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EXISTENCE AND UNIQUENESS OF A REGULAR SOLUTION OF CAUCHY–DIRICHLET PROBLEM FOR EQUATION OF TURBULENT FILTRATION

Dedicated to O. A. Ladyzhenskaya on her jubilee

1. Introduction

In the study of water infiltration through three dimensional porous media in turbulent regimes (in particular, in the case when in view of large velocities the classical Darcy law fails to be true) several authors (see [1]) consider equation

$$\frac{\partial\Theta}{\partial t} - \operatorname{div}\{|\nabla\varphi(\Theta) + K(\Theta)Z|^{m-2}(\nabla\varphi(\Theta) + K(\Theta)Z)\} = 0, \qquad (1.1)$$

where Θ is the volumetric moisture content, $K(\Theta)$ is the hydraulic conductivity, $\varphi(\Theta) = \int_0^\Theta \mathcal{D}(\xi)$, $\mathcal{D}(\Theta) = \Psi'(\Theta)K(\Theta)$, $\Psi(\Theta)$ is the hydrostatical potential, $Z = (0,0,1)$ is the vertical vector, pointing upwards, $m \in (1,2)$.

Equation (1.1) can be derived from the continuity equation

$$\frac{\partial\Theta}{\partial t} + \operatorname{div} v = 0$$

and the following non-classical Darcy law

$$I(v) = -K(\Theta)\operatorname{grad}\Phi,$$
$$\Phi = \Psi(\Theta(x,y,z)) + z, \quad I(v) = |v|^{\omega-2}v, \omega > 2.$$

The functions $\mathcal{D}$ and K are usually determined empirically. For $m = 2$ equation (1.1) coincides with the equation of filtration in the case of laminar regimes. In the latter case Gilding–Peletier [2] motivated that

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for many porous media a resonable analytic expressions for $\mathcal{D}$ and K are given by

$$\mathcal{D} = \mathcal{D}_0\Theta^{s-1},\ K = K_0\Theta^r,\ 1 < s \leqslant r,\ \mathcal{D}_0, K_0 = \text{const} > 0. \tag{1.2}$$

Extending this choice to the case $m \in (1,2)$ we can derive from (1.1), (1.2) equation

$$\mathcal{F}_0[u] := u_t - \operatorname{div}\{|u|^l|\nabla u + c_0|u|^k Z|^{m-2}(\nabla u + c_0|u|^k Z)\}, \tag{1.3}$$

where

$$\begin{gathered} l = (m-1)(s-1),\ k = r - s + 1,\ 1 < s \leqslant r, \\ m \in (1,2),\ c_0 = \text{const} > 0, Z \text{ is the unit vertical vector.} \end{gathered} \tag{1.4}$$

Obviously from (1.4) it follows in particular that $l > 0$, $k \geqslant 1$. For the sake of generality we shall consider equation (1.3), (1.4) for any dimension $n \geqslant 1$. We shall call equation (1.3), (1.4) as the equation of turbulent filtration.

In this paper we shall study equation (1.3) with a more general condition

$$\begin{gathered} n \geqslant 1,\ m > 1,\ l \geqslant 0,\ k \geqslant 0,\ c_0 = \text{const},\ Z \\ \text{is some fixed unit vector in } \mathbb{R}^n. \end{gathered} \tag{1.5}$$

We investigate the existence and uniqueness for Cauchy–Dirichlet problem

$$\mathcal{F}_0[u] = 0 \text{ in } Q_T, \quad u = \varphi \text{ on } \Gamma_T, \tag{1.6}$$

where $Q_T = \Omega \times (0,T]$, Ω is a bounded open set in $\mathbb{R}^n$, $n \geqslant 1$, Γ_T is a parabolic boundary of cylinder Q_T, i.e., $\Gamma_T = S_T \cup (\bar{\Omega} \times \{t = 0\})$, $S_T = \partial\Omega \times (0,T]$, φ is a given function.

Denote

$$v = |u|^\sigma u, \quad \sigma = \frac{l}{m-1}.$$

Then we can write equation (1.3) as

$$\frac{\partial b(v)}{\partial t} - \operatorname{div}\{|\nabla v + G(v)|^{m-2}(\nabla v + G(v)\} = 0, \tag{1.7}$$

where

$$b(v) = |v|^{\frac{1}{\sigma+1}-1}v, \quad G(v) = |v|^{\frac{\sigma+k}{\sigma+1}}Z. \tag{1.8}$$

Equations of the type (1.7) with some $b(v)$ and $G(v)$ of general setting were studied in [1]. From results of [1] it follows, in particular, that

Cauchy–Dirichlet problem (1.6) for equations (1.3), (1.5) has a bounded generalized solution if parameter k satisfies the additional condition $\frac{\sigma+k}{\sigma+1} \leqslant \frac{n}{n-m}$ if $m < n$ (see condition (3.1) in [1]). In the case of equation (1.3), (1.4) of turbulent filtration this restriction means that parameters s and r (see (1.4)) have to satisfy the additional condition

$$\frac{r}{s} \leqslant \frac{n}{n-m} \text{ if } m < n.$$

The main aim of this paper is to prove *existence and uniqueness* of some *regular solution* of Cauchy–Dirichlet problem for the equation (1.3), (1.4) of turbulent filtration *assuming only physical conditions* $1 < s \leqslant r$ (see (1.4)).

The term "regular solution of Gauchy–Dirichlet problem for equation (1.3), (1.5)" means roughly speaking that this solution is a Hölder continuous in $\overline{Q_T}$ and can be represented as a limit of solutions of some regularised problems.

Equation (1.3), (1.5) can be considered as a particular case of equation in the form

$$\mathcal{F}[u] := u_t - \operatorname{div} a(u, \nabla u) = 0, \tag{1.9}$$

where $a = (a^1, \dots, a^n)$ and functions $a^i(u,p)$, $i = 1, \dots, n$, are continuous on $\mathbb{R} \times \mathbb{R}^n$ and satisfy for all $u \in \mathbb{R}$, $p \in \mathbb{R}^n$ inequalities

$$\begin{gathered} a(u,p) \cdot p \geqslant \nu_0 |u|^l |p|^m - \Phi_0(|u|), \ \nu_0 > 0; \\ |a(u,p)| \leqslant \mu_1 |u|^l |p|^{m-1} + \Phi_1(|u|), \ \mu_1 \geqslant 0, \\ m > 1, \ l \geqslant 0, \ \Phi_i(s) \text{ are non-decreasing } (i = 0, 1). \end{gathered} \tag{1.10}$$

Indeed, it is easy that conditions (1.10) are fulfilled for equations (1.3), (1.5) with

$$\Phi_0(s) = cs^{mk+l}, \quad \Phi_1(s) = cs^{(m-1)k+l}, \ c = c(c_0).$$

Existence of a regular solution of Cauchy–Dirichlet problem for equation (1.9), (1.10) is established in [3] and [4] under appropriate growth conditions for functions $\Phi_0(s)$ and $\Phi_1(s)$. An application of these results to equation (1.3), (1.5) with $c_0 \neq 0$ is possible if parameters l and k satisfy inequality $mk + l \leqslant 2$. It gives existence of the regular solution of Cauchy–Dirichlet problem for the equation of turbulent filtration (1.3), (1.4) only if parameters s and r satisfy some additional condition. Fortunately the growth conditions for $\Phi_0(s)$ and $\Phi_2(s)$ are used in [3] and [4] only for establishing L_∞-estimates for solutions of some regularized problems. Using a special character of equation (1.3), (1.5) we establish in this paper uniform á priori L_∞ and Hölder estimates for solutions of appropriate regularized Cauchy–Dirichlet problems by means of the theory of quasilinear parabolic equations of nondivergent type (see [5, 6]). Then, a passage to the limit and a proof of the existence of the regular solution of Cauchy–Dirichlet problem (1.6) can be realized in the same way as in [3] or [4].

The uniqueness results established in this paper are based on approaches of paper [3]. In the case when $m \in (1, 2)$ we use also for the proof of uniqueness some approach of establishing a comparison principle for parabolic equations with non-Lipschitz coefficients (see [2] and references given three).

In such a way we prove the following results.

Theorem 1.1. *Let the boundary set Ω and function φ are sufficient regular (see Sect.* 5 *and* 6*). Then there is at least one regular solution of Cauchy–Dirichlet problem* (1.6) *for equation* (1.3), (1.5)*. If in addition $k \geqslant 1/m$ for $m \in (1, 2)$ than there is exactly one regular solution of Cauchy–Dirichlet problem for equation* (1.3), (1.5)*. In particular, Cauchy–Dirichlet problem for equation of turbulent filtration* (1.3), (1.4) *has exactly one regular solution.*

In this paper we establishe also some weight gradieht estimates for weak solutions of equations (1.3), (1.5). In the end of the paper we discusse this results using some counterexamples.

In the particular case $c_0 = 0$ equation (1.3), (1.5) has a more simple structure

$$\mathcal{F}^0[u] := u_t - \operatorname{div}\{|u|^l |\nabla u|^{m-2} \nabla u\} = 0, \ m > 1, \ l \geqslant 0. \tag{1.11}$$

For equation (1.11) the results stated above were established in [4] and [5].

We note also that for the one-dimensional ($n = 1$) equation (1.3) with $m = 2$ existence and uniqueness of regular solution of Cauchy problem was proved in [2].

We use in this paper the following basic notations:

$$B_\rho(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < \rho\};$$
$$\sup(f, \mathcal{D}) = \operatorname{ess}\sup_{y \in \mathcal{D}} f(y), \quad \inf(f, \mathcal{D}) = \operatorname{ess}\inf_{y \in \mathcal{D}} f(y);$$

$$W^{1,0}(Q_T) = L_m([0,T]; W^1_m(\Omega));$$

$\overset{\circ}{W}{}^1_m(Q_T)$ means the closure of $\overset{\circ}{C}{}^1(Q_T) = \{u \in C^1(\bar{Q}_T) : u = 0 \text{ on } S_T\}$ with respect to the norm $\|u\|_{W^1_m(Q_T)}$.

In this paper the familiar summation convention over twice repeated indices is often used.

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2. Some results for quasilinear parabolic equations of non-divergence type

We consider in Q_T equation

$$\mathcal{L}u := -u_t + a^{ij}(x,t,u,\nabla u)u_{ij} - b(x,t,u,\nabla u) = 0, \tag{2.0}$$

where $\nabla u = (u_1, \dots, u_n)$, $u_i = \frac{\partial u}{\partial x_i}$, $u_{ij} = \frac{\partial^2 u}{\partial x_i, \partial x_j}$, $i,j = 1, \dots, n$.

Let Q be an arbitrary compact set contained in Q_T. We denote by $C_\alpha(Q)$ the set of all functions $w = w(x,t)$ satisfying for some $\alpha \in (0,1)$ the inequality

$$\|w\|_{C_\alpha(Q)} = \sup(|w|, Q) + \langle w \rangle_{\alpha,Q} \leqslant K,$$

where $K = \mathrm{const} > 0$ and

$$\langle w \rangle_{\alpha,Q} := \sup_{(x,t),(x',t') \in Q} \frac{|w(x,t) - w(x',t')|}{(|x-x'| + |t-t'|)^\alpha}.$$

We denote by $C_{1+\alpha}(Q)[C_{2+\alpha}(Q)]$ the set of all functions w for which w and ∇w belong to $C_\alpha(Q)[w, \nabla w, \nabla^2 w$, and w_t belong to $C_\alpha(Q)]$ for some $\alpha \in (0,1)$.

Let now $\widetilde{Q}$ be an arbitrary set contained in $\bar{Q}_T$. We denote by $C^{\mathrm{loc}}_\alpha(\widetilde{Q})$ the set of all functions w such that $w \in C_\alpha(Q')$ for any compact subset $Q' \subset \widetilde{Q}$ with the same exponent $\alpha \in (0,1)$ for every $Q' \subset \widetilde{Q}$ while the constant K in the inequality $\|u\|_{C_\alpha(Q')} \leqslant K$ in general depends on Q'. Analogously, $C^{\mathrm{loc}}_{1+\alpha}(\widetilde{Q})[C^{\mathrm{loc}}_{2+\alpha}(\widetilde{Q})]$ denotes the class of all functions w such that w and ∇w belong to $C^{\mathrm{loc}}_\alpha(\widetilde{Q})[w, \nabla w, \nabla^2 w$, and w_t belong to $C^{\mathrm{loc}}_\alpha(\widetilde{Q})]$.

We denote by $C^{1,\alpha}(\widetilde{Q})[C^{2,1}(\widetilde{Q})]$ the set of all functions w which are continuous in $\widetilde{Q}$ together with $\nabla w[\nabla w, \nabla^2 w$, and $w_t]$.

Let Γ' denotes the following part of the parabolic boundary of Q_T

$$\Gamma' := \Gamma_T \setminus (\partial\Omega \times \{t = 0\}).$$

We say that the boundary $\partial\Omega$ of Ω satisfies the property of positive geometric density if

$$\begin{gathered}\text{there exists } \alpha_* \in (0,1) \text{ and } \rho_* > 0 \text{ such that}\\ \text{for any } x_0 \in \partial\Omega \text{ and any } B_\rho(x_0),\, x_0 \in \partial\Omega,\ \rho \leqslant \rho_0,\\ |\Omega \cap B_\rho(x_0)| \leqslant (1-\alpha_*)|B_\rho(x_0)|,\end{gathered} \tag{PGD}$$

where $|\Sigma|$ denotes the Lebesgue measure of a measurable set Σ.

We say that u be a classical solution of equation (2.0) in Q_T if u belongs to the class $C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ and satisfies equation (2.0) in Q_T.

Let $(x_0, t_0) \in \bar{Q}_T, \rho > 0$. We denote

$$\begin{gathered}Q_\rho(x_0,t_0) := B_\rho(x_0,t_0) \times [t_0-\rho, t_0+\rho];\ \mathcal{P}(x_0,t_0) := Q_\rho(x_0,t_0) \cap Q_T;\\ \mathcal{P}_{\rho,L}(x_0,t_0) := \{(x,t) \in \mathcal{P}_\rho(x_0,t_0) : |\nabla u| > L\},\ L = \text{const} > 0.\end{gathered}$$

Proposition 2.1 ([5]). *Let u be a classical solution of equation* (2.0) *in Q_T such that $\nabla^3 u$ and ∇u_t are continuous on the set $\mathcal{P}_{\rho,L}(x_0,t_0)$. Assume that at this solution on $\mathcal{P}_{\rho,L}(x_0,t_0)$*

$$\begin{gathered}A^\tau := a^{ij}(x,t,u,p)\tau_i\tau_j \geqslant 0,\ \tau \in \mathbb{R}^n,\ |\tau| = 1;\\ E := a^{ij}(x,t,u,p)p_ip_j \geqslant \mu_0^{-1}|p|^{2\varepsilon_1} > 0,\ \varepsilon_1 > 0;\\ |a^{ij}| \leqslant \mu_1 E|p|^{r-2},\ 0 \leqslant r < 2,\ i,j = 1,\dots,n;\\ |pA_p^\tau| \leqslant \mu_2(A^\tau E)^{\frac{1}{2}}|p|^{-1},\ |p||A_p^\tau| \leqslant \mu_3(A^\tau E)^{\frac{1}{2}}|p|^{\frac{r}{2}-1};\\ |\delta A^\tau| \leqslant \mu_4(A^\tau E)^{\frac{1}{2}}; pb_p - b \geqslant -\mu_5 E;\\ |b_p| \leqslant \mu_6 E|p|^{-2\varepsilon_2},\ \varepsilon_2 > 0;\ \delta b \geqslant -\mu_7 E|p|\end{gathered} \tag{2.1}$$

hold, where

$$\begin{gathered}\delta := \frac{p_k}{|p|}\frac{\partial}{\partial x_k} + |p|\frac{\partial}{\partial u},\quad a^{ij} = a^{ij}(x,t,u,p),\ b = b(x,t,u,p),\\ \mu_0 = \mu_0(x,t,u,p) > 0,\ \mu_i = \mu_i(x,t,u,p) \geqslant 0,\ i = 1,2,3,4,6;\\ \mu_j = \mu_j(x,t,u,p),\ j = 5,7;\\ u = u(x,t),\ p = \nabla u(x,t),\ (x,t) \in \mathcal{P}_{\rho,L}(x_0,t_0).\end{gathered} \tag{2.2}$$

Set for some $\beta \in (0,1)$

$$\begin{gathered}
\nu_1 = \mu_2 + 1 + \beta^{-1},\ \nu_2 = 2\mu_7 + 4\mu^2 + \beta(\mu^2 + \mu_5^2 + \mu_6^2),\\
\nu_3 = c\sigma^{-2}(\mu_1 + \mu_3),\ \nu_4 = c\sigma^{-2}\beta^{-1},\ \nu_5 = c\sigma^{-2}\beta^{-1},\ \sigma = \min(\varepsilon_0, \varepsilon_1, \varepsilon_2),\\
\varepsilon_0 = \frac{1}{2} - \frac{r}{4},\ \mu_i = \mu_i(x_*, t_*, u_*, p_*),\ i = 0, 1, \dots, 7,\\
(x_*, t_*)\ \text{be a point of maximum of function}\ u\ \text{in}\ \overline{\mathcal{P}_\rho(x_0,t_0)},\\
u_* = u(x_*, t_*),\ p_* = \nabla u(x_*, t_*),\ c = c(n).
\end{gathered} \tag{2.3}$$

Suppose that at this function u *on the set* $\mathcal{P}_{\rho,L}(x_0,t_0)$ *we have for some constant* $\widehat{k} > 1$

$$4\nu_1 < \widehat{k},\ \nu_2(\widehat{k}+1)\omega^2 < \frac{1}{2},\ \omega := \operatorname{osc}(u, \mathcal{P}_\rho(x_0,t_0)), \tag{2.4}$$

where ν_1 *and* ν_2 *are defined by* (2.3). *Then*

$$|\nabla u(x_0,t_0)| \leqslant \sqrt{2}\,\max(L, \Theta, \sup(|\mathcal{D}u|, \Gamma_T \cap Q_\rho)), \tag{2.5}$$

where

$$\Theta = \max_{i=0,1,2}\left[\frac{12\nu_{3+i}\omega^2(k+1)}{\rho^2}\right]^{\frac{1}{4\varepsilon_i}}, \tag{2.6}$$

and $\nu_3, \nu_4, \nu_5, \varepsilon_0, \varepsilon_1$, *and* ε_2 *are defined by* (2.3).

The close to Proposition 2.1 result, essentially, was obtained in [6]; the version presented by Proposition 2.1 is established in [5]. (See the proofs of Propositions 2.1 and 2.2 taking into account that the last term in (2.10) of paper [5] has to be changed by the term $4\mu_4^2 z\varepsilon\bar{w}$).

Proposition 2.2 ([5]). *Let functions* $a^{ij}(x,t,u,p)$, $i,j = 1,\dots,n$, *and* $a(x,t,u,p)$ *belong to the class* C^1 *on the set* $Q_T \times \mathbb{R} \times \mathbb{R}^n$ *and satisfy the condition*

$$a^{ij}(x,t,u,p)\xi_i\xi_j > 0\ \text{for all}\ (x,t,u,p) \in \bar{Q}_T \times \mathbb{R} \times \mathbb{R}^n. \tag{2.7}$$

Assume that Ω *satisfies the condition*

$$\begin{gathered}
\text{there exists a sequence of open sets}\ \Omega^{(k)}\ \text{such that}\\
\Omega = \bigcup_{k=1}^{\infty}\Omega^{(k)} \subset \Omega^{(k+1)},\ \operatorname{dist}(\Omega^{(k)}, \partial\Omega^{(k+1)}) > 0, \Omega^{(k)} \in C^3, \quad k = 1, 2, \dots;\\
\text{moreover, sets}\ \Omega^{(k)}\ \text{satisfy condition (PGD) uniformly in}\ k = 1, 2, \dots
\end{gathered} \tag{2.8}$$

and let

$$\begin{gathered}\text{there exists a sequence of functions } \varphi_k \text{ such that}\\ \varphi_k \in C^3(\bar{Q}_T),\ \varphi_k \to \varphi \text{ in } C_\beta(\bar{Q}_T) \text{ for some } \beta \in (0,1).\end{gathered} \tag{2.9}$$

Assume also that any solution $v_k \in C^{2,1}(Q_T^{(k)} \cap \Gamma'_{(k)}) \cap C^{1,0}(\bar{Q}_T^{(k)})$ *of the problem*

$$\begin{gathered}-(v_k)_t + a^{ij}(x,t,v_k,\nabla v_k)(v_k)_{ij} - \tau b(x,t,v_k,\nabla v_k) = 0 \text{ in } Q_T^{(k)},\\ v_k = \tau\varphi_k \text{ on } \Gamma_T^{(k)},\ \tau \in [0,1],\ k = 1,2,\dots,\end{gathered} \tag{2.10}$$

where

$$\begin{gathered}Q_T^{(k)} = \Omega^{(k)} \times (\varepsilon_k, T],\ \Gamma_T^{(k)} \text{ is a parabolic boundary of } Q_T^{(k)},\\ \Gamma'_{(k)} = \Gamma_T^{(k)} \setminus (\partial\Omega^{(k)} \times \{t=0\}),\ 0 < \varepsilon_{k+1} < \varepsilon_k,\ k = 1,2,\dots, \varepsilon_k \to 0,\end{gathered} \tag{2.11}$$

satisfies the inequalities

$$\sup(|v_k|, Q_T^{(k)}) \leqslant M_0 \tag{2.12}$$

and

$$\langle v\rangle_{\lambda,\bar{Q}_T^{(k)}} \leqslant K \text{ for some } \lambda \in (0,\beta], \tag{2.13}$$

where numbers λ, M_0 *and* K *are independent of* $v, \tau,$ *and* k.

Let on the set $Q_T \times \{|u| \leqslant M_0\} \times \{|p| > L_0\}$, $L_0 = \mathrm{const} \geqslant 0$, *the conditions*

$$|b(x,t,u,p)| + \mathrm{Tr}\,\|a^{ij}(x,t,u,p)\||p| \leqslant \Psi(|p|)E \tag{2.14}$$

and

$$E\Psi(|p|) \to \infty \text{ as } |p| \to \infty \text{ (uniformly in } (x,t) \in \bar{Q}_T,\ u \in [-M_0, M_0]) \tag{2.15}$$

hold, where $E = a^{ij}(x,t,u,p)p_ip_j$, $\Psi(\rho)$, $0 \leqslant \rho < +\infty$, *is a positive, monotonic, continuous function such that* $\int^{+\infty} \frac{d\rho}{\rho\Psi(\rho\pm c)} = +\infty$ *for any constant* $c \geqslant 0$.

Assume finally that

$$\begin{gathered}\text{on the set } Q_T \times \{|u| \leqslant M_0\} \times \{|p| > L\},\ L = \mathrm{const} > 0,\\ \text{the inequalities 2.1 hold with } a^{ij} = a^{ij}(x,t,u,p),\\ b = b(x,t,u,p), \mu_0 = \mathrm{const} > 0, \mu_i = \mathrm{const} \geqslant 0,\ i = 1,\dots,7,\\ 0 \leqslant r < 2,\ \varepsilon_1 > 0, \varepsilon_2 > 0.\end{gathered} \tag{2.16}$$

Then problem

$$\mathcal{L}[u] = 0 \text{ in } Q_T, \quad u = \varphi \text{ on } \Gamma_T \tag{2.17}$$

has at least one solution $u \in C^{2,1}(Q_T) \cap C_\lambda(\bar{Q}_T)$. *Moreover, for any* $Q'_{\varepsilon,T} = \Omega' \times (\varepsilon, T], \bar{\Omega}' \subset \Omega$, $\varepsilon > 0$,

$$\|u\|_{C_{2+\gamma}(\bar{Q}'_{\varepsilon,T})} \leqslant c, \tag{2.18}$$

where $\gamma \in (0,1)$ *and* $c > 0$ *depend on* $\operatorname{dist}(\Omega', \partial\Omega)$ *and* ε.

Proposition 2.2 is a version of the results of [6] about a classical solvability of Cauchy–Dirichlet problem for quasilinear non-uniformly parabolic equations. The proof of this version is given in [5]. Other results on the Cauchy–Dirichlet problem with decreased regularity on the parabolic boundary for equations mentioned are obtained in [7].

In conclusion of this section, we recall the well-known fact on the á priori estimate of $\sup(|u|, Q_T)$ for classical solutions of equations (2.1).

Proposition 2.3 ([8]). *Let* u *be a classical solution of problem* (2.17) *with* $\varphi \in C(\bar{Q}_T)$. *Assume that* $a^{ij}(x,t,u,p)$ *and* $a(x,t,u,p)$ *are continuous on* $Q_T \times \mathbb{R} \times \mathbb{R}^n$ *and satisfy for any* $(x,t) \in Q_T$, $u \in \mathbb{R}$ *and* $\zeta \in \mathbb{R}^n$ *the inequalities*

$$a^{ij}(x,t,u,\nu)\xi_i\xi_j \geqslant 0, \quad ub(x,t,u,0) \geqslant -b_1u^2 - b_2, \tag{2.19}$$

where b_1 *and* b_2 *are some non-negative constants. Then*

$$\sup(|u|, Q_T) \leqslant \inf_{\lambda > b_1} e^{\lambda T} \max\{\sup(|\varphi|, \Gamma_T), (b_2/(\lambda - b_1))^{\frac{1}{2}}\}. \tag{2.20}$$

The proof of Proposition 2.3 is given in [8].

3. Some results for quasilinear parabolic equations of divergence type

Consider at first in this section equation

$$\mathcal{F}[u] := u_t - \operatorname{div} a(u, \nabla u) = f, \tag{3.1}$$

where function $f = f(x,t)$ belongs $L_1(Q_T)$, $a = (a^1, \dots, a^n)$, $a^i(u,p)$ are continuous on $\mathbb{R} \times \mathbb{R}^n$, $i = 1, \dots, n$.

Definition 3.1. *Let for any $u \in \mathbb{R}$, $p \in \mathbb{R}^n$,*

$$(\widetilde{C}) \quad |a(u,p)| \leqslant \mu(|p|^{m-1}+1), \ \mu = \text{const} \geqslant 0, \ m > 1.$$

Then function u is a generalized solution (subsolution, supersolution) of equation (3.1) if $u \in W_m^{1,0}(Q_T) \cap C([0,T]; L_1(\Omega))$ and for any $\Phi \in \overset{\circ}{W}{}_m^1(Q_T) \cap L_\infty(Q_T)$ ($\Phi \geqslant 0$) and all $t_1, t_2 \in [0,T]$

$$\int_\Omega u\Phi \, dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_\Omega [-u\Phi_t + a(u,\nabla u)\cdot\nabla\Phi - f\Phi]dx\,dt = 0 \ (\leqslant 0, \geqslant 0).$$

Proposition 3.1 (comparision principle, [3])**.** *Assume that condition $(\widetilde{C})$ is fulfilled and let conditions*

$$(M) \quad [a(u,p)-a(u,q)]\cdot(p-q) \geqslant 0 \text{ for any } u \in \mathbb{R}, \ p,q \in \mathbb{R}^n;$$

$$(L) \quad |a(u,p)-a(v,p)| \leqslant \Lambda|u-v|(|p|^{m-1}+1) \text{ for any } u,v \in \mathbb{R}, \ p \in \mathbb{R}^n.$$

Let u_1 and u_2 be a generalized subsolution and a generalized supersolution respectively, such that $\mathcal{F}[u_1] \leqslant f_1$, $\mathcal{F}[u_2] \geqslant f_2$, where $f_1, f_2 \in L_1(Q_T)$. If $u_1 \leqslant u_2$ on $S_T = \partial\Omega \times (0,T]$ then for any $\tau \in (0,T]$ we have

$$\int_\Omega (u_1-u_2)^+ dx\Big|^{t=\tau} \leqslant \int_\Omega (u_1-u_2)^+ dx\Big|^{t=0} + \int_0^\tau\int_\Omega (f_1-f_2)(u_1-u_2)^+ dx\,dt.$$

The proof of Proposition 3.1 is given in [3] (see also [4]).

Corollary 3.1. *Let conditions $(\widetilde{C})$, (M), and (L) hold and let $f \in L_1(Q_T)$, $\varphi \in W_1^1(Q_T)$. Then Cauchy–Dirichlet problem*

$$\mathcal{F}[u] = f \text{ in } Q_T, \quad u = \varphi \text{ on } \Gamma_T \tag{3.2}$$

has at most one generalized solution in the sense of Definition 3.1.

Consider now in Q_T equation in the form

$$u_t - \operatorname{div} a(x,t,u,\nabla u) - b(x,t,u,\nabla u) = f(x,t), \tag{3.3}$$

where $a^i(x,t,u,p)$ and $b(x,t,u,p)$ satisfy the Caratheodory condition, and $f \in L_1(Q_T)$.

Definition 3.2. *Let for any $u \in \mathbb{R}$, $p \in \mathbb{R}^n$,*

$$\begin{gathered} |a(x,t,u,p)| \leqslant \mu_1 |u|^l |p|^{m-1} + \mu(|u|), \\ |b(x,t,u,p)| \leqslant \mu_2 |u|^{lm'} |p|^m + \widetilde{\mu}(|u|), \\ \mu_i = \text{const} \geqslant 0,\ \mu(s) \geqslant 0 \text{ and } \widetilde{\mu}(s) \geqslant 0 \text{ are nondecreasing}, \\ m > 1,\ l > 1-m,\ 1/m + 1/m' = 1. \end{gathered} \tag{3.4}$$

We say that function u is weak solution of equation (3.3), (3.4) if
(i) *u is non-negative in Q_T;*
(ii) $u \in L_\infty(Q_T) \cap C([0,T]; L_2(\Omega))$, $\nabla u^{\sigma+1} \in L_m(Q_T;\mathbb{R}^n)$, $\sigma = \frac{l}{m-1}$;
(iii) *u satisfies equation (3.3), (3.4) in the sense of distributions, i.e., for any $\Phi \in \overset{\circ}{W}{}^1_m(Q_T)$ and all $t_1, t_2 \in [0,T]$*

$$\int\limits_\Omega u\Phi\, dx\Big|_{t_1}^{t_2} + \int\limits_{t_1}^{t_2}\int\limits_\Omega [-u\Phi_t + a(x,t,u,u_x)\nabla\Phi - \\ - b(x,t,u,u_x)\Phi - f(x,t)\Phi] dx\, dt = 0,$$

where by definition

$$u_x = (\sigma+1)^{-1} u^{-\sigma} \nabla u^{\sigma+1} \text{ in } \{Q_T : u > 0\},\ u_x = 0 \text{ in } \{Q_T : u = 0\}.$$

Obviously that equation of the type (3.1) satisfying the growth condition

(G) $\quad |a(u,p)| \leqslant \mu_1 |u^l||p|^{m-1} + \mu(|u|)$, $\mu_1 > 0$, $\mu(s) \geqslant 0$ is non-decreasing

is a particular case of equations of the type (3.3), (3.4).

Definition 3.3. *Let condition (G) is fulfilled, $f \in L_1(Q_T)$, $\varphi \in W_1^1(Q_T)$, $\varphi \geqslant 0$ in Q_T. Then, we say that function u is a weak solution of Cauchy–Dirichlet problem (3.2) if u is a weak solution of eqaution (3.1) in the sense of Definition 3.2 and $u = \varphi$ on the parabolic boundary Γ_T.*

Definition 3.4. *We say that weak solution u of problem (3.2) is a regular solution of this problem if there exists a sequence $\{u_n\}$ of weak solutions $u_n \in C_\alpha(\bar{Q}_T)$ of problems*

$$\mathcal{F}[u_n] = f \text{ in } Q_T, \quad u_n = \varphi + \varepsilon_n \text{ on } \Gamma_T \tag{3.5}$$

such that

$$\inf(u_n, Q_T) \geqslant \varepsilon_n = \text{const} > 0, \quad \varepsilon_n \to 0; \quad u_n \to u \text{ in } C_\alpha(\bar{Q}_T). \tag{3.6}$$

In particular, $u \in C_\alpha(\bar{Q}_T)$.

Definition 3.5. *We say that weak solution u of problem (3.2) is a maximal weak solution of this problem if for any other weak solution v of the same problem we have*

$$u \geqslant v \text{ a.e. in } Q_T. \tag{3.7}$$

Proposition 3.2. *Assume that conditions (G), (M), and the following local Lipshitz condition*

$$\begin{gathered}\text{for any } u, v \in [\varepsilon, M],\ \varepsilon > 0,\ M > \varepsilon \text{ and } p \in \mathbb{R}^n \\ |a(u,p) - a(v,p)| \leqslant \Lambda |u - v|(|p|^{m-1} + 1),\ \Lambda = \Lambda(\varepsilon, M) \geqslant 0,\end{gathered} \tag{$L_{\rm loc}$}$$

hold. Then any regular solution u of problem (3.2) is a maximal solution of this problem.

Proof. Let $\eta \in \overset{\circ}{W}{}^{1,0}_m(Q_T) \cap L_\infty(Q_T)$, $\eta \geqslant 0$, $0 < h < t_1 < t_2 < T - h$, $Q_{t_1,t_2} := \Omega \times [t_1, t_2]$. Then, from conditions of Proposition 3.2 (see also [8, Chap. 5, Sect. 1]) it follows that if u_1 and u_2 are weak solutions of equation (3.1) then

$$\begin{aligned}\iint\limits_{Q_{Pt_1,t_2}} \{(u_1 - u_2)_{\overline{h}t}\eta \\ + [(a(u_1, (u_1)_x)_{\bar h} - (a(u_2(u_2)_x)_{\bar h}] \cdot \nabla\eta\} dx\, dt = 0,\end{aligned} \tag{3.8}$$

where $g_{\bar h} = \frac{1}{h}\int\limits_{t-h}^{t} g(x,\tau)d\tau$ and $(u_i)_x$ are defined by Definition 3.2. We denote

$$H_\delta(s) = \begin{cases} 1 \text{ if } s \geqslant \delta \\ s/\delta \text{ if } 0 < s < \delta \\ 0 \text{ if } s \leqslant 0 \end{cases}, \text{ and } G_\delta(s) = \begin{cases} s - \delta/2 \text{ if } s \geqslant \delta \\ s^2/2\delta \text{ if } 0 < s < \delta \\ 0 \text{ if } s \leqslant 0 \end{cases},$$

so that $G'_\delta(s) = H_\delta(s)$ on $\mathbb{R}$.

Let u be a regular and v be a weak solution of the same problem (3.2). Then there is a sequence $\{u_n\}$ of weak solutions of problems (3.5) such that conditions (3.6) are fulfilled. In particular,

$$u_n \geqslant \varepsilon_n > 0, \quad \varepsilon_n \to 0. \tag{3.9}$$

Consider identity (3.8) in the case $u_1 = v$, $u_2 = u_n$. It is legitimate, because both v and u_n are weak solutions of equation (3.1). Set in (3.8)

$$\eta = H_\delta(v - u_n), \tag{3.10}$$

where $H_\delta(s)$ is defined as above. We have to verify that function (3.10) is admissible as a test-function for (3.8), i.e., $\eta \in \overset{\circ}{W}{}^{1,0}_m(Q_T) \cap L_\infty(Q_T)$ and $\eta \geqslant 0$. Indeed, $\eta \in L_\infty(Q_T)$ and $\eta \geqslant 0$ in view of the definition of function $H_\delta(s)$. On the other hand from the same definition it follows that

$$H_\delta(v-u_n) = H_\delta(v-u_n)\chi(\{v>u_n\}), \tag{3.11}$$

where $\chi(A)$ denotes the characteristic function of the set $A \subset Q_T$. We denote,

$$v_n := \sup(\varepsilon_n, v), \quad n = 1, 2, \ldots. \tag{3.12}$$

In view of (3.9)–(3.12) we have

$$\eta = H_\delta(v_n - u_n), \tag{3.13}$$

because $v = v_n$ on the set $\{v > u_n\}$. To prove that functions u_n and v_n belong to the class $W^{1,0}_m(Q_T)$ we write those as $u_n = g(\widehat{u}_n)$, $v_n = g(\widehat{v}_n)$, where $\widehat{u}_n := u_n^{\sigma+1}$, $\widehat{v}_n := \sup(\widehat{\varepsilon}_n, v^{\sigma+1})$, $\widehat{\varepsilon}_n := \varepsilon_n^{\sigma+1}$, $\sigma = \frac{l}{m-1}$, $g(s) := \sup(\widehat{\varepsilon}_n, s)^{\frac{1}{\sigma+1}}$. Obviously, function $g(s)$ is uniformly Lipshitz on $\mathbb{R}$ and its derivative g' is continuous everywhere on $\mathbb{R}$ with the exception of the point ε_n at which g' has a discontinuity of the first order. Using also that functions $\widehat{u}_n$ and $\widehat{v}_n$ belong to $W^{1,0}_m(Q_T)$ by definition of the weak solution (see Definition 3.2), we can apply Proposition 7.1 of paper [3]. Then, we prove that u_n and v_n belong to $W^{1,0}_m$. Moreover, because on the lateral surface S_T $v_n = \sup(\varepsilon_n, \varphi) \leqslant \varphi + \varepsilon_n = u_n$ we obtain (using again Proposition 7.1 of [3])

$$\eta = H_\delta(v_n - u_n) \in \overset{\circ}{W}{}^{1,0}_m(Q_T), \quad \nabla\eta = H'_\delta(v_n-u_n)(\nabla v_n - \nabla u_n). \tag{3.14}$$

Next, from the definition of symbol u_x in Definition 3.2 it follows that

$$v_x = (\sigma+1)^{-1} v^{-\sigma} \nabla v^{\sigma+1} = (\sigma+1)^{-1} v_n^{-\sigma} \nabla v_n^{\sigma+1} = \nabla v_n \text{ on the set } \{v > \varepsilon_n\}, \tag{3.15}$$

because $v = v_n$, $\nabla v^{\sigma+1} = \nabla v_n^{\sigma+1}$ on $\{v > \varepsilon_n\}$. Thus, we derive from (3.8)–(3.15) that

$$J_1 + J_2 := \iint\limits_{Q_{t_1,t_2}} (v-u_n)_{\bar{h}t} H_\delta(v-u_n)\,dx\,dt +$$
$$+ \iint\limits_{Q_{t_1,t_2}} [(a(v_n,\nabla v_n)_{\bar{h}} - (a(u_n,\nabla u_n))_{\bar{h}}] \cdot H'_\delta(v_n-u_n)(\nabla v_n - \nabla u_n)\,dx\,dt = 0. \tag{3.16}$$

In view of the concavity of function G_δ we have

$$(v-u_n)_{\bar h t} H_\delta(v-u_n) \geqslant (G_\delta(v-u_n))_{\bar h t}$$

and hence

$$J_1 \geqslant \int_\Omega (G_\delta(v-u_n))_{\bar h} dx\Big|_{t_1}^{t_2}. \tag{3.17}$$

On the other hand

$$J_2 = \frac{1}{\delta} \iint_{\{Q_{t_1,t_2}: 0<v_n-u_n<\delta\}} [a(v_n, \nabla v_n))_{\bar h} - (a(u_n)_{\bar h}] \times (\nabla v_n - \nabla u_n) dx\, dt. \tag{3.18}$$

Letting $h \to 0$ we obtain using (3.16)–(3.18) that for any $\tau \in (0, T]$

$$\int_\Omega G_\delta(v-u_n) dx\Big|_0^\tau + \frac{1}{\delta} \iint_{\{Q_{0,\tau}: 0<v_n-u_n<\delta\}} [a(v_n, \nabla v_n) - a(u_n, \nabla u_n)] \cdot (\nabla v_n - \nabla u_n) dx\, dt \leqslant 0. \tag{3.19}$$

Taking into account that

$$G_\delta(v-u_n) \to (v-u_n)^+ \tag{3.20}$$

as $\delta \to 0$ we derive from (3.19), (3.20) and conditions (M) and (L) that

$$\int_\Omega (v-u_n)^+ dx\Big|^{t=\tau} \leqslant \int_\Omega (v-u_n)^+ dx\Big|^{t=0} = 0. \tag{3.21}$$

The integral in the right-hand side of (3.21) is equal to zero because $v = \varphi$, $u_n = \varphi + \varepsilon_n$ on $\Omega \times \{t = 0\}$. Letting $n \to \infty$ in (3.21) we obtain that $v \leqslant u$ a.e. in Q_T, i.e., inequality (3.7) is established. Proposition 3.2 is proved.

Proposition 3.2 is due to the authors of paper [3]. The proof of this proposition given above is close to the proof of Proposition 3.1 which is stated in [3].

Corollary 3.2. *Let conditions* (G), (M), *and* (L_{loc}) *are fulfilled and let* $f \in L_1(Q_T)$, $\varphi \in W_1^1(Q)$, $\varphi \geqslant 0$ *in* Q_T. *Then, there is at most one regular solution of problem* (3.2).

Proof. Because there is at most one maximal weak solution of problem (3.2) uniqueness of the regular solution of problem (3.2) is a direct consequence of Proposition 3.2. Corollary 3.2 is proved.

Consider again equation (3.3), but in the case $f(x,t) \equiv 0$, i.e., equation

$$u_t - \operatorname{div} a(x,t,u,\nabla u) - b(x,t,u,\nabla u) = 0 \tag{3.22}$$

assuming that $a(x,t,u,p)$ and $b(x,t,u,p)$ satisfy the Caratheodory condition and the following conditions

$$\begin{gathered}
\text{for a.e. } (x,t) \in Q_T \text{ and any } u \in \mathbb{R}, p, q \in \mathbb{R}^n : \\
[a(x,t,u,p) - a(x,t,u,q)] \cdot (p-q) \geqslant 0; \\
a(x,t,u,p) \cdot p \geqslant \nu_0 |u|^l |p|^m - \varphi_0, \nu_0 > 0; \\
|a(x,t,u,p)| \leqslant \mu_1 |u|^l |p|^{m-1} + \varphi_1 |u|^\alpha, \ \alpha = \frac{l}{m}; \\
|b(x,t,u,p)| \leqslant \mu_2 |u|^{lm'} |p|^m + \varphi_2, \ 1/m + 1/m' = 1; \\
m > 1, \ l \geqslant 0, \ \mu_i, \phi_i = \mathrm{const} \geqslant 0, \ i = 0,1,2.
\end{gathered} \tag{3.23}$$

Equations of the type (3.22), (3.23) are particular case of equations of the type (3.3), (3.4). Therefore, the notion of weak solution of equation (3.22), (3.23) is given by Definition 3.2.

We say in the next theorem that a constant c depends on the data if this constant depends on $n, m, l, \nu_0, \mu_1, \mu_2, \varphi_0, \varphi_1$, and φ_2.

Theorem 3.1 ([9-11]). *Let* u *be a weak solution of equation* (3.22), (3.23). *Then* u *is a locally Hölder continuous in* Q_T*; moreover, for any* $Q' = \Omega' \times (\varepsilon, T]$, $\bar{\Omega}' \subset \Omega$, $\varepsilon > 0$ *function* u *satisfies the ineqiuality*

$$\langle u \rangle_{\lambda, \bar{Q}'} \leqslant K', \tag{3.24}$$

where $\lambda \in (0,1)$ *depends only on the data while* K' *depends on the data,* $\operatorname{dist}(\Omega', \partial\Omega)$, ε, *and* $\sup(u, Q_T)$. *If the trace of function* u *on the parabolic boundary* Γ_T *satisfies Hölder condition*

$$\langle u \rangle_{\lambda_0, \Gamma_T} \leqslant K_0 \text{ for any } \lambda_0 \in (0,1), \tag{3.25}$$

and the boundary $\partial\Omega$ *satisfies the property* (PGD), *then the function* u *belongs to the class* $C_\lambda(\bar{Q}_T)$ *and*

$$\langle u \rangle_{\lambda, \bar{Q}_T} \leqslant K, \tag{3.26}$$

where $\lambda \in (0,1)$ depends only on the data, λ_0 and K_0, while K depends on the data, λ_0, K_0, and $\sup(u, Q_T)$.

Remark 3.1. We say (see [4]) that equation (3.3), (3.4) is one of the type of slow, normal, fast diffusion if respectively $m+l>2$, $m+l=2$, $m+l<2$. The statement of Theorem 3.1 is unify for all types of equations (3.22), (3.23) but the proofs of Hölder estimates (3.24), and (3.26) given in [9–11]) depend on a type of these equations.

Remark 3.2. In the case of equations (3.22) satisfying conditions (3.23) for $l=0$, i.e., conditions

$$\begin{gathered}
[a(x,t,u,p)-a(x,t,u,q)]\cdot(p-q)\geqslant 0,\\
a(x,t,u,p)\cdot p\geqslant \widetilde{\nu}_0|p|^m-\widetilde{\varphi}_0, \widetilde{\nu}_0>0,\\
|a(x,t,u,p)|\leqslant \widetilde{\mu}_1|p|^{m-1}+\widetilde{\varphi}_1, \widetilde{\mu}_1\geqslant 0,\\
|b(x,t,u,p)|\leqslant \widetilde{\mu}_2|p|^{m}+\widetilde{\varphi}_2, \widetilde{\mu}_2\geqslant 0,\\
m>1,\ \widetilde{\varphi}_i\geqslant 0,\ i=0,1,2,
\end{gathered} \tag{3.27}$$

Hölder estimates (3.24) and (3.26) were established at first by DiBenedetto ([12]) for $m \geqslant 2$ and by Chen–DiBenedetto ([13]) for $m \in (1,2)$. Moreover, these estimates are established in [12] and [13] for bounded generalized solutions of (3.22), (3.27) with no sign restriction (in other words in [12, 13] function u is said to be a bounded generalized solution of (3.22), (3.27) if only conditions (ii) and (iii) of Definition 3.2 in the case $l=0$ are fulfilled).

4. Regularized Cauchy–Dirichlet problem

Consider in Q_T equation

$$\begin{gathered}
\mathcal{F}_0[u]:=u_t-\operatorname{div}\{|u|^l|\nabla u+c_0|u|^kZ|^{m-2}(\nabla u+c_0|u|^kZ)\}=0,\\
m>1,\ l\geqslant 0,\ k\geqslant 0,\ c_0=\mathrm{const},\ |Z|=1.
\end{gathered} \tag{4.1}$$

In the rest of paper we study the Cauchy–Dirichlet problem

$$\begin{gathered}
\mathcal{F}_0[u]=0 \text{ in } Q_T,\ u=\varphi \text{ on } \Gamma_T,\ m>1,\ l\geqslant 0,\ k\geqslant 0,\\
\varphi\in W^1_m(Q_T)\cap C_\beta(\bar{Q}_T) \text{ for some } \beta\in(0,1),\ \varphi\geqslant 0 \text{ in } Q_T.
\end{gathered} \tag{4.2}$$

In this section we investigate the regularized equation

$$\begin{gathered}
\mathcal{F}^{\varepsilon,N}_\delta[u]:=u_t-\operatorname{div} a_\delta(\varkappa(u),\nabla u)=0,\\
a_\delta(u,p):=|u|^l\widehat{a}_\delta(\widehat{p}),\ \widehat{a}_\delta(\widehat{p})=(\delta^2+|\widehat{p}|^2)^{\frac{m-2}{2}}\widehat{p},\\
\widehat{p}=p+c_0|u|^kZ,\ m>1,\ l\geqslant 0,\ k\geqslant 0,\ c_0=\mathrm{const},\ |Z|=1,
\end{gathered} \tag{4.3}$$

and the corresponding Cauchy–Dirichlet problem

$$\begin{gathered}\mathcal{F}_\delta^{\varepsilon,N}[u]=0 \text{ in } Q_T, \quad u=\varphi+\varepsilon \text{ on } \Gamma_T,\\ 0<\delta\leqslant 1, \quad 0<\varepsilon\leqslant 1<N,\end{gathered}\tag{4.4}$$

where

$$\begin{gathered}\varkappa(u)=\varkappa_{\varepsilon,N}(u) \text{ is a smooth non-decreasing function}\\ \text{such that } \frac{\varepsilon}{2}\leqslant\varkappa(u)\leqslant N+1,\ \varkappa(u)=u \text{ if } u\in[\varepsilon,N],\\ \varkappa''(u)\geqslant 0 \text{ if } u\leqslant\varepsilon,\ \varkappa''(u)\leqslant 0 \text{ if } u\geqslant N,\end{gathered}\tag{4.5}$$

and function φ is the same as in (4.2). From (4.5) it follows in particular that

$$0\leqslant\varkappa'(u)\leqslant 1, \quad |\varkappa''(u)|\leqslant c(\varepsilon).\tag{4.6}$$

Equation (4.3) can be written in the non-divergence form

$$\begin{gathered}-u_t+a^{ij}(u,\nabla u)u_{ij}-b(u,\nabla u)=0,\\ a^{ij}(u,p)=\varkappa^l(u)(\delta^2+|\widetilde{p}|^2)^{\frac{m-2}{2}}\left[\delta_i^j+(m-2)\frac{\widetilde{p}_i\widetilde{p}_j}{\delta^2+|\widetilde{p}|^2}\right],\\ \widetilde{p}=p+c_0\varkappa^k(u)Z,\\ -b(u,p)=l\varkappa^{l-1}(u)\varkappa'(u)(\delta^2+|\widetilde{p}|^2)^{\frac{m-2}{2}}|p|^2\\ +c_0l\varkappa^{l+k-1}(u)\varkappa'(u)(\delta^2+|\widetilde{p}|^2)^{\frac{m-2}{2}}p_iZ_i+c_0k\varkappa^{l+k-1}(u)\varkappa'(u)\\ \times(\delta^2+|\widetilde{p}|^2)^{\frac{m-2}{2}}\left[\delta_i^j+(m-2)\frac{\widetilde{p}_i\widetilde{p}_j}{\delta^2+|\widetilde{p}|^2}\right]p_iZ_j.\end{gathered}\tag{4.7}$$

We shall prove existence of the classical solution of problem (4.4) by means of Proposition 2.2. Using notations of Sec. 2 (see Proposition 2.1) and (4.5)–(4.6) we have the following inequalities in the case of equation

(4.7):

$$
\begin{gathered}
E \geqslant \alpha_0 \varkappa^l(u)(\delta^2+|\widetilde{p}|^2)^{\frac{m-2}{2}}|p|^2, \ \alpha_0 := \min(1, m-1), \\
A^\tau \geqslant \alpha_0 \varkappa^l(u)(\delta^2+|\widetilde{p}|^2)^{\frac{m-2}{2}}; (A^\tau E)^{\frac{1}{2}} \geqslant \alpha_0 \varkappa^l(u)(\delta^2+|\widetilde{p}|^2)^{\frac{m-2}{2}}|p|; \\
|a^{ij}| \leqslant c\varkappa^l(u)(\delta^2+|\widetilde{p}|^2)^{\frac{m-2}{2}} \leqslant cE|p|^{-2}; \\
|pA^\tau_p| \leqslant |p|\,|A^\tau_p| \leqslant c\frac{|\widetilde{p}|+c_0\varkappa^k(u)}{(\delta^2+|\widetilde{p}|^2)^{\frac{1}{2}}}(A^\tau E)^{\frac{1}{2}}|p|^{-1}; \\
|\delta A^\tau| \leqslant c(\varkappa^{-1}(u)+\frac{c_0\varkappa^{k-1}(u)}{(\delta^2+|\widetilde{p}|^2)^{\frac{1}{2}}}(A^\tau E)^{\frac{1}{2}}; \\
|b| \leqslant c\left(\varkappa^{-1}(u)+\frac{c_0\varkappa^{k-1}(u)}{|p|}\right)E; \\
|p|\,|b_p| \leqslant c(\varkappa^{-1}(u)+\frac{c_0\varkappa^{k-1}(u)}{|p|}\left(1+\frac{1}{(\delta^2+|\widetilde{p}|^2)^{\frac{1}{2}}}\right)E; \\
pb_p - b \geqslant -c\left(\varkappa^{-1}(u)+\frac{c_0\varkappa^{k-1}(u)}{|p|}\right)\left(1+\frac{1}{(\delta^2+|\widetilde{p}|^2)^{\frac{1}{2}}}\right)E; \\
\delta b \geqslant -\Big\{ l(l-1)\alpha_0^{-1}\varkappa^{-2}(u)(\varkappa'(u))^2+ \\
+c\left(\varkappa^{-1}(u)+\frac{c_0\varkappa^{k-1}(u)}{|p|}\right)\left(1+\frac{c_0\varkappa^{(k-1)}(u)}{(\delta^2+|\widetilde{p}|^2)^{\frac{1}{2}}}\right)\Big\}E|p|;
\end{gathered}
\tag{4.8}
$$

where constant c is independent of ε, δ, N.

From (4.8), (4.6) it follows obviously that for Eq. (4.7) condition (2.7) is fulfilled.

On the other hand on the set $Q_T \times \{|u| \leqslant M_0\} \times \{|p| > L\}$ conditions (2.14) and (2.15) are satisfied with $\Psi(\rho) = c'$ (where c' is some constant depending in particular on ε and N) if $L = L(M_0)$ is sufficient large. Indeed

$$\frac{1}{2}|p| \leqslant |\widetilde{p}| \leqslant 2|p| \text{ if } |p| > L$$

with appropriate $L = L(M_0)$. Then taking into account that for any $m > 1$

$$(\delta^2+|p|^2)^{\frac{m-2}{2}}|p|^2 \geqslant |p|^m - \delta^m$$

(the proof of this inequiality will be given later) we obtain

$$E \geqslant c(\varepsilon)|p|^m \text{ if } |p| \geqslant L$$

with sufficient large $L = L(M_0)$. From here and (4.8) conditions (2.14) and (2.15) with $\Psi(\rho) = c'(\varepsilon, N)$ and $L_0 = L$ (where L is like above) easily follow.

Moreover from (4.8) and the estimate of E from below given above it follows that condition (2.16) with some parameters $\mu_0, \dots, \mu_7$ independent of δ and for $r = 0$, $\varepsilon_1 = \frac{m}{2}$, $\varepsilon_2 = \frac{1}{2}$ is fulfilled on the set $Q_T \times \{|u| \leqslant M_0\} \times \{|p| > L\}$ if $L = L(M_0)$ is chosen like above.

Definition 4.1. *We say that Ω satisfies condition $(\widetilde{PGD})$ if Ω satisfies condition* (2.8).

We shall study problems (4.2) and (4.4) assuming that

$$\begin{gathered}\varphi \in W^1_m(Q_T) \cap C_\beta(\bar{Q}_T) \text{ for some } \beta \in (0,1),\ \varphi \geqslant 0 \text{ in } Q_T;\\ \Omega \text{ satisfies the condition } (\widetilde{PGD}).\end{gathered} \tag{4.9}$$

From (4.9) it follows, in particular, that function φ satisfies condition (2.9) and moreover for functions φ_k from (2.9) we have $\varphi_k \to \varphi$ in $W^1_m(Q_T) \cap C_\beta(\bar{Q}_T)$.

In the case of equation (4.7) problems (2.10) can be written in the form

$$\begin{gathered}(v_k)_t - \operatorname{div} a_\delta(\varkappa(v_k), \nabla v_k) = (1-\tau) b(v_k, \nabla v_k) \text{ in } Q_T^{(k)},\\ v_k = \tau\varphi_k \text{ on } \Gamma_T^{(k)}, \quad \tau \in [0,1], \quad k = 1, 2, \dots,\end{gathered} \tag{4.10}$$

where $Q_T^{(k)} = \Omega^{(k)} \times (\varepsilon_k, T]$, $\varepsilon_k > 0$, $\varepsilon_k \to 0$, $a_\delta(u,p)$ is defined by (4.3) and $b(u,p)$ is defined by (4.7).

In view of Proposition 2.2 to prove existence of solution $u \in C^{2,1}(Q_T) \cap C_\lambda(\bar{Q}_T)$, $0 < \lambda \leqslant \beta$, of problem (4.4) it suffices to establish estimates (2.12) and (2.13) (uniformly in τ, v and k) because conditions (2.7)–(2.11) have been verified already.

Estimate (2.12) follows directly from Proposition 3.3 in the case of functions $a^{ij}(x,t,u,p)$ and $b(x,t,u,p)$ defined by (4.7). Indeed, taking into account that $b(u,0) = 0$ (see (4.7)) we conclude in view of (2.20) that for solution v_k of problem (4.10) we have

$$\sup(|v_k|, \bar{Q}_T^{(k)}) \leqslant \sup(\varphi_k, \bar{Q}_T^{(k)}), \tag{4.11}$$

where the right-hand side is independent of τ, v, and k, because from (4.9) it follows (see also (2.9)) that $\varphi_k \to \varphi$ in $C_\beta(\bar{Q}_T)$ and hence in $C(\bar{Q}_T)$.

To prove uniform (in τ, v, k) Hölder estimate (2.13) we use results by DiBenedetto ([12]) and Chen–DiBenedetto ([13]) (see Remark 4.2). We show first of all that equations (4.10) have a structure defined by formulas (3.22), (3.27). To this end we have to discover some properties of functions $\hat{a}_\delta(\hat{p})$ defined by (4.3).

Because

$$|\hat{p}|^2(\delta^2+|\hat{p}|^2)^{\frac{m-2}{2}} \geqslant \begin{cases} |\hat{p}|^m & \text{if } \geqslant 2, \\ |\hat{p}|^m - \delta^m & \text{if } m \in (1,2), \end{cases} \tag{4.12}$$

we obtain for any $\hat{p} \in \mathbb{R}^n$ inequality

$$\hat{a}_\delta(\hat{p}) \cdot \hat{p} \geqslant |\hat{p}|^m - \gamma_m, \quad \gamma_m = \begin{cases} 0 & \text{if } m \geqslant 2, \\ \delta^m & \text{if } m \in (1.2). \end{cases} \tag{4.13}$$

Remark that (4.12) in the case $m \in (1,2)$ foolows from the inequality

$$\begin{aligned} \frac{|\hat{p}|^2}{(\delta^2+|\hat{p}|^2)^{\frac{2-m}{2}}} &= (\delta^2+|\hat{p}|^2)^{\frac{m}{2}} - \frac{\delta^2}{(\delta^2+|\hat{p}|^2)^{\frac{2-m}{2}}} \\ &\geqslant (\delta^2+|\hat{p}|^2)^{\frac{m}{2}} - \delta^m (m \in (1,2)). \end{aligned}$$

Next, because

$$(\delta^2+|\hat{p}|^2)^{\frac{m-2}{2}}|\hat{p}| \leqslant \begin{cases} (\delta^2+|\hat{p}|^2)^{\frac{m-1}{2}} & \text{if } m \geqslant 2, \\ |\hat{p}|^{m-1} & \text{if } m \in (1,2), \end{cases}$$

we have for any $\hat{p} \in \mathbb{R}^n$

$$\begin{aligned} |\hat{a}_\delta(\hat{p})| \leqslant \hat{\mu}_1(|\widehat{p}|^{m-1} + \tilde{\gamma}_m),\ \hat{\mu}_1 = 2^{(m-2)_+},\ \tilde{\gamma}_m = \\ = \begin{cases} \delta^{m-1} & \text{if } m \geqslant 2, \\ 0 & \text{if } m \in (1,2). \end{cases} \end{aligned} \tag{4.14}$$

Consider now for any $\widehat{p}, \hat{q} \in \mathbb{R}^n$ the expression

$$\begin{aligned} J :&= [\hat{a}_\delta(\widehat{p}) - \hat{a}_\delta(\hat{q})] \cdot (\widehat{p} - \hat{q}) \\ &= [(\delta^2+|\hat{p}|^2)^{\frac{m-2}{2}}\hat{p} - (\delta^2+|\hat{q}|^2)^{m-2}\hat{q}] \cdot (\hat{p}-\hat{q}). \end{aligned} \tag{4.15}$$

Denote

$$f(\tau) := (\delta^2 + |\tau\widehat{p} + (1-\tau)\hat{q}|^2)^{\frac{m-2}{2}}(\tau\hat{p} + (1-\tau)\hat{q}) \cdot (\widehat{p} - \hat{q}). \tag{4.16}$$

Obviously, $J = f(1) - f(0) = \int\limits_0^1 f'(\tau)d\tau$ and

$$f'(\tau) = (\delta^2 + |\tau\widehat{p} + (1-\tau)\hat{q}|^2)^{\frac{m-2}{2}}|\widehat{p} - \hat{q}|^2 \times \left\{1 + (m-2)\frac{[(\tau\widehat{p} + (1-\tau)\hat{q})\cdot(\widehat{p} - \hat{q})]^2}{(\delta^2 + |\tau\widehat{p} + (1-\tau)\hat{q}|^2)|\widehat{p} - \hat{q}|^2}\right\}. \tag{4.17}$$

Using the elementary inequality

$$\int\limits_0^1 |\tau\widehat{p} + (1-\tau)\hat{q}|^{m-2}d\tau \geqslant c(m)|\widehat{p} - \hat{q}|^{m-2} \text{ if } m \geqslant 2 \tag{4.18}$$

and taking into account that for any $\tau \in [0,1]$

$$[\delta^2 + |\tau\widehat{p} + (1-\tau)\hat{q}|^2]^{\frac{m-2}{2}} \geqslant (1 + |\widehat{p}| + |\hat{q}|)^{m-2} \geqslant c(m)(|\widehat{p}|^m + |\hat{q}|^m + 1)^{\frac{m-2}{m}} \text{ if } m \in (1,2) \tag{4.19}$$

we derive from (4.15)–(4.19) inequality

$$\begin{gathered}[\hat{a}_\delta(\widehat{p}) - \hat{a}_\delta(\hat{q})]\cdot(\hat{p} - \hat{q}) \geqslant \hat{\nu}_1|\widehat{p} - \hat{q}|^\kappa(|\hat{p}|^m + |\hat{q}|^m + 1)^{\frac{m-\kappa}{m}}, \\ \kappa = m \text{ if } m \geqslant 2, \quad \kappa = 2 \text{ if } m \in (1.2),\end{gathered} \tag{4.20}$$

where $\hat{\nu}_1$ depends only on m.

Recall that we have (see (4.3), (4.7))

$$\begin{gathered}a_\delta(u,p) = |u|^l\hat{a}_\delta(\hat{p}), \quad \hat{p} = p + c_0|u|^kZ, \\ a_\delta(\varkappa(u),p) = \varkappa^l(u)\hat{a}_\delta(\widehat{p}), \quad \widehat{p} = p + c_0\varkappa^k(u)Z.\end{gathered} \tag{4.21}$$

Using (4.13), (4.14), (4.20), (4.21), taking into account that $|e+d|^m \geqslant 2\gamma_0|e|^m_| d|^m$ for any e, $d \in \mathbb{R}^n$ (with $2\gamma_0 = 2^{1-m}$) and applying Cauchy inequality we obtain for all $u \in \mathbb{R}$ and $p, q \in \mathbb{R}^n$

$$\begin{aligned}a_\delta(u,p)\cdot p &= |u|^l\hat{a}_\delta(\widehat{p})\cdot p = |u|^l\hat{a}_\delta(\widehat{p})\cdot\widehat{p} - |u|^l\hat{a}_\delta(\widehat{p})\cdot c_0|u|^kZ \geqslant \\ &\geqslant |u|^l(|\widehat{p}|^m - \gamma_m) - |u|^l(\hat{\mu}_1|\hat{p}|^{m-1} + \tilde{\gamma}_m)c_0|u|^k \geqslant \\ &\geqslant \frac{1}{2}|u|^l|\widehat{p}|^m - (\gamma_m|u|^l + \tilde{\gamma}_m c_0|u|^{l+k} + 2^{m-1}(\hat{\mu}_1c_0)^m|u|^{l+mk}) \geqslant \\ &\geqslant \gamma_0|u|^l|p|^m - \mu_0(|u|),\end{aligned} \tag{4.22}$$

where

$$\mu_0(s) = c(m,c_0)(s^l + s^{l+k} + s^{l+mk}); \tag{4.23}$$

$$\begin{aligned}|a_\delta(u,p)| = |u|^l|\hat{a}_\delta(\widehat{p})| \leqslant \hat{\mu}_1|u|^l(|\widehat{p}|^{m-1}+\tilde{\gamma}_m) \leqslant \\ \leqslant \mu_1|u|^l|p|^{m-1}+\mu(|u|)|u|^\alpha, \quad \alpha=\frac{l}{m},\end{aligned} \tag{4.24}$$

where

$$\mu_1 = 2^{(m-2)}+\hat{\mu}_1, \mu(s)=c(m,c_0)(s^l+s^{l+k(m-1)})s^{-\frac{1}{m}}; \tag{4.25}$$

$$\begin{aligned}[a_\delta(u,p)-a_\delta(u,q)]\cdot(p-q) = |u|^l[\hat{a}_\delta(\widehat{p})-\hat{a}_\delta(\hat{q})]\cdot(\widehat{p}-\hat{q}) \geqslant \\ \geqslant \hat{\nu}_1|u|^l|\hat{p}-\hat{q}|^\kappa(|\widehat{p}|^m+|\hat{q}|^m+1)^{\frac{m-\kappa}{m}} \geqslant \\ \geqslant \nu_1|u|^l|p-q|^\kappa(|p|^m+|q|^m+1+c_0^m|u|^{mk})^{\frac{m-\kappa}{m}},\end{aligned} \tag{4.26}$$

where κ is the same as in (4.20).

Obviously, from (4.21)–(4.27) it follows that for any $u\in\mathbb{R}$ and $p,q\in\mathbb{R}^n$ we have

$$a_\delta(\varkappa(u),p)\cdot p \geqslant \nu_0\varkappa^l(u)|p|^m-\mu_0(\varkappa(u)); \tag{4.27}$$

$$|a_\delta(\varkappa(u),p)| \leqslant \mu_1\varkappa^l(u)|p|^{m-1}+\mu(\varkappa(u))\varkappa^\alpha(u), \ \alpha=\frac{l}{m}, \tag{4.28}$$

$$\begin{aligned}[a_\delta(\varkappa(u),p)-a_\delta(\varkappa(u),q)]\cdot(p-q) \geqslant \\ \geqslant \nu_1\varkappa^l(u)|p-q|^\kappa(|p|^m+|q|^m+1+c_0^m\varkappa^{mk}(u))^{\frac{m-\varkappa}{m}},\end{aligned} \tag{4.29}$$

where $\nu_0=\gamma_0$ and ν_0, μ_1, $\mu_0(s)$, $\mu(s)$, ν_1, κ are the same as in (4.22)–(4.26).

Obviously also that in the case $\delta>0$ for any $u,v\in\mathbb{R}$ and any $p\in\mathbb{R}^n$ we have in view of (4.21) and (4.14)

$$J := |a_\delta(\varkappa(u),p)-a_\delta(\varkappa(v),p)| \leqslant \Lambda(\delta,\varepsilon,N)|u-v)(|p|^{m-1}+1). \tag{4.30}$$

Indeed,

$$\begin{aligned}J \leqslant |\varkappa^l(u)-\varkappa^l(v)|\,|\hat{a}_\delta(p+c_0\varkappa^k(u)Z| \\ +\varkappa^l(v)|\hat{a}_\delta(p+c_0\varkappa^k(u)Z)-\hat{a}_\delta(p+c_0\varkappa^k(v)Z)|,\end{aligned} \tag{4.31}$$

where

$$\begin{aligned}\sup_{u\in\mathbb{R}}(|\varkappa^l(u))'|,|(\varkappa^k(u))'| \leqslant c(\varepsilon,N), \\ \sup_{\hat{p}\in\mathbb{R}^n}\left(|\hat{a}^i_\delta|,\left|\frac{\partial\hat{a}^i_\delta}{\partial\widehat{p}_j}\right|\right) \leqslant c(\delta,N)(|\widehat{p}|^{m-1}+1).\end{aligned} \tag{4.32}$$

Evidently, (4.31) and (4.32) imply (4.30).

We have to estimate also the growth of function $b(u,p)$ defined by (4.7). It is easy to see that

$$|b(u,p)| \leqslant c(\delta,\varepsilon,N)(|p|^m+1). \tag{4.33}$$

In view of (4.27)–(4.29) and (4.33) we can conclude that equations (4.10) having a divergent structure (3.22) satisfies conditions (3.27) for some numbers

$$\tilde{\nu}_0=\tilde{\nu}_0(\varepsilon),\ \tilde{\varphi}_0=\tilde{\varphi}_0(N),\ \tilde{\mu}_1=\tilde{\mu}_1(N),\ \tilde{\varphi}_1=\tilde{\varphi}_1(N),$$
$$\tilde{\mu}_2=\tilde{\mu}_2(\delta,\varepsilon,N),\ \tilde{\varphi}_2=\tilde{\varphi}_2(\delta,\varepsilon,N). \tag{4.34}$$

Then, applying Hölder estimate (3.26) (see Theorem 3.1 and Remark 3.2) we establish for solutions $v_k\in C^{2,1}(Q_T^{(k)}\cap\Gamma'_{(k)})\cap C^{1,0}(\bar{Q}_T^{(k)})$ of problems of the type (2.10) for equations (4.10) (it is obvious that these solutions are at the same time bounded generalized solutions in the sense of definition given in Remark 3.2) the Hölder estimates

$$\langle v_k\rangle_{\lambda,\bar{Q}_T^{(k)}}\leqslant K,\quad k=1,2,\dots,$$

where $\lambda\in(0,\beta]$ and K are independent of v, τ and K. So for equations (4.10) and Cauchy–Dirichlet problems of the type (2.10) all conditions of Proposition 2.2 are fulfilled. In view of this proposition we prove the following

Proposition 4.1. *Let condition* (4.9) *holds. Then for any* $\delta\in(0,1)$, $\varepsilon\in(0,1)$ *and* $N>1$ *the problem* (4.4) *has at least one classical solution* $u\in C^{2,1}(Q_T)\cap C_\lambda(\bar{Q}_T)$ *with some* $\lambda\in(0,\beta]$. *Moreover, for any* $Q'_{\varepsilon_0,T}=\Omega'\times(\varepsilon_0,T]$, $\bar{\Omega}'\subset\Omega$, $\varepsilon_0>0$ *function* u *belongs to the class* $C^{2+\alpha}(\bar{Q}'_{\varepsilon_0,T})$ *with some* $\alpha\in(0,1)$ *depending on* $\operatorname{dist}(\Omega',\partial\Omega)$ *and* ε_0.

Now we are ready to prove the following

Proposition 4.2. *The solution* u *of problem* (4.4) *defined by Proposition* 4.1 *is at the same time a generalized solution of Eq.* (4.3) *in sense of Definition* 3.1. *Moreover, there exists exactly one bounded generalized solution* u *of problem* (4.4). *This solution belongs to* $W_m^{1,0}(Q_T)\cap C^{2,1}(Q_T)\cap C^\lambda(\bar{Q}_T)$ *with some* $\lambda\in(0,\beta]$ *and satisfies the inequalities*

$$\varepsilon\leqslant u(x,t)\leqslant M_0+\varepsilon \text{ in } Q_T;\quad M_0:=\sup(\varphi,\bar{Q}_T) \tag{4.35}$$

and

$$\varepsilon^l\iint\limits_{Q_T}|\nabla u|^m dx\,dt\leqslant\iint\limits_{Q_T}u^l|\nabla u|^m dx\,dt\leqslant c, \tag{4.36}$$

where constant c depends only on dimension n, parameters m, l, c_0 and k of Eq. (4.1), $\|\varphi\|_{W^1_m(Q_T)}$ *and* M_0.

Proof. From the proof of Proposition 4.1 it follows that for any $k = 1, 2, \dots$, there is a solution $u_k \in C^{\mathrm{loc}}_{2+\gamma}(Q_T^{(k)}) \cap C_{1+\gamma}(\bar{Q}_T^k)$

$$\mathcal{F}_\delta^{\varepsilon,N}[u_k] = 0 \text{ in } Q_T^{(k)}, \quad u_k = \tilde{\varphi}_k \text{ on } \Gamma_T^{(k)}, \quad \tilde{\varphi}_k = \varphi_k + \varepsilon, \tag{4.37}$$

where $Q_T^{(k)} = \Omega^{(k)} \times (\varepsilon_k, T]$. Multiplying equation in (4.37) by $u_k - \tilde{\varphi}_k$, integrating over any cylinder Q', $\bar{Q}' \subset Q_T^{(k)}$, and using (4.27), (4.23) we can in a routine way derive the inequality

$$\iint\limits_{Q_T^{(k)}} k^l(u_k)|\nabla u_k|^m dx\, dt \leqslant c, \tag{4.38}$$

where constant c depends only on $\|\varphi\|_{W^1_m(Q_T)}$, $\sup(\varphi, Q_T)$ and $\sup(\mu_0(\varkappa(u_k)), Q_T)$ (see (4.23)). In particular, from (4.38) it follows that

$$\iint\limits_{Q_T^{(k)}} |\nabla u_k|^m dx\, dt \leqslant c(\varepsilon) \tag{4.39}$$

with $c(\varepsilon) = c\frac{\varepsilon^{-l}}{2}$. Let functions u_k are extended on $Q \backslash Q_T^{(k)}$ by formula

$$u_k = \tilde{\varphi}_k \text{ on } Q_T \setminus Q_T^{(k)},$$

where $\tilde{\varphi}_k \to \varphi + \varepsilon$ in $W^1_m(Q_T) \cap C_\beta(\bar{Q}_T)$ as $k \to \infty$. Because $u_k - \tilde{\varphi}_k \in \overset{\circ}{W}{}^1_m(Q_T^{(k)})$ we can conclude that the extended function u_k belongs to $W^{1,0}_m(Q_T)$. Then from the proof of Proposition 4.1 and (4.39) it follows that

$$u_k \to u \text{ in } C(\bar{Q}_T), \quad \nabla u_k \to \nabla u \text{ weakly in } L_m(Q_T), \tag{4.40}$$

where u is the solution defined by Proposition 4.1. From (4.40) it follows easily that function u is a generalized solution of Eq. (4.3) (obviously Eq. (4.3) satisfies condition (G)). Moreover, in view of (4.28)–(4.30) from Corollary 3.1 it follows easily that there is at most one bounded generalized solution of problem (4.4).

Using (4.11) (in the case $\tau = 1$) we derive that for solution u defined by Proposition 4.1 $\sup(u, Q_T)$ is estimated by a constant which is independent of δ, ε, and N, namely,

$$\sup(u, \bar{Q}_T) \leqslant M_0 + 1. \tag{4.41}$$

Next, because (in view of (4.11) and (4.29), (4.30)) for Eq. (4.3) conditions $(\tilde{G})$, (M), and (L) are fulfilled we can apply Proposition 3.1 for the operator $\mathcal{F}_\delta^{\varepsilon,N}[u]$ and functions $u_1 = \varepsilon$, $u_2 = u$, $f_1 = f_2 = 0$ taking into account that

$$u_1 = \varepsilon \leqslant \varphi + \varepsilon = u_2 \text{ on } S_T \text{ and } (\varepsilon - u)^+ = 0 \text{ on } \Omega \times \{t = 0\}.$$

Then from Proposition 3.1 it follows that $(\varepsilon - u)^+ = 0$ in $\bar{Q}_T$, i.e.,

$$\inf(u, \bar{Q}_T) \geqslant \varepsilon. \tag{4.42}$$

Finally, using the integral identity for bounded generalized solution u of problem (4.4) and inequalities (4.41), (4.42) we obtain inequalities (4.36). Proposition 4.2 is proved.

Beginning since the moment we assume that number N from (4.5) is chosen as $N = M_0 + 1$, where M_0 is defined by (4.35). Then from (4.35) it follows that at the solution u defined by Proposition 4.1 we have

$$\varkappa_{\varepsilon,N}(u) = u \text{ everywhere in } \bar{Q}_T. \tag{4.43}$$

In view of (4.43) we can write problem (4.4) as

$$\mathcal{F}_\delta[u] := u_t - \operatorname{div} a_\delta(u, \nabla u) = 0 \text{ in } Q_T, \quad u = \varphi + \varepsilon \text{ on } \Gamma_T. \tag{4.44}$$

Proposition 4.3. *Let condition* (4.9) *holds. Then for solution u defined by Proposition* 4.1 *we have*

$$\langle u \rangle_{\lambda, \bar{Q}_T} \leqslant K, \tag{4.45}$$

where $\lambda \in (0, \beta]$ depends only on n, m, l, k, c_0, β, and $\langle \varphi \rangle_{\beta, \Gamma_T}$ while K depends on the same data and on $\sup(\varphi, Q_T)$.

Proof. Because functions $a_\delta(u, p)$ satisfy conditions (4.21)–(4.26) and using also (4.35) and (4.43) we can derive that for equation in (4.44) all conditions (3.23) are fulfilled with

$$\nu_0 = \nu_0(m), \quad \varphi_0 = \varphi_0(m, l, k, c_0, M_0),$$
$$\mu_1 = \mu_1(m), \quad \varphi_1 = \varphi_1(m, l, k, c_0, M_0), \quad \mu_2 = 0, \quad \varphi_2 = 0.$$

Then from Theorem 3.1 it follows that (4.45) holds with $\lambda \in (0, \beta]$ and K depending on the data mentioned in the statement of Proposition 4.3. Proposition 4.3 is proved.

In view of (4.35) and (4.43) we can rewrite (4.7) as

$$
\begin{aligned}
&-u_t + a^{ij}(u,\nabla u)u_{ij} - b(u,\nabla u) = 0,\\
&a^{ij}(p) = u^l(\delta^2+|\hat{p}|^2)^{\frac{m-2}{2}}\left[\delta_i^j + (m-2)\frac{\widehat{p}_i\widehat{p}_j}{\delta^2+|p|^2}\right],\ \widehat{p} = p + c_0u^kZ,\\
&b(u,p) = lu^{l-1}((\delta^2+|\widehat{p}|^2)^{\frac{m-2}{2}}|p|^2 + c_0lu^{l+k-1}(\delta^2+|\widehat{p}|^2)^{\frac{m-2}{2}}p_iZ_i+\\
&\quad +c_0ku^{l+k-1}(\delta^2+|\widehat{p}|^2)^{\frac{m-2}{2}}\left[\delta_i^j + (m-2)\frac{\widehat{p}_i\widehat{p}_j}{\delta^2+|\widehat{p}|^2}\right]p_iZ_j.
\end{aligned}
\tag{4.46}
$$

Obviously, that for $\delta > 0$ functions $a^{ij}(u,p)$ and $b(u,p)$ in (4.46) are infinitely differentiable on the set $[\varepsilon, M_0]\times\mathbb{R}^n$. Therefore, from the linear theory for parabolic equations and Proposition 4.1 we can derive the following

Proposition 4.4. *The solution u defined by Proposition 4.1 has continuous in Q_T derivatives $\nabla^3 u$ and ∇u_t.*

Propositions 4.1 and 4.4 give a possibility to apply Proposition 2.1 to solution of problem (4.44).

Proposition 4.5. *Assume that condition (4.9) holds. Let $Q'_{\tau,T} := \Omega' \times (\tau, T]$, $\bar{\Omega}' \subset \Omega$, $0<\tau<T$, and let*

$$
\rho_0 := \frac{1}{2}\inf(\operatorname{dist}(\Omega',\partial\Omega),\tau). \tag{4.47}
$$

Let u be solution of problem (4.44) defined by Proposition 4.1. Then

$$
\sup(|\nabla u|, Q'_{\tau,T}) \leqslant M_1'(\varepsilon), \tag{4.48}
$$

where constant $M_1'(\varepsilon)$ depends only on $\varepsilon, n, m, l, k, c_0$, $\sup(\varphi,\Gamma_T)$, $\operatorname{dist}(\Omega',\partial\Omega)$ and τ. In particular, $M_1'(\varepsilon)$ is independent of $\delta\in(0,1]$.

Proof. Using (4.35), (4.43), (4.12) and assuming that parameter $L = L(M_0)$ is so large that

$$
\begin{gathered}
(\delta^2+|\tilde{p}|^2)^{\frac{m-2}{2}}|p|^2 \geqslant c(m)|p|^m,\\
\frac{1}{(\delta^2+|\tilde{p}|^2)^{\frac{1}{2}}} \leqslant \frac{1}{|\widetilde{p}|} \leqslant 1,\quad \frac{1}{|p|} \leqslant 1 \text{ if } |p| > L
\end{gathered}
\tag{4.49}
$$

we derive from (4.8) that all inequalities in (2.1) are fulfilled with some constants $\mu_0,\dots,\mu_7$ depending on ε but independent of δ and with $r=0$,

$\varepsilon_1 = \frac{m}{2}$, $\varepsilon_2 = \frac{1}{2}$. Set $\hat{k} = 1 + 4\nu_1$ where ν_1 is defined by (2.3) in the case $\beta = 1$. Obviously $\hat{k}$ is independent of δ. Because in view of estimate (4.45) we have, in particular,

$$\omega := \operatorname{osc}(u, Q_\rho) \leqslant K\rho^\lambda, \quad Q_\rho = Q_\rho(x_0, t_0),$$
$$(x_0, t_0) \in Q'_{\tau,T}, \quad \rho \in (0, \rho_0],$$

where $\lambda \in (0,1)$ and K are independent of δ, we can take ρ so small that conditions (2.4) are satisfied for such ρ and k like above. Then applying Proposition 2.1 we obtain (4.48). Proposition 4.5 is proved.

5. Existence of regular solution

In this section we denote solutions of problems (4.44) as $u_{\delta,\varepsilon}$. In view of estimates (4.45) and (4.48) which are independent of $\delta \in (0,1]$ we can realize a passage to the limit as $\delta \to 0$ (for fixed $\varepsilon \in (0,1)$) using the local version of Ladyzhenskaya–Ural'tseva Hölder estimates for quasilinear uniformly parabolic equations of divergent type ([8], p. 497). Then for any cylinder $Q'_{\tau,T}$ (see Proposition 4.5), we have

$$\|u_{\delta,\varepsilon}\|_{C_\lambda(\bar{Q}_T)} + \|\nabla u_{\delta,\varepsilon}\|_{C_\alpha(Q'_{\tau,T})} \leqslant c, \tag{5.1}$$

where $\lambda \in (0,1)$, $\alpha \in (0,1)$ and $c > 0$ are independent of δ. Using (5.1) we can easily realize a passage to the limit as $\delta \to 0$ in the integral identity for problems (4.44) (at first for test-functions $\Phi \in \overset{\circ}{W}{}^1_m(Q_T)$ with a compact support in Q_T) and obtain solution u_t of problem

$$\begin{gathered} \mathcal{F}_0[u_\varepsilon] = (u_\varepsilon)_t - \operatorname{div} a(u_\varepsilon, \nabla u_\varepsilon) = 0 \text{ in } Q_T, \ u_\varepsilon = \varphi + \varepsilon \text{ on } \Gamma_T, \\ a(u,p) := |u|^l |\widehat{p}|^{m-2}\widehat{p}, \quad \widehat{p} = p + c_0|u|^k Z. \end{gathered} \tag{5.2}$$

More precisely in such a way we obtain the following

Lemma 5.1. *There is a Hölder continuous in $\bar{Q}_T$ weak solution u_ε of problem* (5.2) *such that*

$$\varepsilon \leqslant u_\varepsilon(x,t) \leqslant M_0 + \varepsilon \textit{ in } \bar{Q}_T$$

and

$$\iint\limits_{Q_T} u_\varepsilon^l |\nabla u_\varepsilon|^m \, dx\, dt \leqslant c, \tag{5.3}$$

where constant c depends only on the data (in particular, M_0 and c are independent of $\varepsilon \in (0,1]$). Moreover,

$$\langle u_\varepsilon \rangle_{\lambda,\bar{Q}_T} \leqslant c \tag{5.4}$$

with the same $\alpha \in (0,1)$ and $c > 0$ as in (4.45).

Taking into account that function $a(u,p)$ from (5.2) satisfies in view of (4.27)–(4.29) (in the case $\delta = 0$) the inequalities

$$\begin{gathered}
a(u,p)\cdot p \geqslant \nu_0 |u|^l |p|^m - \varphi_0, \quad \nu_0 > 0, \quad \varphi_0 = \text{const} \geqslant 0,\\
|a(u,p)| \leqslant \mu_1 |u|^l |p|^{m-1} + \varphi_1 |u|^\alpha, \quad \mu_1 \geqslant 0, \quad \varphi_1 = \text{const} \geqslant 0, \quad \alpha = \frac{l}{m},\\
[a(u,p) - a(u,q)]\cdot(p-q) \geqslant \nu_1 |u|^l |p-q|^\kappa (|p|^m + |q|^m + 1)^{\frac{m-\kappa}{m}},\\
\kappa = m \text{ if } m \geqslant 2, \quad \kappa = 2 \text{ if } m \in (1,2)
\end{gathered} \tag{5.5}$$

with constants $\nu_0, \mu_1, \varphi_0, \varphi_1, \nu_1$, independent of ε, we can repeat arguments of Sec. 5 in [3] or Sec. 8 in [4] and realize a passage to the limit as $\varepsilon \to 0$ in the integral identity for problems (5.2). In such a way we prove the following

Theorem 5.1. *Let condition* (4.9) *holds. Then there exists at least one regular solution (in sense of Definition* 3.4*) of problem* (1.6) *for equation* (1.3), (1.5).

Remark 5.1. It is easy to see that the result of Theorem 5.1 remains true if parameter k satisfies a more general condition $k \geqslant -\frac{l}{m}$ (cp. (1.5)).

6. Uniqueness of regular solution

Taking into account that in the case $m \geqslant 2$ function $a(u,p)$ defined by (5.2) satisfies the local Lipschitz condition (L_{loc}) we obtain in this case in view of Proposition 3.2 and Corollary 3.2 the following

Theorem 6.1. *Let condition* (4.9) *holds. If in addition $m \geqslant 2$ then there exists at most one regular solution of problem* (1.6) *for equation* (1.3), (1.5). *Moreover this solution is a maximal weak solution (in the sense of Definition* 3.5*) of problem* (1.6).

The aim of this section is to extend uniqueness to the case $m \in (1,2)$. We state first of all the following improvements of results of Sec. 3.

Consider at first the regularized equation

$$
\begin{gathered}
\mathcal{F}_0^{\varepsilon,N}[u_k] := u_t - \operatorname{div} a(\varkappa(u), \nabla u) = f(x,t), \\
a(u,p) = |u|^l |\widehat{p}|^{m-2}\widehat{p}, \quad \widehat{p} = p + c_0|u|^k Z, \\
m \in (1,2),\ l \geqslant 0,\ k \geqslant \frac{1}{m},\ |Z| = 1,\ \varkappa(u) \text{ is defined by (4.5).}
\end{gathered}
\tag{6.1}
$$

Because for Eq. (6.1) the growth condition $(\tilde{G})$ is fulfilled the notion of generalized solution given in Definition 3.1 is applicable to Eq. (6.1).

Proposition 6.1 (comparison principle for Eq. (6.1))**.** *Let u_1 and u_2 be a generalized subsolution and a generalized supersolution respectively, such that $\mathcal{F}_0^{\varepsilon,N}[u_1] \leqslant f_1$, $\mathcal{F}_0^{\varepsilon,N}[u_2] \geqslant f_2$, where $f_1, f_2 \in L_1(Q_T)$. If $u_1 \leqslant u_2$ on $S_T = \partial\Omega \times (0,T]$ then for any $\tau \in (0,T]$ inequality*

$$
\int\limits_\Omega (u_1 - u_2)^+ dx\Big|^{t=\tau} \leqslant \int\limits_\Omega (u_1 - u_2)^+ dx\Big|^{t=0} + \int\limits_0^\tau \int\limits_\Omega (f_1 - f_2) \operatorname{sign}(u_1 - u_2)^+ dx\, dt \tag{6.2}
$$

holds.

Corollary 6.1. *Let $f \in L_1(Q_T)$, $\varphi \in W_1^1(Q_T)$. Then Cauchy–Dirichlet problem*

$$
\mathcal{F}_0^{\varepsilon,N}[u] = f \text{ in } Q_T, \quad u = \varphi \text{ on } \Gamma_T
$$

has at most one generalized solution in the sense of Definition 3.1.

Remark 6.1. From Corollary 6.1 it follows in particular that there is at most one weak solution of problem 5.2.

consider now equation

$$
\begin{gathered}
\mathcal{F}_0[u] = u_t - \operatorname{div} a(u, \nabla u) = f(x,t), \\
a(u,p) \text{ is defined by (6.1).}
\end{gathered}
\tag{6.3}
$$

In view of (4.21)–(4.26) equation (6.3) has a structure of equation of the type (3.1) for which the growth condition (G) is satisfied. The latter is a particular case of equation of the type (3.3), (3.4). Then the notion of weak, regular, and maximal weak solutions of Cauchy–Dirichlet problem

$$
\mathcal{F}_0[u] = f \text{ in } Q_T, \quad u = \varphi \text{ on } \Gamma_T \tag{6.4}
$$

given in Definitions 3.3–3.5 are applicable to Eq. (6.3).

Proposition 6.2. *Any regular solution of Cauchy–Dirichlet problem for Eq.* (6.3) *is a maximal weak solution of this problem.*

We limit ourselves by proving only Proposition 6.2 which can be established as appropriate modification of the proof of Proposition 3.2. Exactly the same modification of the proof of Proposition 3.1 gives then Proposition 6.1.

Note that conditions (M) and $(L_{\rm loc})$ were used in the proof of Proposition 3.2 only for establishing the inequality

$$\liminf_{\delta\to 0}\frac{1}{\delta}\iint\limits_{\{Q_{0,r}:0<v_n-u_n<\delta\}} J\,dx\,dt\geqslant 0,\qquad J:=[a(v_n,\nabla v_n)-a(u_n,\nabla u_n)]\cdot(\nabla v_n-\nabla u_n) \tag{6.5}$$

where functions u_n, v_n are defined in the proof of Proposition 6.2. In particular

$$u_n\geqslant\varepsilon_n,\ v_n\geqslant\varepsilon_n,\ \varepsilon_n>0,\ \varepsilon_n\to 0;\quad u_n,v_n\in W_m^{1,0}(Q_T)\cap L_\infty(Q_T). \tag{6.6}$$

Therefore to prove Proposition 6.2 it suffices to establish (6.5) for function $a(u,p)$ defined by (6.3). To the end we use the appropriate adaptation of the proof of the close inequality which was given by Diaz-de Thelin in [1] (see also [14] where a comparison principle for some quasilinear parabolic equations with non-Lipschitz coefficients was established).

Proof of Proposition 6.2. For the sake of simplicity we prove inequality (6.5) (and hence Proposition 6.2) assuming that $f\equiv 0$ in Q_T. We will use here the following notation

$$\phi_m(s):=|s|^{m-2}s,\quad s\in\mathbb{R}^n,\quad m>1. \tag{6.7}$$

Using notation (6.7) we can write

$$a(u,p)=|u|^l\phi_m(\widehat{p}),\ \widehat{p}=p+G(u),\ G(u):=c_0|u|^kZ,\ m\in(1.2). \tag{6.8}$$

Then

$$\begin{gathered}a(v_n,\nabla v_n)=v_n^l\phi_m(\hat{q}_n),\ \hat{q}_n=\nabla v_n+G(v_n),\\ a(u_n,\nabla u_n)=u_n^l\phi_m(\widehat{p}_n),\ \widehat{p}_n=\nabla u_n+G(u_n),\\ J=(v_n^l-u_n^l)\phi_m(\hat{q}_n)(\nabla v_n-\nabla u_n)+u_n^l[\phi_m(\hat{q}_n)-\phi_m(\widehat{p}_n)]\cdot(\hat{q}_n-\widehat{p}_n)-\\ -u_n^l[\phi_m(\hat{q}_n)-\phi_m(\widehat{p}_n)]\cdot[G(v_n)-G(u_n)]=:J_1+J_2+J_3.\end{gathered} \tag{6.9}$$

Obviously from (6.6)–(6.9) it follows that

$$|J_1| \leqslant c|v_n - u_n|F(x,t) \text{ in } Q_T, \tag{6.10}$$

where constant c and $\|F\|_{L_1(Q_T)}$ are defined by (6.6)–(6.9). Next, in view of (4.20) (with $\delta = 0$) we have

$$J_2 \geqslant 0 \text{ in } Q_T, \tag{6.11}$$

and for any $\varepsilon \in (0,1)$

$$|J_3| \leqslant \varepsilon u_n^l |\phi_m(\hat{q}_n) - \phi_m(\widehat{p}_n)|^{m'} + c(\varepsilon, m) u_n^l |G(v_n) - G(u_n)|^m \text{ in } Q_T. \tag{6.12}$$

We denote now

$$z_n := \phi_m(\hat{q}_n), \quad w_n := \phi_m(\hat{p}_n). \tag{6.13}$$

Because $m \in (1,2)$ we have $m' > 2$. Then using (4.20) (in the case $\delta = 0$) we obtain

$$|z_n - w_n|^{m'} \leqslant c[\phi_{m'}(z_n) - \phi_{m'}(w_n)] \cdot (z_n - w_n), \tag{6.14}$$

where $c = c(m')$. In view of (6.13) and taking into account that

$$\phi_{m'}(\phi_m(\hat{q}_n)) = \hat{q}_n, \quad \phi_{m'}(\phi_m(\widehat{p}_n)) = \widehat{p}_n$$

we can rewrite (6.14) as

$$|\phi_m(\hat{q}_n) - \phi_m(\widehat{p}_n)|^{m'} \leqslant c[\phi_m(\hat{q}_n) - \phi_m(\hat{p}_n)] \cdot (\hat{q}_n - \widehat{p}_n). \tag{6.15}$$

Choosing appropriate $\varepsilon \in (0,1)$ and taking into account condition $k \geqslant \frac{1}{m}$ (see (6.3)) we can derive from (6.12), (6.15), (6.9), (6.11), and (6.6) inequality

$$|J_3| \leqslant \frac{1}{2} J_2 + c|v_n - u_n| \,\text{in}\, Q_T \tag{6.16}$$

where constant c defined by (6.6)–(6.9). Then from (6.9)–(6.11) and (6.16) it follows that

$$\frac{1}{\delta} \iint\limits_{\{Q_{0,r}:0<v_n-u_n<\delta\}} J\,dx\,dt \geqslant -c \iint\limits_{\{Q_{0,r}:0<v_n-u_n<\delta\}} [1 + F(x,t)]dx\,dt, \tag{6.17}$$

where constant c and function $F \in L_1(Q_T)$ are independent of parameter δ. Because, in particular, $v_n - u_n \in L_1(Q_T)$ we can tend $\delta \to 0$ and obtain inequality (6.5)¿ Proposition 6.2 is proved.

Corollary 6.2. *There is at most one regular solution of Cauchy–Dirichlet problem* (6.4) *for equation* (6.3).

Definition 6.1. *Let condition (G) of sect. 3 is fulfilled, $f \in L_1(Q_T)$, $\varphi \in W_1^1(Q_T)$, $\varphi \geqslant 0$ in Q_T. We say that weak solution (in the sense of Definition 3.3) of problem* (3.2) *is a quasistrong solution of this problem if there exists a sequence $\{u_n\}$ of weak solutions $u_n \in W_m^{1,0}(Q_T)$ of problems* (3.5) *such that*

$$\inf(u_n, Q_T) \geqslant \varepsilon_n = \text{const} > 0,\ \varepsilon_n \to 0;\ u_n \to u \text{ in } C([0,T];\ L_1(\Omega)).$$

Proposition 6.3. *Propositions* 3.1, 3.2, 6.1, *and* 6.2 *remain valid if instead of the term "regular solution" in the statements of these propositions one uses the term "quasistrong solution". In particular, there is at most one quasistrong solution of problem* (6.4) *for equation* (6.3).

Obviously from Theorem 5.1 and Corollary 6.2 we can derive following.

Theorem 6.2. *Let condition* (4.9) *holds. Then there is exactly one regular solution of problem* (6.4) *for equation* (6.3). *Moreover this solution is a maximal weak solution of this problem.*

Because in the case of equation of the turbulent filtration parameter k satisfies condition $k \geqslant 1 > \frac{1}{m}$ we are ready now to state the main result of this paper.

Theorem 6.3. *Let condition* (4.9) *holds. Then there is exactly one regular solution of Cauchy–Dirichlet problem* (1.6) *for equation* (1.3), (1.4). *Moreover this solution is a maximal weak solution of this problem.*

Proof. The results of Theorems 6.3 follow directly from Theorem 6.1 if $m \geqslant 2$ and from Theorem 6.2 if $m \in (1,2)$. Theorem 6.3 proved.

7. Gradient estimates

The main results of this section are local and some global weight gradient estimates for solutions of Cauchy–Dirichlet problem for Eqs. (1.3), (1.5). In the case of $m = 0$ (i.e., for Eq. (1.11) similar local waight boundary estimates was given in paper [5]. In [5] we tried also to prove a local boundary estimate in the case $0 \leqslant l < 1$. However some eroop of arihmetic caracter was present in the proof of last estimate in [5]. Really such estimate was proved in [5] only for $l \in [0, l_0)$ with some $l_0 \in (0,1)$. We proved in this section similar result for Eqs. (1.3), (1.5).

Proposition 7.1. *Let u be a solution of problem (4.44) in the case $0 \leqslant l < 1$ defined in Propositions 4.1–4.3. Then there is a number $l_0 =\in (0,1)$ such that for any cylinder $Q'_{\tau,T} = \Omega' \times (\tau, T] = \Omega' \times (\tau, T]$, $\bar{\Omega}' \subset \Omega$, $0 < \tau < T$ solutions u of problem (4.44) with $0 \leqslant l < l_0$ satisfies inequality*

$$\sup(|\nabla u|, Q'_{\tau,T}) \leqslant M'_1$$

where constant M'_1 depends only on $n, m, l, k, \sup(\varphi, \Gamma_T)$, $\operatorname{dist}(\Omega', \partial\Omega)$ and τ.

Proof. Let $(x_0, t_0) \in Q'_{\tau,T}$ and let radius ρ satisfies condition

$$0 < \rho \leqslant \rho_0 = \frac{d_0}{2}, \quad d_0 := \inf(\operatorname{dist}(\Omega', \partial\Omega), \tau). \tag{7.2}$$

Let u be a solution of problem (4.4) defined by Propositions 4.1–4.3. In view of (4.8), (4.35) and (4.43) we can conclude that conditions (2.1) are sarisfies at this solution on the set $P_{\rho,L}(x_0, t_0)$ for

$$\left.\begin{aligned}
&\mu_0^{-1} = \frac{a_0}{2} u^l,\ a_0 := \min(1, m-1);\ \mu_1 = \mu_2 = \mu_3 = c;\\
&\mu_4 = l \max(1, m-1) u^{-1} + \frac{c u^{k-1}}{L};\ \mu_5 = c u^{k-1};\\
&\mu_7 = l(l-1) a_0^{-1} u^{-2} + c\left(u^{-1} + \frac{u^{k-1} + u^{2(k-1)}}{L}\right);\\
&r = 0,\ \varepsilon_1 = \frac{m}{2},\ \varepsilon_2 = \frac{1}{2}
\end{aligned}\right\} \tag{7.3}$$

where $c = c(n, m, l, k, c_0, M_0)$ if constant $L > 0$ is sufficient large. Then from (7.3), (4.8) it follows that for parameters ν_i and ε_i defined in (2.3) we have for appropriate $\beta \in (0,1)$

$$\left.\begin{aligned}
&\nu_1 = c + \beta^{-1}\\
&\nu_2 \leqslant [-2l(1-l) a_0^{-1} + 4l^2 \sup{}^2(1, m-1) + \beta] u^{-2} + c(\beta^{-1})\\
&\nu_3 = c,\ \nu_4 = \nu_5 = c\beta^{-1},\ \varepsilon_0 = \varepsilon_2 = \frac{1}{2},\ \varepsilon_1 = \frac{m}{2}.
\end{aligned}\right\} \tag{7.4}$$

Moreover instead of second inequality in (7.4) we can obtain inequality

$$\nu_2 \leqslant \left[-2l(1-l) a_0^{-1} + \frac{l^2 \sup^2(1, m-1)}{2 - \beta_1} + \varepsilon u^{-2} + \varepsilon\right] u^{-2} + c(\beta^{-1})$$

with any $\beta_1 > 0$. Indeed we can use unstead inequalities (2.10) a more exact estimates, for example

$$|2\sqrt{v}\zeta\delta a^{ij}u_{ij}| \leqslant (2-\beta_1)a^{ij}u_{ki}u_{kj} + \frac{\mu_4^2}{2-\beta_1}Z\mathcal{E}w,\ \beta_1 > 0.$$

Taking into account that $0 \leqslant l < 1$, using (4.35) and choosing appropriate β and β_1 we derive from hear that

$$\nu_i \leqslant c_i, \quad i = 1, \ldots, 5 \tag{7.5}$$

if $l \in [0, l_0]$ with $l_0 = \frac{1}{1+h}$, $h = \frac{\sup^2(1,m-1)}{4\min(1,m-1)}$, with some fixed constants $c_i = c_i(n, m, l, k, M_0)$, $i = 1, \ldots, 5$.

Because in view of (7.2) $\overline{Q_\rho(x_0, t_0)} \subset Q_T$ we can apply the local version of estimate (4.45) (see Theorem 3.1). Then

$$\omega = \operatorname{osc}(u, Q_\rho) \leqslant K\rho^\lambda \tag{7.6}$$

where $\lambda \in (0, 1)$ depends only on n, m, l, k, and c_0 while K depends on the same data and also on d_0 and $\sup(\varphi, Q_T)$. In particular λ and K are independent of ε, δ, and ρ. Set (see (2.4))

$$\hat{\varkappa} = 1 + 4c_1, \quad \rho = \min(\rho_0, \rho_*) \tag{7.7}$$

where ρ_0 is like in (7.2) and ρ_* is defined by the identity

$$c_2(\hat{\varkappa}+1)K^2\rho_*^{2\lambda} = \frac{1}{4}. \tag{7.8}$$

In view of (7.5)–(7.8) conditions (2.4) are fulfilled for $\hat{\varkappa}$ and ρ chosen in (7.7). Then from Proposition 2.1 (see (2.5), (2.6)) we derive estimate (7.1) with a constant M_1' independent of δ and ε. Proposition 7.1 is proved.

Theorem 7.1. *Let u be a regular solution of Cauchy–Dirichlet problem (4.2) in the case $0 \leqslant l < l_0$, where $l_0 \in (0, 1)$ is defined in Proposition 2.1. Then $\nabla u \in L_\infty^{\mathrm{loc}}(Q_T)$ and for any cylinder $Q'_{\tau,T} = \Omega' \times (\tau, T]$, $\Omega' \subset \Omega$, $\tau \in (0, T]$ we have*

$$\sup(|\nabla u|, Q'_{\tau,T}) \leqslant M_1' \tag{7.9}$$

where constant M_1' depends only on the data of problem (4.2), $\operatorname{dist}(\Omega', \partial\Omega)$ and τ. If in addition

$$\sup(|\nabla\varphi(x,0)|, \Omega') \leqslant c_1' \text{ for any } \Omega', \quad \bar{\Omega}' \subset \Omega, \tag{7.10}$$

where $c_1' = c_1'(\mathrm{dist}(\Omega', \partial\Omega))$ *then for every cylinder* $Q_T' = \Omega' \times (0, T]$, $\bar{\Omega}' \subset \Omega$, *we have*

$$\sup(|\nabla u|, Q_T') \leqslant \tilde{M}'_1 \tag{7.11}$$

with a constant $\tilde{M}_1'$ *depending only on the data of problem* (4.2) *and* $\mathrm{dist}(\Omega', \partial\Omega)$.

Proof. In view of (5.1) and (7.1) we have for solutions u_ε pf problems (5.2) in the case $l \in [0, 1)$ the estimates

$$\|u_\varepsilon\|_{C_\lambda(Q'_{\tau,T})} \leqslant K, \quad \sup(|\nabla u_\varepsilon|, Q'_{\tau,T}) \leqslant M_1'$$

where $\lambda \in (0, 1]$, K and M_1' are independent of ε. Then from (7.11) it follows that for some sequence $\{\varepsilon\}$, $\varepsilon \to 0$, and any $p > 1$

$$u_\varepsilon \to u \text{ in } C(\overline{Q_T}), \quad \|\nabla u_\varepsilon\|_{L_p(Q'_{\tau,T})} \leqslant cM_1' \tag{7.12}$$

with a constant c independent of ε and p. Obviously (7.12) implies that the limit function u satisfies inequality (7.1). The proof of validity of inequality (7.11) if condition (7.10) holds can be realized absolutely in the same way. Theorem 7.1 is proved.

As we shall see in the next section in the case $l > 0$ estimate (7.11) for regular solution of problem (4.2) are impossible. However in this case we are able to prove some weight gradient estimates both of local and up to the boundary. For proving such estimates we need to apply the following version of Proposition 2.1.

Proposition 7.2 ([5]). *Let* u *be a classical solution of Eq.* (2.0) *in* Q_T *such that* $\nabla^3 u$ *and* ∇u_t *are continuous on the set* $\mathcal{P}_{\rho,L}(x_0, t_0)$ *defined in Proposition* 2.1. *Assume that at this solution on* $\mathcal{P}_{\rho,L}(x_0, t_0)$ *conditions* (2.1), (2.2) *hold and let*

$$\begin{gathered}
\hat{\nu}_1 = \mu_2 + 1, \ \hat{\nu}_2 = 2\mu_7 + \mu_4 + \mu_5 + c, \\
\hat{\nu}_3 = c\sigma^{-2}(\mu_1 + \mu_3), \ \nu_4 = c\sigma^{-1}\mu_0, \ \nu_5 = c\sigma^{-1}\mu_6, \\
\sigma = \min(\varepsilon_0, \varepsilon_1, \varepsilon_2), \ \varepsilon_o = \frac{1}{2} - = \frac{r}{4}, \\
\mu_i = \mu_i(x_*, t_*, u_*, p_*), \ i = 0, 1, \dots, 7,
\end{gathered} \tag{7.13}$$

(x_*, t_*) *be a point of maximum of function* u *in* $\overline{\mathcal{P}_\rho(x_0, t_0)}$,

$$u_* = u(x_*, t_*), \ p_* = \nabla u(x_*, t_*), \ c = c(n).$$

Suppose that at this function u on the set $\mathcal{P}_{\rho,L}(x_0,t_0)$ we have for some constant $\hat{k}>1$

$$4\hat{\nu}_1<\hat{k},\ \hat{\nu}_2(\hat{k}+1)\omega^2<\frac{1}{2},\ \omega:=\operatorname{osc}(u,\mathcal{P}_\rho(x_0,t_0)), \tag{7.14}$$

where $\hat{\nu}_1$ and $\hat{\nu}_2$ are defined by (7.13). Then

$$|\nabla u(x_0,t_0)|\leqslant\sqrt{2}\,\max(L,\Theta,\sup(|\mathcal{D}u|,\Gamma_T\cap Q_\rho)), \tag{7.15}$$

where

$$\Theta=\max\left\{\left(12\hat{\nu}_3\omega^2\frac{(k+1)}{\rho^2}\right)^{\frac{1}{4\varepsilon_0}},\left(12\hat{\nu}_4\omega^2\frac{(k+1)}{\rho}\right)^{\frac{1}{2\varepsilon_1}},\left(12\hat{\nu}_5\omega^2\frac{(k+1)}{\rho}\right)^{\frac{1}{2\varepsilon_2}}\right\} \tag{7.16}$$

and $\hat{\nu}_3$, $\hat{\nu}_4$, $\hat{\nu}_5$, ε_0, ε_1, and ε_2 are defined by (7.13).

The proof of Proposition 7.2 is given in [5].

Proposition 7.3. *Let u be a solution of problem (4.4) defined by Propositions 4.1–4.3 (in the general case $l\geqslant 0$). Then for any cylinder $Q'_{\tau,T}=\Omega'\times(\tau,T]$, $\bar{\Omega}'\subset\Omega$, we have*

$$\sup(u^\kappa|\nabla u|,Q'_{\tau,T})\leqslant M'_1,\ \kappa:=\max\left(\frac{1}{\lambda}-1,\frac{l-2+\frac{1}{\lambda}}{m}\right) \tag{7.17}$$

where $\lambda\in(0,1)$ is the exponent in (4.45) and constant M'_1 depends on n,m,l,k, and also on $\sup(\varphi,\Gamma_T)$, $\operatorname{dist}(\Omega',\partial\Omega)$ and τ.

Proof. Let $(x_0,t_0)\in Q'_{\tau,T}$ and let radius ρ satisfies condition (7.2). Let u be a solution of problem (4.44) defined by Propositions 4.1–4.3. Then (see the proof of Propositions 7.1) conditions (2.1) are fulfilled for parameters $\mu_0,\dots,\mu_7,r,\varepsilon_1$ and ε_2 of the form (7.3) with $c=c(n,m)$. From (7.3), (4.8), (4.6), (4.35) and (4.43) it follows that numbers $\nu_1,\dots,\nu_5,\varepsilon_0,\varepsilon_1,\varepsilon_2$ defined by (7.13) have in the case of equation (4.46) the following form

$$\begin{gathered}\hat{\nu}_1=c,\ \hat{\nu}_2=cu^{-2}+c,\ \hat{\nu}_3=c,\ \hat{\nu}_4=cu^{-l},\ \hat{\nu}_5=cu^{-1}\\ \varepsilon_0=\varepsilon_2=\frac{1}{2},\ \varepsilon_1=\frac{m}{2},\ c=c(n,m,l,k,c_0,M_0).\end{gathered} \tag{7.18}$$

Denote $\tilde{u}=\min(u,1)$. Then in view of (7.18)

$$\hat{\nu}_2\leqslant\hat{c}\tilde{u}^{-2}. \tag{7.19}$$

Because $\overline{Q_\rho(x_0,t_0)}\subset Q_T$ we can conclude in view of (7.2) that estimate (7.6) holds.

Let $\hat{k} \geqslant 1$ defined as

$$\hat{k} = 1 + 4\hat{c},\ \ \hat{c} \text{ is defined by (7.19).}$$

Then the first inequality in (7.14) is fulfilled.

Denote $u_0 = u(x_0, t_0)$. Obviously $u_0 > 0$ in view of (4.35). Assume that

$$0 < \rho \leqslant \rho_1 := (\tilde{u}_0/2K)^{\frac{1}{\lambda}}, \quad \tilde{u}_0 = \min(u_0, 1), \tag{7.20}$$

where K is the constant in inequality (4.45). Using (7.6) and (7.20) we have for any $(x_*, t_*) \in Q_\rho(x_0, t_0)$

$$|u_* - u_0| \leqslant \frac{u_0}{2},\ \ u_* = u(x_*, t_*).$$

Then for any $(x_*, t_*) \in Q_\rho$ we have

$$\frac{1}{2}u_0 \leqslant u_* \leqslant 2u_0. \tag{7.21}$$

Assume now that

$$0 < \rho \leqslant \rho_2 := (\tilde{u}_0/c_*)^{\frac{1}{\lambda}} \tag{7.22}$$

where $c_* > 1$ is defined by identity

$$c_*^2 = (4K)^2\hat{c}(\hat{k} + 1)$$

with constant $\hat{c}$ from inequality (7.19). Obviously from (7.6), (7.21), (7.22) it follows that the second inequality in (7.19) is fulfilled. Indeed

$$\hat{\nu}_2(\hat{k} + 1)\omega^2 \leqslant \hat{c}4u_0^{-2}(\hat{k} + 1)K^2\rho^{2\lambda} \leqslant \frac{1}{4}.$$

Denote ρ and λ by formulas

$$\rho = \min(\rho_0, \rho_1, \rho_2),\ \ \hat{u}_0 = \rho^\lambda. \tag{7.23}$$

In view of (7.23), (7.2), (7.20), (7.22) and recalling that $\tilde{u} = \min(1, u)$ we have

$$\hat{u}_0 = \min\{(d_0/2)^\lambda, u_0/2K, u_0/c_*, 1/2K, 1/c_*\} < u_0.$$

In particular from here it follows that $\hat{u}_0 \in (0, 1)$. Moreover, using (4.35) we can conclude that

$$u_0\hat{u}_0^{-1} \leqslant c \tag{7.24}$$

where constant c depends on the same data as the constant M_1' in (7.1). In particular c is independent of u_0.

Applying Proposition 7.2 we obtain estimate (7.15) with $\hat{\Theta}$ defined by (7.16). In view of (7.16), (7.23), (7.6), (7.13), (7.7) we obtain

$$\hat{\Theta} \leqslant c \max\left\{ \hat{u}_0^{\frac{2\lambda-2}{2\lambda}}, \hat{u}_0^{\left(\frac{2\lambda-1}{\lambda}-l\right)/m}, \hat{u}^{\frac{2\lambda-1}{\lambda}-1} \right\}$$

where constant c is independent of $\hat{u}_0$. From here, (7.16) and (7.24) it follows that estimate (7.17) holds. Proposition 7.3 is proved.

Theorem 7.2. *Let u be a regular solution of Cauchy–Dirichlet problem (4.2) defined by Theorem 5.1 in the general case $l \geqslant 0$. Let number κ is defined by (7.17). Then $\nabla u^{\kappa+1} \in L_\infty^{\mathrm{loc}}(Q_T)$ and for any cylinder $Q'_{\tau,T} \times (\tau, T]$, $\bar{\Omega}' \subset \Omega$, $\tau \in (0,T)$, the inequality*

$$\sup(|\nabla u^{\kappa+1}|, Q'_{\tau,T}) \leqslant M'_1 \tag{7.25}$$

holds with a constant M'_1 depending only on the data of problem (4.2), $\mathrm{dist}(\Omega', \partial\Omega)$ and τ. If in addition

$$\sup(|\nabla \varphi^{\kappa+1}(x,0)|, \Omega') \leqslant c'_1 \text{ for any } \Omega',\ \bar{\Omega}' \subset \Omega,$$

where $c'_1 = c_1$ $(\mathrm{dist}(\Omega', \partial\Omega))$ then for every cylinder $Q'_T = \Omega' \times (0,T]$, $\bar{\Omega}' \subset \Omega$, we have

$$\sup(|\nabla u^{\kappa+1}|, Q'_T) \leqslant \tilde{M}'_1 \tag{7.26}$$

where constant $\tilde{M}'_1$ depends only on the data of problem (4.2) and $\mathrm{dist}(\Omega', \partial\Omega)$.

Proof. Theorem 7.2 can be proved exactly in the same way as Theorem 7.1 if instead of Proposition 7.1 one uses Proposition 7.3.

Remerk 7.1. In the case of Eqs. (1.9), (1.10) with exponents $m \geqslant 2$, $l > 0$ weight estimates of the type (7.25), (7.26) but for another exponent κ were derived in [15] by means of results of [6] and [16].

Consider now a question about possibility to obtain the global weight estimate for solutions u defined by Theorem 5.1.

Theorem 7.3. *Let u be a regular solution on Cauchy–Dirichlet problem (4.2) defined by Theorem 5.1 in the case $m > 1$, $l \geqslant 0$, $k \geqslant 1$. Assume that $\Omega \in C^2$ and $\varphi \in C^2(\overline{Q}_T)$. Then*

$$\sup(|\nabla u^{\gamma+1}|, Q_T) \leqslant M_1,\ \gamma = \max(\sigma, \kappa), \tag{7.27}$$

where $\sigma = l/(m-1)$, κ is defined by (7.17) and constant M_1 depends only on the data of problem (4.2).

Proof. Let u be a solution of regularized problem (4.44) (in the case $k \geqslant 1$) defined by Propositions 4.1–4.3. For this solution estimates (4.35), (4.36), (4.45) hold and Eq. (4.3) can be written as (see (4.44))

$$u_t - \operatorname{div}\{u^l(\delta^2 + |\nabla u + c_0 u^k Z|^2)^{(m-2)/2}(\nabla u + c_0 u^k Z)\} = 0, \qquad (7.28)$$

where $u \in C_{2,1}(Q_T) \cap C_\lambda(\overline{Q_T}) \cap W_m^{1,0}(Q_T)$. Multiplying (7.28) by u^σ and denoting $v = \frac{u^{\sigma+1}}{\sigma+1}$, we have in the case $m \neq 2$

$$v_t - u^\sigma \operatorname{div}\{(\hat{\delta}^2 + |\nabla v + c_0 v^s Z|^2)^{(m-2)/2}(\nabla v + c_0 v^s Z)\} = 0, \qquad (7.29)$$

where

$$\hat{\delta} = \delta(\sigma+1)^h v^h,\ h := \frac{l(m-1)}{(m-2)(m+l-1)},\ s := \frac{\sigma+k}{\sigma+1},\ \sigma = \frac{l}{m-1}, \qquad (7.30)$$

so that $\hat{\delta} = \hat{\delta}(v)$. Note that in the case $m = 2$ Eq. (7.29) has a more simple form

$$v_t - u^l \operatorname{div}(\nabla v + c_0 v^s Z) = 0.$$

Obviously Eq. (7.29) has almost the same structure as equation in (4.4). Therefore it is easy to see that (7.29) can be written in the following non-divergence form

$$-v_t + a^{ij}(v, \nabla v) v_{ij} - b(v, \nabla v) = 0, \qquad (7.31)$$

where

$$a^{ij}(v,q) = ((\sigma+1)v)^{\frac{\sigma}{\sigma+1}}(\hat{\delta}^2 + |\hat{q}|^2)^{\frac{m-2}{2}}\left[\delta_i^j + (m-2)\frac{\hat{q}_i\hat{q}_j}{\hat{\delta}^2 + |\hat{q}|^2}\right],$$
$$\hat{q} = q + c_0 v^s Z, \quad s \geqslant 1, \qquad (7.32)$$

$$\begin{aligned} &-b(v,q) \\ &= ((\sigma+1)v)^{\frac{\sigma}{\sigma+1}}\Big\{c_0 s v^{s-1}(\hat{\delta}^2 + |\hat{q}|^2)^{\frac{m-2}{2}}\left[\delta_i^j + (m-2)\frac{\hat{q}_i\hat{q}_j}{\hat{\delta}^2 + |\hat{q}|^2}\right] q_i Z_j \\ &\qquad + (m-2)(\hat{\delta}^2 + |\hat{q}|^2)^{\frac{m-2}{2}}\, h\delta^2 v^{2h-1}\frac{q_i\hat{q}_j}{\hat{\delta}^2 + |\hat{q}|^2}\Big\}. \end{aligned} \qquad (7.33)$$

In view of (4.35) for $m \neq 2$ we have

$$\sup(v^{2h-1}, Q_T) \leqslant c(\varepsilon, M_0) \tag{7.34}$$

with some constant $c(\varepsilon, M_0) \geqslant 1$.

Consider in the case $m \neq 2$ the regularized problem (4.44) for fixed $\varepsilon \in (0,1]$ and $\delta \in (0, \delta_0]$ where $\delta_0 = \delta_0(\varepsilon) \in (0,1)$ is defined by identity

$$\delta_0^2 c(\varepsilon, M_0) = 1. \tag{7.35}$$

Then denoting $E = a^{ij} q_i q_j$ we can derive from (7.31)–(7.35) that for all $\delta \in (0, \delta_0(\varepsilon)]$ and any $v \in (0, \hat{M}_0]$, $q \in \mathbb{R}^n, |q| > L$, where $\hat{M}_0 = \left(\frac{M_0+1}{\sigma+1}\right)^{\sigma+1}$ and $L = L(M_0)$ is sufficient large, inequalities

$$\begin{aligned} &E \geqslant \alpha_0((\sigma+1)v)^{\frac{\sigma}{\sigma+1}}(\hat{\delta}^2 + |q|^2)^{\frac{m-2}{2}}|q|^2, \quad \alpha_0 = \min(1, m-1), \\ &E \geqslant c_1((\sigma+1)v)^{\frac{\sigma}{\sigma+1}}|q|^m > 0, \quad |b(v,q)| + \operatorname{Tr}\|a^{ij}(v,q)\|\,|q| \leqslant c_2 E \end{aligned} \tag{7.36}$$

hold where $c_1 > 0$ and $c_2 > 0$ are independent of ε and $\delta \in (0, \delta_0(\varepsilon))$.

Obviously in the case $m = 2$ inequalities (7.36) are also satisfied with similar constants c_1 and c_2. In the proof of these assertions we have used in particular the fact that $s \geqslant 1$. Taking into account that $\frac{\varepsilon^{\sigma+1}}{\sigma+1} \leqslant v^{(\frac{\sigma}{\sigma+1})} \leqslant (M_0+1)^{\frac{\sigma}{\sigma+1}}$ we can derive from (7.33) that after appropriate redefinition of coefficients $a^{ij}(v,q)$ on the set $\{v < (\frac{\varepsilon^{\sigma+1}}{\sigma+1}\} \times \mathbb{R}^n$ we have

$$\begin{gathered} a^{ij}(v,q)\xi^i\xi^j > 0 \\ \text{for any } v \in \mathbb{R},\ q \in \mathbb{R}^n,\ \xi \in \mathbb{R}^n,\ \xi \neq 0. \end{gathered} \tag{7.37}$$

Then in view of (7.35), (7.36) we can apply to Eqs. (7.31)–(7.33) well-known results of the theory of quasilinear parabolic equations (see, for example, Part 3 in [6]). Using the boundary gradient estimates established in [6] we obtain

$$\sup(|\nabla v|, S_T) \leqslant \tilde{M}_0 \tag{7.38}$$

where constant $\tilde{M}_0$ is independent of ε and δ. We can rewrite (7.38) as

$$\sup(|\nabla u^{\sigma+1}|, S_T) \leqslant M_0. \tag{7.39}$$

Together with a version of Theorem 7.2 corresponding to the case $(x_0, t_0) \in \overline{Q_T}$ (see inequalities (2.5), (2.6)) inequality (7.39) implies estimate

$$\sup(|\nabla u^{\gamma+1}|, Q_T) \leqslant M_1 \tag{7.40}$$

where M_1 depends only on the data of problem (4.2) and γ is defined by (7.27). Then using (4.45) and (7.40) we can conclude after a passage to the limit at first as $\delta \to 0$, $\delta \in (0, \delta_0(\varepsilon))$, and then as $\varepsilon \to 0$ that the solution u of problem (4.2) satisfies inequality (7.27). Theorem 7.3 is proved.

Remark 7.1. In the case $m = 2$ estimates (7.39) and (7.27) have respectively forms

$$\begin{gathered} \sup(|\nabla u^{l+1}|, S_T) \leqslant M_0, \\ \sup(|\nabla u^{\gamma+1}|, Q_T) \leqslant M_1, \quad \gamma = \sup(l, \kappa) \end{gathered} \tag{7.41}$$

where κ is defined by (7.17) in the case $m = 2$.

Remark 7.2. The results of this sect. may be applied to the solution u defined by Theorem 6.3. Recall in particular that condition $k \geqslant 1$ of Theorem 7.3 is fulfilled for Eq. (1.3), (1.4).

8. Some counterexamples

In this sect. we use counterexamples constructed in [9] and [5] to show that many regularity results established in the previous sections are in some sense the best possible (even for the simplest Eq. (1.11)).

I. Local regularity for non-negative solutions.

Consider at first the following counterexamples ([9]). The function

$$u = (t+\varepsilon)^{-\alpha n}\{1 - c[|x|/(t+\varepsilon)^{\alpha}]^{\gamma}\}_{+}^{\delta}, \ x = (x_1, \dots, x_n), \ \varepsilon > 0 \tag{8.1}$$

is a weak solution (in the sence of Definition (3.2)) of Eq. (1.11) in $Q_T = \Omega \times (0, T]$, $\Omega \in \mathbb{R}^n$, $n \geqslant 1$, $T > 0$, if

$$\begin{gathered} \alpha^{-1} = n(m+l-2) + m, \ \gamma = \frac{m}{m-1}, \ \delta = \frac{m-1}{m+l-2}, \\ m > 1, \ l > 1 - m, \ m + l > 2 \end{gathered} \tag{8.2}$$

and c is an appropriate constant. In the case $m = 2$ function (8.1), (8.2) coincides with the well-known Barenblatt explicit self-similar solution.

Assume that $0 \in \Omega$ and $\varepsilon > 0$ is sufficient small. Then it is easy to see that gradient ∇u of function (8.1), (8.2) is locally unbounded in the case $l > 1$ because in this case we have $\delta - 1 = \frac{1-l}{m+l-2} < 0$. Moreover it is easy to see that

a) in the case $m > 1$, $l > 1$ (so that $m + l > 2$) we have

$$\begin{gathered} u \in C_{\lambda_*}(Q_T) \quad \text{for} \quad \lambda_* = \frac{m-1}{m+l-2} \in (0,1), \\ u \neq C_\lambda(Q_T) \quad \text{for any} \quad \lambda > \lambda_*; \end{gathered} \tag{8.3}$$

b) in the case $m > 1$, $l > 1 - m$, $m + l > 2$ we have

$$\begin{gathered} \sup(|\nabla u^{\gamma_*}|, Q_T) < +\infty \text{ for } \gamma_* = \frac{1}{\lambda_*} = \frac{m+l-2}{m-1} \\ \sup(|\nabla u^{\gamma}|, Q_T) = +\infty \text{ for any } \gamma < \gamma_*; \end{gathered} \tag{8.4}$$

c) the second derivatives $\nabla^2 u$ of function (8.1), (8.2) are locally unbounded for any $m > 1$, $l > 0$, $m + l > 2$ because $\delta - 2 = \frac{1-m-2l}{m+l-2} < 0$ for m and l under consideration.

In particular from here we can derive the following conclusions concerning the results of given paper.

1) Local holderness is the best possible smoothness of weak solutions of Eqs. (1.3), (1.5) which could be established for the full range of parameters m, l, and k defined in (1.5). Existence of Hölder continuous weak solutions of Cauchy–Dirichlet problem for equations mentioned is proved in sect. 4 and 5. (Recall that the a priori Hölder estimates for weak solutions of Eqs. (1.3), (1.5) can be derived from results [9–11]).

2) Local boundedness of spatial gradient ∇u for weak solutions u of Eqs. (1.3), (1.5) is possible only in the case $l < 1$ (cp. with results of Theorem 7.1).

3) In sect. 7 we proved also existence of Hölder continuous weak solutions u of Cauchy–Dirichlet problem for general Eqs. (1.3), (1.5) with locally bounded $\nabla u^{\kappa+1}$ where

$$\kappa + 1 = \max\left\{\frac{1}{\lambda}, \frac{m+l-2}{m} + \frac{1}{m\lambda}\right\}. \tag{8.5}$$

In particular in the case $l > 1$ we have $m + l > 2$ and hence in view of (8.4) Hölder exponent λ belongs to $(0, \lambda_*]$. Then we can derive from results of sect. 4 and 7 that

$$\begin{gathered} \langle u \rangle_{\lambda, Q'_{\tau,T}} \leqslant K, \quad \lambda \in (0, \lambda_*], \quad \lambda_* = \frac{m-1}{m+l-2}, \\ \sup(|\nabla u^{\kappa+1}|, Q'_{\tau,T}) \leqslant c, \quad \kappa + 1 = \frac{1}{\lambda} \end{gathered} \tag{8.6}$$

because in the case when $\lambda \in (0, \lambda_*]$ and $m + l > 2$ we have $\frac{1}{\lambda} \geqslant \frac{m+l-2}{m} + \frac{1}{m\lambda}$. From (8.6) it follows that exponent $\kappa + 1$ would be as the best possible if Hölder exponent λ in (8.6) is optimal, i.e., $\lambda = \lambda_*$. However to the moment obtaining local Hölder estimates with optimal exponent $\lambda = \lambda_*$ is an open question. Recall that in one-dimensional case ($n = 1$) for equations of the type of slow diffusion (see Remark 3.1) local Hölder estimates

$$\langle u \rangle_{\lambda_*, \lambda_*/(\lambda_*+1), Q'_{\tau,T}} := \\ = \sup_{(x,t),(x',t')\in Q'_{\tau,T}} \frac{|u(x,t) - u(x',t)|}{(|x-x'| + |t-t'|^{1/(\lambda_*+1)})^{\lambda_*}} \leqslant K \quad (8.7)$$

can be derived from results established in papers [17–19]. From here and (8.3) it follows that the estimates (8.7) are optimal only with respect to variable x.

In [9–11] Hölder estimates (3.24) and (3.26) are established for general doubly nonlinear parabolic Eqs. (3.22), (3.23) in the case of arbitrary dimension $n \geqslant 1$. But here one merely guarantees that these estimates hold with some $\lambda \in (0,1)$ depending only on the data.

II. Boundary regularity for non-negative solutions.

Consider the following counteraxample ([5]).

Let $n = 1$, $\Omega = (0,1)$. Function

$$u = x^{\frac{1}{\sigma+1}},\ \sigma = \frac{1}{m-l},\ m > 1,\ l > 0$$

is a Hölder continuous up to the boundary weak solution of Cauchy–Dirichlet problem (see (1.11))

$$\begin{gathered} F^0[u] = 0 \text{ in } (0,1)\times(0,T], \\ u(0,T) = 0,\ u(1,T) = 1,\ u(x,0) = x^{\frac{1}{\sigma+1}}. \end{gathered} \quad (8.8)$$

Note that the initial function $\phi = x^{\frac{1}{\sigma+1}}$ belongs to the class $C_{\frac{1}{\sigma+1}}([0,1])$. Obviously we have for some $\alpha \in \mathbb{R}$

$$u_x = \frac{1}{\sigma+1}\, x^{\frac{1}{\sigma+1}-1},\ u^\alpha u_x = \frac{1}{\sigma+1}\, x^{\frac{\alpha+1}{\sigma+1}-1},\ \sigma = \frac{l}{m-l}. \quad (8.9)$$

Then from (8.9) it follows that the optimal boundary exponent is defined by conditions

$$\begin{aligned} &\sup(|u^{\alpha_*})_x|, S_T) < +\infty \text{ for } \alpha_* = \sigma + 1 \\ &\sup(|(u^{\alpha})_x|, S_T) = +\infty \text{ for any } \alpha < \alpha_* \end{aligned} \quad (8.10)$$

where $S_T = (\{0\} \times (0,T]) \cup (\{1\} \times (0,T])$.

So estimate (7.39) in sect. 7 is proved with the best possible boundary exponent $\alpha_* = \sigma + 1$. In particular in the case $m = 2$ estimate (7.39) is proved with the optimal exponent $\alpha_* = l + 1$.

III. Regularity of solutions with no sign restriction.

There is a difference between regularity for non-negative solutions and those with no sign restriction. Consider the following counterexample ([5]).

Let $n \geqslant 1$, $\Omega = \{|x| < 1\}$, $x = (x_1, \dots, x_n)$. Function

$$u = |x_1|^{\frac{1}{\sigma+1}-1} x_1, \quad \sigma = \frac{l}{m-l}, \quad m > 1, \quad l > 0 \tag{8.11}$$

is a Hölder continuous up to the boundary weak solution of Cauchy–Dirichlet problem (see (1.11))

$$F^0[u] = 0 \text{ in } Q_T, \quad u = \varphi \text{ on } \Gamma_T \tag{8.12}$$

where $\varphi = |x_1|^{\frac{1}{\sigma+1}-1} x_1$. In particular $\varphi \in C_\beta(\overline{Q_T})$ with $\beta = \frac{1}{\sigma+1} \in (0,1)$. The term "weak solution" means here that all conditions of Definition 3.2 with the exception of non-negativity of such solution are fulfilled for function u under consideration. In particular we have $|u|^\sigma u = x_1$. Obviously also that for some $\alpha \in R$ we have

$$u_{x_1} = \frac{1}{\sigma+1} |x_1|^{\frac{1}{\sigma+1}-1}, \quad |u|^\sigma u_{x_1} = \frac{1}{\sigma+1} |x_1|^{\frac{\alpha+1}{\sigma+1}-1}, \quad \sigma = \frac{l}{m-1}. \tag{8.13}$$

From (8.13) it follows that for any cylinder $Q'_{\tau,T} = \Omega' \times (\tau, T], \bar{\Omega}' \subset \Omega$, $\tau \in (0,T)$, we have

$$\begin{aligned} &\sup(|\nabla(|u|^{\alpha_*-1}u)|, Q'_{\tau,T}) < +\infty \text{ for } \alpha_* = \sigma + 1 \\ &\sup(|\nabla(|u|^{\alpha-1}u)|, Q'_{\tau,T}) = +\infty \text{ for any } \alpha < \alpha_*. \end{aligned} \tag{8.14}$$

Those in the case of solutions with no sign restriction we have another optimal exponent α_* than the best possible exponent γ^* (see (8.4)) in the case of non-negative solutions. It is easy to see that $\alpha_* > \gamma^*$ for any $m > 1$, $l > 0$. Really

$$\alpha_* = \frac{m+l-1}{m-1} > \frac{m+l-2}{m-1} = \gamma_*. \tag{8.15}$$

Because for any $m > 1$, $l > 0$ we have $\alpha_* > 1$ from (8.15) it follows that for solution (8.11) of problem (8.12) in the case $m > 1$, $l > 0$ we have

$$\sup(|\nabla u|, Q'_{\tau,T}) = +\infty.$$

So Theorem 7.1 is the best possible in the sence that local boundedness of gradient $|\nabla u|$ fails to be true for solutions u with no sign restriction of couse it is necessary to extend Theorem 7.1 to the case of non-negative weak solutions of Eqs. (1.3), (1.5) with any $l \in [0,1)$.

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Ivanov A. V. and Jäger W. Existence and uniqueness of a regular solution of Cauchy–Dirichlet problem for equation of turbulent filtration.

Existence and uniqueness of a regular solutions of Cauchy–Dirichlet problem for doubly nonlinear parabolic equations are proved as well as inner and boundary weight gradient estimates for these solutions are established. The equation of turbulent filtration is admissible for results obtained.

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