exists $n_i \in \mathbb{N}$ such that $\langle h_i, z_n - z_p\rangle$ is arbitrarily small for $n, p \geqslant n_i$. And since $\langle h_i, x\rangle > c_i$, for n_i suitably chosen,

$$n \geqslant n_i \Rightarrow \langle h_i, x\rangle + \langle h_i, z_n - z_{n_i}\rangle \geqslant c_i.$$

We may now put pieces together and consider

$$n_0 = \max(n_1, \dots, n_q).$$

We get

$$n \geqslant n_0 \Rightarrow \langle h_i, x\rangle + \langle h_i, z_n - z_{n_0}\rangle \geqslant c_i \quad \text{for} \quad i = 1, \dots, q.$$

The existence of $(y_n)_{n\geqslant 0}$ immediately follows. $\square$

Proposition 21. *Let C be an A-polyhedron, let K be the corresponding A-hull and $x \in \operatorname{ext}(K)$. There are finitely many one-dimensional faces of K that contain x.*

Proof. Remark first that there are only finitely many non-compact one-dimensional faces of K that contain x.

Suppose that

$$C = \cap_{i=1}^{q} \{h_i \geqslant c_i\} \quad \text{with} \quad h_1, \dots, h_q \in \mathbb{R}^d \quad \text{and} \quad c_1, \dots, c_q \in \mathbb{R},$$

and assume that there are infinitely many one-dimensional compact faces $[x, z_n]$ of K that contain x, $z_n \in C \cap \mathbb{Z}^d$.

The sequence $(z_n)_{n \geqslant 0}$ must be unbounded since compact sets in $\mathbb{R}^d$ contain finitely many integral points. We may therefore use Lemma 12. There exists $(y_n)_{n \geqslant 0}$ subsequence of $(z_n)_{n \geqslant 0}$ such that:

(1) all y_n are different;
(2) all y_n are different from x;
(3) $\langle h_i, x\rangle + \langle h_i, y_n - y_0\rangle \geqslant c_i$ for $i = 1, \dots, q$.

Consider for the parallelogram $\gamma = [x, y_0, y_n, x + (y_n - y_0)]$. Its vertices belong to $C \cap \mathbb{Z}^d$. If the parallelogram γ is non-degenerate, we get a contradiction because in this case, its diagonal $[x, y_n]$ cannot be a one-dimensional face of K. Hence the points x, y_n, y_0 and $x + (y_n - y_0)$ must lie on the same line. But that again is not possible because all y_n are distinct extremal points of K. We therefore get a contradiction. The number of one-dimensional faces of K that contain x is finite. $\square$

Proof of Theorem 15. Let C be an A-polyhedron and K the corresponding A-hull. Let $h \in \mathbb{R}^d$ be such that $\langle h, u\rangle > 0$ for any non-zero element of $C_{\mathrm{as}}(C)$ and

$$H = \{h = c\} \quad \text{with} \quad c \in \mathbb{R}.$$

The intersection of C with H is compact or empty. We assume for instance that $C \cap \{h \leqslant c\}$ is compact. Consider the convex set $K \cap \{h \leqslant c\}$. It is compact because K is closed, and its extremal points are points of $\operatorname{ext}(K)$, or intersections of 1-dimensional faces of K with H. The number of extremal points of $K \cap \{h \leqslant c\}$ is therefore finite. Moreover

$$C_{\mathrm{as}}(K \cap \{h \leqslant c\}) = \{0\},$$

hence $K \cap \{h \leqslant c\}$ is the convex hull of finitely many extremal points, it is therefore a polytope and thus a polyhedron.

If L is a compact polyhedron, then for c conveniently chosen

$$L \subset \{h \leqslant c\}$$

and

$$K \cap L = K \cap \{h \leqslant c\} \cap L.$$

Since $K \cap \{h \leqslant c\}$ is empty or a polyhedron, $K \cap L$ is also a polyhedron. $\square$

Acknowledgements. It is a pleasure to thank V. I. Arnold who directed carefully this work, E. Korkina and G. Lachaud for proof reading, and F. Napolitano for useful conversation.

References

1. V. I. Arnold, *A-graded Algebras and Continued Fractions.* — Comm. Pure Appl. Math. **XLII** (1989), 993–1000.
2. M. Berger, *Géométrie.* vol. 3, CEDIC – F. Nathan, 1978.
3. F. R. Giles, W. R. Pulleyblank, *Total dual integrality and integer polyhedra.* — Linear Algebra Appl. **25** (1979), 191–196.
4. B. Grünbaum, *Convex Polytopes.* Pure and applied mathematics. New York, Interscience publishers, 1967.
5. E. Korkina, *Classification of A-graded Algebras with 3 generators.* — Indagationes Mathematicae **3** no. 1 (1992), 27–40.
6. G. Lachaud, *Voiles et Polyèdres de Klein.* — Prétirage 95–22, Laboratoire de Mathématiques Discrètes, C. N. R. S. (1995).
7. R. R. Meyer, *On the existence of optimal solutions to integer and mixed-integer programming problems.* — Math. Programming **7** (1974), 223–235.
8. A. Pajor, *Cours d'optimisation de l'université Denis Diderot.*
9. H. E. Scarf, *Integral polyhedra in three space.* — Math. Oper. Res. **10** no. 3 (1985), 403–438.
10. A. Schrijver, *Theory of linear and integer programming.* Chichester, John Wiley & Sons Ltd., 1986, a Wiley-Interscience Publication.
11. B. Sturmfels, *Gröbner bases and convex polytopes.* Amer. Math. Soc., 1996.

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Поступило 10 октября 1999 г.