Записки научных
семинаров ПОМИ
Том 259, 1999 г.

G. A. Seregin

PARTIAL REGULARITY FOR SOLUTIONS TO THE MODIFIED NAVIER–STOKES EQUATIONS

Abstract The initial-boundary value problem for the modified Navier–Stokes equations is considered in the case of the homogeneous Dirichlet boundary conditions. Under some assumptions partial regularity for its solution is proved. It is shown that Hausdorff's dimension of the set of singular points is not greater than three.

1991 Mathematical subject classification (Amer. Math. Soc.): 35K, 76D.

1. Inroduction

In [1] sufficient conditions of local regularity for solutions to the 3D modified Navier–Stokes equations were established. These equations were inroduced by O. A. Ladyzhenskaya in [2–4] for the description of incompressible fluids under large values of the velocity gradient. With the help of the criterition mentioned above partial regularity for solutions to initial-boundary value problems for the modified Navier–Stokes equations was proved in the case of periodical boundary conditions. It was also shown in [1] that Hausdorff's dimension of the set of singular points is not greater that three. In the present paper we are going to demonstrate that the same statement remains to be valid in the case of the homogeneous Dirichlet boundary conditions. For the Navier–Stokes equations, partial regularity of so-called suitable weak solutions was established in [5].

2. Main Results

We consider the following initial-boundary value problem for the modified Navier–Stokes equations. It includes the equations of motion

$$\left.\begin{aligned} &\partial_t v + \operatorname{div}(v\otimes v) - \operatorname{div}\sigma = g - \nabla p,\\ &\operatorname{div} v = 0,\\ &\sigma = \Phi(\varepsilon(v)), \end{aligned}\right\} \quad \text{in} \quad Q_T \tag{2.1}$$

This research was partially supported by INTAS, grant No. 96-0835, and by RFFI, grants Nos. 96-15-96121, 99-01-00108.

the boundary condition

$$v = 0 \quad \text{on} \quad \partial\Omega \times [0, T], \tag{2.2}$$

and the initial condition

$$v|_{t=0} = a \quad \text{in} \quad \Omega. \tag{2.3}$$

Here Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary, T is a positive parameter, $Q_T \equiv \Omega \times]0, T[$ is the space-time cylinder, $v : Q_T \to \mathbb{R}^3$ and $p : Q_T \to \mathbb{R}$ are unknown fields of velocity and pressure, σ is the deviator of the stress tensor, $g : Q_T \to \mathbb{R}^3$ and $\Phi : \mathbb{S}^3 \to \mathbb{S}^3$ are given functions, $\mathbb{S}^3$ is the space of all real symmetric 3×3 matrices, $\varepsilon(v)$ is the symmetric part of the velocity gradient ∇v.

Concerning the function Φ we assume that

$$\left.\begin{aligned} &\Phi : \mathbb{S}^3 \to \mathbb{S}^3 \quad \text{is a function of class} \quad C^1, \\ &\Big|\frac{\partial \Phi}{\partial \varepsilon}(\varepsilon)\Big| \le \nu_1(1 + |\varepsilon|^{m-2}), \quad \nu_1 = \text{constant} > 0, \\ &\nu(1 + |\varepsilon|^{m-2})|\kappa|^2 \le \Big(\frac{\partial \Phi}{\partial \varepsilon}(\varepsilon)\kappa\Big) : \kappa, \quad \nu = \text{constant} > 0, \end{aligned}\right\} \tag{2.4}$$

for all $\varepsilon \in \mathbb{S}^3$ and $\kappa \in \mathbb{S}^3$ and for some number

$$m \ge 2.$$

We recall that the scalar product of two 3×3 matrices A and B is denoted by $A : B \equiv \operatorname{tr} A^* B = A_{ij}B_{ij}$ ($|A| \equiv \sqrt{A : A}$), and we adopt summation over repeated Latin indices, running from 1 to 3. The scalar product of two vectors a and b is denoted by $a \cdot b \equiv a_i b_i$ ($|a| \equiv \sqrt{a \cdot a}$).

To formulate the theorem on global unique solvability of initial-boundary value problem (2.1)–(2.3) we introduce the following fuctional spaces:

$$\dot{C}^\infty(\Omega) \equiv \{ v \in C_0^\infty(\Omega; \mathbb{R}^3) \,||\, \quad \operatorname{div} v = 0 \quad \text{in} \quad \Omega \};$$

$J(\Omega)$ is the closure of the set $\dot{C}^\infty(\Omega)$ in the Lebesgue space $L_2(\Omega; \mathbb{R}^3)$;

$\overset{\circ}{J}{}^1_m(\Omega)$ is the closure of the set $\dot{C}^\infty(\Omega)$ in the Sobolev space $W^1_m(\Omega; \mathbb{R}^3)$.

We also assume that

$$a \in \overset{\circ}{J}{}^1_m(\Omega), \tag{2.6}$$

$$g \in L_2(\Omega;\mathbb{R}^3) \tag{2.7}$$

and there is a potential $D_0 : \mathbb{R}_+ \to \mathbb{R}_+$ of class C^2, satisfying the conditions

$$\left.\begin{array}{l} \Phi(\varepsilon) = 2D_0'(|\varepsilon|^2)\varepsilon, \quad \forall \varepsilon \in \mathbb{S}^3, \\ D_0(0) = 0, \quad \Phi(0) = 0. \end{array}\right\} \tag{2.7}$$

It is known the following result (see [6])

Theorem 2.1. *Suppose that conditions* (2.4)–(2.7) *hold and let, in addition,*

$$m \geq \frac{9}{4}. \tag{2.8}$$

Then there is a unique pair of functions v and p, possessing the properties:

$$\left.\begin{array}{l} v \in C([0,T]; J(\Omega)) \cap L_\infty(0,T; \overset{\circ}{J}{}^1_m(\Omega)), \\ \partial_t v \in L_2(Q_T;\mathbb{R}^3), \quad p \in L_{\frac{m}{m-1}}(Q_T), \\ \int\limits_\Omega p(t)\,dx = 0, \quad t \in]0,T[, \end{array}\right\} \tag{2.9}$$

$$\int\limits_{\Omega' \times]0,T[} (1 + |\varepsilon(v)|^{m-2})|\nabla\,\varepsilon(v)|^2 dxdt < +\infty \tag{2.10}$$

for all $\Omega' \Subset \Omega$ and satisfying initial-boundary conditions (2.2), (2.3) *and the variational identity*

$$\begin{array}{l} \int\limits_\Omega \left[\partial_t v(t)\cdot w + \Phi(\varepsilon(v(t))) : \varepsilon(w)\right] dx = \int\limits_\Omega \big[g(t)\cdot w + \\ +(v(t)\otimes v(t)) : \nabla\, w + p(t)\mathrm{div}\, w\big]\, dx, \quad \forall \in \overset{\circ}{W}{}^1_m(\Omega;\mathbb{R}^3), \end{array} \tag{2.11}$$

for a.a. $t \in [0,T]$.

We shall use a "parabolic" variant of Hausdorff's measure. Let A be an arbitrary set in $\mathbb{R}^3 \times \mathbb{R}$. Then we define

$$\mathcal{P}_k(A) \equiv \lim_{\delta\to 0} \mathcal{P}_k^\delta(A),$$

where

$$\mathcal{P}_k^\delta(A) \equiv \inf\Big\{\sum_i R_i^k \;||\quad A \subset \bigcup_i Q(x_i,t_i,R_i), \quad R_i < \delta\Big\},$$

$$Q(x,t,R) \equiv B(x,R) \times]t-R^2,t[,$$

$$B(x,R) \equiv \{y \in \mathbb{R}^3 \mid\mid \quad |y-x| < R\}.$$

We shall prove the following statement

Theorem 2.2. *Suppose that all conditions of Theorem 2.1 hold. Let, in addition,*

$$\partial_t g \in L_2(Q_T;\mathbb{R}^3) \tag{2.12}$$

and there are positive numbers ν_0 *amd* α_0 *such that*

$$\text{ess sup}_{t_0-R^2\leq t\leq t_0} \frac{1}{|B(x_0,R)|} \int\limits_{B(x_0,R)} |g(t)|^2\,dx \leq \nu_0 R^{2\alpha_0-2} \tag{2.13}$$

for all $Q(x_0,t_0,R) \Subset Q_T$.

Suppose also that

$$\frac{12}{5} \leq m < \frac{10}{3}. \tag{2.14}$$

Then, for an arbitrary number $\alpha \in]2, \frac{m+\frac{4}{3}}{m-1}[$, *the set*

$$Q_T^*(\alpha) \equiv \{(x_0,t_0) \in Q_T \mid\mid$$

there is a sequence of numbers $R_k \to 0$ *such that:*

$$\sup_{k\in\mathbb{N}} \left(|(v)_{x_0,t_0,R_k})| + |(\nabla\, v)_{x_0,t_0,R_k})|\right) < +\infty, \quad Y(x_0,t_0,R_k) \to 0\}$$

is open. Moreover, the function $(x,t) \longmapsto \nabla\, v(x,t)$ *is Hölder continuous in some neighborhood of each point of the set* $Q_T^*(\alpha)$, *and Hausdorff's dimension of the set* $Q_T - Q_T^*(\alpha)$ *is not greater than three, i.e.*

$$\mathcal{P}_{3+\gamma}(Q_T - Q_T^*(\alpha)) = 0 \tag{2.15}$$

for all $\gamma > 0$.

To formulate Theorem 2.2 we exploited the following notation

$$Y(x_0,t_0,R) \equiv \Big(\quad \fint\limits_{Q(x_0,t_0,R)} |\nabla\, v - (\nabla\, v)_{x_0,t_0,R}|^m\,dxdt\Big)^{\frac{1}{m}} +$$

$$+\Big(\quad \fint\limits_{Q(x_0,t_0,R)} |v - (\nabla\, v)_{x_0,t_0,R}(x-x_0) - (v)_{x_0,t_0,R}|^m\,dxdt\Big)^{\frac{1}{m}} +$$

$$+\Big(\frac{1}{R^2}\int\limits_{t_0-R^2}^{t_0} dt\Big(\quad\fint\limits_{B(x_0,R)} |p(t)-(p(t))_{x_0,R}|^2\,dx\Big)^{\frac{\alpha}{2}}\Big)^{\frac{1}{\alpha}},$$

$$(g)_{x_0,t_0,R}\equiv\fint\limits_{Q(x_0,t_0,R)} g\,dxdt\equiv\frac{1}{|Q(x_0,t_0,R)|}\int\limits_{Q(x_0,t_0,R)} g\,dxdt,$$

$$(f)_{x_0,R}\equiv\fint\limits_{B(x_0,R)} f\,dx\equiv\frac{1}{|B(x_0,R)|}\int\limits_{B(x_0,R)} f\,dx.$$

3. Sufficient Conditions for Local Regularity of Solutions to the Modified Navier–Stokes Equations

The proof of Theorem 2.2 is based on application of sufficient conditions for local regularity of solutions to the modified Navier–Stokes equations that were established in [1].

Let us consider a pair of functions $u : Q_T \to \mathbb{R}^3$ and $q : Q_T \to \mathbb{R}$, which have the following differential properties

$$\left.\begin{aligned}&u\in L_\infty(0,T;W^1_2(\Omega;\mathbb{R}^3))\cap L_m(0,T;W^1_m(\Omega;\mathbb{R}^3)),\\&(1+|\varepsilon(u)|)\frac{m-2}{2}|\nabla\,\varepsilon(u)|\in L_2(Q_T),\end{aligned}\right\}\tag{3.1}$$

$$q\in L_{\bar\alpha}(0,T;L_2(\Omega)),\quad \bar\alpha>2,\tag{3.2}$$

and satisfy the modified Navier–Stokes equations

$$\left.\begin{aligned}&\partial_t u+\operatorname{div}(u\otimes u)-\operatorname{div}\Phi(\varepsilon(u))=g-\nabla\,q,\\&\operatorname{div}u=0,\end{aligned}\right\}\quad\text{in}\quad Q_T\tag{3.3}$$

in the sense of distributions.

Theorem 3.1. *Suppose that conditions* (2.4), (2.6), (2.13), (2.14) *hold. Let a pair of functions u and q satisfy conditions* (3.1)–(3.3).

Assume that, for some $\alpha\in]0,\bar\alpha]\cap]2,\frac{m+\frac{4}{3}}{m-1}[$ and for some point $(x_0,t_0)\in Q_T$, there is sequence $R_k\to 0$ such that:

$$\sup_{k\in\mathbb{N}}\big(|(u)_{x_0,t_0,R_k})|+|(\nabla\,u)_{x_0,t_0,R_k})|\big)<+\infty\tag{3.4}$$

and

$$\Big(\quad\fint\limits_{Q(x_0,t_0,R_k)} |u-(\nabla\,u)_{x_0,t_0,R_k}(x-x_0)-(u)_{x_0,t_0,R_k}|^m\,dxdt\Big)^{\frac{1}{m}}+$$

$$+\Big(\ \ \fint\limits_{Q(x_0,t_0,R_k)} |\nabla u-(\nabla u)_{x_0,t_0,R_k}|^m\,dxdt\Big)^{\frac{1}{m}}+ \tag{3.5}$$

$$+\Big(\frac{1}{R_k^2}\int\limits_{t_0-R_k^2}^{t_0} dt\Big(\ \ \fint\limits_{B(x_0,R_k)} |p(t)-(p(t))_{x_0,R_k}|^2\,dx\Big)^{\frac{\alpha}{2}}\Big)^{\frac{1}{\alpha}}\to 0.$$

Then the function $(x,t)\longmapsto\nabla u(x,t)$ *is Hölder continuous in some neighborhood* $\mathcal{O}$ *of the point* (x_0,t_0).

Remark 3.2. Analyzing the proof of Theorem 3.1 (see [1]), one can observe that relations (3.4) and (3.5) are fulfilled for all points of the neighborhood $\mathcal{O}$.

4. Proof of Theorem 2.2. Part I

In this section we shall prove that, for any $\alpha\in]2,\frac{m+\frac{4}{3}}{m-1}$, the set $Q_T^*(\alpha)$ is open.

So, let a point $(x_0,t_0)\in Q_T^*(\alpha)$ is chosen arbitrarily and fix. Further, we choose subdomain $\Omega'\Subset\Omega$ with smooth boundary and a number δ such that $x_0\in\Omega'$ and $0<\delta<t_0$. If we show that

$$p'\equiv p-\frac{1}{|\Omega'|}\int\limits_{\Omega'} p\,dx\in L_{\frac{m+\frac{4}{3}}{m-1}}(\delta,T;L_2(\Omega')), \tag{4.1}$$

then our claim will be proved. Indeed, by Theorem 2.1, we get from (4.1) that the pair of functions v and p' meet all conditions of Theorem 3.1 in the cylinder $Q'_{\delta,T}\equiv\Omega'\times]0,T[$ for $\bar\alpha=\frac{m+\frac{4}{3}}{m-1}$. Therefore, the function $(x,t)\longmapsto\nabla v(x,t)$ is Hölder continuous in some neighborhood of the point (x_0,t_0), and, by Remark 3.1, this neighborhood belongs to the set $Q_T^*(\alpha)$.

Now, let us prove (4.1). Since Ω' has the smooth boundary, one can use Korn's inequality and get

$$\Big(\int\limits_{\Omega'}|v(t)|^{3m}\,dx\Big)^{\frac{1}{6}}\le c_1(m,\Omega')\Big\{\Big(\int\limits_{\Omega'}|\varepsilon(v(t))|^{3m}\,dx\Big)^{\frac{1}{6}}+$$

$$+\Big(\int\limits_{\Omega'}|v(t)|^2\,dx\Big)^{\frac{m}{4}}\Big\}.$$

Next, Gagliardo's inequality implies

$$\Big(\int\limits_{\Omega'}|\varepsilon(v(t))|^{\frac{m}{2}6}\,dx\Big)^{\frac{1}{6}}\le c_2(m,\Omega')\Big[\Big(\int\limits_{\Omega'}|\nabla\left(|\varepsilon(v(t))|^{\frac{m}{2}}\right)|^2\,dx\Big)^{\frac{1}{2}}+$$

$$+\Big(\int\limits_{\Omega'}|\varepsilon(v(t))|^{m}\,dx\Big)^{\frac{1}{2}}\Big]\le$$

$$\le c_2'(m,\Omega')\Big[\Big(\int\limits_{\Omega'}(1+|\varepsilon(v(t))|^{m-2})|\nabla\,\varepsilon(v(t))|^2\,dx\Big)^{\frac{1}{2}}+$$

$$+\Big(\int\limits_{\Omega'}|\varepsilon(v(t))|^{m}\,dx\Big)^{\frac{1}{2}}\Big].$$

Taking into account statements of Theorem 2.1 (see (2.9)), we get

$$\Big(\int\limits_{\Omega'}|\nabla\,v(t)|^{3m}\,dx\Big)^{\frac{1}{3}}\le$$

$$\le c_3(m,\Omega',v)\Big[1+\int\limits_{\Omega'}(1+|\varepsilon(v(t))|^{m-2})|\nabla\,\varepsilon(v(t))|^2\,dx\Big]. \tag{4.2}$$

Hölder's inequality leads to the estimate

$$\int\limits_{\Omega'}|\nabla\,v(t)|^{\frac{5}{3}m}\,dx\le\Big(\int\limits_{\Omega'}|\nabla\,v(t)|^{m}\,dx\Big)^{\frac{2}{3}}\Big(\int\limits_{\Omega'}|\nabla\,v(t)|^{3m}\,dx\Big)^{\frac{1}{3}}.$$

Making use of Theorem 2.1 (see (2.9), (2.10)) and (4.2), we establish the inequality

$$\int\limits_{\Omega'\times]0,T[}|\nabla\,v(t)|^{\frac{5}{3}m}\,dx\,dt\le c_4(m,\Omega',v). \tag{4.3}$$

Conditions (2.4) and (2.7), in particular, lead to the relation

$$\Phi(\varepsilon)\le\bar\nu_1(|\varepsilon|+|\varepsilon|^{m-1}),\quad\forall\varepsilon\in\mathbb{S}^3, \tag{4.4}$$

for some positive constant $\bar\nu_1$. From this and also from (2.14) and (4.3) it follows the estimate

$$\int\limits_{\Omega'\times]0,T[}|\Phi(\varepsilon(v(t)))|^2\,dx\,dt\le c_5(m,\Omega',v). \tag{4.5}$$

Let us remark, in addition, that the following multiplicative inequality takes place

$$\Big(\int\limits_{\Omega} |v(t)|^4\,dx\Big)^{\frac{1}{4}} \le c_6(m)\Big(\int\limits_{\Omega} |v(t)|^2\,dx\Big)^{\frac{1}{5}}\Big(\int\limits_{\Omega} |\nabla\, v(t)|^{\frac{12}{5}}\,dx\Big)^{\frac{1}{4}}. \qquad (4.6)$$

For the pressure p', we have the identity

$$\left.\begin{aligned}&\int\limits_{\Omega'} p'(t)\mathrm{div}\, w\,dx = \int\limits_{\Omega'}\Big\{\partial_t v(t)\cdot w - (v(t)\otimes v(t)) : \nabla\, w+\\ &+\Phi(\varepsilon(v(t))) : \varepsilon(w)\Big\}\,dx, \quad \forall \in C_0^\infty(\Omega').\end{aligned}\right\} \qquad (4.7)$$

Combining relations (4.5)–(4.7), (2.9), we claim (see [7]) that

$$p' \in L_2(Q'_T), \quad Q'_T \equiv \Omega'\times]0,T[,$$

and, moreover, for a.a. $t\in]0,T[$, the inequality

$$\left.\begin{aligned}&\int\limits_{\Omega'} |p'(t)|^2\,dx \le c_8(m,\Omega')\int\limits_{\Omega'}\Big(|\Phi(\varepsilon(v(t)))|^2 + |\partial_t v|^2+\\ &+|v(t)|^4 + |g(t)|^2\Big)\,dx\end{aligned}\right\} \qquad (4.8)$$

is valid.

Repeating the proof of Lemma 6.1 in [1], one can derive the following estimate

$$\left.\begin{aligned}&\text{ess sup}_{\delta\le t\le T}\int\limits_{\Omega} |\partial_t v(t)|^2\,dx+\\ &+\int\limits_{\Omega\times]\delta,T[}(1+|\varepsilon(v)|^{m-2})|\varepsilon(\partial_t v)|^2\,dx\,dt+\\ &+\int\limits_{\Omega\times]\delta,T[}|\partial_t v|^{\frac{10}{3}}\,dx\,dt \le c_9(\delta,v,g) < +\infty, \quad \forall\delta\in]0,T[.\end{aligned}\right\} \qquad (4.9)$$

Taking into account inequality (4.4), we obtain from (4.8) the relation

$$\int\limits_{\delta}^{T} dt\Big(\int\limits_{\Omega'} |p'(t)|^2\,dx\Big)^{\frac{m+\frac{4}{3}}{m-1}} \le c_{10}(m,\Omega')\Big\{\int\limits_{Q'_{\delta,T}}\Big[|\nabla\, v|^{m+\frac{4}{3}}+$$

$$+|\nabla\, v|^{\frac{m+\frac{4}{3}}{m-1}} + |\partial_t v|^{\frac{m+\frac{4}{3}}{m-1}} + |v|^{2\frac{m+\frac{4}{3}}{m-1}}\Big]\, dx\, dt +$$

$$+T\mathrm{ess\ sup}_{0\le t\le T}\Big(\int\limits_\Omega |g(t)|\, dx\Big)^{\frac{1}{2}\frac{m+\frac{4}{3}}{m-1}}\Big\}.$$

It remains to remark that $\frac{m+\frac{4}{3}}{m-1} \le \frac{10}{3} \le \frac{5}{3}m$ and $m+\frac{4}{3}\le\frac{5}{3}m$ if $m\ge 2$,

$$\int\limits_\Omega |v(t)|^{2\frac{m+\frac{4}{3}}{m-1}}\, dx \le c_{11}(m)\Big(\int\limits_\Omega |\nabla\, v(t)|^2\, dx\Big)^{\frac{7}{2(m-1)}}\Big(\int\limits_\Omega |v(t)|^2\, dx\Big)^{\frac{6m-13}{6(m-1)}} \le$$

$$\le (\text{see } (2.9)) \le c_{12}(m,v) < +\infty$$

for a.a. $t\in]0,T[$, and to deduce (4.1) from (4.9) and (4.3). So, the first part of Theorem 2.2 is proved.

5. Proof of Theorem 2.2. Part II

In this section we shall prove identity (2.15).

It is easy to see that

$$Q_T - Q^*_T(\alpha) \subset \bigcup_{i=1}^{4} A_i, \tag{5.1}$$

where

$$A_1 \equiv \Big\{(x_0,t_0)\in Q_T \,||\quad \limsup_{R\to 0}\Big[|(v)_{x_0,t_0,R}| + |(\nabla\, v)_{x_0,t_0,R}|\Big] = +\infty\Big\},$$

$$A_2 \equiv \Big\{(x_0,t_0)\in Q_T \,||\quad \limsup_{R\to 0}\quad \fint\limits_{Q(x_0,t_0,R)} |\nabla\, v - (\nabla\, v)_{x_0,t_0,R}|^m\, dx\, dt > 0\Big\},$$

$$A_3 \equiv \Big\{(x_0,t_0)\in Q_T \,||$$

$$\limsup_{R\to 0}\quad \fint\limits_{Q(x_0,t_0,R)} |v - (\nabla\, v)_{x_0,t_0,R}(x-x_0) - (v)_{x_0,t_0,R}|^m\, dx\, dt > 0\Big\},$$

$$A_4 \equiv \Big\{(x_0,t_0)\in Q_T \,||$$

$$\limsup_{R\to 0}\frac{1}{R^2}\int\limits_{t_0-R^2}^{t_0} dt\Big(\fint\limits_{B(x_0,R)} |p(t) - (p(t))_{x_0,R}|^2\, dx\Big)^{\frac{\alpha}{2}} > 0\Big\}.$$

So, by (5.1), we have

$$\mathcal{P}_{3+\gamma}(Q_T - Q_T^*(\alpha)) \leq \sum_{i=1}^{4} \mathcal{P}_{3+\gamma}(A_i). \tag{5.2}$$

Let us start with evaluation of $\mathcal{P}_{3+\gamma}(A_1)$. It is known that if $f \in W^1_{2,loc}(Q_T)$ and, for some $\gamma > 0$,

$$\limsup_{R\to 0} \frac{1}{R^{3+\gamma}} \int\limits_{Q(x_0,t_0,R)} \Big(|\nabla f|^2 + R^2|\partial_t f|^2\Big)\,dx\,dt < +\infty,$$

then there exixts

$$\lim_{R\to 0} \fint\limits_{Q(x_0,t_0,R)} f\,dx\,dt.$$

This can be derived from the inequality

$$\Big|\frac{d}{dR}(f)_{x_0,t_0,R}\Big| \leq \int\limits_{Q(x_0,t_0,R)} \Big(|\nabla f| + 2R|\partial_t f|\Big)\,dx\,dt.$$

But then (see, for instance, [5]) one may claim that

$$\mathcal{P}_{3+\gamma}(A_1) = 0, \quad \forall \gamma > 0. \tag{5.3}$$

Since $m < \frac{10}{3}$, the embedding theorem implies (see [LSU])

$$\left.\begin{aligned} &\Big(\fint\limits_{Q(x_0,t_0,R)} |\nabla v - (\nabla v)_{x_0,t_0,R}|^m\,dx\,dt\Big)^{\frac{1}{m}} \leq \\ &\leq c_{14}(m)R\Big(\fint\limits_{Q(x_0,t_0,R)} [|\nabla^2 v|^2 + |\partial_t v|^2]\,dx\,dt\Big)^{\frac{1}{2}}. \end{aligned}\right\} \tag{5.4}$$

From (5.4) it follows that

$$\mathcal{P}_3(A_2) = 0. \tag{5.5}$$

By (5.4) and Poincaré's inequlity, we have

$$\Big(\fint\limits_{Q(x_0,t_0,R)} |v - (\nabla v)_{x_0,t_0,R}(x-x_0) - (v)_{x_0,t_0,R}|^m\,dx\,dt\Big)^{\frac{1}{m}} \leq$$

$$\leq c_{15}(m)R^2\Big(\ \fint\limits_{Q(x_0,t_0,R)} [|\nabla^2 v|^2 + |\partial_t v|^2]\,dx\,dt\Big)^{\frac{1}{2}}$$

and, therefore,

$$\mathcal{P}_1(A_3) = 0. \tag{5.6}$$

Let $B(x_0, R) \Subset \Omega$. Arguing as in the previous section (see the proof of (4.1)), we show that $p(t)-(p(t))_{x_0,R} \in L_2(B(x_0,R))$ and, moreover, (see [7]) a function $u_R(t) \in \overset{\circ}{W}{}^1_2(B(x_0,R);\mathbb{R}^3)$ exists such that

$$\left.\begin{aligned}
&\operatorname{div} u_R(t) = p(t) - (p(t))_{x_0,R} \quad \text{in} \quad B(x_0,R),\\
&\frac{1}{R^2}\int\limits_{B(x_0,R)} |u_R(t)|^2\,dx + \int\limits_{B(x_0,R)} |\nabla u_R(t)|^2\,dx \leq\\
&\leq c_{16}\int\limits_{B(x_0,R)} |p(t) - (p(t))_{x_0,R}|^2\,dx.
\end{aligned}\right\} \tag{5.7}$$

We insert $w = u_R(t)$ into (2.11) and make use of (5.7). As a result, we get

$$\left.\begin{aligned}
&\int\limits_{B(x_0,R)} |p(t) - (p(t))_{x_0,R}|^2\,dx \leq\\
&\leq \Big\{c_{17}\int\limits_{B(x_0,R)} |\Phi(\varepsilon(v(t))) - \Phi((\varepsilon(v))_{x_0,t_0,R})|^2\,dx +\\
&+ R^2\int\limits_{B(x_0,R)} (|\partial_t v(t)|^2 + |v(t)|^2|\nabla v(t)|^2 + |g(t)|^2)\,dx\Big\}.
\end{aligned}\right\} \tag{5.8}$$

Recalling (2.4), we see that

$$\Big|\Phi(\varepsilon(v(t))) - \Phi((\varepsilon(v))_{x_0,t_0,R})\Big| =$$

$$= \Big|\int\limits_0^1 \frac{\partial\Phi}{\partial\varepsilon}\Big((\varepsilon(v))_{x_0,t_0,R} + \theta_1(\varepsilon(v(t)) - (\varepsilon(v))_{x_0,t_0,R})\Big)(\varepsilon(v(t))-$$

$$-(\varepsilon(v))_{x_0,t_0,R})\,d\theta_1\Big| \leq \nu_1\Big[1 + (|(\varepsilon(v))_{x_0,t_0,R}|+$$

$$+|\varepsilon(v(t)) - (\varepsilon(v))_{x_0,t_0,R}|)^{m-2}\Big]|\varepsilon(v(t)) - (\varepsilon(v))_{x_0,t_0,R}| \leq$$

$$\leq 2\nu_1(1+|(\varepsilon(v))_{x_0,t_0,R}|^{m-2})(1+|\varepsilon(v(t))-$$
$$-(\varepsilon(v))_{x_0,t_0,R}|^{m-2})|\varepsilon(v(t))-(\varepsilon(v))_{x_0,t_0,R}|,$$

and thus

$$\left.\begin{aligned}
&Y_3(x_0,t_0,R)\equiv\Big(\frac{1}{R^2}\int\limits_{t_0-R^2}^{t_0}dt\Big(\fint\limits_{B(x_0,R)}|p(t)-(p(t))_{x_0,R}|^2\,dx\Big)^{\frac{\alpha}{2}}\Big)^{\frac{1}{\alpha}}\\
&\leq c_{18}(\alpha,m,\nu_1)\Big\{(1+|(\varepsilon(v))_{x_0,t_0,R}|^{m-2})\times\\
&\Big[\Big(\fint\limits_{Q(x_0,t_0,R)}|\varepsilon(v(t))-(\varepsilon(v))_{x_0,t_0,R}|^{\alpha}\,dx\,dt\Big)^{\frac{1}{\alpha}}+\\
&+\Big(\fint\limits_{Q(x_0,t_0,R)}|\varepsilon(v(t))-(\varepsilon(v))_{x_0,t_0,R}|^{(m-1)\alpha}\,dx\,dt\Big)^{\frac{1}{\alpha}}\Big]+\\
&+R\Big(\fint\limits_{Q(x_0,t_0,R)}|\partial_t v|^{\alpha}\,dx\,dt\Big)^{\frac{1}{\alpha}}+R\Big(\fint\limits_{Q(x_0,t_0,R)}|v|^{\alpha}|\nabla v|^{\alpha}\,dx\,dt\Big)^{\frac{1}{\alpha}}+\\
&+R\Big(\frac{1}{R^2}\int\limits_{t_0-R^2}^{t_0}dt\Big(\fint\limits_{B(x_0,R)}|g(t)|^2\,dx\Big)^{\frac{\alpha}{2}}\Big)^{\frac{1}{\alpha}}\Big\}
\end{aligned}\right\}\tag{5.9}$$

for all $\alpha\in]2,\frac{m+\frac{4}{3}}{m-1}[$. Condition (2.13) give us

$$\left.\begin{aligned}
&R\Big(\frac{1}{R^2}\int\limits_{t_0-R^2}^{t_0}dt\Big(\fint\limits_{B(x_0,R)}|g(t)|^2\,dx\Big)^{\frac{\alpha}{2}}\Big)^{\frac{1}{\alpha}}\leq\\
&\leq R\ \mathrm{ess\ sup}_{t_0-R^2\leq t\leq t_0}\Big(\fint\limits_{B(x_0,R)}|g(t)|^2\,dx\Big)^{\frac{\alpha}{2}}\Big)^{\frac{1}{\alpha}}\leq\sqrt{\nu_0}R^{\alpha_0}.
\end{aligned}\right\}\tag{5.10}$$

We let

$$Y_+(x_0,t_0,R)\equiv\Big(\fint\limits_{Q(x_0,t_0,R)}|\varepsilon(v)-(\varepsilon(v))_{x_0,t_0,R}|^{m+\frac{4}{3}}\,dx\,dt\Big)^{\frac{1}{m+\frac{4}{3}}}.$$

Since $\frac{m+\frac{4}{3}}{m-1}\leq\frac{2}{3}$ if $m\geq\frac{12}{5}$, we derive from (5.9) and (5.10) the following relation

$$Y_3(x_0,t_0,R)\leq c_{18}\Big\{(1+|(\nabla v)_{x_0,t_0,R}|^{m-2})\Big[Y_+(x_0,t_0,R)+$$

$$+Y_+^{m-1}(x_0,t_0,R)\Big] + R\Big(\ \ \fint\limits_{Q(x_0,t_0,R)} |\partial_t v|^{\frac{8}{3}}\,dx\,dt\Big)^{\frac{3}{8}} + \tag{5.11}$$

$$+R\Big(\ \ \fint\limits_{Q(x_0,t_0,R)} |v|^{\frac{8}{3}}|\nabla\, v|^{\frac{8}{3}}\,dx\,dt\Big)^{\frac{3}{8}} + \sqrt{\nu_0}R^{\alpha_0}\Big\}.$$

Now, we remark

$$A_4 \subset A_1 \cup B_1 \cup B_2,$$

where

$$B_1 \equiv \Big\{(x_0,t_0) \in Q_T \ ||\quad \limsup_{R\to 0} Y_+(x_0,t_0,R) > 0\Big\},$$

$$B_2 \equiv \Big\{(x_0,t_0) \in Q_T \ ||$$

$$\limsup_{R\to 0} \frac{1}{R^{\frac{7}{3}}} \int\limits_{Q(x_0,t_0,R)} \Big(|\partial_t v|^{\frac{8}{3}} + |v|^{\frac{8}{3}}|\nabla\, v|^{\frac{8}{3}}\Big)\,dx\,dt > 0\Big\}$$

and, therefore,

$$\mathcal{P}_{3+\gamma}(A_4) \le \mathcal{P}_{3+\gamma}(A_1) + \mathcal{P}_{3+\gamma}(B_1) + \mathcal{P}_{3+\gamma}(B_2). \tag{5.12}$$

Hölder inequality implies

$$\int\limits_\Omega |v|^{\frac{8}{3}}|\nabla\, v|^{\frac{8}{3}}\,dx \le \Big(\int\limits_\Omega |v|^{\frac{8}{3}\frac{5m}{5m-8}}\,dx\Big)^{\frac{5m-8}{5m}}\Big(\int\limits_\Omega |\nabla\, v|^{\frac{5}{3}m}\,dx\Big)^{\frac{8}{5m}}.$$

Next, we make use of the multiplicative inequality

$$\Big(\int\limits_\Omega |v(t)|^{\frac{40m}{3(5m-8)}}\,dx\Big)^{\frac{3(5m-8)}{40m}} \le c_{19}(m)\Big(\int\limits_\Omega |\nabla\, v(t)|^{\frac{12}{5}}\,dx\Big)^{\frac{5}{12}\delta}\times$$

$$\times\Big(\int\limits_\Omega |v(t)|^2\,dx\Big)^{\frac{1}{2}(1-\delta)}, \quad \delta = 3\frac{5m+24}{50m}.$$

But then, by (2.9)and (4.3),

$$\int\limits_{\Omega'\times]0,T[} |v|^{\frac{8}{3}}|\nabla\, v|^{\frac{8}{3}}\,dx\,dt \le c_{20}(\Omega') < +\infty$$

for any $\Omega' \Subset \Omega$ and, therefore, according to (2.9), we have

$$\mathcal{P}_{\frac{7}{3}}(B_2) = 0. \tag{5.13}$$

Now, let us remark that the following three relations take place:

$$\int\limits_{B(x_0,R)} |\varepsilon(v(t)) - (\varepsilon(v))_{x_0,t_0,R}|^{m+\frac{4}{3}}\,dx \leq$$

$$\leq \Big(\int\limits_{B(x_0,R)} |\varepsilon(v(t)) - (\varepsilon(v))_{x_0,t_0,R}|^2\,dx\Big)^{\frac{2}{3}} \times$$

$$\times \Big(\int\limits_{B(x_0,R)} |\varepsilon(v(t)) - (\varepsilon(v))_{x_0,t_0,R}|^{3m}\,dx\Big)^{\frac{1}{3}},$$

$$\Big(\int\limits_{B(x_0,R)} |\varepsilon(v(t)) - (\varepsilon(v))_{x_0,t_0,R}|^{3m}\,dx\Big)^{\frac{1}{3}} \leq$$

$$\leq c_{21}(m)\Big\{ \int\limits_{B(x_0,R)} |\nabla\,(|\varepsilon(v(t)) - (\varepsilon(v))_{x_0,t_0,R}|^{\frac{m}{2}})|^2\,dx +$$

$$+\frac{1}{R^2} \int\limits_{B(x_0,R)} |\varepsilon(v(t)) - (\varepsilon(v))_{x_0,t_0,R}|^m\,dx\Big\} \leq$$

$$\leq c_{21}'(m)\Big\{ \int\limits_{B(x_0,R)} |\varepsilon(v(t)) - (\varepsilon(v))_{x_0,t_0,R}|^{m-2} |\nabla\,\varepsilon(v)|^2\,dx + \ldots\Big\}$$

and

$$\sup_{t_0-R^2\leq t\leq t_0} \int\limits_{B(x_0,R)} |\varepsilon(v(t)) - (\varepsilon(v))_{x_0,t_0,R}|^2\,dx \leq$$

$$\leq 2\Big\{\frac{1}{R^2} \int\limits_{Q(x_0,t_0,R)} |\varepsilon(v) - (\varepsilon(v))_{x_0,t_0,R}|^2\,dx\,dt +$$

$$+R^2 \int\limits_{Q(x_0,t_0,R)} |\varepsilon(\partial_t v)|^2\,dx\,dt\Big\}.$$

They together with (5.4) give:

$$Y_+^{m+\frac{4}{3}}(x_0,t_0,R)\leq c_{22}(m)\times$$

$$\times\Big[\int\limits_{Q(x_0,t_0,R)}(|\nabla^2 v|^2+|\partial_t v|^2)\,dx\,dt+R^2\int\limits_{Q(x_0,t_0,R)}|\varepsilon(\partial_t v)|^2\,dx\,dt\Big]^{\frac{2}{3}}\times$$

$$\times\Big[\ \fint\limits_{Q(x_0,t_0,R)}|\varepsilon(v(t))-(\varepsilon(v))_{x_0,t_0,R}|^{m-2}|\nabla\,\varepsilon(v)|^2\,dx\,dt+$$

$$+\frac{1}{R^2}\fint\limits_{Q(x_0,t_0,R)}|\nabla\,v-(\nabla\,v)_{x_0,t_0,R}|^m\,dx\,dt\Big]\leq$$

$$\leq c_{22}'(m)\Big[\int\limits_{Q(x_0,t_0,R)}(|\nabla^2 v|^2+|\partial_t v|^2+R^2|\varepsilon(\partial_t v)|^2)\,dx\,dt\Big]^{\frac{2}{3}}\times$$

$$\times\Big[(1+|(\nabla\,v)_{x_0,t_0,R}|^{m-2})\fint\limits_{Q(x_0,t_0,R)}(1+|\varepsilon(v)|^{m-2})|\nabla\,\varepsilon(v)|^2\,dx\,dt+$$

$$+R^{m-2}\Big(\fint\limits_{Q(x_0,t_0,R)}(|\nabla^2 v|^2+|\partial_t v|^2)\,dx\,dt\Big)^{\frac{m}{2}}\Big].$$

Setting

$$h\equiv(1+|\varepsilon(v)|^{m-2})|\nabla\,\varepsilon(v)|^2+|\nabla^2 v|^2+|\partial_t v|^2+R^2|\varepsilon(\partial_t v)|^2,$$

we get

$$Y_+^{m+\frac{4}{3}}(x_0,t_0,R)\leq c_{23}(m)\Big[(1+|(\nabla\,v)_{x_0,t_0,R}|^{m-2})\times$$

$$\times\Big(\frac{1}{R^3}\int\limits_{Q(x_0,t_0,R)}h\,dx\,dt\Big)^{\frac{5}{3}}+\Big(\frac{1}{R^3}\int\limits_{Q(x_0,t_0,R)}h\,dx\,dt\Big)^{\frac{3m+4}{6}}\Big].$$

The latter means that

$$B_1\subset A_1\cup B_3,$$

where

$$B_3\equiv\Big\{(x_0,t_0)\in Q_T\ ||\quad\limsup_{R\to 0}\frac{1}{R^3}\int\limits_{Q(x_0,t_0,R)}h\,dx\,dt>0\Big\}.$$

It is clear that $\mathcal{P}_3(B_3) = 0$ and

$$\mathcal{P}_{3+\gamma}(B_1) \leq \mathcal{P}_{3+\gamma}(A_1) + \mathcal{P}_{3+\gamma}(B_3).$$

From this and also from (5.2), (5.3), (5.5), (5.6), (5.12), (5.13) statement (2.15) of Theorem 2.2 follows.

References

1. G. A. Seregin, *Interior regularity for solutions to the modified Navier–Stokes equations.* to appear in Journal of Mathematical Fluid Mechanics.
2. O. A. Ladyzhenskaya, *On nonlinear problems of continuum mechanics.* Proc. Internat. Congr. Math. Moscow (1966), "Nauka", Moscow, 1968, pp. 560-573; English transl. in Amer. Math. Soc. Transl. (2), 70 (1968).
3. O. A. Ladyzhenskaya, *New equations for the description of motion of viscous incompressible fluids and global solvability of boundary value problems for them.* Trudy Mat. Inst. Steklov, **102** (1967), 85-104; English transl. in Proc. Steklov Inst. Math., **102** (1967).
4. O. A. Ladyzhenskaya, *On some modifications of the modified Navier–Stokes equations for the large gradients of velocity.* Zap. Nauchn. Semin. LOMI, **7** (1968), 126–154; English transl. in Sem. Math. V.A. Steklov Math. Inst. Leningrad 7 (1968).
5. L. Caffarelli, R.-V. Kohn, L. Nirenberg, *Partial regularity of suitable weak solutions of the Navier–Stokes equations.* Comm. Pure Appl. Math., **XXXV** (1982), 771–831.
6. J. Málek, J. Nečas, M. Růžička, *On weak solutions to a class of non-Newtonian incompressible fluids in bounded three-dimesional domains. The case* $\geqslant 2$. to appear in Advances in Differential Equations.
7. O. A. Ladyzhenskaya, V. A. Solonnikov, *Some problems of vector analysis, and generalized formulations of boundary value problems for the Navier–Stokes equations.* Zap. Nauchn. Semin. LOMI, **59** (1976), 81–116; English transl. in J. Soviet Math. 10(1978), No. 2.
8. O. A. Ladyzhenskaya, V. A. Solonnikov, N. N. Uraltseva,, *Linear and quasilinear equations of parabolic type.* "Nauka", Moscow, 1967; English transl., Transl. Math. Monogr., vol. 23, Amer. Math. Soc., Providence, RI, 1968.

V. A. Steklov Institute of Mathematics
at Saint Petersburg,
Fonatka 27, 191011 St.-Petersburg ,
Russia

Поступило 19 июля 1999 г.