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$C^{1,\alpha}$-SOLUTIONS TO A CLASS OF NONLINEAR FLUIDS IN TWO DIMENSIONS-STATIONARY DIRICHLET PROBLEM

ABSTRACT. We prove the global existence of $C^{1,\alpha}$-solutions to a system of nonlinear equations describing steady planar motions of a certain class of non-Newtonian fluids including in particular various variants of the power-law models. We study the Dirichlet problem. The nonlinear operator has a p-potential structure. If $p > 3/2$ we construct global $C^{1,\alpha}$-solutions up to the boundary, while for $p > 6/5$ solutions with interior $C^{1,\alpha}$-regularity are obtained. A proof of global higher regularity is outlined. Uniqueness of $C^{1,\alpha}$-solutions within the class of weak solutions is also proved assuming the smallness of data.

1. Introduction

We investigate the following problem of nonlinear fluid mechanics:[1] to construct $v = (v_1, v_2) : \Omega \to \mathbb{R}^2$ and $\pi : \Omega \to \mathbb{R}$ solving

$$\begin{gathered} v_k \frac{\partial v}{\partial x_k} - \operatorname{div}(T(D(v))) + \nabla \pi = f \quad \text{in} \quad \Omega, \\ \operatorname{div} v = 0 \quad \text{in} \quad \Omega, \qquad v = 0 \quad \text{at} \quad \partial\Omega, \end{gathered} \tag{1.1}$$

and satisfying for a certain $\alpha > 0$

$$\begin{aligned} &(\alpha) \qquad v \in C^{1,\alpha}(\overline{\Omega}), \quad \pi \in C^{0,\alpha}(\overline{\Omega}) \ \ \text{(global regularity result)} \\ &\text{or} \\ &(\beta) \qquad v \in C^{1,\alpha}_{\text{loc}}(\Omega), \quad \pi \in C^{0,\alpha}_{\text{loc}}(\Omega) \ \ \text{(interior regularity result).} \end{aligned} \tag{1.2}$$

We put the following assumptions on data Ω, f, and T:

(a) $\Omega \subset \mathbb{R}^2$ is a bounded open set; the smoothness of the boundary $\partial\Omega$ is stated precisely in the formulation of theorems below.

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[1] The summation convention is used throughout the whole paper.

(b) $f = (f_1, f_2) : \Omega \to \mathbb{R}^2$ is given; for a given $p \in (1,2)$ we usually assume $f \in L^{p'}(\Omega)$, where $p' = p/(p-1)$; for a given $p \geqslant 2$ we assume $f \in L^{2+\varepsilon_0}(\Omega)$ for a certain $\varepsilon_0 > 0$.

(c) Let $\mathbb{S}$ be the set of symmetric matrices of the type 2×2. Then $D(v) \in \mathbb{S}$ denotes the symmetrized gradient of v with the components $\frac{1}{2}\big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\big)$, $i, j = 1, 2$. The nonlinear tensor-valued function $T : \mathbb{S} \to \mathbb{S}$ is assumed to satisfy[2]

(i) there exists $F : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that $F \in C^2(\mathbb{R}_0^+)$ and for all i, $j = 1, 2$

$$T_{ij}(\eta) = \partial_{ij} F(|\eta|^2) = 2F'(|\eta|^2)\eta_{ij},$$
$$\forall\, \eta \in \mathbb{S}, \quad F(0) = \partial_{ij} F(0) = 0; \tag{1.3}$$

(ii) for a certain $p \in (1, \infty)$ there exists C_1, $C_2 > 0$ such that

$$\partial_{ij}\partial_{kl} F(|\eta|^2)\xi_{ij}\xi_{kl} \geqslant C_1(1 + |\eta|^2)^{\frac{p-2}{2}}|\eta|^2, \quad \forall\, \eta, \quad \xi \in \mathbb{S}; \tag{1.4}$$

and (for all i, j, k, $l = 1, 2$)

$$\big|\partial_{ij}\partial_{kl} F(|\eta|^2)\big| \leqslant C_2(1 + |\eta|^2)^{\frac{p-2}{2}}, \quad \forall\, \eta \in \mathbb{S}. \tag{1.5}$$

It is not difficult to observe consequences of (1.3)–(1.5) (see [16], Sec. 5.1 for a proof of it): there exist C_i, $i = 3, 4, 5$, such that[3]

$$T(\eta) \cdot \eta \geqslant C_4\left((1 + |\eta|^2)^{\frac{p}{2}} - 1\right), \quad \forall\, \eta \in \mathbb{S}, \tag{1.6}$$

$$|T(\eta)| \leqslant C_5\left(1 + |\eta|^2\right)^{\frac{p-2}{2}} |\eta|, \quad \forall\, \eta \in \mathbb{S}, \tag{1.7}$$

$$(R(\eta) - T(\xi)) \cdot (\eta - \xi) \geqslant C_5|\eta - \xi^2|^2, \quad \forall\, \xi \in \mathbb{S}$$

$$\text{with} \quad C_5 = C_5(\eta, \xi) \equiv C_1 \int_0^1 \left(1 + |\xi + s(\eta - \xi)|^2\right)^{\frac{p-2}{2}} \mathrm{d}s \tag{1.8}$$

and trivially also

$$T(0) = 0 \quad \text{and} \quad \partial_{ij} T_{kl}(0) = \nu\delta_{ik}\delta_{jk} \quad \text{with} \quad \nu > 0. \tag{1.9}$$

Assuming only (1.9) (but not any of the conditions (1.3)–(1.8)) then it is not difficult to observe that for a certain $\delta > 0$ there are (v, π) satisfying

[2]Notation: if $Z : \mathbb{S} \to \mathbb{R}$ (or $\mathbb{S}$, then $\partial_{ij} Z \equiv \frac{\partial Z}{\partial \eta_{ij}}$ denotes partial derivative of Z with respect to the independent variable η_{ij}, i, $j = 1, 2$.

[3]If A, $B \in \mathbb{S}$, then $A \cdot B \equiv A_{ij}B_{ij}$.

$(1.2)_1$ provided that $\|f\|_{2+\varepsilon_0} < \delta$. The proof is based on the application of the inverse function theorem. The interested reader can find the proof in [1], where a more difficult evolutionary variant of (1.1) is studied. In this paper we prove (1.2) without any restriction on the size of f, assuming, however, the potential structure of the nonlinearity T.

We primarily deal with the case $p \in (1, 2]$ as it is more interesting case from the point of view of analysis (the convective term $v_k \frac{\partial v}{\partial x_k}$ completes with nonlinear elliptic term, which brings the restriction on p's-results dealing with the analysis of these models for $p < 2$ appeared only in the nineties), and it has also important engineering applications in many areas of engineering, as it captures the shear thinning phenomenon exhibited by various fluids/materials as chemical suspensions, colloids, ice, glacier, blood, tectonic plates, etc. Here, we do not provide any list of contributions in which the nonlinear (shear dependent) viscosities are found to be essential to model of properties of materials. Instead we feref to [17] or [10] where such lists of references are given. (In [17] more details on physical and engineering background of the problem is also discussed.) The results for the case $p > 2$ are also stated in the formulations of theorems and reader might consult [11] where details are presented, however, the periodic problem is analyzed, for simplicity.

We show two results here:

$$\begin{aligned} &\text{(i)} \qquad \text{if} \quad p > \frac{6}{5} \quad \text{then} \quad (1.2)\ (\beta)\ \text{holds;} \\ &\text{(ii)} \qquad \text{if} \quad p > \frac{3}{2} \quad \text{then} \quad (1.2)\ (\alpha)\ \text{is proved.} \end{aligned} \tag{1.10}$$

Let us note that if the Dirichlet boundary condition is replaced by the requirement that v and π are periodic at each variable x_1 and x_2 (i.e., Ω is a two-dimensional cube), then (1.2) (α) has been proved for

$$p > 1 \tag{1.11}$$

in [12]. Thus we see that there is a significant difference in the results for Dirichlet and space periodic problem, respectively.

The bounds stated in (1.10) and (1.11) occur while deriving $W^{2,p}$-estimates and estimating the convective term. In the periodic case $\int_\Omega v_k \frac{\partial v_i}{\partial x_k} \Delta v_i \, \mathrm{d}x = 0$. The analogous calculations for the Dirichlet problem yield (ξ is a cut-off function with support inside of Ω needed to fulfil the boundary conditions)

$$-\int_\Omega v_k \frac{\partial v_i}{\partial x_k}\left(\frac{\partial}{\partial x_s}\xi^2\frac{\partial v_i}{\partial x_s}\right)\mathrm{d}x$$
$$= \int_\Omega \frac{\partial v_i}{\partial x_k}\frac{\partial v_k}{\partial x_s}\frac{\partial v_i}{\partial x_s}\xi^2\mathrm{d}x + \int_\Omega v_k\frac{\partial}{\partial x_k}\left(\frac{|\nabla v|^2}{2}\right)\xi^2\mathrm{d}x$$
$$= -\int_\Omega v_k\frac{|\nabla v|^2}{2}\frac{\partial \xi^2}{\partial x_k}\mathrm{d}x, \quad (1.12)$$

where we use the known fact that

$$\frac{\partial v_i}{\partial x_k}\frac{\partial v_k}{\partial x_s}\frac{\partial v_i}{\partial x_s} = 0, \quad (1.13)$$

which follows from the divergence-free conditions (see [23] or [16], p. 226 for the proof of it).

The remaining term in (1.12) can be controlled if $p > 6/5$ and we obtain

$$v\xi \in W^{p,2}(\Omega). \quad (1.14)$$

Then we apply the procedure developed in [12] and based on ideas from [20–22]. See Sec. 2 for a brief description. Then, in Sec. 3, we deal with a nonlinear case when $p = 2$.

For $p < 2$, we construct approximations (v^λ, π^λ) to quadratic problems, i.e., $p^\lambda = 2$, derive (1.14) for v^λ uniformly, and then show that (1.14) leads to (1.2) (β). This is presented in Sec. 4.

If we want to get global regularity (1.2) (α) the cancellation (1.13) does not work near the boundary. Then we have to estimate also the term $\int_\Omega |\nabla v|^3\xi^2\mathrm{d}x$, which is possible if $p > 3/2$. For such p's, we are able to show (1.2) (α), see Sec. 5. In the last section the assertion "$C^{1,\alpha}$-refularity implies C^∞-regularity" is indicated.

Besides of the results in [12] we are aware only of the results in [7, 8], where the question of interior $C^{1,\alpha}$-regularity (in two dimensions) and partial $C^{1,\alpha}$-regularity (up to dimension 4) is investigated; however, the authors restrict themselves to the stress tensor T of the form given by $\mu_\infty D(v) + \mu_0(|D(v)|)D(v)$ with $\mu_\infty > 0$. On the contrary, we can cover also the case $\mu_\infty = 0$ in this paper and in addirion we deal with $C^{1,\alpha}$-regularity near the boundary. (Note that is μ_∞ is positive that the model falls into the case $p = 2$ in this paper.) We wish to emphasize that we do not show that (for $p \neq 2$) every weak solution belongs to $C^{0,\alpha}$; we just present an existence proof. However, if $p > 1$ and $\|f\|_{2+\varepsilon}$ is small ($\varepsilon_0 > 0$) then weak and smooth solutions coincide; uniqueness of this type is shown is Sec. 6.

To our knowledge, nothing is known on $C^{0,\alpha}$-regularity if dimensional $d > 2$ (except of the announced partial regularity result of a particular model mentioned in [7]). We can recall two recent results regarding the global existence theory. In [6] the existence of weak solutions is proved for $p \geqslant \frac{2d}{d+1}$ using the strict monotonicity of the elliptic term and a special construction of bounded test function, while in [4] the authors study (1.1) with non-homogeneous boundary conditions, and under a certain smallness condition on the size of boundary data they prove the existence of weak solutions if $p \geqslant \frac{3d}{d+2}$, and the existence of very weak solutions (satisfying a weak inequality) if $p \geqslant \frac{2d}{d+2}$. Mathematical analysis of steady flows of power-law fluids through a thin slab is presented in [18], including the derivation of the system that captures the behavior of fluid-flow when the thickness of the slab vanishes.

Throughout the whole paper we use a standard notation for Lebesgue spaces $L^p(\Omega)$ and Sobolev spaces $W^{k,p}(\Omega)$, and their norms $\|\cdot\|_p$ and $\|\cdot\|_{k,p}$, respectively. By $W^{-1,p'}(\Omega)$ we mean the dual space to $W_0^{1,p}(\Omega)$, $p' = p/(p-1)$. We do not distinguish notation of spaces of vector-valued, tensor-valued and scalar functions as it should be clear from the context. By V_p we mean the space $\{u \in W_0^{1,p}(\Omega); \operatorname{div} u = 0 \text{ in } \Omega\}$, V_p^* is its dual space.

2. Preliminaries

In this section we deal with weak solutions of a generalized Stokes system, i.e., we assume that $u \in V_2$ and $\widetilde{\pi} \in L^2(\Omega)$, $\int_\Omega \widetilde{\pi} \mathrm{d}x = 0$, solve

$$\begin{aligned}\int_\Omega A_{ij}^{kl}(x) D_{kl}(u(x)) D_{ij}(\varphi(x)) \mathrm{d}x - \int_\Omega \widetilde{\pi}(x) \operatorname{div} \varphi(x) \mathrm{d}x \\ = \int_\Omega G_{ij}(x) D_{ij}(\varphi(x)) \mathrm{d}x, \quad \forall\, \varphi \in W_0^{1,2}(\Omega),\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}&A_{ij}^{kl} \in L^\infty(\Omega), \quad A_{ij}^{kl} = A_{kl}^{ij} \text{ for } i,\, j,\, k,\, l = 1,2, \\ &\exists\, \gamma_1,\, \gamma_2 > 0 \quad \forall\, \eta \in \mathbb{S} \text{ and for a.e.} \\ &x \in \Omega : \gamma_1 |\eta|^2 \leqslant A_{ij}^{kl}(x) \eta_{ij}\, \eta_{kl} \leqslant \gamma_2 |\eta|^2.\end{aligned} \tag{2.2}$$

Note that $\widetilde{\pi}$ vanishes in (2.1) if $\varphi \in V_2$. Observe also that if $A_{ij}^{kl} = \delta_{ik}\,\delta_{jl}$ then (2.1) is a weak form of the Stokes system.

Let us first consider this auxiliary form of the Stokes system, i.e.,

$$\int_\Omega D_{ij}(u(x))D_{ij}(\varphi(x))\mathrm{d}x = \int_\Omega G_{ij}(x)D_{ij}(\varphi(x))\mathrm{d}x \quad \forall\,\varphi \in V_2 \tag{2.3}$$

and estimate certain L^q-norm of the symmetrized gradient $D(u)$ by corresponding norms of G (which is a matrix with components g_{ij}, $i,\,j = 1,2$). Taking $\varphi = u$ in (2.3), we clearly have

$$\|D(u)\|_2 \leqslant \|g\|_2.$$

Using the L^q-theory for the Stokes system (see [2] or [9] for example) and Riesz–Thorin interpolation (cf. [3]) we obtain the following lemma (for a fixed positive ε_0).

Lemma 2.4. *There is a constant $K = K(r) > 1$ such that for any $q \in [2, 2+r)$ and $G_{ij} \in L^q(\Omega)$, $i,\,j = 1,2$, the (unique) weak solution $u \in V_2$ of (2.3) belongs to $W^{1,q}(\Omega)$ and*

$$\|D(u)\|_q \leqslant K^{1-2/q}\|G\|_q. \tag{2.5}$$

Now, we return to the generalized Stokes system (2.1).

Lemma 2.6. *Let $\varepsilon > 0$ be defined through*

$$2 + \varepsilon = 2\left(1 - \log\left[\frac{1 - \frac{\gamma_1}{2\gamma_2}}{1 - \frac{\gamma_1}{\gamma_2}}\right] \Big/ \log(2K)\right)^{-1}, \tag{2.7}$$

where γ_1, γ_2 occur in (2.2) and K is taken from (2.5), Let $q \in (2, 2+\varepsilon]$ be arbitrary. Assume that $G_{ij} \in L^q(\Omega)$, $i,\,j = 1,2$. Then the correspondong solution u of (2.1) belongs to $W^{1,q}(\Omega)$ and

$$\|D(u)\|_q \leqslant \frac{2}{\gamma_1}K^{1-2/q}\|G\|_q.$$

Proof. We rewrite (2.1) into the form ($\operatorname{div}\varphi = 0$)

$$\int_\Omega D_{ij}(u)D_{ij}(\varphi)\mathrm{d}x = \int_\Omega B_{ij}^{kl}D_{kl}(u)D_{ij}(\varphi)\mathrm{d}x + \frac{1}{\gamma_2}\int_\Omega G_{ij}D_{ij}(\varphi)\mathrm{d}x, \tag{2.9}$$

where $B_{ij}^{kl}(x) \equiv \delta_{ik}\delta_{jl} - 1/\gamma^2 A_{ij}^{kl}(x)$. As A is symmetric we have

$$|(B\eta,\xi)| \leqslant |\xi|\ |B\eta| \leqslant |\xi|\ |\eta|\left(1-\frac{\gamma_1}{\gamma_2}\right).$$

This implies that the L^q-norm of the right-hand side of (2.9) can be estimated by $\left(q' = \frac{q}{q-1}\right)$

$$\frac{1}{\gamma^2}\|G\|_q + \sup_{\substack{\psi_{ij}\in L^{q'}(\Omega)\\ \sum_{i,j=1}^2 \|\psi_{ij}\|_{q'=1}^{q'}}} \left(1-\frac{\gamma_1}{\gamma_2}\right)$$

$$\int_\Omega \left(\sum_{k,l=1}^2 |D_{kl}(u)|^2\right)^{1/2}\left(\sum_{i,j=1}^2 |\psi_{ij}|^2\right)^{1/2} \mathrm{d}x.$$

By Hölder inequality, the second term yields

$$\sup_{\substack{\psi_{ij}\in L^{q'}(\Omega)\\ \sum_{i,j=1}^2 \|\psi_{ij}\|_{q'=1}^{q'}}} \left(1-\frac{\gamma_1}{\gamma_2}\right)$$

$$\left(\int_\Omega \left(\sum_{k,l=1}^2 |D_{kl}(u)|^2\right)^{q/2}\mathrm{d}x\right)^{1/q}\left(\int_\Omega\left(\sum_{i,j=1}^2 |\psi_{ij}|^2\right)^{q'/2}\mathrm{d}x\right)^{1/q'}$$

$$\leqslant \left(1-\frac{\gamma_1}{\gamma_2}\right)2^{1-2/q}\left(\int_\Omega \sum_{k,l=1}^2 |D_{kl}(u)|^q \mathrm{d}x\right)^{1/q},$$

where we used $\left(\sum_{i,j=1}^2 \|\psi_{ij}\|^2\right)^{q'/2} \leqslant \sum_{i,j=1}^2 \|\psi_{ij}\|^{q'}$ as $q' \leqslant 2$. By (2.5) we have

$$\|D(u)\|_q$$
$$\leqslant \left(2K(\varepsilon_0)\right)^{1-2/q}\left(1-\frac{\gamma_1}{\gamma_2}\right)\|D(u)\|_q + (K(\varepsilon_0))^{1-2/q}\,\frac{1}{\gamma_2}\|G\|_q.$$

Then condition (2.7) is equivalent to

$$\left(2K(\varepsilon_0)\right)^{1-2/q}\left(1-\frac{\gamma_1}{\gamma_2}\right) = 1-\frac{\gamma_1}{2\gamma_2}.$$

This implies assertion (2.8). $\square$

3. $C^{1,\alpha}$-regularity for $p = 2$

The aim of this section is to prove $C^{1,\alpha}$-regularity for the solution of (1.1) if $p = 2$, see Theorem 3.19 below.

Note first that the operator $\widetilde{T}$ defined by $\langle \widetilde{T}u, \varphi \rangle = \int_\Omega \partial_{ij} F(|D(u)|^2)\, D_{ij}(\varphi)\mathrm{d}x$ maps $W_0^{1,2}(\Omega)$ into $W^{1,2}(\Omega)$ by (1.7). Moreover, due to (1.8), $\widetilde{T}$ is monotone. Thus applying the standard monotone operator theory we obtain the existence of a weak solution stated in the following lemma.

Lemma 3.1. *Let $f \in V_2^*$. Then there is a weak solution $v \in V_2$ to (1.1) satisfying ($\Phi \in V_2$)*

$$\int_\Omega \partial_{ij} F\big(|D(v)|^2\big) D_{ij}(\Phi)dx + \int_\Omega v_k \frac{\partial v_i}{\partial x_k} \Phi_i \, dx = \int_\Omega f_i \, \Phi_i \, dx. \tag{3.2}$$

The corresponding pressure π, normalized by $\int_\Omega \pi \, dx = 0$, belongs then to $L^2(\Omega)$.

Moreover, if Ω is of class C^2 and $f \in L^2(\Omega)$ then every weak solution $v \in V_2$ to (1.1) satisfies

$$v \in W^{2,2}(\Omega) \cap V_2, \quad \pi \in W^{1,2}(\Omega). \tag{3.3}$$

Proof. As stated above, the existence of a weak solution $v \in V_2$ follows from standard monotone operator theory, generally applicable for $p \geqslant \frac{3d}{d+2}$, d is dimension. (The proof can be derived following steps in [13] or [14], combined with $\|\nabla u\|_2 = \sqrt{2}\,\|D(u)\|_2$.)

To prove (3.3), one can incorporate the difference method. We skip the proof, and we feref the reader to [12], where this step is performed for the periodic problem or to [15], Sec. 3, where the same result (including estimates near the boundary) was proved even for a time-dependent model. $\square$

Now, we introduce some notation related to the domain $\Omega \in C^2$. This means that $\partial\Omega$ is locally described by C^2-maps $a_1, a_2, \dots, a_k$, $k \in \mathbb{N}$. In the corresponding ℓth coordinate system ($\ell = 1, 2, \dots, k$), we suppose that for a fixed positive α and $x_1 \in (-\alpha, \alpha)$

$$\begin{aligned} &\text{(i)} \quad (x_1, x_2) \in \partial\Omega \ \text{ if and only if } \ x_2 = a_\ell(x_1),\\ &\text{(ii)} \quad U_\ell^+ = \{(x_1, x_2); x_1 \in (-\alpha, \alpha), x_2 \in (a_\ell(x_1), a_\ell(x_1) + \alpha)\} \subset \Omega,\\ &\text{(iii)} \quad U_\ell^- = \{(x_1, x_2); x_1 \in (-\alpha, \alpha), x_2 \in (a_\ell(x_1) - \alpha, a_\ell(x_1))\} \subset \mathbb{R}^2 \setminus \overline{\Omega}. \end{aligned} \tag{3.4}$$

We also assume that

$$\frac{\partial a_\ell(0)}{\partial x_1} = 0. \tag{3.5}$$

Further, we denote $U_\ell = \{(x_1, x_2);\ x_1 \in (-\alpha, \alpha),\ x_2 \in (a_\ell(x_1) - \alpha, a_\ell(x_1) + \alpha)\}$ and choose an open smooth set U_0 so that

$$\bigcup_{\ell=0}^{k} U_\ell \supset \Omega. \tag{3.6}$$

Let ξ_ℓ ($\ell = 0, 1, \dots, k$) be a partition of unity corresponding to the covering $U_0, U_1, \dots, U_k$ of Ω.

Next, using (3.2) we write the equations of type (2.1) for u playing a role of derivative of v, multiplied by a cut-off function ξ_ℓ, and corrected by z so that u is divergence-free. We distinguish the case of interior regularity and regularity at tangential direction at the boundary. Let $\varphi \in V_2$.

(a) In the case of interior regularity we test (3.2) by

$$\Phi \equiv -\frac{\partial \varphi}{\partial x_r}\xi_0 + \psi^\varphi, \quad r = 1, 2, \tag{3.7}$$

where ψ^φ is such that $\operatorname{div}\Phi = 0$, i.e., (see [2, 5] or [9] for details)

$$\begin{gathered} \operatorname{div}\psi^\varphi = \frac{\partial \varphi_i}{\partial x_r}\frac{\partial \xi_0}{\partial x_i} \ \text{ in } \Omega, \quad \psi^\varphi = 0 \ \text{ at } \partial\Omega, \\ \|\psi^\varphi\|_{1,s} \leqslant C\|\nabla\varphi\|_s \quad s \in (1, \infty). \end{gathered} \tag{3.8}$$

Putting (3.7) into (3.2) leads (after certain manipulations including integration by parts) to the equation (for simplicity, we abreviate $F(|D(v)|^2)$ by $F()$)

$$\begin{aligned} &\int_\Omega \partial_{ij}\partial_{kl}F()D_{kl}\left(\frac{\partial v}{\partial x_r}\xi_0\right)D_{ij}(\varphi)\mathrm{d}x \\ &= \int_\Omega \left\{\xi_0\left(v_k\frac{\partial v_i}{\partial x_k} - f_i\right) + \partial_{ij}F()\frac{\partial \xi_0}{\partial x_j}\right\}\frac{\partial \varphi_i}{\partial x_r}\mathrm{d}x \\ &- \int_\Omega \partial_{ij}F()\frac{\partial \xi_0}{\partial x_r}D_{ij}(\varphi)\mathrm{d}x + \int_\Omega \partial_{ij}\partial_{kl}F()\frac{\partial v_k}{\partial x_r}\frac{\partial \xi_0}{\partial x_l}D_{ij}(\varphi)\mathrm{d}x \\ &+ \int_\Omega \left(f_i - v_k\frac{\partial v_i}{\partial x_k}\right)\psi_i^\varphi\,\mathrm{d}x - \int_\Omega \partial_{ij}F()D_{ij}(\psi^\varphi)\mathrm{d}x. \end{aligned} \tag{3.9}$$

Since $\frac{\partial v}{\partial x_r}\xi_0$ is not divergence-free we set

$$u \equiv \frac{\partial v}{\partial x_r}\xi_0 + z \tag{3.10}$$

where

$$\operatorname{div} x = -\frac{\partial v_i}{\partial x_r}\frac{\partial \xi_0}{\partial x_i} \quad \text{in} \quad U_0, \quad z = 0 \quad \text{at} \quad \partial U_0, \quad z \equiv 0 \quad \text{at} \quad \Omega \setminus \overline{U_0},$$

$$\|z\|_{1,s} \leqslant c\|\nabla v\|_s \quad s \in (1,\infty). \tag{3.11}$$

Thus, u satisfies the equation

$$\int_\Omega \partial_{ij}\partial_{kl}F()D_{kl}(u)D_{ij}(\varphi)\mathrm{d}x$$

$$= \text{RHS } of(3.9) + \int_\Omega \partial_{ij}\partial_{kl}F()D_{kl}(z)D_{ij}(\varphi)\mathrm{d}x. \tag{3.12}$$

(b) In case of tangential derivative, defined by

$$\frac{\partial v}{\partial t} = \frac{\partial v}{\partial x_1} + \frac{\partial a}{\partial x_1}\frac{\partial v}{\partial x_2}. \tag{3.13}$$

We use a similar procedure on $U_1, U_2, \dots, U_k$ with cut-off functions $\xi_1, \xi_2, \dots, \xi_k$. Thus we test (3.2) by ($\ell \in \{1,2,\dots,k\}$)

$$\Phi \equiv -\frac{\partial \varphi}{\partial t}\xi_\ell + \psi^\varphi, \tag{3.14}$$

where ψ^φ is defined by

$$\operatorname{div}\psi^\varphi = \frac{\partial \varphi_i}{\partial t}\frac{\partial \xi_\ell}{\partial x_i} + \frac{\partial^2 a}{\partial x_1^2}\frac{\partial \varphi_1}{\partial x_2}\xi_\ell \quad \text{in} \quad U_\ell^+,$$

$$\psi^\varphi = 0 \quad \text{at} \quad \partial U_\ell^+, \quad \|\psi^\varphi\|_{1,s} \leqslant c\|\nabla\varphi\|_s \quad s \in (1,\infty). \tag{3.15}$$

In this case we set

$$u \equiv \frac{\partial v}{\partial t}\xi_\ell + z, \tag{3.16}$$

where

$$\operatorname{div} z = -\frac{\partial v_i}{\partial t}\frac{\partial \xi_\ell}{\partial x_i} - \frac{\partial^2 a}{\partial x_1^2}\frac{\partial v_1}{\partial x_2}\xi_\ell \quad \text{in} \quad U_\ell^+,$$

$$z = 0 \ \text{ at } \ \partial U_\ell^+, \quad z \equiv 0 \ \text{ in } \ \Omega\backslash\overline{U_\ell^+}, \|z\|_{1,s} \leqslant c\|\nabla v\|_s \ \ s \in (1,\infty). \tag{3.17}$$

and we see that u satisfies the equation

$$\begin{aligned}
&\int\limits_\Omega \partial_{ij}\partial_{kl}F()D_{kl}(u)D_{ij}(\varphi)\mathrm{d}x \\
&= \int\limits_\Omega \partial_{ij}\partial_{kl}F()D_{kl}(z)D_{ij}(\varphi)\mathrm{d}x \\
&+ \int\limits_\Omega \left\{\xi_\ell\left(v_k\frac{\partial v_i}{\partial x_k} - f_i\right) + \partial_{ij}F()\frac{\partial\xi_\ell}{\partial x_j}\right\}\frac{\partial\varphi_i}{\partial t}\mathrm{d}x \\
&- \int\limits_\Omega \partial_{ij}F()\frac{\partial\xi_\ell}{\partial t}D_{ij}(\varphi)\mathrm{d}x + \int\limits_\Omega \partial_{i1}F()\frac{\partial^2 a}{\partial x_1^2}\frac{\partial\varphi_i}{\partial x_2}\xi_\ell\mathrm{d}x \\
&+ \int\limits_\Omega \partial_{ij}\partial_{kl}F()\left\{\frac{\partial v_k}{\partial t}\frac{\partial\xi_\ell}{\partial x_l} + \frac{\partial v_k}{\partial x_2}\frac{\partial^2 a}{\partial x_1\partial x_l}\xi_\ell\right\}D_{ij}(\varphi)\mathrm{d}x \\
&+ \int\limits_\Omega \left(f_i - v_k\frac{\partial v_i}{\partial x_k}\right)\psi_i^\varphi\mathrm{d}x - \int\limits_\Omega \partial_{ij}F()D_{ij}(\psi^\varphi)\mathrm{d}x.
\end{aligned} \tag{3.18}$$

Theorem 3.19. *Let Ω be C^2. Let $v \in V_2$ and $\pi \in L^2(\Omega)$, $\int_\Omega \pi dx = 0$, be a weak solution to* (1.1). *Assume that $f \in L^{2+\varepsilon_0}(\Omega)$, $\varepsilon_0 > 0$, and* (1.4), (1.5) *holds with $p = 2$. Then there is $q \in (2, 2+\varepsilon_0)$ such that*

$$v \in W^{2,q}(\Omega) \cap C^{1,\alpha}(\overline{\Omega}) \ \ \text{and} \ \ \pi \in W^{1,q}(\Omega) \cap C^{0,\alpha}(\overline{\Omega}). \tag{3.20}$$

Proof. By Lemma 3.1, we know that if $f \in L^2(\Omega)$ then every weak solution (v, π) satisfies (3.3), and consequently we can derive Eqs. (3.12) and (3.18), respectively, for which we are going to apply Lemma 2.6 and show (3.20) in interior of Ω (Step 1) and at the tangential direction at the boundary (Step 2). Finally, we deal with derivatives in normal directions (Step 3).

Step 1 (Interior regularity). We first look at the solution of an auxiliary problem

$$\int\limits_\Omega D_{ij}(\widetilde{u})D_{ij}(\varphi)\mathrm{d}x = \text{RHS of (3.12)}. \tag{3.21}$$

Since $f \in L^{2+\varepsilon_0}(\Omega)$, we consider $q \in (2, 2+\varepsilon_0]$. Set $q' = \frac{q}{q-1}$. Due to (3.3), and the fact that $|\partial_{ij}\partial_{kl}F(|D(v)|^2)| \leqslant C$, one can observe that

RHS of (3.12) is bounded by

$$c\,(\|v\|_{1,q}+\|f\|_{2+\varepsilon_0})\,\|\nabla\varphi\|_{q'}\leqslant C\,(1+\|f\|_{2+\varepsilon_0})\,\|\nabla\varphi\|_{q'}.$$

Applying the L^q-theory of the Stokes system we have $\widetilde{u}\in V_q$. Thus setting $G_{ij}\equiv D_{ij}(\widetilde{u})$, we see that (3.12) is of the form (2.1). Then, all assumptions of Lemma 2.6 are fulfilled and we obtain that for all $q\in(2,2+\varepsilon]$ (ε is given by (2.5)) it holds

$$\|u\|_{1,q}\leqslant K. \tag{3.22}$$

Comparing (3.22) with the definition of u (cf. (3.10)) and using (3.11) with help of (3.3) we can conclude that

$$\left\|\frac{\partial v}{\partial x_r}\xi_0\right\|_{1,q}\leqslant K,\quad r=1,2. \tag{3.23}$$

Step 2. (Tangential derivative). The procedure is analogous. We first look at

$$\int\limits_\Omega D_{ij}(\widetilde{u})D_{ij}(\varphi)\mathrm{d}x=\text{RHS of (3.18)}$$

and again we observe that

$$|\text{ RHS of (3.18) }|\leqslant C\,(1+\|f\|_{2+\varepsilon_0})\,\|\nabla\varphi\|_{q'}.$$

Applying Lemma 2.6 for u in combination with (3.16), (3.17) yields ($q\in(2,2+\varepsilon]$)

$$\left\|\frac{\partial v}{\partial t}\right\|_{1,q,U_\ell^+}\leqslant K,\quad \ell=1,2,\dots,k. \tag{3.24}$$

Step 3 (Normal derivative). In order to obtain (3.20) it remains to show that the normal derivatives of ∇v, i.e., derivatives with respect to x_2 in a local coordinate system, belong to $L^{2+\varepsilon_0}(U_\ell^+)$. Because of (3.24) we can concentrate on $\frac{\partial^2 v_1}{\partial x_2^2}$ and $\frac{\partial^2 v_2}{\partial x_2^2}$. The second term, however, can be expressed directly from $\operatorname{div} v=0$. This yields

$$\frac{\partial^2 v_2}{\partial x_2^2}=-\frac{\partial^2 v_1}{\partial x_1\partial x_2}. \tag{3.25}$$

The missing information on $\frac{\partial^2 v_1}{\partial x_2^2}$ will be extracted from (1.1). To avoid pressure we take curl[4] of (1.1), which gives

$$\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_1}\partial_{21}F() + \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\partial_{22}F() - \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1}\partial_{11}F()$$
$$-\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_2}\partial_{12}F() = \operatorname{curl}\left(v_k\frac{\partial v}{\partial x_k}\right) - \operatorname{curl} f. \tag{3.26}$$

Set $G \equiv \frac{\partial}{\partial x_2}\partial_{12}F(|D(v)|^2)$. Then clearly ($\ell = 1, 2, \dots, k$)

$$\|\xi_\ell\, G\|_{-1,2+\varepsilon_0} \leqslant C.$$

Next, by (1.5) and (3.3),

$$\left\|\frac{\partial}{\partial x_1}(\xi_\ell\, G)\right\|_{-1,2+\varepsilon_0} \leqslant C + C_0\sum_{i=1}^{2}\left\|\frac{\partial^2 v}{\partial x_1\partial x_i}\right\|_{2+\varepsilon_0}.$$

Finally, from (3.26), we have

$$\left\|\frac{\partial}{\partial x_2}(\xi_\ell\, G)\right\|_{-1,2+\varepsilon_0} \leqslant C + C_0\sum_{i=1}^{2}\left\|\frac{\partial^2 v}{\partial x_1\partial x_i}\right\|_{2+\varepsilon_0}.$$

Now, we can apply Nečas' theorem on "negative" Sobolev spaces (cf. [19], for a formulation of theorem see also [16], p. 196) to obtain

$$\|\xi_\ell\, G\|_{2+\varepsilon_0} \leqslant C + C_0\sum_{i=1}^{2}\left\|\frac{\partial^2 v}{\partial x_1\partial x_i}\right\|_{2+\varepsilon_0}. \tag{3.27}$$

From the definition of G we observe

$$\partial_{12}\partial_{12}F()D_{12}\left(\frac{\partial v}{\partial x_2}\right)$$
$$= \frac{G}{2} - \frac{1}{2}\partial_{11}\partial_{12}F()\frac{\partial D_{11}(v)}{\partial x_2} - \frac{1}{2}\partial_{22}\partial_{12}F()\frac{\partial D_{22}(v)}{\partial x_2}. \tag{3.28}$$

Using (3.25) for the last term and the facts that

$$D_{12}\left(\frac{\partial v}{\partial x_2}\right) = \frac{\partial^2 v_1}{\partial x_2^2} + \frac{\partial^2 v_2}{\partial x_1\partial x_2} \quad\text{and}\quad \partial_{12}\partial_{12}F(|D(v)|^2) \geqslant C_1$$

[4]Recall that $\operatorname{curl} z = \frac{\partial z_2}{\partial x_1} - \frac{\partial z_1}{\partial x_2}$ for $z : \Omega \to \mathbb{R}^2$.

by (1.4) we can conclude from (3.27) and (3.28) that

$$\left\| \xi_\ell \frac{\partial^2 v_1}{\partial x_2^2} \right\|_{2+\varepsilon_0} \leqslant C + C_0 \sum_{i=1}^{2} \left\| \frac{\partial^2 v}{\partial x_1 \partial x_i} \right\|_{2+\varepsilon_0}. \tag{3.29}$$

Then (3.25) and (3.29) together with (3.24) lead to

$$\sum_{j=1}^{2} \left\| \xi_\ell \frac{\partial^2 v_j}{\partial x_2^2} \right\|_{2+\varepsilon_0} \leqslant C + \left\| \frac{\partial \nabla v}{\partial t} \right\|_{2+\varepsilon_0}$$
$$+\widetilde{c} \sup_{x_1 \in (-\alpha,\alpha)} \left| \frac{\partial a(x_1)}{\partial x_1} \right| \sum_{j=1}^{2} \left\| \xi_\ell \frac{\partial^2 v_j}{\partial x_2^2} \right\|_{2+\varepsilon_0}. \tag{3.30}$$

For α small enough (2.13) implies that

$$\widetilde{c} \max_\ell \sup_{x_1 \in (-\alpha,\alpha)} \left| \frac{\partial a_\ell(x_1)}{\partial x_1} \right| \leqslant \frac{1}{2}.$$

Thus the last term in (3.30) can be moved to the left-hand side and assertion (3.20) easily follows.

Finally, since v and π solve (1.1) almost everywhere due to $(3.3)_1$, we see that the fact $v \in W^{2,q}(\Omega) \cap C^{1,\alpha}$ implies $\pi \in W^{1,q}(\Omega)$, and consequently $\pi \in C^{0,\alpha}$. The proof pf Theorem 3.19 is complete. $\square$

4. Interior $C^{1,\alpha}$-regularity for $p > 6/5$

While for $p = 2$ we have showed in Sec. 3 that every weak solution of (1.1) satisfies (1.2) (α), for $p \neq 2$ we directly construct a solution having $C^{1,\alpha}$-regularity.

Let $p \in (1,2)$ in the sequel. For $\lambda \in (0,1)$ we consider the following quadratic λ-aproximation of (1.1): to find $(v,\pi) : (v^\lambda, \pi^\lambda)$ solving

$$\begin{aligned} v_k \frac{\partial v}{\partial x_k} - \operatorname{div} \left(\left(1 + \lambda |D(v)|^2\right)^{\frac{2-p}{p}} T(D(v)) \right) &= -\nabla \pi + f \quad \text{in} \quad \Omega, \\ \operatorname{div} v = 0 \quad \text{in} \quad \Omega&, \\ v = 0 \quad \text{at} \quad \partial\Omega&. \end{aligned} \tag{4.1}$$

The corresponding potential F^λ to (4.1) given by

$$F^\lambda \left(|D(v)|^2\right) \equiv \int_0^{|D(v)|^2} (1 + \lambda s)^{\frac{2-p}{2}} F'(s) \, \mathrm{d}s, \tag{4.2}$$

satisfies (see [12] for calculations) for all i, j, k, $l = 1, 2$

$$
\begin{aligned}
\partial_{ij}\partial_{kl}F^{\lambda}(|\eta|^2)\xi_{ij}\xi_{kl} &\geqslant C_1(1+\lambda|\eta|^2)^{\frac{2-p}{2}}(1+|\eta|^2)^{\frac{2-p}{2}}|\xi|^2 \\
&\geqslant C_1\lambda^{\frac{2-p}{2}}|\xi|^2 \equiv \widetilde{C}_1(\lambda)|\xi|^2, \\
\partial_{ij}\partial_{kl}F^{\lambda}(|\eta|^2) &\leqslant C_6(1+\lambda|\eta|^2)^{\frac{2-p}{2}}(1+|\eta|^2)^{\frac{2-p}{2}}|\xi|^2 \leqslant C_6,
\end{aligned} \tag{4.3}
$$

where C_6 is independent of λ.

As $(1+|\eta|^2)^{\frac{2-p}{2}} \geqslant 1$ we also have from $(4.3)_1$

$$
\partial_{ij}\partial_{kl}F^{\lambda}(|\eta|^2)\xi_{ij}\xi_{kl} \geqslant C_1(1+\lambda|\eta|^2)^{\frac{2-p}{2}}|\xi|^2. \tag{4.4}
$$

By (4.3), the coefficients $\partial_{ij}\partial_{kl}F^{\lambda}(|\eta|^2) \in L^{\infty}(\Omega)$. Theorem 3.19 then implies the existence of $q^{\lambda} > 2$ and a couple $(v^{\lambda}, \pi^{\lambda})$ such that

$$
v^{\lambda} \in W^{2,q^{\lambda}}(\Omega) \cap C^1(\overline{\Omega}) \quad \text{and} \quad \pi^{\lambda} \in W^{1,q^{\lambda}}(\Omega) \cap C(\overline{\Omega}). \tag{4.5}
$$

Our goal is to estimate $(v^{\lambda}, \pi^{\lambda})$ in $W^{2,q_0}(\Omega) \times W^{1,q_0}(\Omega)$ *uniformly* with respect to λ, for some $q_0 > 2$.

Lemma 4.6. *Let v^{λ} satisfying* (4.5) *be a solution of* (4.1). *Let $\Omega_0 \subset \overline{\Omega}_0 \subset \Omega$. If $p > \frac{6}{5}$ then there is a $C \in (0,\infty)$ and $q > 2$ independent of λ such that*

$$
\|v^{\lambda}\|_{2,q,\Omega_0} \leqslant C < \infty. \tag{4.7}
$$

Proof. We will first derive uniform estimates for v^{λ} in the space $W^{2,p}(\Omega_1)$ (Step 1), where $\Omega_1 \subset \Omega$ and $\overline{\Omega}_1 \supset \overline{\Omega}_0$. Then, we will use Lemma 2.6 to find uniform estimates in $W^{2,q}(\Omega_0)$ for a certain $q > 2$ (Step 2).

Step 1 (Estimates in $W^{2,p}(\Omega_1)$). Let us first multiply (4.1) by v^{λ} and integrate over Ω. As $\int_{\Omega} v_k^{\lambda}\, \frac{\partial v_i^{\lambda}}{\partial x_k}\, v_k^{\lambda}\, \mathrm{d}x$ vanishes and (1.6) holds due to (4.4) we obtain (also with help of Korn's inequality proved in [19], for example)

$$
\int_{\Omega} \left(1+\lambda|D(v^{\lambda})|^2\right)^{\frac{2-p}{2}} |D(v^{\lambda})|^p \, \mathrm{d}x + \|v^{\lambda}\|_{1,p} \leqslant C < \infty. \tag{4.8}
$$

The next step is to show that for $p \in \left(\frac{6}{5}, 2\right)$ there is a $k \in \mathbb{N}$ such that

$$
\|v^{\lambda}\,\tau^k\|_{2,p} \leqslant C < \infty, \tag{4.9}
$$

where τ is a usual cut-off function with compact support in Ω, $\tau \equiv 1$ in Ω_1. Recall that (cf. [16], p. 227)

$$\left\| \nabla^{(2)} v^\lambda \tau^k \right\|_p^2 \leqslant C(1 + \|\nabla v^\lambda\|_p)^{2-p} \int_\Omega \left(1 + |D(v^\lambda)|^2\right)^{\frac{p-2}{2}} |D(\nabla v^\lambda)|^2 \tau^{2k} \, \mathrm{d}x,$$

which combined with (4.8) gives

$$\left\| \nabla(\nabla v^\lambda \tau^k) \right\|_p^2 \leqslant C \int_\Omega \left(1 + |D(v^\lambda)|^2\right)^{\frac{2-p}{2}} |D(\nabla v^\lambda)|^2 \tau^{2k} \, \mathrm{d}x + C. \quad (4.10)$$

Let $\psi : \Omega \to \mathbb{R}$, $u : \Omega \to \mathbb{R}^2$. We denote rot $\psi \equiv \left(-\frac{\partial \psi}{\partial x_2}, \frac{\partial \psi}{\partial x_1}\right)$ and recall curl $u \equiv \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$. Note that rot ψ is divergence-free. If u satisfies $\operatorname{div} u = 0$ then, in two dimensions, it holds

$$\operatorname{rot}(\tau^{2k} \operatorname{curl} u) = (\Delta u)\tau^{2k} + (\operatorname{rot} \ \tau^{2k})(\operatorname{curl} u).$$

Thus we can multiply $(4.1)_1$ by $\operatorname{rot}(\tau^{2k} \operatorname{curl} v^\lambda)$, which eliminates the pressure and all boundary terms occurring while integrating by parts. After long, but elementary calculations, where we use (4.3), (4.8), (1.12) with $\xi = \tau^k$, Hölder's and Young's inequalities and (4.10) we obtain $(v := v^\lambda)$

$$\|\nabla(\nabla v \tau^k)\|_p^2 \leqslant C \int_\Omega |v| \ |\nabla v|^2 \ |\nabla(\tau^{2k})| dx + C\|f\|_{p'}^2.$$

This together with the embedding $W^{1,p}(\Omega) \hookrightarrow L^{2p/2-p}(\Omega)$ leads to

$$\|\nabla v \tau^k\|_{2p/2-p}^2 \leqslant C\|v\|_{2p/2-p} \left(\int_\Omega |\nabla v|^{2r} \, \tau^{(2k-1)r} \, \mathrm{d}x \right)^{1/r} + C\|f\|_{p'}^2, \quad (4.11)$$

where $r = \frac{2p}{2p-2}$. For $a \in (0,1)$, $\delta > 1$ and $\delta' = \frac{\delta}{\delta - 1}$ we have

$$\int_\Omega |\nabla v|^{2r} \, \tau^{(2k-1)r} \, \mathrm{d}x = \int_\Omega |\nabla v|^{2ra} \, \tau^{(2k-1)r} \, |\nabla v|^{2r(1-a)} \, \mathrm{d}x$$

$$\leqslant \Big(\int\limits_{\Omega} |\nabla v|^{2ra\delta}\, \tau^{(2k-1)r\delta}\, \mathrm{d}x \Big)^{1/\delta} \Big(\int\limits_{\Omega} |\nabla v|^{2ra(1-a)\delta'}\, \mathrm{d}x \Big)^{1/\delta'} . \tag{4.12}$$

The requirements

$$2ra\delta = \frac{2p}{2-p} \quad \text{and} \quad 2r(1-a)\frac{\delta}{\delta - p} = p \tag{4.13}$$

imply

$$\delta = \frac{p(3p-2)}{3(2-p)^2} \quad \text{and} \quad a = \frac{3(2-p)}{2p}.$$

Note that $\delta > 1$ and $a \in (0,1)$ for $p > \frac{6}{5}$. Consequently, (4.11)–(4.13) and (4.8) give

$$\big\| \nabla v \tau^k \big\|^2_{\frac{2p}{2-p}} \leqslant C \Big\| \nabla v \tau^{(2k-1)\frac{p}{3(2-p)}} \Big\|^{\frac{3(2-p)}{p}}_{\frac{2p}{2-p}} + C \|f\|^2_{p'}.$$

Thus (4.9) is provided that

$$\frac{3(2-p)}{p} \quad \text{and} \quad (2k-1)\frac{p}{3(2-p)} \geq k. \tag{4.14}$$

However, the first inequality holds for $p \geqslant \frac{6}{5}$ and the second one is equivalent to $p \geq \frac{6}{5-1/k}$. So, we see that for every $p > \frac{6}{5}$ we always find a $k \in \mathbb{N}$ such that (4.14), and consequently (4.9) are fulfilled. Denoting $\theta = \theta_\lambda \equiv (1 + |D(v^\lambda)|^2)^{1/2}$ we have proved

$$\big\| \tau^k \nabla \theta^{p/2} \big\|_2^2 \leqslant C \int\limits_{\Omega} \big(1 + |D(v)|^2\big)^{\frac{p-2}{2}} \, |D(\nabla v)|^2\, \tau^{2k}\, \mathrm{d}x \leqslant C$$

uniformly with respect to λ. The choice of τ and embedding theorems give that

$$\nabla v, \;\; \theta^{p/2} \in L^s(\Omega_1) \;\; \text{for all} \;\; s \in (1, \infty). \tag{4.15}$$

Step 2 (Estimates in $W^{2,q}(\Omega_0)$). We are going to use a localized variant of the procedure developed in Sec. 3.

For $x_0 \in \Omega_1$ we consider $B_{x_0,R} \equiv \{x; |x - x_0| < R\}$ such that for all $x \in B_{x_0,R}$

$$R < \operatorname{dist}(x, \partial\Omega_1) < 5R. \tag{4.16}$$

Clearly set of such balls cover Ω_1, and $B_{x_0,2R} \subset \Omega_1$ due to (4.16).

Let ξ_0 be a cut-off function with support in $B_{x_0,2R} \equiv \{x; |x-x_0| < 2R\}$, $\xi_0 \equiv 1$ on $B_{x_0,R}$ and $|\nabla \xi_0| \le C/R$.

Now, denote $u^\lambda \equiv \frac{\partial v^\lambda}{\partial x_r}\xi_0 + z$, where the correction term z is a solution of $\operatorname{div} z = -\frac{\partial v^\lambda}{\partial x_r} \cdot \nabla \xi_0$ in $B_{x_0,2R}$, $z = 0$ on $\partial B_{x_0,2R}$ such that $\|z\|_{1,s} \le c\|\nabla v \nabla \xi_0\|_s$ for all $s \in (1,\infty)$; we define $z \equiv 0$ on $\Omega \setminus \overline{B}_{x_0,2R}$.

Clearly, u^λ satisfies Eq. (3.12), where we replace u, v, F by u^λ, v^λ, F^λ, and ξ_0 has (sligthly) different meaning. Notice that all integrals over Ω can be viewed as integrals over $B_{x_0,2R}$, which we use in the sequel.

We focus on regularity of solutions of the auxiliary problem

$$\int_\Omega D_{ij}(\widetilde{u}) D_{ij}(\varphi)\,\mathrm{d}x = \text{RHS} \quad \text{of (3.12)}.$$

Typically we obtain that

$$\big|\text{RHS} \quad \text{of (3.12)}\big| \le c\big(\|\nabla\xi_0 \nabla v\|_q + \|f\|_{p'}\big)\|\nabla\varphi\|_{q'} + \|\nabla v\|_q\,\|\nabla \psi^\varphi\|_{q'}.$$

However, due to (4.15),

$$\|\nabla \xi_0 \nabla v\|_q \le \frac{C}{R}\|\nabla v\|_q \le \frac{C}{R}.$$

Similarly, using also (3.8), we see that

$$\|\nabla v\|_q\,\|\nabla\psi^\varphi\|_{q'} \le \frac{C}{R}\|\nabla\varphi\|_{q'}.$$

Thus RHS of (3.12) belongs to $L^q(\Omega)$, which implies $\widetilde{u} \in V_q$. Setting $G_{ij} \equiv D_{ij}(\widetilde{u})$, we have rewritten (3.12) into the equation of the form (2.3) (for $\varphi \in V_{q'}$) with

$$\|G\|_q \le \frac{C}{R}.$$

The coefficients $A_{ij}^{kl} = \partial_{ij}\partial_{kl}F^\lambda(|D(v^\lambda)|^2)$ are measurable and bounded (cf. (4.3)). Denoting

$$V_R = V_R(\lambda, x_0) \equiv \sup_{x \in B_{x_0,R}} \big(1 + |D(v^\lambda(x))|^2\big)^{1/2},$$

then one observes, from $(4.3)_3$ and (4.4), that

$$C_1\,V_{2R}^{p-2}\,|\xi|^2 \le \frac{\partial^2 F^\lambda(|D(v^\lambda)|^2)}{\partial\eta_{ij}\,\partial\eta_{kl}}\xi_{ij}\,\xi_{kl} \le C_6|\xi|^2 \tag{4.17}$$

is valid on $B_{x_0,2R}$ for all $\lambda > 0$. We see that all assumptions of Lemma 2.6 are fulfilled and therefore

$$\|u^\lambda\|_{1,q,B_{x_0,R}} \le \frac{C}{R} V_{2R}^{2-p} \tag{4.18}$$

with

$$q = 2\left(1 - \log\left[\frac{1-\frac{1}{2}\frac{C_1}{C_2}V_{2R}^{p-2}}{1-\frac{C_1}{C_2}V_{2R}^{p-2}}\Big/\log\gamma_3\right)^{-1}, \quad \gamma_3 > 1. \tag{4.19}$$

By (4.8) and (4.9)

$$\left\|\theta^{p/2}\right\|_{1,2,B_{x_0,R}} < C, \tag{4.20}$$

and by (4.18)

$$\left\|\theta^{p/2}\right\|_{1,q,B_{x_0,R}} < \frac{C}{R} V_{2R}^{2-p}. \tag{4.21}$$

For $p_0 \in (2,q)$ the interpolation inequality implies (all norms are on $B_{x_0,R}$)

$$\left\|\theta^{p/2}\right\|_{1,p_0} \le \left\|\theta^{p/2}\right\|_{1,2}^{1-a}\left\|\theta^{p/2}\right\|_{1,q}^{a} \le \frac{C}{R^a} V_{2R}^{a(2-p)},$$

where a is given by $\frac{1}{p_0} = \frac{1-a}{2} + \frac{a}{q}$. Then due to (4.19) we see

$$p_0 = 2\left(1 - \delta\log\left[\frac{1-\frac{1}{2}\frac{C_1}{C_2}V_{2R}^{p-2}}{1-\frac{C_1}{C_2}V_{2R}^{p-2}}\right]\Big/\log\gamma_3\right)^{-1} \quad \text{for a certain} \quad \delta \in (0,1).$$

This implies

$$1 - \frac{2}{p_0} = \frac{\delta}{\log\gamma_3}\log\left(\frac{1-\frac{1}{2}\frac{C_1}{C_2}V_{2R}^{p-2}}{1-\frac{C_1}{C_2}V_{2R}^{p-2}}\right) = \frac{\delta}{\log\gamma_3}\log\left(1 + \frac{\frac{1}{2}\frac{C_1}{C_2}V_{2R}^{p-2}}{1-\frac{C_1}{C_2}V_{2R}^{p-2}}\right)$$

$$\ge \begin{cases} \delta\dfrac{\log 2}{\log\gamma_3} & \text{if } \dfrac{3}{2}\dfrac{C_1}{C_2}V_{2R}^{p-2} \ge 1, \\ \delta\dfrac{1}{4\log\gamma_3}\dfrac{C_1}{C_2}V_{2R}^{p-2} & \text{if } \dfrac{3}{2}\dfrac{C_1}{C_2}V_{2R}^{p-2} < 1, \end{cases}$$

as $\log(1+x) \ge \frac{x}{2}$ on $[0,1]$. Thus setting $L \equiv \min\left(\frac{\delta\log 2}{\log\gamma_3}, \frac{C_1\delta}{4C_2\log\gamma_3}\right)$ yields $(1 \ge V_{2R}^{p-2})$

$$1 - \frac{2}{p_0} \ge L V_{2R}^{p-2}. \tag{4.22}$$

Finally, from the Morrey imbedding inequality (cf. [24], p. 58), (4.21) and (4.22), we see

$$V_R^{p/2} \le \left\| \theta^{p/2}\xi_0 \right\|_{C(\overline{B}_{x_0,2R})} \le \frac{C}{R^a}\left(\frac{p_0-1}{p_0-2}\right)^{1-1/p_0} V_{2R}^{a(2-p)}$$

$$\le \frac{C}{R^a}\left(\frac{1}{1-2/p_0}\right)^{1-1/p_0} V_{2R}^{a(2-p)} \le \frac{C}{R^a} V_{2R}^{a(2-p)a+(2-p)(1-1/p_0)}. \quad (4.23)$$

We need

$$(2-p)\left(1-\frac{1}{p_0}+a\right) < \frac{p}{2}, \quad (4.24)$$

which can be rewritten (if p_0 is close to 2 than a is small) as

$$p > \frac{2(a+1-1/p_0)}{3/2-1/p_0+a} \xrightarrow[a\to 0]{p_0\to 2} 1.$$

Therefore, for a given $p \in (6/5, 2)$ we find p_0 (and corresponding a) so that (4.24) holds. Under these circumstances

$$V_R \le \frac{C}{R^\gamma} V_{2R}^\alpha \quad \text{with} \quad \alpha = \frac{2}{p}(2-p)\left(1-\frac{1}{p_0}+a\right), \quad \gamma = \frac{2a}{p}, \quad (4.25)$$

and a constant K is independent of x_0, R, λ. Denote $d(x) \equiv \operatorname{dist}(x, \partial\Omega_1)$. Multiplying (4.25) by R^δ, $\delta = \frac{\gamma}{1-\alpha}$, and using (4.16) one can conclude that

$$\sup_{B_R(x_0)} \left([d(x)]^\delta (1+|D(v(x))|^2)^{1/2}\right)$$

$$\le \widehat{K} \sup_{B_{2R}(x_0)} \left([d(x)]^\delta (1+|D(v(x))|^2)^{1/2}\right)^\alpha .$$

As $\{B_{x_0,R}; x_0 \in \Omega_1, R < R_0\}$ covers Ω_1 and $B_{x_0,2R} \subset \Omega_1$ we obtain

$$\sup_{\Omega_1} \left([d(x)]^\delta (1+|D(v(x))|^2)^{1/2}\right) \le \widehat{K} \sup_{\Omega_1} \left([d(x)]^\delta (1+|D(v(x))|^2)^{1/2}\right)^\alpha$$

which implies ($\alpha < 1$) that

$$\sup_{\Omega_1} \left([d(x)]^\delta (1+|D(v(x))|^2)^{1/2}\right) \le K.$$

Since $\overline{\Omega}_0 \subset \Omega_1$ we finally conclude that

$$\sup_{\Omega_0} \left(1+|D(v(x))|^2)^{1/2}\right) \le K$$

uniformly with respect to λ. This estimate together with (4.18) gives (4.7). The proof of Lemma 4.6 is complete. $\square$

Let us conclude this section by formulating the final result.

Theorem 4.26. *Let $p \in (6/5, 2)$ and $f \in L^{p'}(\Omega)$. Then there is a $q > 2$ and a solution (v, π) of the problem* (1.1) *such that*

$$v \in V_p \cap W^{2,q}_{\mathrm{loc}}(\Omega) \quad \pi \in W^{1,q}_{\mathrm{loc}}(\Omega)$$

and (1.2) (β) *holds.*

5. Global $C^{1,\alpha}$-regularity for $p > \frac{3}{2}$

We proceed in the way similar to the previous section. We use the same λ-approximations (4.1). Thus we already know that approximative solutions v^λ belong to $W^{2,q_\lambda}(\Omega)$ for $q_\lambda > 2$. Moreover, by the results of Sec. 4, v^λ are uniformly bounded in $W^{2,q}(U_0)$ (in the interior) for a certain $q > 2$. Our aim is to show uniform estimates for v^λ up to the boundary, which allow us to transfer higher regularity to the weak limit solution v.

As in Sec. 3 we work first with tangential derivatives of solutions and then we estimate the normal derivatives using (1.1). While estimating tangential derivatives we meet two different points (in comparison with the interior estimates in Sec. 4):

(1) We cannot use $\mathrm{rot}(\xi_\ell^2 \mathrm{curl}\ v)$ as a test function since it does not satisfy the boundary conditions;
(2) Since we work with tangential derivatives only, the cancellation (1.12) does not take place

We use the notation of Sec. 3. In addition, we set

$$\rho_\lambda \equiv \left(1 + \lambda |D(v^\lambda)|^2\right)^{-1/2} \left(1 + |D(v^\lambda)|^2\right)^{1/2}$$

and recall $\theta_\lambda \equiv \left(1 + |D(v^\lambda)|^2\right)^{1/2}$. Note that $\rho_\lambda^{p-2} \geq \theta_\lambda^{p-2}$.

Lemma 5.1. *Let Ω be C^2 and $p > 3/2$. Then for $\ell = 1, \dots, k$ it holds*

$$\left\| \xi_\ell \nabla \frac{\partial v^\lambda}{\partial t} \right\|_p^2 + \int_\Omega \rho_\lambda^{p-2} \left| D\left(\frac{\partial v^\lambda}{\partial t} \xi_\ell \right) \right|^2 dx \leq C + C \|\nabla v^\lambda\|_{p'}^2. \tag{5.2}$$

Proof. We use Eq. (3.18) with $\varphi = u^\lambda$. Using $(4.3)_3$, (4.4), (4.8), (4.10) together with (3.10), (3.11) and (3.16), (3.17) yields directly

$$\|\nabla u^\lambda\|_p^2 + \int_\Omega \rho_\lambda^{p-2}|D(u^\lambda)|^2 \mathrm{d}x \leqslant C\|\nabla v^\lambda\|_{p'}\,\|\nabla u^\lambda\|_p$$

$$+C\|f\|_{p'}\,\|\nabla u^\lambda\|_p + C\left|\int_\Omega \xi_\ell\, v_k^\lambda \frac{\partial v_i^\lambda}{\partial x_k}\,\frac{\partial v_i^\lambda}{\partial t}\,\mathrm{d}x\right|. \tag{5.3}$$

The last term brings the restriction on the range of p's. After some calculations (corresponding to integration by parts with respect to tangential derivative) we can observe that

$$\left|\int_\Omega v_k^\lambda \frac{\partial v_i^\lambda}{\partial x_k}\,\frac{\partial v_i^\lambda}{\partial t}\,\mathrm{d}x\right| \leq \left|\int_\Omega u_k^\lambda \frac{\partial v_i^\lambda}{\partial x_k}\,v_i^\lambda\,\mathrm{d}x\right| + C\|\nabla v^\lambda\|_2^2 + C. \tag{5.4}$$

The Hölder inequality together with the interpolation inequality

$$\|z\|_{\frac{2p}{3p-4}} \leq \|z\|_p^{\frac{2(2p-3)}{p}}\,\|z\|_{\frac{2p}{2-p}}^{\frac{3(2-p)}{p}}$$

then gives

$$\left|\int_\Omega u_k^\lambda \frac{\partial v_i^\lambda}{\partial x_k}\,v_i^\lambda\,\mathrm{d}x\right| \leqslant \|\nabla u^\lambda\|_p\;\|v^\lambda\|_{\frac{2p}{2-p}}\;\|u^\lambda\|_{\frac{2p}{3p-4}}$$

$$\leq C\|\nabla u^\lambda\|_p^{1+\frac{3(2-p)}{p}}\;\|v^\lambda\|_{1,p}^{1+\frac{4p-6}{p}} \leq C\|\nabla u^\lambda\|_p^{\frac{6-2p}{p}}. \tag{5.5}$$

As $\frac{6-2p}{p} \leq 2$ for $p \geq \frac{3}{2}$, we obtain the assertion from (5.3)–(5.5), combined with (2.16), (3.17). $\square$

The further observation brings global uniform estimates on second derivatives of v^λ.

Lemma 5.6. *If $p > \frac{8}{5}$ then*

$$\|\nabla^2 v^\lambda\|_{\frac{p}{3-p}} + \|\theta_\lambda^{p-2}\nabla^2 v^\lambda\|_p \leq C, \tag{5.7}$$

$$\|\theta_\lambda^{p-1}\|_{1,p} + \|\nabla\, v^\lambda\|_{\frac{2p(p-1)}{2-p}} \leq C. \tag{5.8}$$

Proof. As $\frac{p}{3-p} \leq p$ for $p \leq 2$, we see that (5.8) holds in interior of Ω due to (4.9).

It follows from (5.2) that

$$\left\| \nabla \frac{\partial v^\lambda}{\partial t} \right\|_p \le C + C\|\nabla v^\lambda\|_{p'}. \tag{5.9}$$

Next, from (3.28) and $\partial_{12}\partial_{12}F^\lambda(|D(v^\lambda)|^2) \ge C_1\rho_\lambda^{p-2}$ we see

$$\rho_\lambda^{p-2}\left|\frac{\partial^2 v_1^\lambda}{\partial x_2^2}\right| \le C\left\{\rho_\lambda^{p-2}\left|\frac{\partial^2 v_1^\lambda}{\partial x_1 \partial x_2}\right| + \rho_\lambda^{p-2}\left|\frac{\partial^2 v_1^\lambda}{\partial x_1^2}\right|\right\} + |G^\lambda|$$

$$\le C\rho_\lambda^{p-2}\left|\nabla \frac{\partial v^\lambda}{\partial t}\right| + |G^\lambda| + C\max_\ell \sup_{x_1\in(-\alpha,\alpha)}\left|\frac{\partial a(x_1)}{\partial x_1}\right|\rho_\lambda^{p-2}\left|\frac{\partial^2 v_1^\lambda}{\partial x_2^2}\right|. \tag{5.10}$$

In addition we have

$$\frac{\partial^2 v_1^\lambda}{\partial x_1 \partial x_2} = \frac{\partial^2 v_1^\lambda}{\partial x_2 \partial t} + \frac{\partial a}{\partial x_1}\,\frac{\partial^2 v_1^\lambda}{\partial x_2^2},$$

$$\frac{\partial^2 v_1^\lambda}{\partial^2 x_1} = \frac{\partial^2 v_1^\lambda}{\partial x_1 \partial t} + \frac{\partial a}{\partial x_1}\,\frac{\partial^2 v_1^\lambda}{\partial x_1 \partial x_2},$$

$$\frac{\partial^2 v_2^\lambda}{\partial x_2^2} = -\frac{\partial^2 v_1^\lambda}{\partial x_1 \partial x_2} = -\frac{\partial^2 v_1^\lambda}{\partial x_2\, \partial t} - \frac{\partial a}{\partial x_1}\,\frac{\partial^2 v_1^\lambda}{\partial x_2^2},$$

$$\frac{\partial^2 v_2^\lambda}{\partial x_1 \partial x_2} = -\frac{\partial^2 v_1^\lambda}{\partial x_1^2} = -\frac{\partial^2 v_1^\lambda}{\partial x_2 \partial t} - \frac{\partial a}{\partial x_1}\,\frac{\partial^2 v_1^\lambda}{\partial x_1 \partial x_2},$$

$$\frac{\partial^2 v_2^\lambda}{\partial x_1^2} = \frac{\partial^2 v_2^\lambda}{\partial x_1 \partial t} + \frac{\partial a}{\partial x_1}\,\frac{\partial^2 v_2^\lambda}{\partial x_1 \partial x_2}.$$

Using these identities, we can conclude from (5.10) that

$$\rho_\lambda^{p-2}|\nabla^2 v^\lambda|\xi_\ell \le C\rho_\lambda^{p-2}\left|\nabla\frac{\partial v^\lambda}{\partial t}\right|\xi_\ell + |G^\lambda|\xi_\ell$$

$$+c\max_\ell \sup_{x_1\in(-\alpha,\alpha)}\left|\frac{\partial a(x_1)}{\partial x_1}\right|\rho_\lambda^{p-2}|\nabla^2 v^\lambda|\xi_\ell. \tag{5.11}$$

We want to take the L^p-norm of (5.11). Before doing it, let us estimate $\|G^\lambda\xi_\ell\|_p$ by means of the theorem of Nečas on 'negative' Sobolev spaces and (3.26). Thus we obtain

$$\|\xi_\ell\, G^\lambda\|_p \le \left\|v_k^\lambda \frac{\partial v^\lambda}{\partial x_k}\right\|_p + \|f\|_p + \left\|\rho_\lambda^{p-2}\nabla\left(\frac{\partial v^\lambda}{\partial x_1}\right)\xi_\ell\right\|_p + C$$

$$\leq \|f\|_p + c\|\nabla v^\lambda\|_2 + \left\|\nabla\left(\frac{\partial v^\lambda}{\partial t}\right)\xi_\ell\right\|_p + C \tag{5.12}$$

$$+\max_\ell \sup_{x_1\in(-\alpha,\alpha)}\left|\frac{\partial a(x_1)}{\partial x_1}\right| \left|\rho_\lambda^{p-2}\,\nabla^2\, v^\lambda\,\xi_\ell\right\|_p,$$

where we used $\rho_\lambda^{p-2}\leq 1$ and the inequality ($p<2$)

$$\left\|v_k^\lambda\frac{\partial v^\lambda}{\partial x_k}\right\|_p \leq \|v^\lambda\|_{\frac{2p}{2-p}}\,\|\nabla v^\lambda\|_2 \leq c\|\nabla v^\lambda\|_p\,\|\nabla v^\lambda\|_2 \leq C\|\nabla v^\lambda\|_2.$$

Now, taking the L^p-norm of (5.11), and using (5.9), (5.12) and

$$(C+1)\max_\ell \sup_{x_1\in(-\alpha,\alpha)}\left|\frac{\partial a(x_1)}{\partial x_1}\right|\leq\frac{1}{2},$$

we finally obtain ($\rho_\lambda^{p-2}\geq\theta_\lambda^{p-2}, p'>2$)

$$\|\theta_\lambda^{p-2}\nabla^2 v^\lambda\|_p \leq C + C\|\nabla v^\lambda\|_{p'}. \tag{5.13}$$

Since $\|\nabla\theta_\lambda^{p-2}\|_p\leq\|\theta_\lambda^{p-2}\nabla^2 v^\lambda\|_p$, it follows from (5.13) that

$$\|\theta^{p-1}\|_{1,p}\leq C\|\nabla v^\lambda\|_{p'},$$

which yields (using also the imbedding theorem and Rkrn's inequality)

$$\|\nabla v^\lambda\|^{p-1}_{\frac{2p(p-1)}{2-p}} \leq c_0 + c\|\nabla v^\lambda\|_{p'} \leq c_0 + c\|\nabla v^\lambda\|_p^{1-\alpha}\,\|\nabla v^\lambda\|^\alpha_{\frac{2p(p-1)}{2-p}}$$

with

$$\alpha = \frac{2(p-1)(2-p)}{3p-4}.$$

Since $\alpha<p-1$ for $p>\frac{8}{5}$, the assertion folows. The proof is finished. □

We are now ready to prove global $C^{1,\alpha}$-regularity to (1.1) if $p>\frac{8}{5}$.

Theorem 5.14. *Let $p\in(8/5,2)$ and $f\in L^{p'}(\Omega)$. There exist $q>2$ and a solution (v,π) to (1.1) such that*

$$v\in W^{2,q}(\Omega),\quad \pi\in W^{1,q}(\Omega), \tag{5.15}$$

and (1.2) (α) holds.

Proof. Note first that due to (5.7), (5.8) and the analysis within the proof of the previos lemma, we have $f - v_k^\lambda\frac{\partial v^\lambda}{\partial x_k}\in L^{\frac{2p(p-1)}{2-p}}(\Omega)$. Consequently, the solution $\widetilde{u}$ of a problem

$$\int_\Omega D_{ij}(\widetilde{u})D_{ij}(\varphi)\,\mathrm{d}x = \text{RHS}\quad\text{of (3.18)}$$

belongs to $W^{1,\frac{2p(p-1)}{2-p}}(\Omega)$. Setting $G_{ij} \equiv D_{ij}(\widetilde{u})$, we can apply Lemma 2.6 to conclude that for all $q \in (2, 2+\varepsilon)$ (here ε is given through (2.7)) we have

$$\|u^\lambda\|_{1,q} \leq KV^{2-p}, \tag{5.16}$$

where $V = V_\lambda \equiv \|\theta_\lambda\|_\infty$. By the definition of u^λ we see that (5.16) implies

$$\left\| \frac{\partial v^\lambda}{\partial t}\xi_\ell \right\|_{1,q} \leq KV^{2-p}, \quad (\ell - 1, \dots, k). \tag{5.17}$$

Similarly to the proof of Lemma 5.6, using (3.26) and (3.28), we have

$$\rho_\lambda^{p-2}\left|\frac{\partial^2 v_1}{\partial x_2^2}\right|\xi_\ell \leq C\rho_\lambda^{p-2}\left|D\left(\frac{\partial v}{\partial t}\right)\right|\xi_\ell + |G^\lambda|\xi_\ell$$
$$+\widetilde{c}\max_\ell \max_{x_1\in(-\alpha,\alpha)}\left|\frac{\partial a(x_1)}{\partial x_1}\right|\rho_\lambda^{p-2}\left|\frac{\partial^2 v_1}{\partial x_2^2}\right|\xi_\ell. \tag{5.18}$$

As $\rho_\lambda^{p-2} \leq 1$, it follows from (5.17) and (5.18) that

$$\|\theta_\lambda^{p-2}\nabla^2 v\|_q \leq KV^{2-p}, \tag{5.19}$$

which yields

$$\|\theta_\lambda^{p-1}\|_{1,q} \leq KV^{2-p} \quad \text{for} \quad q \in (2, 2+\varepsilon]. \tag{5.20}$$

Now, we proceed in the similar way as in the proof of Lemma 4.6 with the difference that we directly use (5.20) with $q \in (2, 2+\varepsilon]$, i.e.

$$1 - \frac{2}{q} = \frac{\delta \log\left(\frac{1-\frac{C_1}{2C_2}V^{p-2}}{1-\frac{C_1}{C_2}V^{p-2}}\right)}{\log\gamma_3} \geqslant LV^{p-2},$$
$$L \equiv \min\left(\frac{\delta\log 2}{\log\gamma_3}, \frac{C_1}{4C_2\log\gamma_3}\right). \tag{5.21}$$

Again, using (5.21) and the Morrey imbedding inequality (compare with (4.23)), we obtain

$$V^{p-1} = \|\theta^{p-1}\|_{C(\overline{\Omega})} \leq \left(\frac{1}{1-\frac{2}{q}}\right)^{1-1/q} V^{p-2} \leq CV^{(2-p)(2-1/q)}. \tag{5.22}$$

However, if $p > 8/5$, we can always find $q > 2$ such that

$$(2-p)\left(2 - \frac{1}{q}\right) < p - 1. \tag{5.23}$$

With (5.23) we see that (5.22) implies

$$\|\theta_\lambda\|_{C(\overline{\Omega})} \leq K.$$

Consequently, there is a K_0 such that

$$K_0 \leq \max_{x\in\overline{\Omega}} \theta_\lambda^{p-2}(x) \leq 1.$$

Assertion (5.15) then follows from (5.19). Theorem 5.14 is proved. □

Starting from quadratic approximations we have shown the existence of $C^{1,\alpha}$-solutions for $p > 8/5$ and it is observable that this bound comes from the fact that quadratic approximations were used. In order to extend the result for p below $8/5$ we use the following approximative problems to (1.1) for $\lambda \in (0,1)$ and $w \in (p,2]$: to find $(v,\pi) := (v^\lambda, \pi^\lambda)$ solving

$$\begin{aligned} v_k \frac{\partial v}{\partial x_k} - \operatorname{div}\left(\partial F^{\lambda,\omega}\left(|D(v)|^2\right)\right) &= -\nabla\pi + f \quad \text{in} \quad \Omega, \\ \operatorname{div} v &= 0 \quad \text{in} \quad \Omega, \\ v &= 0 \quad \text{at} \quad \partial\Omega, \end{aligned} \tag{5.24}$$

where the $F^{\lambda,\omega}$ is given through

$$F^{\lambda,\omega}\left(|D(v)|^2\right) := \int_0^{|D(v)|^2} (1+\lambda s)^{\frac{\omega-p}{2}} F'(s)\,\mathrm{d}s. \tag{5.25}$$

It is not difficult to verify that $T^{\lambda,\omega} := \partial F^{\lambda,\omega}$ and $F^{\lambda,\omega}$ satisfy the following estimates

$$\begin{aligned} \partial_{ij}\partial_{kl}F^{\lambda,\omega}(|\eta|^2)\,\xi_{ij}\xi_{kl} &\geqslant C_1(1+\lambda|\eta|^2)^{\frac{\omega-p}{2}}(1+|\eta|^2)^{\frac{p-2}{2}}|\xi|^2 \\ &\geqslant C_1 \max\left\{\lambda^{\frac{\omega-p}{2}}(1+|\xi|^2)^{\frac{\omega-p}{2}}, (1+|\eta|^2)^{\frac{p-2}{2}}\right\}|\xi|^2; \end{aligned} \tag{5.26}$$

$$\begin{aligned} \left|\partial_{ij}\partial_{kl}F^{\lambda,\omega}(|\eta|^2)\right| &\leqslant C_2(1+\lambda|\eta|^2)^{\frac{\omega-p}{2}}(1+|\eta|^2)^{\frac{p-2}{2}} \\ &\leqslant C_2(1+|\eta|^2)^{\frac{\omega-2}{2}}; \end{aligned} \tag{5.27}$$

$$\begin{aligned} \partial_{ij}F^{\lambda,\omega}(|\eta|^2)\eta_{ij} &\geqslant C_3(1+\lambda|\eta|^2)^{\frac{\omega-p}{2}}(1+|\eta|^2)^{\frac{p-2}{2}}|\eta|^2 \\ &\geqslant C_3\lambda^{\frac{\omega-2}{2}}(|\eta|^\omega - 1); \end{aligned} \tag{5.28}$$

$$\left|\partial F^{\lambda,w}(|\eta|^2)\right| \leqslant C_4(1+|\eta))^{w-1}, \tag{5.29}$$

where C_i, $i=1,2,3,4$ are independent of λ

Note that if $w=2$ then we obtain the approximations introduced in (4.1)–(4.4) that were used to show the existence of $C^{1,\alpha}$-solutions for $p>8/5$. Now, taking $w\in(8/5,2)$ we can ask the following question: Is it possible to construct via the w-approximations the $C^{1,\alpha}$-solutions for $p\leqslant 8/5$? The answer is positive: to prove the result for $p>3/2$ we have to proceed iteratively as shown below.

Theorem 5.30. *Let $p\in(3/2,2)$ and $f\in L^{p'}(\Omega)$. There exist $q>2$ and a solution (v,π) to (1.1) such that*

$$v\in W^{2,q}(\Omega),\quad \pi\in W^{1,q}(\Omega), \tag{5.31}$$

and (1.2) (α) holds.

Proof. By Theorem 5.14 we know that (5.31) holds for $p>8/5$. Now, using the approximations (5.25)–(5.29), we are going to list changes in the proofs of Lemma 5.1, Lemma 5.6 and Theorem 5.14 that these new approximations bring.

(1) In Lemma 5.1 the only change is caused by (5.29). Consequently, instead of (5.2) we have

$$\left\|\xi_\ell\nabla\frac{\partial v^\lambda}{\partial t}\right\|_p^2+\int\limits_\Omega\rho_\lambda^{p-2}\left|D\left(\frac{\partial v^\lambda}{\partial t}\xi_\ell\right)\right|^2\mathrm{d}x\leqslant C+C\|\nabla v^\lambda\|^2_{p'(w-1)}. \tag{5.32}$$

We wish to emphasize that the bound $p>3/2$ comes from (5.5), which is independent of choosen approximations.

(2) Lemma 5.6 holds also for $p>3/2$. To see it we first redefine ρ^λ as $\rho^\lambda:=(1+\lambda|D(v)|^2)^{\frac{w-p}{2(p-2)}}(1+|D(v)|^2)^{1/2}$. Because of (5.32), p' is replaced in (5.9) by $p'(w-1)$, and (5.13) reads

$$\left\|\theta_\lambda^{p-2}\,\nabla^2\,v^\lambda\right\|_p\leqslant C+C\left(\|\nabla v^\lambda\|_2+\|\nabla v^\lambda\|_{p'(w-1)}\right). \tag{5.33}$$

This leads to

$$\begin{aligned}\|\nabla v\|^{p-1}_{\frac{2p(p-1)}{2-p}}&\leqslant C+C\left(\|\nabla v^\lambda\|_2+\|\nabla v^\lambda\|_{p'(w-1)}\right)\\ &\leqslant C\left(1+\|\nabla v^\lambda\|_p^{1-\alpha}\|\nabla v^\lambda\|^\alpha_{\frac{2p(p-1)}{2-p}}+\|\nabla v^\lambda\|_p^{1-\beta}\|\nabla v^\lambda\|^\beta_{\frac{2p(p-1)}{2-p}}\right)\end{aligned}$$

with

$$\alpha = \frac{2(w-p)(p-1)}{(3p-4)(w-1)}, \quad \beta = \frac{(2-p)(p-1)}{3p-4}.$$

Observe that $\beta < p-1$ if $p > 3/2$. The requirement $\alpha < p-1$ leads to the restriction

$$p > 2\frac{3w-2}{3w-1}. \tag{5.34}$$

(3) In the proof of Theorem 5.14, we replace (5.21) by

$$1 - \frac{2}{q} \geqslant LV^{p-w} \tag{5.35}$$

with L defined in (5.21). (See also lines between (4.19)) and (4.22) for derivation of (5.35).) Thus, because of (5.26) and (5.27) we obtain instead of (5.22)

$$V^{p-1} = \|\theta^{p-1}\|_{C(\overline{\Omega})} \leqslant C\left(\frac{1}{1-\frac{2}{q}}\right)^{1-1/q} V^{2-p} \leqslant CV^{2-p+(w-p)(1-1/q)}. \tag{5.36}$$

It means that we need

$$2 - p + (w-p)\left(1 - \frac{1}{q}\right) < p - 1, \tag{5.37}$$

which is equivalent to

$$p > \frac{w(1-1/q)+3}{3-\frac{1}{q}} \to \frac{\omega+6}{5} \quad \text{for} \quad q \to 2. \tag{5.38}$$

As $w \in (p, 2]$ one can check that condition (5.24) is fulfilled authomatically if (5.38) holds. However, setting $w_0 = 2$ and $w_i = \frac{w_{i-1}+6}{5}$ for $i = 1, 2, \dots$ we see that $w_i \to 3/2$ as $i \to \infty$. Therefore, for every $p > 3/2$ we find after doing $(i-1)$ iteration steps w_{i-1}-approximations that are $C^{1,\alpha}$ so that using them we observe that (5.26) and (5.37) are satisfied, and (5.31) holds.

Theorem 5.30 is proved. □

6. Higher regularity and uniqueness

For the sake of completness we first formulate the theorem which deals with the case $p > 2$.

Theorem 6.1. *Let $p > 2$ and $f \in L^s(\Omega)$, $s > 2$. Then there exist $q > 2$ and a solution (v, π) to (1.1) such that*

$$v \in W^{2,q}(\Omega), \quad \pi \in W^{1,q}(\Omega), \tag{5.15}$$

and (1.2) (α) holds.

Proof. Proof is omitted. It can be done similarly, see [11] for periodic case or [22] where parabolic systems of this type are studied.

Further we show in brief that if data are smooth the solution is also smooth.

Theorem 6.3. *Let $p > 3/2$, $k \in \mathbb{N}$, $k \geqslant 1$. Assume $F \in C^{k+2}(\mathbb{R}_0^+)$, $f \in W^{k,2}(\Omega)$ and $\partial\Omega \in C^{k+2}$. Then*

$$v \in W^{k+2,2}(\Omega), \quad \pi \in W^{k+1,2}(\Omega). \tag{6.4}$$

Proof. A proof reduces to a more or less standard application of the technique of the regularity theory for linear elliptic systems. The only term that requires circumspection is the term of the type

$$J \equiv \int_\Omega \frac{\partial^3 F(|D(v)|^2)}{\partial p_{ij}\partial p_{kl}\partial p_{rs}} D_{rs}(\nabla v) D_{kl}(\nabla v) D_{ij}(\nabla^{(2)} v)\, \mathrm{d}x,$$

which appears while deriving $W^{3,2}$-regularity. In such a case we have the second power of L^2-norm of the third derivatives of v on the left-hand side and we know (see (1.14)) that $v \in W^{2,p_0}_{\mathrm{loc}}(\mathbb{R}^2)$ for $p_0 > 2$. Write $p_0 = 2 + \rho$. Then the interpolation inequality

$$\|u\|_4 \leqslant \|u\|_{2+\rho}^{\frac{2+\rho}{4}} \|\nabla u\|_2^{\frac{2-\rho}{4}}$$

leads to the following estimate of J (using also (1.14), the Hölder and Young inequalities):

$$|J| \leqslant C \int_\Omega |\nabla^{(2)} v|^2\, |\nabla^{(3)} v| \leqslant C \|\nabla^{(2)} v\|_4^2\, \|\nabla^{(3)} v\|_2$$

$$\leqslant \|\nabla^{(2)} v\|_{p_0}^{\frac{2+\rho}{2}}\, \|\nabla^{(3)} v\|_2^{2-\frac{\rho}{2}} \leqslant \frac{\varepsilon}{2} \|\nabla^{(3)} v\|_2^2 + C.$$

Choosing $\varepsilon < 2C_1$ we move $\varepsilon/2\|\nabla^{(3)} v\|_2^2$ to the left-hand side and we conclude $W^{3,2}$-regularity for v. The higher regularity is then straighforward. □

Now, we are going to show that for $p \geqslant 3/2$ the $C^{1,\alpha}$-solution is unique in the class a weak solutions provided that $\|f\|_{V_p^*}$ is small.

By a weak solution to (1.1) we mwan a function $u \in V_p$ satisfying

$$\int_\Omega T_{ij}(D(u))D_{ij}(\varphi)\,\mathrm{d}x + \int_\Omega u_k \frac{\partial u_i}{\partial x_k}\,\varphi_i\,\mathrm{d}x = \int_\Omega f_i\varphi_i\,\mathrm{d}x \quad \forall\,\varphi \in V_p. \tag{6.5}$$

It is easy to check that if $p \geqslant 3/2$ we can put $\varphi = u$ in (6.5). By a strong solution we will mean $v \in C^{1,\alpha}(\Omega)$ solving (1.1). Clearly, such that a solution fulfils

$$\int_\Omega T_{ij}(D(v))D_{ij}(\varphi)\,\mathrm{d}x + \int_\Omega v_k \frac{\partial v_i}{\partial x_k}\,\varphi_i\,\mathrm{d}x = \int_\Omega f_i\varphi_i\,\mathrm{d}x \quad \forall\,\varphi \in V_p. \tag{6.6}$$

Theorem 6.7. *Let $p \geqslant 3/2$. Assume that u and v are weak and strong solutions to (1.1), respectively. Then u and v coincide provided that $\|f\|_{V_p^*} < \delta$ for δ small enough.*

Proof. We subtract, as usual, (6.6) from (6.5) and put $\varphi = u - v$ in the so-obtained equation. Let $r = p$ if $p \in [3/2, 2]$ and $r = 2$ if $p > 2$. Using the monotonicity we obtain

$$\|\nabla(u-v)\|_r^2 \leqslant C\int_\Omega |u-v|^2\,|\nabla u|\,\mathrm{d}x. \tag{6.8}$$

The Hölder inequality and the imbedding $W^{1,p}(\Omega) \hookrightarrow L^{\frac{2p}{2-p}}(\Omega)$ yield $\big(\frac{2p}{p-1} \leqslant \frac{2p}{2-p}$ if and only if $p \geqslant \frac{3}{2}\big)$

$$\|\nabla(u-v)\|_r^2 \leqslant C\|u-v\|^2_{\frac{2p}{p-1}}\|\nabla u\|_p \leqslant C\|\nabla(u-v)\|_r^2\,\|\nabla u\|_p.$$

Since weak solution satisfies

$$\|\nabla u\|_p \leqslant C\big(\|f\|_{V_p^*}\big) \quad \text{with} \quad C\big(\|f\|_{V_p^*}\big) \to 0,$$

the uniqueness follows. $\square$

Finally, we deal with the question of uniqueness if $p > 1$. Then we have to modify the definition of weak solution in the following sense: a function $u \in V_p$ is called a weak solution to (1.1) if

$$\int_\Omega T_{ij}(D(u))D_{ij}(\varphi)\,\mathrm{d}x - \int_\Omega u_k u_i \frac{\partial \varphi_i}{\partial x_k}\,\mathrm{d}x = \int_\Omega f_i\varphi_i\,\mathrm{d}x$$

$$\forall\, \varphi \in C^{1,\alpha}(\overline{\Omega}) \tag{6.9}$$

and the energy inequality

$$\int_\Omega T_{ij}(D(u))D_{ij}(u)\,\mathrm{d}x \leqslant \int_\Omega f_i\, u_i\,\mathrm{d}x \tag{6.10}$$

holds. Remark that strong solution clearly satisfies the energy equality (taking $\varphi = v$ in (6.6))

$$\int_\Omega T_{ij}(D(v))D_{ij}(v)\,\mathrm{d}x = \int_\Omega f_i\, v_i\,\mathrm{d}x, \tag{6.11}$$

and we know from the construction performed in Sec. 4 that

$$\|v\|_{C^1(\overline{\Omega})} \leqslant C_0(\|f\|_{2+\varepsilon}) \quad \text{and} \quad C_0(\|f\|_{2+\varepsilon}) \to 0 \quad \text{if} \quad \|f\|_{2+\varepsilon} \to 0. \tag{6.12}$$

Under such a situation we can prove the following result.

Theorem 6.13. *Let $p > 1$. Assume that u and v are weak and strong solutions to (1.1), respectively. Then u and v coincide provided that $\|f\|_{2+\varepsilon} < \delta$ for sufficiently small δ.*

Proof. Taking $\varphi = v$ in (6.9) and $\varphi = u$ in (6.6) we obtain two identities which combined with (6.10) and (6.11) imply

$$\begin{aligned}\int_\Omega &\big(T(D(u)) - T(D(v))\big)\cdot(D(u) - D(v))\,\mathrm{d}x \\ &\leqslant \int_\Omega \left(v_k \frac{\partial v_i}{\partial x_k} u_i - u_k\, u_i \frac{\partial v_i}{\partial x_k}\right)\mathrm{d}x.\end{aligned} \tag{6.14}$$

As

$$\int_\Omega v_k \frac{\partial v_i}{\partial x_k}\, v_i\,\mathrm{d}x = 0 \quad \text{and} \quad \int_\Omega u_k\, v_i \frac{\partial v_i}{\partial x_k}\,\mathrm{d}x = 0,$$

we see that the right-hand side of (6.14) is equal to

$$-\int_\Omega (v_k - u_k)\frac{\partial v_i}{\partial x_k}\,(v_i - u_i)\,\mathrm{d}x.$$

Consequently, by strong monotonicity, we have from (6.14) (for $p \in (1,2)$)

$$\|\nabla(u-v)\|_p^2 \leqslant C \int_\Omega |u-v|^2\,|\nabla x|\,\mathrm{d}x \leqslant C\|v\|_{C^{1,\alpha}(\overline{\Omega})}\,\|u-v\|_2^2$$

$$\leqslant C_0(\|f\|_{2+\varepsilon})\,\|\nabla(u-v)\|_p^2,$$

where we have used the imbedding $W^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$ $(p \geqslant 1)$ and (6.12). If $\|f\|_{2+\varepsilon} < 1/C_0$, then uniqueness follows. $\square$

References

1. H. Amann, *Stability of the rest state of a viscous incompressible fluid.* — Arch. Rat. Mech. Anal. **126** (1994), 231–242.
2. Ch. Amrouche, V. Girault, *Decomposition of vector spaces and application to the Stokes problem in arbitrary dimension.* — Czechoslovak Math. **44** (1994), 109–141.
3. J. Bergh, J. Löfström, *Interpolation Dpaces. An Introduction.* Springer-Verlag, Berlin–New York (1976).
4. E. Blavier, A. Mikelić, *On the stationary Quasi-Newtonian Flow Obeying a Power-law.* — Math. Meth. Appl. Sci. **18** (1995), 927–948.
5. M. E. Bogovskii, *Solution of some vector analysis problems connected with operators div and grad.* — Trudy Seminar S. L. Sobolev Akad. Nauk SSSR **1** (1980), 5–40.
6. J. Frehse, J. Málek, M. Steinhauer, *An existence result for fluids with shear dependent viscosity-steady flows.* — Nonlinear Analysis, Theory Meth. Appl. **30** (1997), 3041–3049.
7. M. Fuchs, *On quasi-static non-Newtonian fluids with power-law.* — Math. Methods Appl. Sci. **19** (1996), 1225–1232.
8. M. Fuchs, G. Seregin, *Regularity results for the quasi-static Bingham variational inequality in dimensions two and three.* — Math. Z. **227** (1998), 525–541.
9. P. G. Galdi, *An introduction to the mathematical theory of the Navier–Stokes equations.* in: *Linearized Steady Problems*, Vol. 1, Springer-Verlag, New York (1994).
10. J. Hron, J. Málek, S. Turek, *A numerical investigation of flows of shear-thinning fluids with applications to blood rheology.* — submitted Internat. J. Numer. Methods Fluids (1998).
11. P. Kaplický,, *On existence of $C^{1,\alpha}$-solutions to a class of nonlinear fluid in 2D – stationary, periodic case.* submitted to *Proceedings of International Conference on Navier–Stokes equation, Theory and Numerical Methods*, to appear in Pitman Research in Mathematics Series (1998).
12. P. Kaplický, J. Málek, J. Stará, *Full regularity of weak solutions to a class of nonlinear fluids on two dimensions – stationary, periodic problem.* — Comment. Math. Univ. Carolinae **38**, (4) (1997), 681–695.
13. O. A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow.* Gordon and Beach, New York (1969).
14. J. L. Lions, *Quelques Méthodes de Résolution des Pronlèmes aux Limites Non Linéaires.* Dunod, Paris (1969).
15. J. Málek, J. Nečas, M. Růužička, *On weak solutions to a class of non-Newtonian Incompressible fluids in bounded three-dimensional domains. The case $p \leq 2$.* to appear in Adv. in Diff. Eqs., also Preprint SFB, No. 481 (1996).

16. J. Málek, J. Nečas, M. Růužička, *Weak and Measure-Valued Solutions to the Evolutionary PDE's.* Chapman& Hall, London (1996).
17. J. Málek, K. R. Rajagopal, M. Růužička, *Existence and Regularity of solutions and stability of the rest state for fluids with shear dependent viscosity.* — Math. Models Methods Appl. Sci. **6** (1995), 789–812.
18. A. Mikelić, R. Tapiéro, *Mathematical derivation of the power law describing plymer flow through a thin slab.* — RAIRO Modél Anal. Numér. **29** (1995), 3–21.
19. J. Nečas, *Sur les normes équivalentes dans $W^{k,p}(\Omega)$ et sur la coercivité des formes formellement positives.* Les Presses de l'Université de Montréal, Montréal, 1996 pp. 102–128.
20. J. Nečas, *Sur la régularité des solutions faibles des équations elliptiques non-linéaires.* — Comment. Math. Univ. Carolinae **9**, No. 3 (1968), 365–413.
21. J. Nečas, *Sur la régularité des solutions variationnelles des équations elliptiques non-linéaires d'ordre 2k en deux dimensions.* — Annali della Scuola Normale Superiore di Pisa **XXI**, III (1967), 427–457.
22. J. Stará, *Regularity results for non-linear elliptic systems in two-dimensions.* — Annali della Scuola Normale Superiore di Pisa **XXV**, I (1971), 163–190.
23. T. Temam, *Navier–Stokes Equations and Nonlinear Functional Analysis.* Second Edition, Philadelphia, Pennsylvania (1995).
24. W. P. Ziemer, *Weakly Differentiable Functions.* Springer-Verlag, New York Inc. (1989).

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