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INVARIANTS OF B–TYPE LINKS VIA AN EXTENSION OF THE KAUFFMAN BRACKET

Abstract. In this paper we find a method for constructing solid torus (or: **B**–type) link invariants by means of extension of the Kauffman bracket for such links. This method can be viewed as the combination of the ideas proposed in [5, 7] and [8] with **B**–type braid group theoretic approach to solid torus links which is given in [18]. Such method is shown to be compatible with skein relations of **B**–type and leads to state sum expression for link invariant (in the case of the special diagrams).

Key words and phrases: Kauffman bracket, solid torus links, Markov trace

Introduction

The study of new (or quantum) invariants of links in 3-manifolds is the field of the growing interest both for mathematicians and physicists during last time. The main ingredient for constructing such invariants, from the physical point of view, is Chern–Simons (CS) theory in three dimensions, where these invariants appear as the expectation values of Wilson loop operators in perturbative CS theory. In mathematical terms, the construction of new (or quantum) link invariants is given mainly for the case of the trivial topology of the underlying space (i.e., for the case of $\mathbf{R}^3$ or for three-dimensional sphere $\mathbf{S}^3$). There are three different constructions.

The first one is due to V. Jones [10] which uses the methods of operator algebras. This construction is, in fact, closely related to limiting behavior of integrable models of statistical physics. (This construction can be interpreted via state models and skein categories, see [11]).

The second one is due to V. G. Turaev and N. Yu. Reshetikhin [23, 24, 26] and is combinatorial. Such construction uses the methods of category theory and quasi-Hopf algebras. This also gives the method of constructing 3-manifold invariants (see also [27]).

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The third construction is due to V. A. Vassiliev [28], M. Kontsevich [14] and D. Bar-Natan [3] and is based on the notion of the algebras of chord diagrams. Roughly speaking, such method can be considered as the mathematical formulation of the physical idea of the perturbative expansion of CS invariants over the Feynman diagrams (e.g. [13]). In this way one can obtain so called finite type invariants of links and the corresponding finite type invariants of 3-manifolds [19–22]. Finite type invariants of link can be combined into one generating function, the Kontsevich integral for this link, taking its values in the chord algebra. The Kontsevich integral gives an analytic expression for link invariants. The relationship of such integral to combinatorial expression for link invariants (at least, for the case of knots) is given in [1].

The problem of evaluation of link invariants can be posed for the link in spaces with topology less trivial than $\mathbf{R}^3$, or $\mathbf{S}^3$ (e.g. [2, 29]). In this paper we deal with links in solid torus (handlebody of genus 1) and give the construction of the invariants of links in solid torus (or $\mathbf{B}$-type links) via the extension of Kauffman bracket (see [11, 12]). Such approach seems to be very natural from the point of view of the combinatorial method in constructing link invariants.

The paper is organized as follows.

In Sect. 1 the background on Kauffman bracket on knots and links in three-dimensional sphere is given after [12].

In Sect. 2 we find $\mathbf{B}$-type Hecke algebra solutions of the reflection equation related to the given R-matrix solution of Yang–Baxter equation (defining via the Kauffman bracket).

In Sect. 3 we deal with the special solution of the reflection equation with coefficients from the ring $\mathbf{Z}[A^{1/2}, A^{-1/2}]$. Such solution is interpreted as the extension of Kauffman bracket. We prove the version of Kauffman theorem for invariants of links in solid torus making use of the results of [18] and the method of [26].

In Sect. 4 we derive the skein relations for $\mathbf{B}$-type link invariant defined by the extension of the Kauffman bracket. The extra skein relation — *clasp rule* is very reminiscent to that in [18].

Acknowledgments

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1. Background on Kauffman Bracket

Recall the definition of the Kauffman bracket polynomial after [12]. Consider an unoriented link diagram $\mathfrak{K}$ and associate to each crossing of $\mathfrak{K}$ the smoothing of two types as shown in Fig. 1 weighted by the factor A and A^{-1}, respectively.

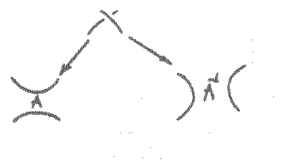

Fig. 1.

By *state* $\mathfrak{S}$ of the diagram $\mathfrak{K}$ we mean the choice of the smoothing for each crossing in $\mathfrak{K}$. The number of disjoint Jordan curves obtained in such way is denoted by $|| \mathfrak{S} ||$. The *weight* $< \mathfrak{K}|\mathfrak{S} >$ is called the product of the vertex weights of $\mathfrak{S}$. The bracket polynomial $< \mathfrak{K} > \in \mathbf{Z}[A, A^{-1}]$ is defined by the following state sum formula:

$$< \mathfrak{K} > := \sum_{\mathfrak{S}} < \mathfrak{K}|\mathfrak{S} > d^{||\mathfrak{S}||}, \tag{1.1}$$

where the sum is taken over all states of diagram $\mathfrak{K}$, and $d = -(A^2+A^{-2})$. Recall that the bracket polynomial is an invariant of the regular isotopy of link diagrams. Furthermore, if $\mathfrak{K}$ is an oriented link diagram and $w(\mathfrak{K})$ is the writhe of this diagram (the definition of the writhe will be given below), then the function

$$f_{\mathfrak{K}}(A) := (-A^3)^{-w(\mathfrak{K})} < \mathfrak{K} > / < O > \tag{1.2}$$

is the invariant of the ambient isotopy of link diagrams. The value $< O >$ here is the value of the bracket polynomial on the unknot, $< O > = -(A^2 + A^{-2})$. In other words, the value $< \mathfrak{K} >$ is the invariant with respect of the second and the third Reidemeister moves, and the value $f_{\mathfrak{K}}(A)$ is the invariant with respect of the first, the second and the third Reidemeister moves. The Reidemeister moves are shown in Fig. 2.

The *writhe* of the oriented link diagram $\mathfrak{K}$ is the algebraic sum of the crossings in the diagram. The rules for the signs of the crossings are given in Fig. 3.

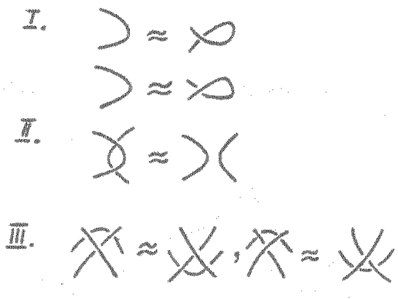

Fig. 2.

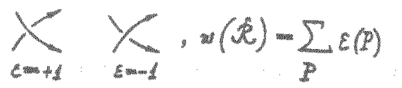

Fig. 3.

Remark. If $V_{\mathfrak{K}}(t)$ is the Jones polynomial [10], then

$$V_{\mathfrak{K}}(t) = f_{\mathfrak{K}}(t^{-1/4}).$$

Now we translate the bracket polynomial to the language of the Temperley–Lieb algebras (of **A**-type). Moreover, such translation provides us with an alternative description of the invariance of the bracket polynomial.

Definition 1 (see [12]). *The Temperley–Lieb algebra of* **A**-*type on* n *strands* $\mathbf{TA_n}$ *over* $\widetilde{\mathbf{Z}}[A, A^{-1}]$ *is the free additive algebra with multiplicative generators* 1_n, U_1, ... , U_{n-1} *subject to the the following relations:*

$$U_i^2 = dU_i, \qquad \text{for any} \quad i = 1, \ldots, n-1; \tag{1.3}$$

$$U_i U_j U_i = U_i, \qquad \text{for any} \quad |i-j| = 1, \quad 1 \le i, j \le n-1; \tag{1.4}$$

$$U_i U_j = U_j U_i, \qquad \text{for any} \quad |i-j| > 1, \quad 1 \le i, j \le n-1. \tag{1.5}$$

The diagrammatic calculus of $\mathbf{TA_n}$ *is shown in Fig. 4 in the case* $n = 4$.

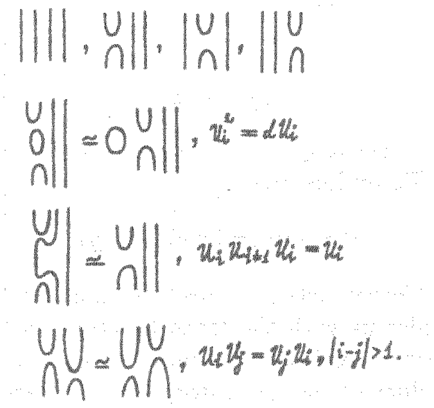

Fig. 4.

We take the value d equal to $-(A^2+A^{-2})$, and A, A^{-1} commute with all elements of $\mathbf{TA_n}$. (Here $\widetilde{\mathbf{Z}}[A, A^{-1}]$ denotes the set of rational functions P/Q with $P, Q \in \mathbf{Z}[A, A^{-1}]$).

The Temperley–Lieb algebra $\mathbf{TA_n}$ can be generalized to the tangle algebra $\mathbf{T}_n$ (over $\widetilde{\mathbf{Z}}[A, A^{-1}]$) which is generated multiplicatively by the regular isotopy classes of all n-strand tangles. Each n-strand tangle (or: simply, n-tangle) has n inputs and n outputs (see Fig. 5).

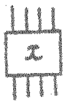

Fig. 5

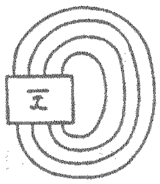

Fig. 6.

The Temperley–Lieb algebra can be viewed as the subalgebra of the tangle algebra.

The *trace* function on $\mathbf{T}_n$ is defined as follows. Let $\overline{x}$ be the usual closure of the tangle $x \in \mathbf{T}_n$ obtained by the attaching the kth input strand and the kth output strand as in Fig. 6.

Then the trace

$$\mathrm{tr}: \mathbf{T}_n \to \mathbf{Z}[A, A^{-1}]$$

is defined by:

$$\mathrm{tr}(x) := < \overline{x} >,$$

where $<>$ is the bracket polynomial, for each tangle $x \in \mathbf{T}_n$. This trace prolongs to whole $\mathbf{T}_n$ by linearity:

$$\mathrm{tr}(x+y) := \mathrm{tr}(x) + \mathrm{tr}(y),$$

for any $x, y \in \mathbf{T}_n$. The restriction of this trace function to the Temperley–Lieb algebra provides us with the trace function on this algebra. Such trace can be described explicitly. Namely, for each $x \in \mathbf{TA_n}$, which is written as the product of the generators U_i, the trace $\mathrm{tr}(x) := < \overline{x} > = d^{||x||}$, where $||x||$ is the number of disjoint Jordan curves (and $\bar{x}$ is the disjoint union of these curves).

Remark. The trace function on $\mathbf{TA_n}$ constructed as above corresponds to the Jones trace (see [9]).

Recall the notion of the braid group (of **A**-type) B_n. The elements of this group (the braids) are generated by the following basic elements (elementary braids):

$$1_n, \sigma_1, \ldots, \sigma_{n-1}, \sigma_1^{-1}, \ldots, \sigma_{n-1}^{-1}$$

(see Fig. 7 for $n = 4$), subject to the following relations:

$$\begin{aligned} \sigma_i \sigma_j \sigma_i &= \sigma_j \sigma_i \sigma_j, \quad \text{for} \quad |i-j| = 1, \quad 1 \le i, j \le n-1, \\ \sigma_i \sigma_j &= \sigma_j \sigma_i, \quad \text{for} \quad |i-j| > 1, \quad 1 \le i, j \le n-1. \end{aligned}$$

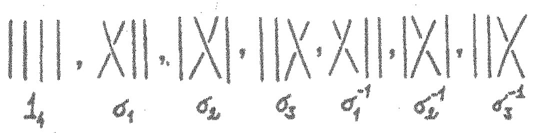

Fig. 7.

Braids are related to oriented links and their invariants due to Alexander's theorem which states that any oriented link is isotopic to the closure of some braid. Such presentation of oriented link is not unique. The Markov theorem gives the braid analogue of Reidemeister's theorem. Namely, the braid equivalence in $\bigcup_{n=1}^{\infty} B_n$ is as follows:

(1) Conjugation: If $\alpha, \beta \in B_n$, then $\alpha \simeq \beta^{-1}\alpha\beta$,

(2) Markov moves: If $\alpha \in B_n$,

then $\alpha \simeq \alpha\sigma_n \in B_{n+1}$, and $\alpha \simeq \alpha\sigma_n^{-1} \in B_{n+1}$.

Remark. Sometimes these equivalences are called type I and type II Markov moves. The graphical expressions for these moves are shown in Fig. 8 (see [4]).

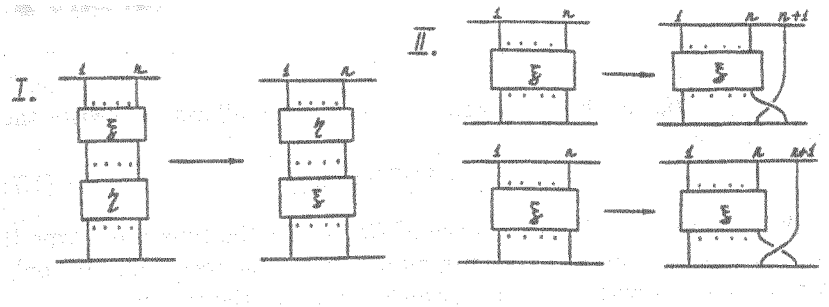

Fig. 8.

Hence the strategy in constructing link invariant consists in two steps. *Step one* is to construct a representation of the braid group B_n. *Step two* is to construct a Markov move invariant defined on this representation. Our aim here is to consider link invariant constructing from Kauffman bracket from the point of view of this strategy.

Step 1. The reader could verify easily making use of the definition of the Temperley–Lieb algebra of **A**-type that the Kauffman bracket which is defined (on the level of braids) as:

$$\rho(\sigma_i) := AU_i + A^{-1}\cdot 1_n,$$
$$\rho(\sigma_i^{-1}) := A^{-1}U_i + A\cdot 1_n,$$

forms an underlying representation with values in $\mathbf{T}_n$

$$\rho : B_n \longrightarrow \mathbf{T}_n$$

(with loop value $d = -(A^2 + A^{-2})$).

This is an easy exercise from Temperley–Lieb algebras which is left to the reader.

Step 2. Following to our strategy we have to construct the Markov move invariant from the representation of the braid group described as above. We claim that this invariant can be defined by the formula:

$$I(\xi) = (-A^3)^{e(\xi)} < \overline{\xi} >, \tag{1.6}$$

where $e(\xi)$ means the algebraic sum of the exponents in the expression for $\xi \in B_n$ (with the convention that $e(\sigma_i^{\pm 1}) = \pm 1$), and $\overline{\xi}$ is a link representing by a braid $\xi \in B_n$[1].

Remark. From the discussion given above it follows that (see [12]):

$$\mathrm{tr}\, \rho(\xi) =< \overline{\xi} >,$$

for any $\xi \in B_n$. Such expression for the bracket allows to rewrite the invariant $I(\xi)$ as:

$$I(\xi) = (-A^3)^{e(\xi)}\, \mathrm{tr}\, \rho(\xi). \tag{1.7}$$

We have to verify the invariance of $I(\xi)$ under the type I and type II Markov moves. This is an exercise on diagrammatic technique. We only sketch the proof without details (which are left to the reader).

1. Invariance under the conjugation (or: type I Markov move).

Since $e(\eta^{-1}\xi\eta) = e(\xi)$, for any braids $\xi, \eta \in B_n$, then one has to verify that:

$$< \overline{\eta^{-1}\xi\eta} >=< \overline{\xi} >,$$

for any $\xi, \eta \in B_n$, or, in the notations involving the representation ρ:

$$\mathrm{tr}\, \rho(\eta^{-1}\xi\eta) = \mathrm{tr}\, \rho(\xi),$$

for any two braids $\xi, \eta \in B_n$. Moreover, representing the element $\eta \in B_n$ as the product of σ_i and their inverses, it follows by induction that it is sufficient to verify that:

$$< \overline{\sigma_i^{-1}\xi\sigma_i} >=< \overline{\xi} >,$$

for any $\xi \in B_n$, and $1 \leq i \leq n-1$.

[1]It should be noted that $e(\xi)$ is equivalent to $w(\overline{\xi})$, namely, $e(\xi) = -w(\overline{\xi})$, see [4] for details.

This equality can be derived directly from the definition of the Kauffman bracket and the proof is left to the reader.

2. Invariance under the type II Markov moves.

We can verify the invariance of $I(\xi)$ under the move $\xi \simeq \xi\sigma_n \in B_{n+1}$, the other case $\xi \simeq \xi\sigma_n^{-1} \in B_{n+1}$ can be considered analogously. The behavior of the Kauffman bracket under the type II Markov moves can be derived directly in the graphical form. This exercise is left to the reader. We just indicate the final result:

$$< \overline{\xi\sigma_n} >= (-A^{-3}) < \overline{\xi} >,$$

for any $\xi \in B_n$. But $e(\xi\sigma_n^{+1}) = e(\xi) + 1$, then

$$\begin{aligned} I(\xi\sigma_n^{+1}) =&(-A^3)^{e(\xi\sigma_n^{+1})} < \overline{\xi\sigma_n^{+1}} >= \\ &(-A^3)^{(e(\xi)+1)}(-A^{-3}) < \overline{\xi} >= (-A^3)^{e(\xi)} < \overline{\xi} >= I(\xi). \end{aligned}$$

Thus we obtain the braid-theoretic proof of the invariance of the well-known expression for link invariant basing on the Kauffman bracket.

We conclude our discussion on Kauffman bracket remarking that usually one can use the modifying invariant $\widetilde{I(\xi)}$ inolving the factor $\frac{1}{<O>}$, where $< O >$ is the value of the bracket polynomial on the unknot. This factor can be viewed as the purely normalization of the link invariant.

2. B-Type Hecke Algebra Solutions of the Reflection Equation

The approach to constructing invariants of links in solid torus is based on the relevant analogues of the Alexander's and Markov's theorems in this case as given in [18]. Recall some basic results from [18] which are related to our discussion.

The oriented **B**-type links (or: oriented links in solid torus **M**) can be viewed as the *mixed links* in three-dimensional sphere $\mathbf{S}^3$ containing the complementary solid torus $\hat{\mathbf{I}}$, since the solid torus **M** can be considered as the complement in $\mathbf{S}^3$ of the another solid torus $\hat{\mathbf{I}}$. In other words, any link L in solid torus **M** can be represented uniquely as the mixed link $\hat{\mathbf{I}} \cup L$ in three-dimensional sphere $\mathbf{S}^3$, "consisting of a *standard* link in $\mathbf{S}^3$ linking with the fixed complementary *solid torus part* $\hat{\mathbf{I}}$".

The topology of **B**-type links was considered in [18] in details. For our combinatorial approach for constructing the invariants of links in solid torus is essential that the analogues of Alexander's and Markov's theorems are hold. (These theorems will be formulated below). This motivates a search of a construction of the representation of braid group $B_{1,n}$ of n strings in solid torus taking its values in Temperley–Lieb algebra of **B**-type. Recall the notion of braid group $B_{1,n}$ of n strings in the solid handlebody of genus 1 (solid torus) which is defined via the generators and relations (see e.g. [15–18]):

$$\sigma_1, \dots, \sigma_{n-1}, \tau : \quad \sigma_i\sigma_j = \sigma_j\sigma_i, \quad \text{for} \quad |i-j| \geq 2,$$
$$\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j, \quad \text{for} \quad |i-j| = 1,$$
$$\tau\sigma_1\tau\sigma_1 = \sigma_1\tau\sigma_1\tau, \qquad \sigma_i\tau = \tau\sigma_i, \quad \text{for} \quad 2 \leq i \leq n-1.$$

The elements of $B_{1,n}$ will be called *mixed braids* (in the terminology of [18]), or *braids of* **B**-*type*.

The graphical interpretation of the generators τ, σ_i $(i = 1, \dots, n-1)$ as well as the relation $\tau\sigma_1\tau\sigma_1 = \sigma_1\tau\sigma_1\tau$ (so called "reflection equation", see [15–17]) is given in Fig. 9.

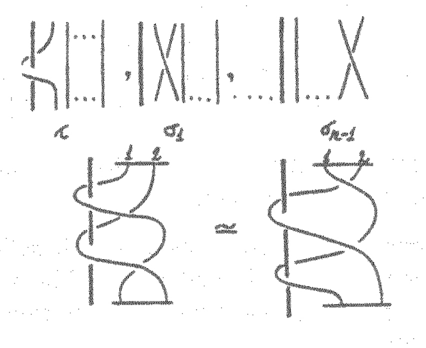

Fig. 9.

Correspondingly, the links in solid torus will be called *mixed links*, or *links of* **B**-*type*.

We are in a position to formualate the analogues of Alexander's and Markov's theorems for solid torus links following by [18].

Alexander's theorem for mixed links. *Any oriented mixed link is isotopic to the closure of some mixed braid.*

Markov's theorem for solid torus link. *Let* $\mathbf{M} = \mathbf{S}^3 \backslash \hat{\mathbf{I}}$ *be a solid torus with* $\hat{\mathbf{I}}$ *be also a solid torus; let* L_1, L_2 *be two oriented links in*

$\mathbf{M}$ and $\mathbf{I}\bigcup B_1$, $\mathbf{I}\bigcup B_2$ be mixed braids in $\bigcup_{n=1}^{\infty} B_{1,n}$, corresponding to L_1, L_2. Then L_1 is isotopic to L_2 in $\mathbf{M}$ if and only if $\mathbf{I}\bigcup B_1$ is equivalent to $\mathbf{I}\bigcup B_2$ in $\bigcup_{n=1}^{\infty} B_{1,n}$ under equivalence generated by the augmented braid relations together with the following two moves:

(i) *Conjugation:* *If $\alpha, \beta \in B_{1,n}$, then $\alpha \simeq \beta^{-1}\alpha\beta \in B_{1,n}$;*
(ii) *Markov moves:* *If $\alpha \in B_{1,n}$, then $\alpha \simeq \alpha\sigma_n^{\pm 1} \in B_{1,n+1}$.*

Remark. These equivalences will be called the type I and the type II Markov moves for $B_{1,n}$. The graphical expressions for these moves are shown in Fig. 10.

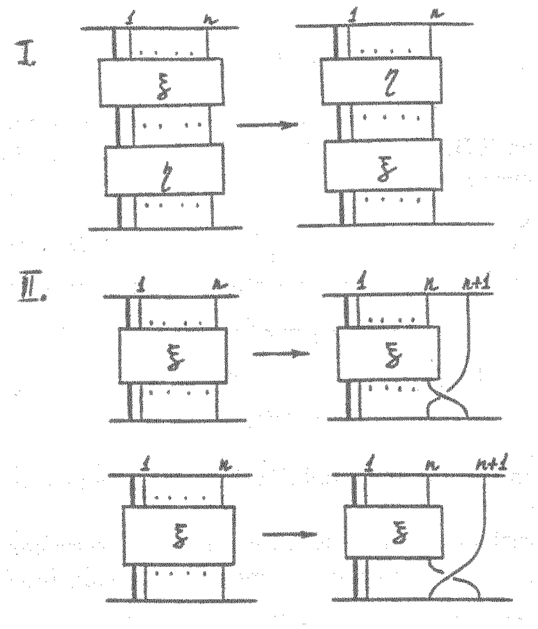

Fig. 10.

Hence from the point of view of these two theorems the constructing **B**-type link invariant consists in two steps. *Step one* is to construct a representation of $B_{1,n}$ taking its values in the relevant analogue of the Temperley–Lieb algebra for **B**-type links. *Step two* is to construct the Markov move invariant defined on this reperesentation. Such Markov move invariant regarding as the trace function on the corresponding

Hecke algebra of **B**-type can be considered as the natural extension of the Kauffman bracket to the case of **B**-type links in the spirit of the ideas of R. Häring–Oldenburg and T. tom Dieck [5, 7, 8].

To give a construction of the Temperley–Lieb algebra of **B**-type $\mathbf{TB_n}$ one must "to add" to the set of the generators 1_n, U_1, ... , U_{n-1} of $\mathbf{TA_n}$ an additional generator U_0, whose graphical interpretation is shown in Fig. 11.

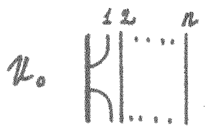

Fig. 11.

Hence at the level of the *formal*[2] calculations the Temperley–Lieb algebra of **B**-type $\mathbf{TB_n}$ is generated by 1_n, U_0, U_1, ... , U_{n-1}[3], with respect to the following rules:

$$U_iU_jU_i = U_i, \qquad \text{for} \quad |i-j| = 1, \quad 1\le i, j\le n-1; \tag{2.1}$$

$$U_iU_j = U_jU_i, \qquad \text{for} \quad |i-j|\ge 2, \quad 0\le i, j\le n-1; \tag{2.2}$$

$$U_1U_0U_1 = D^{'}U_1; \tag{2.3}$$

$$U_i^2 = dU_i, \qquad \text{for} \quad 1\le i\le n-1; \tag{2.4}$$

$$U_0^2 = DU_0, \tag{2.5}$$

where D, $D^{'}$ are indeterminates, and d is the value assigned to the unknot (closed loop).

We will work with an extended Kauffman bracket, so we have to choose the parameters in the presentation of Temperley–Lieb algebra (of **B**-type) as follows:

$$d = -(A^2 + A^{-2}),$$

and the formulas for the σ_i, σ_i^{-1} (for $1\le i\le n-1$) are remained unchanged:

$$\rho(\sigma_i) = AU_i + A^{-1}\cdot 1_n, \quad \rho(\sigma_i^{-1}) = A^{-1}U_i + A\cdot 1_n. \tag{2.6}$$

[2]This means that we are doing the calculations involving the formal linear combinations of diagrams.

[3]We will denote the generators of $\mathbf{TB_n}$ by the same letters as for **A**-type.

The bracket polynomial for link having no winding around the hole (up to the normalization factor) is equivalent to the Jones polynomial of the link.

Let us derive a calculus on **B**-type links which involves an additional generator τ appearing in the definition of braid group in solid torus. For this aim we define an *extension* of Kauffman bracket in the form proposed in [8]:

$$\rho(\tau) = xU_0 + y\cdot 1_n, \quad \rho(\tau^{-1}) = x^{'}U_0 + y^{'}\cdot 1_n, \tag{2.7}$$

where x, $x^{'}$, y, $y^{'}$ are the additional parameters which we have to choose in order to satisfy:

(1) reflection equation:

$$\rho(\tau)\rho(\sigma_1)\rho(\tau)\rho(\sigma_1) = \rho(\sigma_1)\rho(\tau)\rho(\sigma_1)\rho(\tau);$$

(2) consistency property:

$$\rho(\tau)\rho(\tau^{-1}) = \rho(\tau^{-1})\rho(\tau) = 1_n.$$

Using this property one can see that we must seek the solution in the form (2.7), where x, $x^{'}$, y (with $y^{'} = y^{-1}$) are related to the additional parameter D by the equation:

$$D = -\frac{y^{-1}}{x^{'}} - \frac{y}{x}. \tag{2.8}$$

Of course, as mentioned above, the parameters x, $x^{'}$, y cannot be independent variables since they are related to each other by the reflection equation (which can be solved by means of the relations in Temperley–Lieb algebra of **B**-type). Before solving the reflection equation we have to note that the solution $\rho(\tau)$ which we want to find must have the following *Hecke algebra* property:

$$x^{'}\rho(\tau) - x\rho(\tau^{-1}) = (x^{'}y - xy^{-1})\cdot 1_n, \tag{2.9}$$

which means the quadratic relation on $\rho(\tau)$ **4** which is nothing but the characteristic equation for the solution of the reflection equation (see [15, 16]).[4] In other words, we seek the solution $\rho(\tau)$ of the reflection equation which lies in the Hecke algebra of **B**-type (see [7, 8]).

[4] The quadratic relation on $\rho(\tau)$ immediately follows from the definition of the extended Kauffman bracket.

To find a dependence between x, y, D, D'[5] we have to substitute the general expressions for $\rho(\sigma_1)$ and $\rho(\tau)$:

$$\rho(\sigma_1) = AU_1 + A^{-1}\cdot 1_n, \quad \rho(\tau) = xU_0 + y\cdot 1_n,$$

into the reflection equation:

$$\begin{aligned}(xU_0 + y\cdot 1_n)(AU_1 + A^{-1}\cdot 1_n)(xU_0 + y\cdot 1_n)(AU_1 + A^{-1}\cdot 1_n) = \\ (AU_1 + A^{-1}\cdot 1_n)(xU_0 + y\cdot 1_n)(AU_1 + A^{-1}\cdot 1_n)(xU_0 + y\cdot 1_n).\end{aligned} \tag{2.10}$$

One can deduce easily from (2.10) that the sufficient condition for $\rho(\tau)$ to be the solution of the reflection equation is:

$$x^2 A^2 D' + x^2 D + xyA^2 d + 2xy = 0. \tag{2.11}$$

Substituting $d = -(A^2 + A^{-2})$, we have the relation between x, y, D, D' and A:

$$y = x\cdot\frac{A^2 D' + D}{A^4 - 1}. \tag{2.12}$$

Below we discuss two different regimes of the behavior of $\rho(\tau)$.

1. $\mathbf{R^2}$-matrix solution. In this case we have $D' = 1$, and $D := d = -(A^2 + A^{-2})$. For special parameters we have:

$$y = \frac{xA^{-2}}{1 - A^4}, \qquad \text{and} \quad x = 1 - A^4, \quad y = A^{-2}.$$

Thus for $\rho(\tau)$, $\rho(\tau^{-1})$ one can find:

$$\rho(\tau) = (1 - A^4)U_0 + A^{-2}\cdot 1_n, \quad \rho(\tau^{-1}) = (1 - A^{-4})U_0 + A^2\cdot 1_n. \tag{2.13}$$

Remark. The reader can find easily that these formulas agree with the Kauffman bracket considered above. Of course, one can construct **B**-type link invariant starting from the $\mathbf{R^2}$-matrix solution of the reflection equation. But in this case one can not find essentially new results. In fact, one can find that such invariants coincide (up to the normalization) with the link invariant derived from the Kauffman bracket using the well known embedding $B_{1,n} \subset B_{n+1}$.

[5] We have to note that x' is related to x, y, D via (2.8).

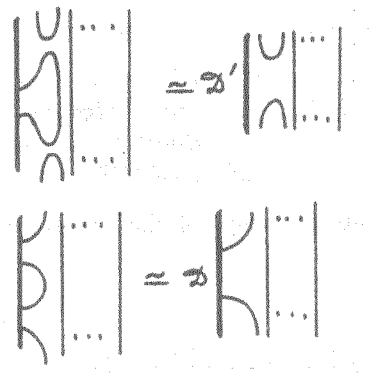

Fig. 12.

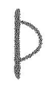

Fig. 13.

2. Deformation of $\mathbf{R}^2$-matrix solution. In this case we can take $D = D'$ which is motivated by the observation that the rules (2.3) and (2.5) (involving the parameters D' and D) can be rewritten graphically as shown in Fig. 12.

In such sense the graph with bonds attaching to the boundary[6] as shown in Fig. 13
acts as the multiplication by D. In this case our previous formula is rewritten as follows:

$$y = \frac{xD}{A^2 - 1}. \tag{2.14}$$

If we take $x = 1 - A$, then $y = A^{-1/2}$. Thus we obtain the concrete solution of the reflection equation:

$$\rho(\tau) = (1 - A)U_0 + A^{-1/2}\cdot 1_n, \quad \rho(\tau^{-1}) = (1 - A^{-1})U_0 + A^{1/2}\cdot 1_n, \tag{2.15}$$

with coefficients from the polynomial ring $\mathbf{Z}[A^{1/2}, A^{-1/2}]$. The generator

[6] The usefulness of the reflection equation for the integrable systems with non-periodic boundary conditions was discovered in [25].

U_0 satisfies the deformed rules (2.3) and (2.5) with $D = D' = -(A^{1/2} + A^{-1/2})$.

Such extension of the Kauffman bracket does not coincide with the $\mathbf{R}^2$-matrix solution. It will be shown in the next Section that this extended Kauffman bracket leads to the construction of **B**-type link invariant.

3. Constructing B-Type Link Invariants

In this Section we deal with the special solution of the reflection equation with coefficients from the ring $\mathbf{Z}[A^{1/2}, A^{-1/2}]$. Our objective here is to construct **B**-type link invariant starting from this solution.

To give such construction we need to formulate (once again) the notion of the Temperley–Lieb algebras of **B**-type.

Definition 2. *The Temperley–Lieb algebra of* **B**-*type on n strands* $\mathbf{TB_n}$ *over* $\widetilde{\mathbf{Z}}[A^{1/2}, A^{-1/2}]$ *is the free additive algebra with multiplicative generators* 1_n, U_0, U_1, $\ldots$, U_{n-1} *subject to the relations (2.1)–(2.5) with* $d := -(A^2 + A^{-2})$ *and* $D = D' := -(A^{1/2} + A^{-1/2})$.

The graphical interpretation of the generators of $\mathbf{TB_n}$ is given in Fig. 14 after [8].

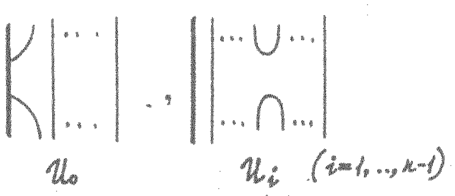

Fig. 14.

We will assume that $A^{1/2}$, $A^{-1/2}$ commute with all elements of $\mathbf{TB_n}$. (Here $\widetilde{\mathbf{Z}}[A^{1/2}, A^{-1/2}]$ denotes the set of rational functions P/Q with $P, Q \in \mathbf{Z}[A^{1/2}, A^{-1/2}]$).

Now we are in a position to give an extension of Kauffman bracket for mixed link. Consider unoriented mixed link diagram $\mathfrak{K}$ representing as the closure of some mixed braid as shown in Fig. 15.

Recall that the closure of the first string **I** is the fixed component $\hat{\mathbf{I}}$ of such mixed link $\mathfrak{K}$. Divide the set of crossings of $\mathfrak{K}$ into two parts: the crossings relating to σ_i and their inverses (the crossings of **A**-type) and

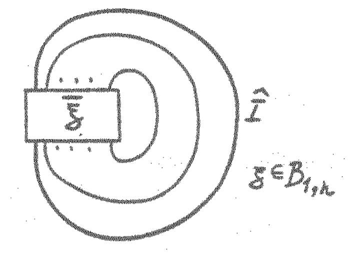

Fig. 15.

the crossings relating to the generator τ and its inverse (the crossings of **B**-type)[7]. Then we associate to each crossing of **A**-type the smoothing of two types as earlier and to each crossing of **B**-type the smoothing of two types as shown in Fig. 16.

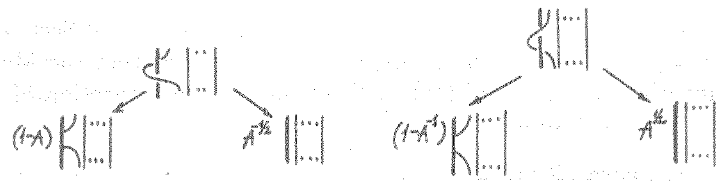

Fig. 16.

By a state $\mathfrak{S}$ of diagram $\mathfrak{K}$ we mean the choice of smoothing for each crossing (of **A**- or **B**-type) in $\mathfrak{K}$. The *weight* $<\mathfrak{K}|\mathfrak{S}>$ is called the product of the vertex weights. The number of disjoint Jordan curves having no bonds on $\hat{\mathbf{I}}$ (which is called the *boundary*) obtained in such way is denoted by $||\mathfrak{S}||_{\mathbf{A}}$, and the number of the "half-unknots" having two bonds lying on the boundary is denoted by $||\mathfrak{S}||_{\mathbf{B}}$. The *extended bracket polynomial* $<\mathfrak{K}>_{ext} \in \mathbf{Z}[A^{1/2}, A^{-1/2}]$ is defined by the following state sum formula:

$$<\mathfrak{K}>_{ext} = \sum_{\mathfrak{S}} <\mathfrak{K}|\mathfrak{S}> d^{||\mathfrak{S}||_{\mathbf{A}}} D^{||\mathfrak{S}||_{\mathbf{B}}}, \tag{3.1}$$

[7] We will assume that the directions of arrows in the figures representing mixed links are from the top to the bottom.

where the sum is taken over all states of diagram $\mathfrak{K}$, and $d = -(A^2+A^{-2})$, $D = -(A^{1/2}+A^{-1/2})$. The extended bracket polynomial takes its values in polynomial ring $\mathbf{Z}[A^{1/2}, A^{-1/2}]$.

Following the strategy developing for constructing invariants of **A**-type links our method in constructing **B**-type link invariant consists in two steps. *Step 1* is to construct a representation of the braid group $B_{1,n}$ in a solid handlebody. *Step 2* is to construct a Markov move invariant defined on this representation.

Step 1. The extended Kauffman bracket (on the level of braids) is defined by the formulas:

$$\rho(\sigma_i) := AU_i + A^{-1}\cdot 1_n, \quad \rho(\sigma_i^{-1}) := A^{-1}U_i + A\cdot 1_n,$$
$$\rho(\tau) := (1-A)U_0 + A^{-1/2}\cdot 1_n, \quad \rho(\tau^{-1}) := (1-A^{-1})U_0 + A^{1/2}\cdot 1_n.$$

Such defined extended Kauffman bracket forms an underlying representation with values in $\mathbf{TB_n}$

$$\rho : B_{1,n} \longrightarrow \mathbf{TB_n}$$

(with parameters $d = -(A^2+A^{-2})$, $D = -(A^{1/2}+A^{-1/2})$). This fact immediately follows from the discussion given in Sect. 1 and Sect. 2.

Step 2. Following to our strategy we have to construct the Markov move invariant (in sense of [18]) defined on the representation of $B_{1,n}$ which is described above.

Main Theorem. *The value*

$$I_{ext}(\xi) := (-A^3)^{e(\xi)} < \overline{\xi} >_{ext} \tag{3.2}$$

is Markov move invariant (in the sense of solid torus braid group), and, henceforth, is **B***-type link invariant*[8]*. Here $e(\xi)$ means an algebraic sum of exponents (involving σ_i and their inverses only: $e(\tau) = 0$) in the expression for $\xi \in B_{1,n}$, and $\overline{\xi}$ is a link representing as the closure of a braid $\xi \in B_{1,n}$.*

Proof. We have to verify the invariance of $I_{ext}(\xi)$ under the type I and the type II Markov moves for $B_{1,n}$.

1. Invariance under the conjugation (or: under the type I Markov move).

[8] We will call this invariant on the level of oriented **B**-type links by the same name I_{ext}.

Since $w(\eta^{-1}\xi\eta) = w(\xi)$, for any braids $\xi, \eta \in B_{1,n}$, then one have to verify that:

$$< \overline{\eta^{-1}\xi\eta} >_{ext} = < \overline{\xi} >_{ext},$$

for any two braids $\xi, \eta \in B_{1,n}$. Representing the element $\eta \in B_{1,n}$ as the product of σ_i, τ and their inverses it follows by induction that it is sufficient to verify that:

$$< \overline{\sigma_i^{-1}\xi\sigma_i} >_{ext} = < \overline{\xi} >_{ext}, \tag{3.3}$$

for any $\xi \in B_{1,n}$ and $1 \leq i \leq n-1$;

$$< \overline{\tau^{-1}\xi\tau} >_{ext} = < \overline{\xi} >_{ext}, \tag{3.4}$$

for any $\xi \in B_{1,n}$.

The graphical proof of (3.3) is essentially the same as for the case of the group B_n and is straightforward. The proof of (3.3) is left to the reader. The graphical proof of (3.4) using the definition of the special solution of the reflection equation is shown in Fig. 17[9].

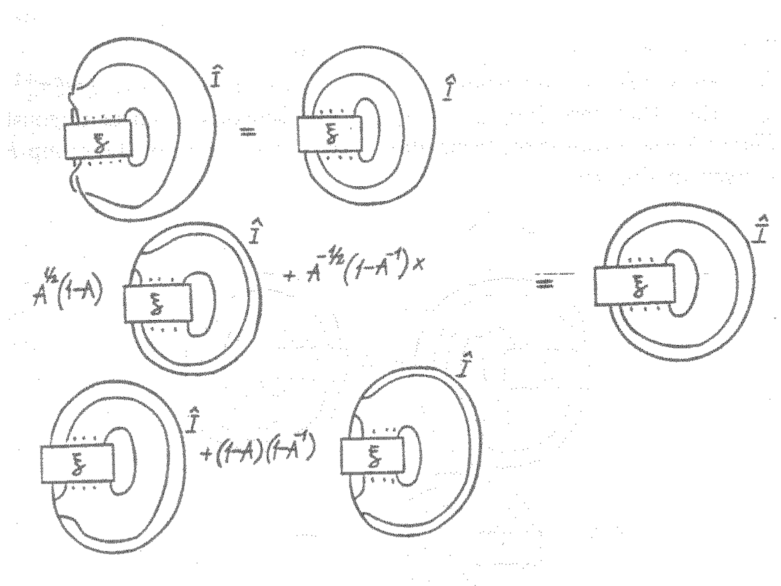

Fig. 17.

[9] We will omit the bars in the graphical proofs.

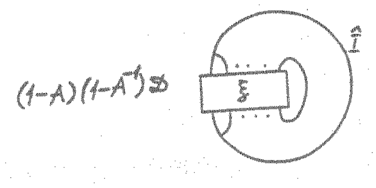

Fig. 18.

since the last term in the right hand side is shown in Fig. 18
and

$$A^{-1/2}(1-A) + A^{1/2}(1-A) + (1-A)(1-A^{-1})D = 0,$$

since $D = -(A^{1/2} + A^{-1/2})$.

2. Invariance under the type II Markov moves.

We can verify the invariance of $I_{ext}(\xi)$ under the move $\xi \simeq \xi\sigma_n^{+1} \in B_{1,n+1}$; the other case $\xi \simeq \xi\sigma_n^{-1} \in B_{1,n+1}$ can be considered analogously.

The pictorial proof is quite similar to that in the case of the group B_n. It is given in Fig. 19.

The arguments for the proof are the same as in Sect. 1. Really, $e(\xi\sigma_n^{+1}) = e(\xi) + 1$, then

$$\begin{aligned} I_{ext}(\xi\sigma_n^{+1}) =& (-A^3)^{e(\xi\sigma_n^{+1})} < \overline{\xi\sigma_n^{+1}} >_{ext} = \\ & (-A^3)^{(e(\xi)+1)} \cdot (-A^{-3}) < \overline{\xi} >_{ext} = \\ & (-A^3)^{e(\xi)} < \overline{\xi} >_{ext} = I_{ext}(\xi). \end{aligned}$$

This completes the proof of Main Theorem. •

Main Theorem can be viewed as the ground for constructing **B**-type link invariants via the state sum expansion.

The invariant for solid torus link which is given by Main Theorem satisfies the skein relations which will be discussed below.

4. Skein Relations for $\mathbf{I_{ext}}$

The representation $\rho : B_{1,n} \rightarrow \mathbf{TB_n}$ which defines the extended bracket has the following skein properties:

$$A^{-1}\rho(\sigma_i) - A\rho(\sigma_i^{-1}) = (A^{-2} - A^2) \cdot 1_n, \tag{4.1}$$

$$A^{-1/2}\rho(\tau) + A^{1/2}\rho(\tau^{-1}) = (A^{-1} + A) \cdot 1_n. \tag{4.2}$$

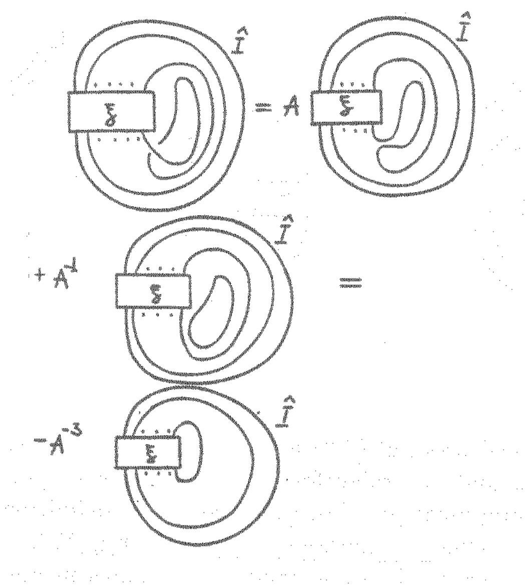

Fig. 19.

These relations are basic tool in deriving the skein relations for I_{ext}.

The property (4.1) leads to the skein relation satisfied by the Jones polynomial. Rewrite the equation (4.1) in terms of I_{ext}:

$$-A^{-4}I_{ext}\Big(\qquad\Big)+A^{4}I_{ext}\Big(\qquad\Big)=(A^{-2}-A^{2})I_{ext}\Big(\qquad\Big). \quad (4.3)$$

Rewriting this relation in terms of variable $t=A^{-4}$:

$$t^{-1}I_{ext}\Big(\qquad\Big)-tI_{ext}\Big(\qquad\Big)=(\sqrt{t}-\frac{1}{\sqrt{t}})I_{ext}\Big(\qquad\Big), \quad (4.4)$$

one can deduce that I_{ext} satisfies the same skein relation as the Jones polynomial (see [11]).

The property (4.2) leads to the so called *clasp rule* (in terminology of [18]) for **B**-type link invariant. Rewrite the equation (4.2) in terms of I_{ext}:

$$A^{-1/2}I_{ext}(\mathfrak{M}_+)+A^{1/2}I_{ext}(\mathfrak{M}_-)=(A^{-1}+A)I_{ext}(\mathfrak{M}_0), \quad (4.5)$$

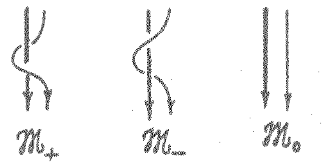

Fig. 20.

where $\mathfrak{M}_+$, $\mathfrak{M}_-$, $\mathfrak{M}_0$ are the oriented mixed links having the identical diagrams, except the regions which are depicted in Fig. 20.

Such skein relation is very reminiscent to the clasp rule

$$x^{-1}W_{\widetilde{\mathfrak{M}}_+} + xW_{\widetilde{\mathfrak{M}}_-} = \alpha W_{\mathfrak{M}_0}, \tag{4.6}$$

derived by J. Hoste and M. Kidwell (see [6]) for a "new chromatic skein invariant"[10].

Remark. The relationship between Hoste–Kidwell approach and the aspects of constructing **B**-type link invariant is mentioned in [18].

To relate their approach with our case $W = I_{ext}$ we find immediately that one must take:

$$x = A^{-1/2}, \qquad \alpha = A^{-1} + A.$$

We have to remark that the invariant W constructed by J. Hoste and M. Kidwell [6] has an extra skein relation, namely, *connected sum rule.* In the Lampropoulou approach such skein relation does not appear since "the component $\hat{\mathbf{I}}$ of a mixed link as well as the string $\mathbf{I}$ of a mixed braid remain always pointwise fixed" (see [18]).

To obtain the complete picture of our invariant we need to calculate the initial data. Recall that I_{ext} takes its values in the polynomial ring $\mathbf{Z}[A^{1/2}, A^{-1/2}]$.

We have:

$$I_{ext}(\mathbf{1}) = -(A^2 + A^{-2}).$$

The value of I_{ext} on τ can be calculated graphically as shown in Fig. 21.

$$I_{ext}(\tau) = (1 - A)(-(A^{1/2} + A^{-1/2})) + A^{-1/2}(-(A^2 + A^{-2})) = \\ - A^{-3/2}(A + A^{-1}).$$

[10] In (4.6) we use the notations suggesting in [18] for clasp rule deriving in [6]. Our notations for $\mathfrak{M}_+$, $\mathfrak{M}_-$ differ from $\widetilde{\mathfrak{M}}_+$, $\widetilde{\mathfrak{M}}_-$, namely, $\mathfrak{M}_+ = \widetilde{\mathfrak{M}}_-$, $\mathfrak{M}_- = \widetilde{\mathfrak{M}}_+$.

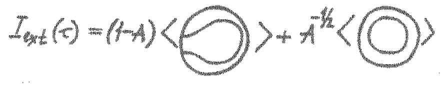

Fig. 21.

Thus the initial data for this **B**-type link invariant are:

$$I_{ext}(\mathbf{1}) = -(A^2 + A^{-2}), \qquad I_{ext}(\tau) = -A^{-3/2}(A + A^{-1}). \tag{4.7}$$

With these initial data and skein relations this **B**-type link invariant is defined by the induction, and it can be viewed as the analogue of the Jones polynomial for solid torus links.

Conclusion

In this paper we find a method for constructing solid torus (or: **B**-type) link invariants by means of the Kauffman bracket extension for such links. This method can be viewed as the combination of the ideas proposed in [5, 7, 8] with **B**-type braid group theoretic approach which is given in [18]. Such method is shown to be compatible with skein relations of **B**-type and leads to state sum expression for link invariant (in the case of the special diagrams).

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