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**Dedicated to N.N. Ural'tseva
on the occasion of her 65th birthday**

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REGULARITY FOR DUAL SOLUTIONS AND FOR WEAK CLUSTER POINTS OF MINIMIZING SEQUENCES OF VARIATIONAL PROBLEMS WITH LINEAR GROWTH

ABSTRACT. The minimum problem $\int_\Omega f(\nabla u)dx \longrightarrow \min$ among mappings $u : \mathbb{R}^n \supset \Omega \to \mathbb{R}^N$ with prescribed Dirichlet boundary data and for integrands f of *linear* growth in general fails to have solutions in the Sobolev space W_1^1. We therefore concentrate on the dual variational problem which admits a unique maximizer σ and prove partial Hölder continuity of σ. Moreover, we study smoothness properties of L^1-limits of minimizing sequences of the original problem.

1. INTRODUCTION

For strictly convex integrands f: $\mathbb{R}^{nN} \to \mathbb{R}$ of class $C^2(\mathbb{R}^{nN}, \mathbb{R})$ we discuss the minimization problem

$$J(u) := \int_\Omega f(\nabla u)dx \longrightarrow \min \tag{1.1}$$

among mappings $u : \Omega \to \mathbb{R}^N$ with prescribed Dirichlet boundary data u_0 where Ω denotes a bounded domain in $\mathbb{R}^n$. We investigate (1.1) under the assumption that f is of linear growth at infinity, i.e.

$$\lambda\,|X| \le f(X) \ \le \ \Lambda\,|X| \tag{1.2}$$

is satisfied with positive constants λ, Λ and for all matrices X whose Euclidean norm is sufficiently large (compare Section 2 for precise definitions). Prominent examples of integrands f with (1.2) are the minimal surface case $f(\nabla u) = \sqrt{1+|\nabla u|^2}$ (with $N = 1$) and various densities from the theory of perfect plasticity. For an intensive discussion of the theory of minimal surfaces including a variety of references we mention the monographs [12] and [6], the behaviour of plastic materials is described in the recent book [8].

As a matter of fact it is in general not possible to solve problem (1.1) in the affine class $u_0 + u_0 + \overset{\circ}{W}{}_1^1(\Omega, \mathbb{R}^N)$. One way to overcome this difficulty is to introduce suitable relaxed versions of (1.1) and to work in spaces like $\mathbf{v}$ (see, for instance, [9] or [5]) by the way producing very weak solutions whose derivatives are just measures. As a second approach one can look at the dual variational problem which has several advantages. First of all the solution of the dual problem is unique. Secondly, the maximizer σ has a clear geometrical or physical meaning: for minimal surfaces σ corresponds to the normal vector field (see [7], pp. 116), in the case of plastic materials σ represents the stress tensor (compare [8]).

Regularity of the dual solution in the setting of plasticity theory has been mainly investigated by Seregin in a series of papers (see [15–19]), a more complete list of references and also a survey of the history is given in the recent monograph [8]. The purpose of our note is to extend some of Seregin's results to the general variational setting where the integrand not necessarily depends on the modulus of the gradient. For example, we prove partial regularity of the dual solution σ by showing that

$$\sigma = \frac{\partial f}{\partial X}(\nabla u^*) \quad \text{almost everywhere on} \quad \Omega \tag{1.3}$$

for some generalized solution u^* of the original problem. The main contribution of our paper however is the result that all weak cluster points of J–minimizing sequences are partially of class C^1 which follows from the results of [2] combined with a new construction poviding suitable comparison functions (see Section 7).

Once having established this, the regularity of σ is a direct consequence of (1.3): in order to get (1.3) we could of course follow the lines of [15–19] by using the relaxed minimax inequality which gives (1.3) immediately on the regular set of u^*. Instead of this we present a slight variant which is based on the properties of a special minimizing sequence.

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2. Notation and Statement of the Main Results

Given boundary values $u_0 \in W_1^1(\Omega, \mathbb{R}^N)$, we consider the problem $(\mathcal{P})$, that is to minimize the functional

$$J(u) = \int_\Omega f(\nabla u)dx \quad \text{in the class} \quad u_0 + \overset{\circ}{W}{}_1^1(\Omega, \mathbb{R}^N),$$

where Ω denotes a bounded Lipschitz domain in $\mathbb{R}^n$. Throughout this paper we assume that the integrand satisfies the following general hypothesis:

Assumption 2.1. *The function f is smooth, strictly convex and of linear growth in the following sense:*

(i). $f \in C^2(\mathbb{R}^{nN}, \mathbb{R})$.

(ii). *There is a real number M such that for all X, $Y \in \mathbb{R}^{nN}$, $Y \neq 0$,*

$$0 < D^2 f(X)(Y,Y) \;\leq\; M \frac{1}{\sqrt{1+|X|^2}} |Y|^2.$$

(iii). *There is a real number A such that $|\nabla f(X)| \leq A$ for all $X \in \mathbb{R}^{nN}$.*

(iv). *For numbers $a > 0$ and $b \in \mathbb{R}$ we have $f(X) \geq a|X| + b$ for all $X \in \mathbb{R}^{nN}$.*

Remark 2.2 (*i.*) We need $D^2 f$ to be positive in order to apply the results of [2].

(*ii.*) Without loss of generality $\nabla f(0) = 0$ can be assumed. This implies

$$\lim_{|Y| \to \infty} (\nabla f(Y) - \nabla f(X)) : (Y - X) = \infty \quad \text{for all} \quad X \in \mathbb{R}^{nN}.$$

Now let us recall the notion of the conjugate function (see, for instance, [7, 14] and [20]).

Definition 2.3. *Consider a Banach space V, its dual V^* and a function $G : V \to \overline{\mathbb{R}}$. Then the polar or conjugate function of G is defined for all $v^* \in V^*$ by*

$$G^*(v^*) := \sup_{v \in V} \left\{ \langle v, v^* \rangle - G(v) \right\}.$$

If the subdifferential of G is denoted by ∂G (see [7] pp. 20) and if $\partial G(u) \neq \emptyset$, then the duality relation holds true:

$$u^* \in \partial G(u) \Leftrightarrow G(u) + G^*(u^*) = \langle u, u^* \rangle.$$

This gives for our smooth integrand $f : \mathbb{R}^{nN} \to \mathbb{R}$:

$$f(u) + f^*(\nabla f(u)) = u : \nabla f(u). \tag{2.1}$$

We recall the following representation formula (see [7])

$$J(u) = \sup_{\tau \in L^\infty(\Omega, \mathbb{R}^{nN})} \left\{ \int_\Omega \tau : \nabla u dx - \int_\Omega f^*(\tau) dx \right\}. \tag{2.2}$$

The representation formula (2.2) for J can be used to define an appropriate dual problem. We first define the Lagrangian $l(u, \tau)$ for all $(u, \tau) = (u_0 + \varphi, \tau)$ in the class $u_0 + \overset{\circ}{W}{}^1_1(\Omega, \mathbb{R}^N) \times L^\infty(\Omega, \mathbb{R}^{nN})$ by the formula

$$l(u, \tau) := \int_\Omega \tau : \nabla u dx - \int_\Omega f^*(\tau) dx \;=\; l(u_0, \tau) + \int_\Omega \tau : \nabla \varphi dx.$$

Then the dual functional R: $L^\infty(\Omega, \mathbb{R}^{nN}) \to \overline{\mathbb{R}}$ is by definition

$$R(\tau) := \inf_{u \in u_0 + \overset{\circ}{W}{}^1_1(\Omega, \mathbb{R}^N)} l(u, \tau) \;=\; \begin{cases} -\infty & \text{if} \quad \operatorname{div} \tau \neq 0 \\ l(u_0, \tau) & \text{if} \quad \operatorname{div} \tau = 0 \end{cases},$$

and we have

$$\inf_{u \in u_0 + \overset{\circ}{W}{}^1_1(\Omega, \mathbb{R}^N)} J(u) = \sup_{\tau \in L^\infty(\Omega, \mathbb{R}^{nN})} R(\tau). \tag{2.3}$$

We now consider the dual problem

$$(\mathcal{P}^*) : \quad R(\tau) \longrightarrow \max \quad \text{among all functions} \quad \tau \in L^\infty(\Omega, \mathbb{R}^{nN}).$$

Then, our main result reads as follows (for details we refer to the Theorems 4.1, 5.1, 6.1):

Main Theorem. a.) *Let u denote any weak cluster point of a J-minimizing sequence. Then we have $u \in C^{1,\alpha}(\Omega_0, \mathbb{R}^N)$ for an open set Ω_0 of full measure.*

b.) *The dual problem $(\mathcal{P}^*)$ has a (unique) solution $\sigma \in W^1_{2,loc}(\Omega, \mathbb{R}^{nN})$ which is Hölder continuous up to a relatively closed set of measure 0.*

Remark 2.4. Uniqueness of the maximizer has been proved in the preliminary version [4] of our paper under the additional assumption that the closure of the image of ∇f is a strictly convex set. In the meantime the first author obtained uniqueness of the maximal solution even under weaker assumptions as stated in 2.1, the proof of this fact is given in the separate paper [3].

3. Approximation

Perturbing the original problem in a well known way (compare, e.g. [17]), we now construct a special maximizing sequence for the dual problem. We discuss this sequence in detail on account of its very convenient properties which will be useful in proving the second part of the main theorem. In this section the boundary values u_0 are assumed to be of class $W_2^1(\Omega,\mathbb{R}^N)$ and problem (1.1) will be approximated in the following way: consider for $\delta \in (0,1)$ the functional

$$J_\delta(w) := \frac{\delta}{2}\int_\Omega |\nabla w|^2 dx + J(w), \quad w \in u_0 + \overset{\circ}{W}{}_2^1(\Omega,\mathbb{R}^N), \tag{3.1}$$

and denote by u_δ the unique solution of the problem $(\mathcal{P}_\delta)$, that is u_δ is the minimizer of J_δ in the class $u_0 + \overset{\circ}{W}{}_2^1(\Omega,\mathbb{R}^N)$. Setting

$$f_\delta := \frac{\delta}{2}|\cdot|^2 + f, \quad \tau_\delta := \nabla f(\nabla u_\delta), \quad \sigma_\delta := \delta\nabla u_\delta + \tau_\delta = \nabla f_\delta(\nabla u_\delta), \tag{3.2}$$

$$\int_\Omega \sigma_\delta : \nabla\varphi dx = 0 \quad \text{for all} \quad \varphi \in \overset{\circ}{W}{}_2^1(\Omega,\mathbb{R}^N). \tag{3.3}$$

is seen to be true. The minimality of u_δ implies $J_\delta(u_\delta) \le J_\delta(u_0) \le J_1(u_0)$, and

$$\delta\int_\Omega |\nabla u_\delta|^2 dx \le c_1, \quad \int_\Omega f(\nabla u_\delta)dx \le c_1 \tag{3.4}$$

follows for some real number c_1. Since ∇f is bounded,

$$\|\tau_\delta\|_{L^\infty(\Omega,\mathbb{R}^{nN})} \le c_2$$

is true for some real number c_2, and choosing a subsequence we may assume the existence of $\tau \in L^\infty(\Omega,\mathbb{R}^{nN})$ and of $\sigma \in L^2(\Omega,\mathbb{R}^{nN})$ such that

$$\tau_\delta \overset{*}{\rightharpoondown} \tau \text{ in } L^\infty(\Omega,\mathbb{R}^{nN}) \quad \text{and} \quad \sigma_\delta \rightharpoondown \sigma \text{ in } L^2(\Omega,\mathbb{R}^{nN}) \quad \text{as} \quad \delta \to 0. \tag{3.5}$$

We have the identity $\tau = \sigma$. To see the last equation, observe that τ is also the weak L^2-limit of the sequence $\{\tau_\delta\}_\delta$ (if necessary choose a subsequence) and that

$$\delta\nabla u_\delta \rightharpoondown 0 \quad \text{in} \quad L^2(\Omega,\mathbb{R}^{nN}) \quad \text{as} \quad \delta \to 0. \tag{3.6}$$

The following lemma proves that σ is the maximizer of the dual variational problem.

Lemma 3.1.

(i). *The limit σ defined above is admissible in the sense that* $\operatorname{div}\sigma = 0$.

(ii). *The limit σ maximizes the dual variational Problem* $(\mathcal{P}^*)$.

Remark 3.2. Lemma 3.1 corresponds to Lemma 2 in [17] where the case of integrands depending on the modulus of the gradient is considered. Similar results were obtained in [18], Lemma 3.2, and [19], Lemma 3.1.

Proof. Equation (3.3) can be written as (see (3.2))

$$\int_\Omega \tau_\delta : \nabla\varphi dx + \int_\Omega \delta\nabla u_\delta : \nabla\varphi dx = 0 \quad \text{for all} \quad \varphi \in C_0^\infty(\Omega, \mathbb{R}^N),$$

thus implying (i.) by (3.5) and (3.6). To complete the proof, observe that the duality relation (compare (2.1)) $\tau_\delta : \nabla u_\delta - f^*(\tau_\delta) = f(\nabla u_\delta)$ yields

$$\begin{aligned}
J_\delta(u_\delta) &= \frac{\delta}{2}\int_\Omega |\nabla u_\delta|^2 dx + \int_\Omega (\tau_\delta : \nabla u_\delta - f^*(\tau_\delta))\, dx \\
&= -\frac{\delta}{2}\int_\Omega |\nabla u_\delta|^2 dx + \int_\Omega (\sigma_\delta : \nabla u_\delta - f^*(\tau_\delta))\, dx \\
&= -\frac{\delta}{2}\int_\Omega |\nabla u_\delta|^2 dx + \int_\Omega (\sigma_\delta : \nabla u_0 - f^*(\tau_\delta))\, dx,
\end{aligned}$$

where we used (3.3) to get the last equation. We have

$$\begin{aligned}
\sup\{R(\kappa) : \kappa \in K\} \le \inf\{J(v) :\ & \\
& v \in u_0 + \overset{\circ}{W}{}_1^1(\Omega, \mathbb{R}^N)\} \ \le\ J(u_\delta) \ <\ J_\delta(u_\delta) \\
&= -\frac{\delta}{2}\int_\Omega |\nabla u_\delta|^2 dx + \int_\Omega (\sigma_\delta : \nabla u_0 - f^*(\tau_\delta))\, dx \\
&= -\frac{\delta}{2}\int_\Omega |\nabla u_\delta|^2 dx + \int_\Omega (\tau_\delta : \nabla u_0 - f^*(\tau_\delta))\, dx \\
&+ \delta \int_\Omega \nabla u_\delta : \nabla u_0 dx.
\end{aligned}$$

Passing to the limit, using the upper semicontinuity of $-\int_\Omega f^*(\cdot)dx$ with respect to weak-$*$ convergence and observing that the last integral tends to 0 as $\delta \to 0$, we obtain

$$\delta \int\limits_\Omega |\nabla u_\delta|^2 dx \to 0 \quad \text{as} \quad \delta \to 0,$$

and the maximality of σ follows. □

Remark 3.3. Replacing the first inequality in (3.8) by a strict one also the inf-sup relation (2.3) follows immediately by contradiction.

4. $W^{\mathbf{1}}_{\mathbf{2},\mathrm{loc}}$-REGULARITY FOR THE DUAL PROBLEM

The δ-approximation introduced before leads us to a first regularity result for σ:

Theorem 4.1. *The maximizer σ of the dual problem $(\mathcal{P})^*$ satisfies*

$$\sigma \in W^1_{2,\mathrm{loc}}(\Omega, \mathbb{R}^{nN}).$$

Remark 4.2. Again we benefit from arguments outlined in [16–19].

Proof of Theorem 4.1. We prove this under the assumption $u_0 \in W^1_2(\Omega, \mathbb{R}^N)$. As in the previous section, we have the equation (3.3) being true for all $\delta \in (0,1)$. Given a compactly supported $\varphi \in W^2_2(\Omega, \mathbb{R}^N)$, we may replace φ by $\partial_\gamma \varphi$ for all $\gamma = 1, \dots, n$. Furthermore u_δ and $\sigma_\delta = \nabla f_\delta(\nabla u_\delta)$ are by construction smooth enough to perform an integration by parts and we obtain

$$\int\limits_\Omega \partial_\gamma \sigma_\delta : \nabla \varphi dx = \int\limits_\Omega D^2 f_\delta(\nabla u_\delta)(\nabla \partial_\gamma u_\delta, \nabla \varphi) dx \;=\; 0, \quad \gamma = 1, \dots n. \tag{4.1}$$

In order to use this equation, the following inequality is needed: there is a real number c, not depending on δ, such that

$$|\nabla \sigma_\delta|^2 \le c\, \partial_\gamma \sigma_\delta : \partial_\gamma \nabla u_\delta\,, \tag{4.2}$$

where we always take the sum with respect to γ from 1 to n. (4.2) is just a consequence of the definition of σ_δ together with the Cauchy–Schwarz inequality. Now the Euler equation (4.1) with $\varphi = \eta^2 \partial_\gamma u_\delta$, $\eta \in C^\infty_0(\Omega, \mathbb{R})$, becomes

$$I := \int\limits_\Omega \eta^2 \partial_\gamma \sigma_\delta : \nabla \partial_\gamma u_\delta dx \;=\; -2 \int\limits_\Omega \eta \partial_\gamma \sigma_\delta : (\nabla \eta \otimes \partial_\gamma u_\delta) dx.$$

Notice that $\partial_\gamma \sigma_\delta : \nabla \partial_\gamma u_\delta = D^2 f_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)$. Again the Cauchy–Schwarz inequality and the growth conditions for f and $D^2 f$ show the existence of a real number c_2, not depending on δ, such that

$$\begin{aligned}
&-2 \int_\Omega \eta \partial_\gamma \sigma_\delta : (\nabla \eta \otimes \partial_\gamma u_\delta) dx \; = \\
&-2 \int_\Omega \eta D^2 f_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \nabla \eta \otimes \partial_\gamma u_\delta) dx \\
&\leq 2\, I^{\frac{1}{2}} \cdot \left(\int_\Omega D^2 f_\delta(\nabla u_\delta)(\nabla \eta \otimes \partial_\gamma u_\delta, \nabla \eta \otimes \partial_\gamma u_\delta) dx \right)^{\frac{1}{2}} \\
&\leq 2\, I^{\frac{1}{2}} \cdot \left(\int_\Omega |D^2 f_\delta|(\nabla u_\delta)\, |\nabla \eta|^2\, |\nabla u_\delta|^2 dx \right)^{\frac{1}{2}} \\
&\leq c_2\, I^{\frac{1}{2}} \cdot \left(\int_\Omega |\nabla \eta|^2\, |\nabla u_\delta|^2 \left[\delta + \frac{1}{\sqrt{1 + |\nabla u_\delta|^2}} \right] dx \right)^{\frac{1}{2}} \\
&\leq c_2\, I^{\frac{1}{2}} \cdot \left(\int_\Omega |\nabla \eta|^2 \left[|\nabla u_\delta| + \delta |\nabla u_\delta|^2 \right] dx \right)^{\frac{1}{2}} .
\end{aligned}$$

The last integral is bounded uniformly in δ (see (3.4) and (3.9)) and we obtain

$$\int_\Omega \eta^2 \partial_\gamma \sigma_\delta : \nabla \partial_\gamma u_\delta dx \;\leq\; c_3(\nabla \eta).$$

This together with (4.2) implies that σ_δ is uniformly (with respect to δ) bounded in $W^1_{2,\mathrm{loc}}(\Omega, \mathbb{R}^{nN})$. Thus, the weak L^2-convergence $\sigma_\delta \rightharpoonup \sigma$ yields the conclusion $\sigma \in W^1_{2,\mathrm{loc}}(\Omega, \mathbb{R}^{nN})$. □

5. $\mathbf{L}^1$–CLUSTER POINTS OF $\mathbf{J}$–MINIMIZING SEQUENCES

Now we prove the following smoothness theorem which will also give higher regularity for the maximizer σ.

Theorem 5.1. *Consider a J-minimizing sequence $\{u_m\}_{m \in \mathbb{N}}$, $u_m \in u_0 + \overset{\circ}{W}{}^1_1(\Omega, \mathbb{R}^N)$ for all $m \in \mathbb{N}$, and $u^* \in L^1(\Omega, \mathbb{R}^N)$ such that*

$$u_m \longrightarrow u^* \quad in \quad L^1(\Omega, \mathbb{R}^N).$$

Then there exists an open set $\Omega_0 \subset \Omega$ such that

$$|\Omega - \Omega_0| = 0 \text{ and } u^* \in C^{1,\alpha}(\Omega_0, \mathbb{R}^N) \quad \text{for all} \quad \alpha \in (0,1).$$

The proof of Theorem 5.1 is organized in three steps: at first, we introduce following for example [11], [1] or [5] an abstract relaxation of J on the space of functions of bounded variation and collect some properties which are needed in what is following. The second step is to show the local minimality of u^* with respect to the abstract relaxation. Then, in the last step, the results of [2] will give the conclusion of the theorem.

Here and in the following $\hat{\Omega}$ always denotes a bounded Lipschitz domain in Ω.

Definition 5.2. *For all $w \in BV(\hat{\Omega}, \mathbb{R}^N)$ the functional $\hat{J}(w, \hat{\Omega})$ is given by*

$$\hat{J}(w,\hat{\Omega}) :=$$
$$= \inf\left\{ \liminf_{k\to\infty} J(w_k) : w_k \in C^1(\hat{\Omega}, \mathbb{R}^N),\ w_k \to w \quad \text{in} \quad L^1_{\mathrm{loc}}(\hat{\Omega}, \mathbb{R}^N) \right\}.$$

The following properties of $\hat{J}$ are needed in our context:

Proposition 5.3.

(i). *The functional $\hat{J}$ is lower semicontinuous with respect to $L^1_{\mathrm{loc}}(\hat{\Omega}, \mathbb{R}^N)$-convergence.*

(ii). *On $W^1_1(\hat{\Omega}, \mathbb{R}^N)$, $\hat{J}$ and $J_{|\hat{\Omega}}$ coincide.*

(iii). *With the above assumptions $\hat{J}(u^*, \Omega) \leq \inf\{J(u) : u \in u_0 + \overset{\circ}{W}{}^1_1(\Omega, \mathbb{R}^N)\}$.*

Proof of Proposition 5.3. Using L^1-approximations for BV-functions, the first part follows immediately from the definition of $\hat{J}$. The lower semicontinuity of J on $W^1_1(\hat{\Omega}, \mathbb{R}^N)$ with respect to the L^1_{loc}-topology (see [1]) implies part (ii). To prove the third statement, consider $\{u_m\}_{m\in\mathbb{N}}$ and u^* as in Theorem 5.1 and assume without loss of generality that $u_m \in C^1(\Omega, \mathbb{R}^N)$ for all $m \in \mathbb{N}$. Then the strong L^1-convergence $u_m \to u$ yields

$$\hat{J}(u^*, \Omega) =$$
$$= \inf\left\{ \liminf_{k\to\infty} J(w_k) : w_k \in C^1(\Omega, \mathbb{R}^N),\ w_k \to u^* \quad \text{in} \quad L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \right\}$$
$$\leq \liminf_{m\to\infty} J(u_m) \;=\; \inf\{J(u) : u \in u_0 + \overset{\circ}{W}{}^1_1(\Omega, \mathbb{R}^N)\},$$

and the proposition is proved. □

A deeper result is the following representation formula of Goffman and Serrin (see [11]):

Proposition 5.4. *The representation formula*

$$\hat{J}(u,\hat{\Omega}) = \int\limits_{\hat{\Omega}} f(\nabla^a u)dx + \int\limits_{\hat{\Omega}} f_\infty\left(\frac{\nabla^s u}{|\nabla^s u|}\right) d|\nabla^s u|$$

is true for all $u \in BV(\hat{\Omega}, \mathbb{R}^N)$*, where* f_∞ *is the recession function of* f *defined by*

$$f_\infty(X) = \limsup_{t\to+\infty} \frac{f(tX)}{t}.$$

The absolutely continuous part of Du *with respect to the Lebesgue measure is here denoted by* $\nabla^a u$*, the singular part by* $\nabla^s u$ *and* $\nabla^s u/|\nabla^s u|$ *is the Radon–Nikodym derivative.*

For a proof we also refer to [1], where f is only required to be quasiconvex. The next proposition follows from [2], Theorem 2.1 respectively Proposition 2.2, p. 247 (see also [9] and [13]).

Proposition 5.5. *Consider* $u \in BV(\hat{\Omega}, \mathbb{R}^N)$ *and a sequence* $\{u_m\}_{m\in\mathbb{N}} \subset W^1_1(\hat{\Omega}, \mathbb{R}^N)$ *such that:*

(i). $u_m \to u$ *in* $L^1(\hat{\Omega}, \mathbb{R}^N)$,
(ii). $\int_{\hat{\Omega}} |\nabla u_m|dx \to \int_{\hat{\Omega}} |\nabla u|$,
(iii). $\nabla u_m \rightharpoonup \nabla u$ *in the sense of measures.*

Then $\hat{J}$ *is continuous with respect to this kind of convergence, i.e.,*

$$\hat{J}(u_m,\hat{\Omega}) \to \hat{J}(u,\hat{\Omega}) \quad \text{as} \quad m \to \infty.$$

We now come to the Proof of Theorem 5.1: consider $\{u_m\}_{m\in\mathbb{N}} \subset u_0 + \overset{\circ}{W}{}^1_1(\Omega, \mathbb{R}^N)$ with

$$\begin{aligned} &J(u_m) \longrightarrow \inf\{J(u) : u \in u_0 + \overset{\circ}{W}{}^1_1(\Omega, \mathbb{R}^N)\} \\ &\text{and} u_m \longrightarrow u^* \quad \text{in} \quad L^1(\Omega, \mathbb{R}^N) \quad \text{as} \quad m \to \infty. \end{aligned}$$

Obviously the claim is established as soon as we can show that for any point $x_0 \in \Omega$ there is a ball B centered at x_0 and an open subset B_0 of B such that $|B - B_0| = 0$ and $u^* \in C^{1,\alpha}(B_0, \mathbb{R}^N)$, $0 < \alpha < 1$. To do this, we improve the properties of $\{u_m\}_{m\in\mathbb{N}}$ by fixing traces, i.e. given $x_0 \in \Omega$ we choose $B_R(x_0)$, $B_{2R}(x_0) \Subset \Omega$, and a sequence $\{w_m\}_{m\in\mathbb{N}} \subset u_0 + \overset{\circ}{W}{}^1_1(\Omega, \mathbb{R}^N)$ such that:

(i). $w_m \longrightarrow u^*$ in $L^1(\Omega, \mathbb{R}^N)$ as $m \to \infty$,
(ii). $\lim_{m\to\infty} J(w_m) \le \lim_{m\to\infty} J(u_m)$,
(iii). $w_{m|\partial B_R(x_0)} = u^*_{|\partial B_R(x_0)}$, where the traces are well defined functions of class $L^1(\partial B_R(x_0), \mathbb{R}^N)$.

A sequence like that can be found according to the properties of traces in the space BV. A rigorous proof is given in the Section 7. Setting

- $I : W^1_1\big(B_R(x_0), \mathbb{R}^N\big) \to \mathbb{R}\,, \quad I(w) := \int_{B_R(x_0)} f(\nabla w)dx$,
- $\mathbb{K} := \Big\{ w \in W^1_1(B_R, \mathbb{R}^N) : w_{|\partial B_R} = u^*_{|\partial B_R} \Big\}$,

one immediately verifies with $v_m = w_{m|B_R(x_0)}$ that

$$\inf_{\mathbb{K}} I = \liminf_{m\to\infty} I(v_m). \tag{5.1}$$

In fact, we argue by contradiction and assume that there is $w \in \mathbb{K}$ satisfying

$$I(w) \le I(v_m) - \delta$$

for some $\delta > 0$ and for all m sufficiently large. By the definition of $\mathbb{K}$, the function

$$\tilde{w}_m := \begin{cases} w & \text{on} \quad B_R(x_0) \\ w_m & \text{on the rest of} \quad \Omega \end{cases}$$

is of class W^1_1 and as a direct consequence we have

$$\limsup_{m\to\infty} J(\tilde{w}_m) \le \lim_{m\to\infty} J(w_m) - \delta,$$

which gives the contradiction to (ii.). Now we complete the proof of Theorem 5.1 by showing partial regularity on $B_R(x_0)$. To this purpose consider $\tilde{u} \in BV(B_R(x_0), \mathbb{R}^N)$ such that $\operatorname{spt}(\tilde{u} - u^*) \subset\subset B_R(x_0)$. According to [10], Remark 2.12, we choose an approximation $\tilde{u}_m \in W^1_1(B_R(x_0), \mathbb{R}^N)$ satisfying $\tilde{u}_m \to \tilde{u}$ in $L^1(B_R, \mathbb{R}^N)$, $\int_{B_R} |\nabla \tilde{u}_m| dx \to \int_{B_R} |\nabla \tilde{u}|$ and $\tilde{u}_{m|\partial B_R} = \tilde{u}_{|\partial B_R}$. Then $\tilde{u}_m \in \mathbb{K}$, hence by the Propositions (5.3) and (5.5)

$$\begin{aligned} \hat{J}(\tilde{u}, B_R) = \lim_{m\to\infty} \hat{J}(\tilde{u}_m, B_R(x_0)) \;&=\; \lim_{m\to\infty} I(\tilde{u}_m) \\ \ge \inf_{\mathbb{K}} I \;\overset{(5.1)}{=}\; \liminf_{m\to\infty} I(v_m) \;&\ge\; \hat{J}(u^*, B_R(x_0)), \end{aligned} \tag{5.2}$$

where the last inequality follows from $v_m \longrightarrow u^*$ in L^1 (see (i.)) and the definition of $\hat{J}(\cdot, B_R(x_0))$. Quoting results of [2] concerning partial regularity of $\hat{J}(\cdot, B_R(x_0))$–minimizers, Theorem 5.1 is proved. □

Remark 5.6.

(i). Consider the δ–approximation $\{u_\delta\}_\delta$ of Section 3 in the case $u_0 \in W_2^1(\Omega, \mathbb{R}^N)$. Then the conclusion of Theorem 5.1 in particular holds for each L^1-limit of this sequence.

(ii). On account of the approximation lemma presented in Section 7 the above arguments are of local character, i.e. we do not have to involve the boundary values which would lead to the study of the functional

$$\hat{J}(u, \Omega) + \int_{\partial\Omega} f^\infty\big((u_0 - u) \otimes \nu\big)\, d\mathcal{H}^{n-1}.$$

6. Partial Regularity of the Dual Solution

Now we can prove the second part of our main result, that is partial regularity for the dual solution σ. We prefer to give a "direct" proof just relying on our δ-approximation, an alternative way is outlined in [15–19]: by using the relaxed minimax inequality one can show that $\nabla u^* = \frac{\partial f^*}{\partial E}(\sigma)$ holds on the regular set of any cluster point u^* of a J-minimizing sequence, hence we have $\sigma = \frac{\partial f}{\partial E}(\nabla u^*)$ which implies Theorem 6.1.

Theorem 6.1. *Consider the solution σ of the dual variational problem. Then there exists an open set $\Omega_0 \subset \Omega$ such that*

$$|\Omega - \Omega_0| = 0 \quad \textit{and} \quad \sigma \in C^{0,\alpha}(\Omega_0, \mathbb{R}^{nN}) \quad \textit{for all} \quad 0 < \alpha < 1.$$

Remarks 6.2. Of course, this theorem is not optimal in the following sense:

(i). Are there better results in the case $n = 2$ which means: does $\Omega_0 = \Omega$ hold?

(ii). In the scalar case we expect $\Omega_0 = \Omega$ for any dimension n. If stronger ellipticity according to [9] is assumed and if $u_0 \in L^\infty$, then this conjecture is true.

Proof of Theorem 6.1. We prove the theorem for the case $u_0 \in W_2^1(\Omega, \mathbb{R}^N)$. Consider the δ-approximation from Section 3. With $\sigma_\delta :=$

$\delta\nabla u_\delta + \nabla f(\nabla u_\delta)$ we have the equation

$$\int_\Omega \sigma_\delta : \nabla\varphi dx = 0 \quad \text{for all} \quad \varphi \in C_0^1(\Omega, \mathbb{R}^N). \tag{6.1}$$

By Theorem 5.1, the L^1-limit u^* of the approximating sequence $\{u_\delta\}$ satisfies $u^* \in C^{1,\alpha}(\Omega_0, \mathbb{R}^N)$ for an open set $\Omega_0 \subset \Omega$, $|\Omega_0 - \Omega| = 0$. Now for every open set G such that $\overline{G} \subset \Omega_0$, we have $\nabla u^* \in L^\infty(G, \mathbb{R}^{nN})$ and additionally

$$\int_G \nabla f(\nabla u^*) : \nabla\varphi dx = 0 \quad \text{for all} \quad \varphi \in C_0^1(G, \mathbb{R}^N). \tag{6.2}$$

This is the Euler equation with respect to the abstract relaxation $\hat{J}$: by (5.2) u^* is a local minimizer of $\hat{J}(\cdot, B_R(x))$ for a suitable ball around each point $x \in \Omega$. The representation formula from Proposition 5.4 then easily shows that variations on the regular set imply (6.2). The Euler equations (6.1) and (6.2) together yield

$$\nabla u_\delta(x) \longrightarrow \nabla u^*(x) \quad \text{for almost every} \quad x \in G \quad \text{as} \quad \delta \to 0. \tag{6.3}$$

In fact, inserting $\eta^2[u_\delta - u^*]$, $\eta \in C_0^1(G, \mathbb{R}^N)$, $0 \le \eta \le 1$, and subtracting the above equations we obtain

$$\int_G (\sigma_\delta - \nabla f(\nabla u^*)) : \nabla\left(\eta^2[u_\delta - u^*]\right) dx = 0.$$

The definition of σ_δ and the computation of the derivative yield

$$\begin{aligned} &\int_G (\nabla f(\nabla u_\delta) - \nabla f(\nabla u^*)) : \\ &: (\nabla u_\delta - \nabla u^*)\,\eta^2 dx + \int_G \delta\nabla u_\delta : (\nabla u_\delta - \nabla u^*)\,\eta^2 dx \\ &= -\int_G \sigma_\delta : (\nabla\eta \otimes [u_\delta - u^*])\, 2\eta dx + \\ &+ \int_G \nabla f(\nabla u^*) : (\nabla\eta \otimes [u_\delta - u^*])\, 2\eta dx. \end{aligned} \tag{6.4}$$

Now $\nabla f(\nabla u^*) \in L^\infty(\Omega, \mathbb{R}^{nN})$ and $u_\delta - u^* \to 0$ in $L^1(\Omega, \mathbb{R}^{nN})$ prove

$$\int_G \nabla f(\nabla u^*) : (\nabla \eta \otimes [u_\delta - u^*])\, 2\eta dx \longrightarrow 0 \quad \text{as} \quad \delta \to 0,$$

whereas

$$\int_G \delta \nabla u_\delta : (\nabla u_\delta - \nabla u^*)\, \eta^2 dx \longrightarrow 0 \quad \text{as} \quad \delta \to 0$$

follows from $\delta \nabla u_\delta \rightharpoondown 0$ in $L^2(\Omega, \mathbb{R}^{nN})$ and $\delta \int_\Omega |\nabla u_\delta|^2 dx \to 0$. The first integral on the right hand side of (6.4) can be written as

$$-\int_G \delta \nabla u_\delta : (\nabla \eta \otimes [u_\delta - u^*])\, 2\eta dx -$$
$$-\int_G \nabla f(\nabla u_\delta) : (\nabla \eta \otimes [u_\delta - u^*])\, 2\eta dx := I_1 + I_2.$$

As above we immediately get $I_2 \to 0$ as $\delta \to 0$ and the same is true for I_1. This implies

$$\int_G (\nabla f(\nabla u_\delta) - \nabla f(\nabla u^*)) : (\nabla u_\delta - \nabla u^*)\, \eta^2 dx \longrightarrow 0 \quad \text{as} \quad \delta \to 0.$$

Choosing a subsequence, we get

$$(\nabla f(\nabla u_\delta) - \nabla f(\nabla u^*)) : (\nabla u_\delta - \nabla u^*) \longrightarrow 0 \quad \text{a.e. in } G \text{ as} \quad \delta \to 0. \tag{6.5}$$

(6.5) first implies – recalling Remark 2.2 (ii.) – that $\nabla u_\delta(x)$ is bounded for almost every $x \in G$, but then (6.5) immediately yields (6.3), and we have

$$\nabla f(\nabla u_\delta(x)) \longrightarrow \nabla f(\nabla u^*(x)) \quad \text{for almost every} \quad x \in G \quad \text{as} \quad \delta \to 0\,.$$

Keeping (6.3) and $\nabla f(\nabla u_\delta) \rightharpoondown \sigma$ in $L^2(\Omega, \mathbb{R}^{nN})$ as $\delta \to 0$ in mind, we have shown

$$\sigma(x) = \nabla f(\nabla u^*(x)) \quad \text{in} \quad G, \tag{6.6}$$

which implies (6.6) for x in Ω_0, and the theorem is proved. □

Remark 6.3. The above results can be extended to the case of $u_0 \in W^1_1(\Omega, \mathbb{R}^N)$.

Proof of Remark 6.3. Consider boundary values $u_0 \in W_1^1(\Omega, \mathbb{R}^N)$ and a sequence $\{u_0^m\}_{m \in \mathbb{N}} \subset C^\infty(\overline{\Omega}, \mathbb{R}^N)$ such that

$$u_0^m \longrightarrow u_0 \quad \text{in} \quad W_1^1(\Omega, \mathbb{R}^N) \quad \text{as} \quad m \to \infty.$$

The definition of the δ-approximation of Section 3 was the crucial point to assume smooth boundary values. There are two possibilities to modify this construction:

I: Denote by u_δ^m the unique solution of $(\mathcal{P}_\delta^m)$, that is to minimize the functional J_δ (see 3.1) in the class $u_0^m + \overset{\circ}{W}{}_2^1(\Omega, \mathbb{R}^N)$. With the obvious changes in notation, the approximation of Section 3 can be carried over, if $\delta = \delta(m)$ is chosen small enough for each fixed $m \in \mathbb{N}$ and if we then pass to the limit $m \to \infty$.

II: The second possibility is a bit more complicated in formulation, however here δ and m can be chosen independent of each other. Consider for all $\delta \in (0,1)$ the functional

$$J_\delta^m(w) := \frac{\delta}{2} \int_\Omega |\nabla w|^2 dx + \int_\Omega f(\nabla(u_0^m + w)) dx \quad \text{for all} \quad w \in \overset{\circ}{W}{}_2^1(\Omega, \mathbb{R}^N).$$

Denote by w_δ^m the unique solution of the problem $\tilde{\mathcal{P}}_\delta^m$, that is to minimize J_δ^m in the class $\overset{\circ}{W}{}_2^1(\Omega, \mathbb{R}^N)$. The minimality of w_δ^m implies $J_\delta^m(w_\delta^m) \leq J_\delta^m(0) \leq \int_\Omega f(\nabla u_0^m) dx$, and

$$\delta \int_\Omega |\nabla w_\delta^m|^2 dx \leq c_1, \ \int_\Omega f(\nabla u_0^m + w_\delta^m) dx \ \leq \ c_1$$

follows for some real number c_1, which is not depending on m and δ. Letting

$$\begin{aligned} f_\delta^m &:= \frac{\delta}{2} |\cdot|^2 + f(\nabla u_0^m + \cdot), \quad \tau_\delta^m \ := \ \nabla f(\nabla(u_0^m + w_\delta^m)), \\ \text{and} \quad \sigma_\delta^m &:= \delta \nabla w_\delta^m + \tau_\delta^m \ = \ \nabla f_\delta^m(\nabla w_\delta^m) \end{aligned}$$

and considering any sequence $(m_k, \delta_k) \to (\infty, 0)$, the calculations of Section 3 can be carried over if we observe that $-\delta \int_\Omega \nabla w_\delta^m : \nabla u_0^m dx$ occurs in the counterparts of (3.7) and (3.8) with different signs.

Both modifications give the same results as above since u_0 can be replaced by a function $\tilde{u}_0$ which has the same trace as u_0 on $\partial\Omega$ and which is vanishing on a fixed ball inside Ω. In order to apply Poincaré's inequality during the discussion of the integrals in the proof of Theorem 6.1 again choose $\delta = \delta(m)$ sufficiently small. $\square$

7. An approximation lemma

We finish this paper giving a rigorous proof that uniformly bounded sequences of class W^1_1 can be replaced in some sense by sequences with fixed traces on an appropriate ball: setting

$$J(u, B_R(x)) = \int_{B_R(x_0)} f(\nabla u)dx$$

we have the following

Lemma 7.1. *Consider a sequence $\{u_m\}_{m\in\mathbb{N}} \subset W^1_1(\Omega, \mathbb{R}^N)$ such that:*

(i). $u_m \longrightarrow u^*$ *in* $L^1(\Omega, \mathbb{R}^N)$ *as* $m \to \infty$,
(ii). $\sup_{m\in\mathbb{N}} \|u_m\|_{W^1_1(\Omega,\mathbb{R}^N)} < \infty$.

Then a subsequence (still denoted by $\{u_m\}_{m\in\mathbb{N}}$) has the following properties: For any $x_0 \in \Omega$ there is a ball $B_R(x_0)$, $B_{2R}(x_0) \Subset \Omega$, and a sequence $\{w_m\}_{m\in\mathbb{N}} \subset W^1_1(\Omega, \mathbb{R}^N)$ with the following properties:

(i). $w_m \longrightarrow u^*$ *in* $L^1(\Omega, \mathbb{R}^N)$ *as* $m \to \infty$,
(ii). $\limsup_{m\to\infty} J\big(w_m, B_R(x_0)\big) \le \liminf_{m\to\infty} J\big(u_m, B_R(x_0)\big)$,
(iii). $\limsup_{m\to\infty} J(w_m, \Omega) \le \liminf_{m\to\infty} J(u_m, \Omega)$,
(iv). $w_{m|\partial B_R(x_0)} = u^*_{|\partial B_R(x_0)}$, *where the traces are well defined functions of class $L^1(\partial B_R(x_0), \mathbb{R}^N)$,*
(v). *The sequence $\{w_{m|\Omega - B_{2R}(x_0)}\}_{m\in\mathbb{N}}$ is a subsequence of $\{u_{m|\Omega - B_{2R}(x_0)}\}_{m\in\mathbb{N}}$. In particular the boundary values are preserved.*

Proof. We have $u^* \in \mathbf{v}$ and we may also assume that

$$\nabla u_m \rightharpoonup \nabla u^*, \quad |\nabla u_m| \rightharpoonup \mu \quad \text{as} \quad m \to \infty$$

in the sense of measures where μ denotes a Radon measure of finite mass. Without loss of generality let $x_0 = 0$ and choose a radius $R > 0$ according to

$$B_{2R} \Subset \Omega \text{and} \mu(\partial B_R) = 0 = |\nabla u^*|(\partial B_R), \tag{7.1}$$

where we have set $B_R = B_R(0)$. This implies (see [10], Remark 2.13) that $u^*_{|\partial B_R}$ is well defined. With $T_\varepsilon := \{x : R - \varepsilon < |x| < R + \varepsilon\}$ we further obtain:

$$\lim_{\varepsilon\to 0}\left\{\limsup_{m\to\infty}\int\limits_{T_\varepsilon}|\nabla u_m|dx\right\} = 0, \tag{7.2}$$

$$\lim_{\varepsilon\to 0}\int\limits_{T_\varepsilon}|\nabla u^*| = 0. \tag{7.3}$$

For (7.3) we just observe using (7.1)

$$\int\limits_{T_\varepsilon}|\nabla u^*| \stackrel{\varepsilon\to 0}{\longrightarrow} |\nabla u^*|\left(\bigcap_{\delta>0}T_\delta\right) = |\nabla u^*|(\partial B_R) = 0.$$

Next, let $\varphi_\varepsilon \in C_0(T_{2\varepsilon},[0,1])$, $\varphi_\varepsilon = 1$ on T_ε. Then

$$\int\limits_{T_\varepsilon}|\nabla u_m|dx \le \int\limits_{T_{2\varepsilon}}\varphi_\varepsilon\,|\nabla u_m|dx \stackrel{m\to\infty}{\longrightarrow} \int\limits_{T_{2\varepsilon}}\varphi_\varepsilon\,d\mu,$$

$$\text{hence}\quad \limsup_{m\to\infty}\int\limits_{T_\varepsilon}|\nabla u_m|dx \le \mu(T_{2\varepsilon}) \stackrel{\varepsilon\to 0}{\longrightarrow} \mu(\partial B_R) = 0,$$

thus (7.2) holds. With R fixed we let as in Section 5

$$\mathbb{K} := \left\{w\in W_1^1(B_R,\mathbb{R}^N):\ w_{|\partial B_R} = u^*_{|\partial B_R}\right\}.$$

Note that $\mathbb{K}\neq\emptyset$ on account of [10], Remark 2.12 and Theorem 2.16. We first claim that there exists a sequence $\{v_k\}_{k\in\mathbb{N}}\subset\mathbb{K}$ for which conclusion (ii) holds:

$$\limsup_{k\to\infty} J(v_k,B_R) \le \liminf_{m\to\infty} J(u_m,B_R). \tag{7.4}$$

For proving (7.4) consider $\tilde u_m \in C^\infty(B_R,\mathbb{R}^N)$ with (compare again [10], Remark 2.12)

$$\int\limits_{B_R}|\tilde u_m - u^*|dx \longrightarrow 0,\quad \int\limits_{B_R}|\nabla\tilde u_m|dx \longrightarrow \int\limits_{B_R}|\nabla u^*| \quad\text{as}\quad m\to\infty$$

and such that $\tilde u_{m|\partial B_R} = u^*_{|\partial B_R}$. Then

$$\lim_{\varepsilon\to 0}\left(\limsup_{m\to\infty}\int\limits_{B_R\cap T_\varepsilon}|\nabla\tilde u_m|dx\right) = 0. \tag{7.5}$$

Let $A_\varepsilon := \{x : R-\varepsilon < |x| < R\}$. We have (see [10], Prop. 1.13)

$$\limsup_{m\to\infty} \int\limits_{B_R\cap T_\varepsilon} |\nabla \tilde{u}_m| dx = \limsup_{m\to\infty} \int\limits_{B_R\cap A_\varepsilon} |\nabla \tilde{u}_m| dx$$

$$\leq \int\limits_{B_R\cap \overline{A}_\varepsilon} |\nabla u^*| \leq \int\limits_{T_{2\varepsilon}} |\nabla u^*| \longrightarrow 0$$

as $\varepsilon \to 0$ and (7.5) follows. We define

$$v_m^\varepsilon := \begin{cases} u_m & \text{on } \quad B_R - T_\varepsilon \\ u_m + \varepsilon^{-1}(\tilde{u}_m - u_m)(|x| - R + \varepsilon) & \text{on } \quad A_\varepsilon = B_R \cap T_\varepsilon \end{cases} .$$

Both parts of v_m^ε induce the same trace on $\partial B_{R-\varepsilon}$, thus $v_m^\varepsilon \in W_1^1(B_R, \mathbb{R}^N)$ and in addition $v_m^\varepsilon \in \mathbb{K}$. Since f is of linear growth, the behaviour of $\int_{A_\varepsilon} f(\nabla v_m^\varepsilon)dx$ follows from the discussion of $\int_{A_\varepsilon} |\nabla v_m^\varepsilon| dx$:

$$\int\limits_{A_\varepsilon} |\nabla v_m^\varepsilon| dx \leq 2 \int\limits_{A_\varepsilon} |\nabla u_m| dx + \int\limits_{A_\varepsilon} |\nabla \tilde{u}_m| dx + \frac{1}{\varepsilon} \int\limits_{B_R} |u_m - \tilde{u}_m| dx.$$

According to (7.2) and (7.5) it is possible to define a sequence $\varepsilon_k \to 0$ such that

$$\limsup_{m\to\infty} \int\limits_{A_{\varepsilon_k}} |\nabla u_m| dx \leq \frac{1}{k} \quad \text{and} \quad \limsup_{m\to\infty} \int\limits_{A_{\varepsilon_k}} |\nabla \tilde{u}_m| dx \leq \frac{1}{k}$$

for all $k \in \mathbb{N}$, in conclusion , for any $k \in \mathbb{N}$, there is $m_k \in \mathbb{N}$ such that

$$\int\limits_{A_{\varepsilon_k}} |\nabla u_m| dx \leq \frac{2}{k} \quad \text{and} \quad \int\limits_{A_{\varepsilon_k}} |\nabla \tilde{u}_m| dx \leq \frac{2}{k}$$

for all $m \geq m_k$. Recalling the L^1-convergences u_m, $\tilde{u}_m \to u^*$ on B_R, we assume in addition

$$\frac{1}{\varepsilon_k} \int\limits_{B_R} |u_m - \tilde{u}_m| dx \leq \varepsilon_k$$

for all $m \geq m_k$, $k \in \mathbb{N}$. Putting together these estimates we get

$$J(v_m^{\varepsilon_k}, B_R) \leq \int\limits_{B_R - T_{\varepsilon_k}} f(\nabla u_m) dx + \alpha_k, \quad m \geq m_k, \quad k \in \mathbb{N},$$

for a sequence $\alpha_k \geq 0$, $\alpha_k \to 0$ as $k \to \infty$. By enlarging m_k, if necessary, we can arrange

$$\lim_{k\to\infty} \int_{B_R} f(\nabla u_{m_k})dx = \liminf_{l\to\infty} \int_{B_R} f(\nabla u_l)dx.$$

Finally $v_k := v_{m_k}^{\varepsilon_k}$ is introduced. Then

$$\limsup_{k\to\infty} J(v_k, B_R) \leq \limsup_{k\to\infty} \int_{B_R} f(\nabla u_{m_k})dx,$$

and (7.4) is established. From the definition of v_k it also follows that

$$\lim_{k\to\infty} \int_{B_R} |v_k - u^*|dx = 0. \tag{7.6}$$

An analoguous construction is needed in the exterior domain: choose some radius $\rho > R$, $B_\rho \Subset \Omega$, and a sequence $\hat{u}_m \in C^\infty(B_\rho - B_R, \mathbb{R}^N)$ satisfying

$$\hat{u}_{m|\partial(B_\rho - B_R)} = u^*_{|\partial(B_\rho - B_R)}, \quad \hat{u}_m \longrightarrow u^* \quad \text{in} \quad L^1(B_\rho - B_R, \mathbb{R}^N)$$

$$\text{and} \quad \int_{B_\rho - B_R} |\nabla \hat{u}_m|dx \longrightarrow \int_{B_\rho - B_R} |\nabla u^*| \quad \text{as} \quad m \to \infty.$$

For small enough $\varepsilon > 0$ and for $\{v_m\}_{m\in\mathbb{N}}$ given in (7.4) we then let

$$w_m^\varepsilon := \begin{cases} v_m & \text{on} \quad B_R \\ \hat{u}_m + \varepsilon^{-1}(u_m - \hat{u}_m)(|x| - R) & \text{on} \quad T_\varepsilon - B_R \\ u_m & \text{on the rest of } \Omega \end{cases}.$$

Again w_m^ε is seen to be of class W_1^1 and we have

$$J(w_m^\varepsilon, \Omega) = J(v_m, B_R) + \int_{T_\varepsilon - B_R} f(\nabla w_m^\varepsilon)dx + \int_{\Omega\cap[|x|>R+\varepsilon]} f(\nabla u_m)dx$$

$$= J(v_m, B_R) + \int_{\Omega - B_R} f(\nabla u_m)dx + \int_{T_\varepsilon - B_R} f(\nabla w_m^\varepsilon)dx - \int_{T_\varepsilon - B_R} f(\nabla u_m)dx. \tag{7.7}$$

As before we deduce from (7.2) the existence of sequences $\varepsilon_k \to 0$ and $m_k \to \infty$ such that

$$\int\limits_{T_{\varepsilon_k}-B_R} |\nabla u_m|dx \le \frac{2}{k} \quad \text{and} \quad \int\limits_{T_{\varepsilon_k}-B_R} |\nabla \hat{u}_m|dx \ \le \ \frac{2}{k}$$

for all $m \ge m_k$, for all $k \in \mathbb{N}$ and we get

$$\int\limits_{T_{\varepsilon_k}-B_R} |\nabla w_m^{\varepsilon_k}|dx \le \frac{6}{k} + \frac{1}{\varepsilon_k} \int\limits_{T_{\varepsilon_k}-B_R} |u_m - \hat{u}_m|dx \ \le \ \frac{6}{k} + \varepsilon_k ,$$

again for all $k \in \mathbb{N}$ and for all $m \ge m_k$. Thus decomposition (7.7) gives together with (7.4) for m_k chosen sufficiently large

$$\begin{aligned} J(w_m^{\varepsilon_k}, \Omega) &\le J(v_m, B_R) + \int\limits_{\Omega - B_R} f(\nabla u_m)dx + \beta_k \\ &\le \liminf_{m\to\infty} J(u_m, B_R) + \int\limits_{\Omega - B_R} f(\nabla u_m)dx + 2\beta_k \end{aligned}$$

for any $k \in \mathbb{N}$, for all $m \ge m_k$ and with a sequence $\beta_k \to 0$ as $k \to \infty$. Again enlarging m_k if necessary we may assume that

$$\begin{aligned} J(w_{m_k}^{\varepsilon_k}, \Omega) &\le \liminf_{m\to\infty} J(u_m, B_R) + \liminf_{m\to\infty} J(u_m, \Omega - B_R) + 3\beta_k \\ &\le \liminf_{m\to\infty} J(u_m, \Omega) + 3\beta_k . \end{aligned}$$

Setting $w_k = w_{m_k}^{\varepsilon_k}$ the lemma is proved observing that as in (7.6) the definition of w_k also implies L^1 convergence on the whole domain Ω. □

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