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THE REGULARITY THEORY FOR (M, L)-LAPLACIAN PARABOLIC EQUATION

Посвящается памяти
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1. Introduction

Consider (m, l)-Laplacian parabolic equation

$$\mathcal{F}[u] := u_t - \operatorname{div}\{|u|^l |\nabla u|^{m-2} \nabla u\} = 0, \quad m > 1,\ l > 1 - m \tag{1.1}$$

where $\nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right)$ is a spatial gradient of solution u. Similar equations arise in the study of turbulent filtration of a gas or a fluid through porous media, in the theoretical glassiology, in the theory of non-Newtonians fluids.

Equation (1.1) is a main model for, the so-called, doubly nonlinear parabolic equations (DNPE). Existence of generalized solution of Cauchy–Dirichlet problem for equation (1.1) was proved at first by Raviart and J.-L. Lions in the end of 60-s. The regularity for DNPE has been investigated only recently.

The aim of this paper is to present the latter theory for DNPE of arbitrary dimension ($n \geqslant 1$).

We say that (1.1) (and similar DNPE) is an equation of the type of slow, normal, and fast diffusion if $m + l > 2$, $m + l = 2$, and $m + l < 2$ respectively.

In this paper we always consider DNPE in cylinder $Q_T = \Omega \times (0, T)$ where Ω is a bounded open set in $\mathbb{R}^n$, $n \geqslant 1$.

Denote the full set of admissible parameters m, l as

$$\mathcal{D} := \{m > 1, l > 1 - m\} \subset \mathbb{R}^2.$$

Obviously

$$\mathcal{D} = S \cup N \cup \mathcal{F}$$

where S, N and $\mathcal{F}$ are subsets of $\mathcal{D}$ corresponding to the cases of slow, normal, and fast diffusion equations respectively.

In the one-dimensional case ($n = 1$) some regularity results for generalized solutions of Cauchy and Cauchy–Dirichlet problems for equation

$$u_t = \frac{d}{dx}(|u|^l |u_x|^{m-2} u_x), \quad m + l > 2,$$

were obtained in papers [1, 2] (see also references in these papers).

The regularity results for DNPE of arbitrary dimension presented in this paper are based mainly on the results of papers [3–13]. Note that L_∞ and Hölder estimates established in the latter papers can be considered as appropriate development of well-known results for quasilinear parabolic and degenerate parabolic equations obtained by Ladyzhenskaya–Ural'tseva (see [14]) and DiBenedetto and DiBenedetto–Chen (see [15]).

In the end of given paper we give also a proof of the local L_∞-estimate for generalized solutions of equation (1.1) under the exact condition on parameters m, l which was discovered at first in [3].

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2. L_∞-ESTIMATES

Definition 2.1. *Function u is a generalized solution of equation* (1.1) *if*

a) $u \geqslant 0$ *in* Q_T, $u \in C([0,T]; L_{\sigma+2}(\Omega))$, $\nabla u^{\sigma+1} \in L_m(Q_T; \mathbb{R}^n)$, $\sigma = \frac{l}{m-1}$;

b) u *satisfies* (1.1) *in the distributional sense, i.e.,*

$$\int_\Omega u\Phi\, dx\Big|_{t_1}^{t_2} + \iint_{Q_{t_1,t_2}} (-u\Phi_t + |\nabla u^{\sigma+1}|^{m-2} \nabla u^{\sigma+1} \cdot \nabla\Phi) dx\, dt = 0$$

$$\text{for any} \quad \Phi \in \overset{\circ}{W}{}^1_m(Q_T) \cap L_\infty(Q_T)$$

where $Q_{t_1,t_2} = \Omega \times [t_1, t_2]$.

The problem of establishing of local L_∞-estimates for generalized solutions of equation (1.1) discovers (see [3]) the patological set ω which can be defined by

$$\mathcal{D} \setminus \omega := \{(m,l) \in \mathcal{D} : \overset{\circ}{W}{}^1_m(\Omega) \quad \text{is compactly imbedded in} \quad L_{\frac{\sigma+2}{\sigma+1}}(\Omega)\},$$

where $\sigma = \frac{l}{m-1}$.

It is obvious that

1) the analytical version of definition of set ω is

$$\omega = \left\{(m,l) \in \mathcal{D} : \frac{\sigma+1}{\sigma+2} \leqslant \frac{1}{m} - \frac{1}{n},\ \sigma = \frac{l}{m-1}\right\};$$

2) $\omega \subset \mathcal{F}$;

3) $\omega = \varnothing$ in the case $n = 1$.

Theorem 2.1. *If $(m,l) \in \mathcal{D} \setminus \omega$ then any generalized solution u of equation (1.1) is locally bounded; moreover for any $Q' = \Omega' \times [\varepsilon, T]$, $\overline{Q'} \subset \Omega$, $\varepsilon > 0$*

$$\sup(u, Q') \leqslant c \tag{2.1}$$

with some constant c depending only on n, m, l, $\operatorname{dist}(\Omega', \partial\Omega)$, ε, $\sup\limits_{t\in[0,T]} \int\limits_{\Omega} u^{\sigma+2} dx$ *and* $\iint\limits_{Q_T} |\nabla u^{\sigma+1}|^m \,dx\,dt$ *where* $\sigma = \frac{l}{m-1}$.

If $(m,l) \in \omega$ then estimate (2.1) does not hold.

Theorem 2.1 was stated in [3] where also impossibility of estimate (2.1) in the case $(m,l) \in \omega$ was proved. However instead of a proof of local L_∞-estimate in the case $(m,l) \in D \setminus \omega$ we prefered to prove in [3] the global version of maximum modulus estimate for some regularisations of DNPE including Eq. (1.1) with $(m,l) \in \mathcal{D} \setminus \omega$. We shall show in Sect. 9 that the proof of the local L_∞-estimate in the case $(m,l) \in \mathcal{D} \setminus \omega$ can be presented as appropriate version of arguments given in [3].

Remark 2.1. The main meaning of Theorem 2.1 is that the local L_∞-estimate for solutions of equation (1.1) is established for the best possible condition on parameters m, l. Local estimate of maximum modulus of solutions of equation (1.1) for other (not best possible) condition on m, l was obtained in [16].

Remark 2.2. A question about discovery of the exact condition on m, l guaranteeing the global estimate

$$\sup(u, Q_T) \leqslant c, c = c(\sup(u, \Gamma_T)), \tag{2.2}$$

where Γ_T is a parabolic boundary of Q_T, is more complicated. For example, we proved (see [4]) existence of a bounded in Q_T generalized solution of Cauchy–Dirichlet problem

$$\mathcal{F}[u] = 0 \quad \text{in} \quad Q_T, \quad u = \varphi \quad \text{on} \quad \Gamma_T \tag{2.3}$$

for equation (1.1) with any $m > 1$, $l \geqslant 0$ if initial-boundary function φ and boundary $\partial\Omega$ are sufficient smooth. Probably condition

$$(m,l) \in \mathcal{D} \setminus \omega = \left\{ \mathcal{D} : \frac{\sigma+1}{\sigma+2} > \frac{1}{m} - \frac{1}{n},\ \sigma = \frac{l}{m-1} \right\} \tag{2.4}$$

is sharp for establishing estimate (2.2) for solutions of DNPE of "essentially non-homogeneous structure." In paper [3] we proved estimate (2.2) for some classes of DNPE satisfying condition (2.4).

3. Hölder estimates

Definition 3.1. *We say that generalized solution u of equation* (1.1) *is a weak solution of* (1.1) *if u is bounded in Q_T. In other words function u is a weak solution of equation* (1.1) *if*

a) $u \geqslant 0$ *in* Q_T, $u \in L_\infty(Q_T) \cap C([0,T]; L_2(\Omega))$, $\nabla u^{\sigma+1} \in L_m(Q_T; \mathbb{R}^n)$, $\sigma = \frac{l}{m-1}$;

b) u *satisfies* (1.1) *in the distributional sense.*

From Hölder estimates established in [5–8] we can derive in particular the following

Theorem 3.1. *Let $m > 1$, $l \geqslant 0$. Then any weak solution u of equation* (1.1) *is locally Hölder continuous in Q_T. Moreover for any $Q' = \Omega' \times (\varepsilon, T]$, $\overline{\Omega'} \subset \Omega$, $\varepsilon > 0$, we have*

$$\langle u \rangle_{\lambda, Q'} := \sup_{(x,t),(x',t') \in Q'} \frac{|u(x,t) - u(x',t')|}{(|x-x'| + |t-t'|)^\lambda} \leqslant K' \tag{3.1}$$

for some $\lambda \in (0,1)$, $\lambda = \lambda(n,m,l)$, and $K' = K'(n,m,l,\operatorname{dist}(\Omega',\partial\Omega),\varepsilon$, $\sup(u,Q_T))$.

Remark 3.1. If the trace of weak solution u of equation (1.1) on the parabolic boundary Γ_T is Hölder continuous and the boundary $\partial\Omega$ satisfies the property of positive geometrical density

$(\mathcal{PGD})$ there exist $\alpha_* \in (0,1)$ and $\rho_* > 0$ such that for any $x_0 \in \partial\Omega$ and any $B_\rho(x_0)$, $\rho \leqslant \rho_*$, $|\Omega \cap B_\rho(x_0)| \leqslant (1-\alpha_*)|B_\rho(x_0)|$,

then the global version of Hölder estimate

$$\langle u \rangle_{\lambda, \overline{Q}_T} \leqslant K$$

holds with some $\lambda \in (0,1)$ and K depending only on the data of problem (2.3).

Remark 3.2. The statement of Theorem 3.1 is unify for all types of equation (1.1) but the proofs of Hölder estimates (3.1) given in [5–8] depend on a type of this equation. Similar Hölder estimates are established in [5–8] for general classes DNPE.

Remark 3.3. The proof of Hölder estimates given in [5–8] use the leading idea of papers by Ladyzhenskaya–Uraltseva (see [14]) and Di Benedetto and Di Benedetto–Chen (see [15]).

Remark 3.4. Other results concerning Hölder estimates for DNPE were obtained in papers [17, 18].

Consider the following counterexample. Function

$$u(x,t) = (t+\varepsilon)^{-\alpha n}\left[1 - c\left(\frac{|x|}{(t+\varepsilon)^{\alpha}}\right)^{\gamma}\right]_+^{\delta}, \quad x=(x_1,\dots,x_n),\ \varepsilon>0, \tag{3.2}$$

is a weak solution of equation (1.1) in Q_T if

$$\alpha^{-1} = n(m+l-2)+m, \quad \gamma = \frac{m}{m-1}, \quad \delta = \frac{m-1}{m+l-2},\ m>1,\ m+l>2, \tag{3.3}$$

and c is an appropriate constant. Assume that $0 \in \Omega$ and $\varepsilon > 0$ is sufficient small. It is easy to see that gradient ∇u of function (3.2), (3.3) is unbounded if $l > 1$ because in this case $\delta - 1 = \frac{1-l}{m+l-2} < 0$. Moreover in the case $l > 1$ we have

$$\left.\begin{array}{l} u \in C_{\lambda_*}(Q_T) \text{ with } \lambda_* = \dfrac{m-1}{m+l-2} \in (0,1), \\ u \notin C_{\lambda}(Q_T) \text{ for any } \lambda > \lambda_*. \end{array}\right\} \tag{3.4}$$

So holderness is the best possible smoothness of weak solutions of equation (1.1) which can be established for the full range of parameters $(m,l) \in \mathcal{D}$. On the other hand because in our counterexample $\lambda_* \in (0,1)$ only in the case $l > 1$ the following question arises: can we prove some higher regularity for weak solutions of equation (1.1) in the case $l \leqslant 1$?

We give some answer to this question in Sec. 5. The next sect. is devoted to establishing existence and uniqueness of a regular solution of Cauchy–Dirichlet problem (2.3).

4. Existence and uniqueness of a regular solution of Cauchy–Dirichlet problem

Consider Cauchy–Dirichlet problem (2.3) with initial-boundary function φ satisfying condition

$$\varphi \in W_1^{1,0}(Q_T) \cap C([0,T]; L_1(\Omega)), \quad \varphi \geqslant 0 \quad \text{in} \quad Q_T. \tag{4.1}$$

Definition 4.1. *Function u is a weak solution of Cauchy–Dirichlet problem* (2.3) *if u is a weak solution of equation* (1.1) *and $u = \varphi$ on Γ_T.*

Definition 4.2. *Weak solution u of Cauchy–Dirichlet problem* (2.3) *is a maximal weak solution of this problem if for any weak solution v of the same problem we have*

$$u \geqslant v \text{ a.e. in } Q_T.$$

Definition 4.3. *Weak solution u of Cauchy–Dirichlet problem* (2.3) *is a regular solution of this problem if there exists a sequence $\{u_n\}$ of weak solutions $u_n \in C_\alpha(\overline{Q}_T)$ (with some $\alpha \in (0,1)$) of problems*

$$\mathcal{F}[u_n] = 0 \text{ in } Q_T, \quad u_n = \varphi_n \text{ on } \Gamma_T$$

such that

$$\varphi_n \geqslant \max(\varphi, \varepsilon_n), \quad \varphi_n \in C_\alpha(\overline{Q}_T), \quad \varphi_n \to \varphi \text{ in } C_\alpha(\overline{Q}_T),$$
$$\inf(u_n, Q_T) \geqslant \varepsilon_n = \text{const} > 0, \quad \varepsilon_n \to 0, \quad u_n \to u \quad \text{in} \quad C_\alpha(\overline{Q}_T).$$

In particular $u \in C_\alpha(\overline{Q}_T)$.

Proposition 4.1. *Let $m > 1$, $l \geqslant 0$. Then any regular solution of Cauchy–Dirichlet problem* (2.3) *is a maximal weak solution of this problem.*

The proof of Proposition 4.1 is given in [9] (see Proposition 3.4 in [9]). Actually it follows from the proof of Proposition 4.1 in paper [10].

Corollary 4.1. *There is at most one regular solution of Cauchy–Dirichlet problem* (2.3).

Assume that Ω satisfies condition

$$(\widetilde{PGD}) \quad \begin{cases} \text{there is a sequence of open sets } \Omega^{(k)} \text{ such that} \\ \Omega = \bigcup\limits_{k=1}^{\infty} \Omega^{(k)}, \ \Omega^{(k)} \subset \Omega^{(k+1)}, \ \operatorname{dist}(\Omega^{(k)}, \partial\Omega^{(k+1)}) > 0, \\ \Omega^{(k)} \in C^3, \ k = 1, 2, \dots; \text{ moreover sets } \Omega^{(k)} \text{ satisfy} \\ \text{condition (PGD) uniformly in } k = 1, 2, \dots \end{cases}$$

and the initial boundary function φ satisfies the following assumption

$$(BI)\quad \begin{cases} \varphi \in C_\beta(\overline{Q}_T) & \text{for some } \beta \in (0,1),\ \varphi \geqslant 0 \text{ in } Q_T \\ \varphi = \psi \text{ on } S_T = \Omega \times (0,T] & \text{for some } \psi \in W_m^{1,0}(Q_T) \\ & \text{such that } \psi_t \in L_1(Q_T). \end{cases}$$

Theorem 4.1. *Assume that $(\widetilde{PGD})$ and (BI) hold. Then there is exactly one regular solution u of Cauchy–Dirichlet problem* (2.3) *for equation* (1.1) *in the case $m > 1$, $l \geqslant 0$.*

The result stated by Theorem 4.1 follows easily from the proof of Theorem 2.1 in [9] and Theorem 1.1 in [10].

5. Local gradient estimates

In this sect. we show that the spatial gradient of a regular solution of Cauchy–Dirichlet problem (2.3) is locally bounded in Q_T if $0 \leqslant l < l_*$ with some $\lambda_* \in (0,1)$ (see Sec. 3).

Theorem 5.1. *Let u be a regular solution of Cauchy–Dirichlet problem* (2.3) *(see Definition* 4.3*) in the case $0 \leqslant l < l_*$, $l_* := \frac{2\min(1,m-1)}{2\min(1,m-1)+\frac{n}{4}}$. Then $\nabla u \in L_\infty^{\mathrm{loc}}(Q_T)$ and, moreover, for any cylinder $Q'_{\varepsilon,T} = \Omega' \times (\varepsilon, T]$, $\bar{\Omega} \subset \Omega$, $\varepsilon \in (0,T)$, we have*

$$\sup(|\nabla u|, Q'_{\varepsilon,T}) \leqslant M'_1 \tag{5.1}$$

where constant M'_1 depends only on the data of problem (2.3)*, dist $(\Omega', \partial\Omega)$ and ε.*

The proof of Theorem 5.1 is given in [4], but in [4] we did some error in writing that leaded to the wrong conclusion that (5.1) holds for any $l \in [0,1)$. Unfortunately the proof stated in [4] gives (5.1) only for $l \in [0, l_*)$ with l_* defined above.

Remark 5.1. Theorem 5.1 is sharp in the following aspects:

a) extension of estimate (5.1) to the cylinders $Q_{\varepsilon,T} = \Omega \times (\varepsilon, T]$ is impossible;

b) estimate (5.1) does not hold for solutions of (2.3) with no sign restriction (i.e., assumption of non-negative of initial-boundary function φ is essential).

We show in this sect. the validity only of assertion b) because a counterexample concerning assertion a) will be given in the next sect.

Let $\Omega = \{|x| < 1\}$, $x = (x_1, \dots, x_n)$, $n \geqslant 1$. Function

$$u = |x_1|^{\frac{1}{\sigma+1}-1} x_1, \quad \sigma = \frac{l}{m-1}, \ \ m > 1, \ \ l > 0 \tag{5.2}$$

is a regular solution of Cauchy–Dirichlet problem (2.5) with $\varphi = |x_1|^{\frac{1}{\sigma+1}-1} x_1$. Note that such initial-boundary function satisfies all conditions in (BI) with the exception of assumption of non-negativity of this function. Of course, the term "regular solution of problem (2.3)" means here that all conditions of Definition 4.3 with the exception of nonnegativity of such solution are fulfilled for function (5.2).

In particular, because $|u|^\sigma u = x_1$ we have $\nabla(|u|^\sigma u) \in L_\infty(Q_T; \mathbb{R}^n) \subset L_m(Q_T; \mathbb{R}^n)$. Moreover we have for some fixed $\alpha \in \mathbb{R}$

$$\frac{\partial u}{\partial x_1} = \frac{1}{\sigma+1} |x_1|^{\frac{1}{\sigma+1}-1}, \quad |u|^\alpha \frac{\partial u}{\partial x_1} = \frac{1}{\sigma+1} |x_1|^{\frac{\alpha+1}{\sigma+1}-1}, \ \sigma = \frac{l}{m-1}. \tag{5.3}$$

From (5.3) it follows that for any cylinder $Q'_{\varepsilon,T} = \Omega' \times (\varepsilon, T]$, $\bar{\Omega}' \subset \Omega$, $\varepsilon > 0$, we have

$$\left.\begin{aligned} &\sup(|\nabla(|u|^\sigma u)|, Q'_{\varepsilon,T}) < +\infty, \\ &\sup(|\nabla(|u|^\alpha u)| Q'_{\varepsilon,T}) = +\infty \text{ for any } \alpha < \sigma. \end{aligned}\right\} \tag{5.4}$$

In particular we can take $\alpha = 0$ in the second condition in (5.4) because $0 < \sigma = \frac{l}{m-1}$ for any $m > 1$, $l > 0$, i.e., estimate (5.1) is impossible for function (5.2).

Remark 5.2. Recall that from counterexample given in Sec. 2 it follows that estimate (5.1) fails to be true in the case $l > 1$. Moreover from the same counterexample we can derive that for any $m > 1$, $m + l > 2$ we have

$$\left.\begin{aligned} &\sup(|\nabla u^{\varkappa_*+1}|, Q_T) < +\infty \text{ for } \varkappa_* = \frac{l-1}{m-1}, \\ &\sup(|\nabla u^{\varkappa+1}|, Q_T) = +\infty \text{ for any } \varkappa < \varkappa_*. \end{aligned}\right\} \tag{5.5}$$

In particular if $l > 1$ we have (5.5) with $\varkappa_* > 0$; more precisely $\varkappa_* + 1 = \frac{1}{\lambda_*}$ where $\lambda_* = \frac{m-1}{m+l-2}$ (see Sec. 2).

Conclusions (5.5) stimulate to obtain the weight gradient estimates for solutions of equation (1.1).

6. Local weight gradient estimates

Theorem 6.1. *Let u be a regular solution of Cauchy–Dirichlet problem (2.3) in the case $m > 1$, $l \geqslant 0$. Let*

$$\varkappa = \varkappa_\lambda; \quad \varkappa_\lambda + 1 = \max\left(\frac{1}{\lambda}, \frac{m+l-2}{m} + \frac{1}{m\lambda}\right) \tag{6.1}$$

and $\lambda \in (0,1)$ be the Hölder exponent in estimate (3.1). Then $\nabla u^{\varkappa+1}$ is locally bounded in Q_T and moreover for any $Q'_{\varepsilon,T} = \Omega' \times (\varepsilon, T)$, $\bar{\Omega}' \subset \Omega$, $\varepsilon \in (0,T)$, we have

$$\sup(|\nabla u^{\varkappa+1}|, Q'_{\varepsilon,T}) \leqslant M'_1 \tag{6.2}$$

where M'_1 depends only on the data of problem (2.3), dist $(\Omega', \partial\Omega)$, and ε.

The proof of Theorem 6.1 is given in [4]. In the case $m \geqslant 2$ the local weight gradient estimate with another exponent $\varkappa$ was derived in [11] from results of [12] and [5].

Remark 6.1. If $l > 1$ we have $\varkappa_\lambda + 1 = \frac{1}{\lambda}$ and hence from (5.2), (6.1), (6.2) it follows that estimate (6.2) would be optimal if local Hölder estimate (3.1) is proved with optimal exponent $\lambda = \lambda_*$. However to the moment obtaining local Hölder estimate with the optimal exponent is an open question for equation (1.1) of arbitrary dimension $n > 1$.

Remark 6.2. There is a big difference between local and boundary regularity of non-negative solutions of Cauchy–Dirichlet problem (2.3). Really, consider the following counterexample.

Let $n = 1$, $\Omega = (0,1)$. Function

$$u = x^{\frac{1}{\sigma+1}}, \quad \sigma = \frac{l}{m-1}, \quad m > 1, \quad l > 0 \tag{6.3}$$

is a regular solution of Cauchy–Dirichlet problem

$$\left.\begin{aligned} &\mathcal{F}[u] = 0 \quad \text{in} \quad (0,1) \times (0,T], \\ &u(0,t) = 0, \quad u(1,t) = 1, \quad u(x,0) = x^{\frac{1}{\sigma+1}}. \end{aligned}\right\} \tag{6.4}$$

Note that initial-boundary function in problem (6.4) satisfies all conditions (BI). Obviously for some $\alpha \in \mathbb{R}$ we have

$$u_x = \frac{1}{\sigma+1} x^{\frac{1}{\sigma+1}-1}, \quad u^\alpha u_x = \frac{1}{\sigma+1} x^{\frac{\alpha+1}{\sigma+1}-1}, \quad \sigma = \frac{l}{m-1}. \tag{6.5}$$

Then from (6.3)–(6.5) it follows that

$$\left.\begin{aligned} &\sup(|(u^{\sigma+1})x|, S_T) < +\infty \\ &\sup(|(u^{\alpha+1})x|, S_T) = +\infty \text{ for any } \alpha < \sigma, \end{aligned}\right\} \tag{6.6}$$

where $S_T = (\{0\} \times (0,T]) \cup (\{1\} \times (0,T])$.

Comparing (5.5) and (6.6) we can conclude that optimal exponents in the local and boundary weight gradient estimates are equal to $\varkappa_* + 1$ and $\sigma + 1$ respectively. Moreover for any $m > 1$, $m + l > 2$ we have

$$\varkappa_* + 1 = 1 + \frac{l-1}{m-1} = \frac{m+l-2}{m-1} < \frac{m+l-1}{m-1} = \sigma + 1.$$

7. Boundary weight gradient estimates

Theorem 7.1 (A. V. Ivanov and W. Jäger)**.** *Let u be a regular solution of Cauchy–Dirichlet problem* (2.3) *in the case $m > 1$, $l \geqslant 0$. Assume that $\Omega \in C^2$ and $\varphi \in C^2(\overline{Q}_T)$. Then*

$$\sup_{\substack{x_0 \in \partial\Omega,\, x \in \Omega \\ t \in [0,T]}} \frac{|u^{\sigma+1}(x,t) - u^{\sigma+1}(x_0,t)|}{|x - x_0|} \leqslant \widetilde{M}_1, \quad \sigma = \frac{l}{m-1}, \tag{7.1}$$

where $\widetilde{M}_1$ depends only on the data of problem (2.3)*. Moreover let*

$$\gamma = \max(\sigma, \varkappa), \quad \sigma = \frac{l}{m-1}, \quad \varkappa = \varkappa_\lambda \text{ is defined by } (6.1). \tag{7.2}$$

Then $\nabla u^{\gamma+1}$ is bounded in Q_T and

$$\sup(|\nabla u^{\gamma+1}|, \overline{Q}_T) \leqslant M_1 \tag{7.3}$$

with some constant M_1 depending only on the data of problem (2.3)*.*

The proof of Theorem 7.1 is given in [13].

Remark 7.1. The counterexample given in sect. 6 shows that exponent $\sigma + 1$ in estimate (7.1) is the best possible.

Remark 7.2. In the case $l > 1$ the best possible value $\gamma_* + 1$ of exponent $\gamma + 1$ in the estimate (7.3), (7.2) is equal to $\sigma + 1$ because

$$\gamma_* = \max(\sigma, \varkappa_*) = \max\left(\frac{l}{m-1}, \frac{l-1}{m-1}\right) = \frac{l}{m-1} = \sigma.$$

8. Some extensions

All regularity results mentioned above are actually established for some classes of DNPE in the form

$$u_t - \operatorname{div} a(x,t,u,\nabla u) + b(x,t,u,\nabla u) = 0 \tag{8.1}$$

where functions $a(x,t,u,p)$ and $b(x,t,u,p)$ satisfy the Caratheodory condition and inequalities of the following type

$$\left.\begin{aligned}
&[a(x,t,u,p) - a(x,t,u,q)]\cdot(p-q) \geqslant 0,\\
&a(x,t,u,p)\cdot p \geqslant \nu|u|^l|p|^m - \phi_0(|u|),\\
&|a(x,t,u,p)| \leqslant \mu_1|u|^l|p|^{m-1} + \phi_1(|u|),\\
&|b(x,t,u,p)| \leqslant \phi_2(|u|,|p|),\ m>1,\ l>1-m,\\
&\phi_i \quad \text{are some nonnegative functions}\\
&\text{of their arguments.}
\end{aligned}\right\} \tag{8.2}$$

For example Hölder estimates for weak solutions (see Definition 3.1) of DNPE are proved in [6–8] for equations (8.1), (8.2) with $m>1$, $l\geqslant 0$ and

$$\phi_0(s) = \varphi_0; \quad \phi_1(s) = \varphi_1 s^{\alpha}, \quad \alpha = \frac{l}{m}; \quad \phi_2(s,r) = \mu_2 s^{lm'} + \varphi_2 \tag{8.3}$$

where φ_2 and μ_i are some nonnegative constants. Moreover these estimates are proved even in the case when φ_i in (8.3) are some nonnegative functions belonging to Lebesque classes $L_{q,q_0}(Q_T)$ with appropriate exponents.

Extensions of results given in sect. 2–7 can be found in papers [3–13].

9. The proof of local L_∞-estimates

Proposition 9.1. *Let function $g(s)$ satisfies the Lipschitz condition uniformly on $\mathbb{R}$ and its derivative $g'(s)$ is continuous everywhere on $\mathbb{R}$ with the possible exception of finitely many points at which $g'(s)$ has a continuity of the first kind. Let u be a generalized solution of equation* (1.1), $\widehat{u} = u^{\sigma+1}$, $\sigma = \frac{l}{m-1}$. *Note that the primitive*

$$G(u) = \int g(\widehat{u})\,du$$

satisfies inequality

$$|G(u)| \leqslant c(|u|^{\sigma+2} + 1), \quad u \in \mathbb{R}, \tag{9.1}$$

with some constant c. Then for any $\psi \in \overset{\circ}{W}{}^{1,1}_{\infty}(Q_T)$, *where* $\overset{\circ}{W}{}^{1,1}_{\infty}(Q_T) = \{v \in W^{1,1}_{\infty}(Q_T) : v = 0 \text{ on } S_T\}$, *and all* $t_1, t_2 \in [0,T]$ *we have*

$$\int\limits_{\Omega} G(u)\psi\, dx\Big|_{t_1}^{t_2} +$$

$$+\int\limits_{t_1}^{t_2}\int\limits_{\Omega}\{-G(u)\psi_t + |\nabla\widehat{u}|^{m-2}\nabla\widehat{u}\cdot[g'(\widehat{u})(\nabla\widehat{u})\psi + g(\widehat{u})\nabla\psi)]\}\, dx\, dt = 0. \tag{9.2}$$

Proposition 9.1 is some slight modification of Lemma 3.3 in [6] (see [6], p. 49).

Proposition 9.2. *Let* Ω *is a bounded Lipschitz open set in* $\mathbb{R}^n$ *and let function u satisfies conditions*

$$\left.\begin{aligned} &u \in L_\infty([0,T]; L_q(\Omega)),\ \nabla(|u|^s u) \in L_m(Q_T;\mathbb{R}^n),\ s = \left(q - \frac{\sigma+2}{\sigma+1}\right)/m,\\ &q > \frac{1}{\sigma+1}, \quad \sigma = \frac{l}{m-1}, \quad m > 1, \quad l > 1-m. \end{aligned}\right\} \tag{9.3}$$

Then

$$u \in L_\beta(Q_T), \quad \beta = \left(1 + \frac{m}{n}\right) q + m - \frac{\sigma+2}{\sigma+1}.$$

Moreover

$$\iint\limits_{Q_T} |u|^\beta\, dx\, dt \leqslant c \left(\iiint\limits_{Q_T} |\nabla(|u|^s u)|^m dx\, dt + \int\limits_0^T \left(\int\limits_\Omega |u|^{s+1} dx\right)^m dt\right) \times$$
$$\times \left(\sup_{t\in[0,T]} \int\limits_\Omega |u|^q\, dx\right)^{\frac{m}{n}} \tag{9.4}$$

where β *is defined by* (9.4) *and constant c depends only on* n, m, l *and* Ω. *If in addition* $u = 0$ *on* $S_T = \partial\Omega \times (0,T]$ *then for any bounded* Ω *we have*

$$\iint\limits_{Q_T} |u|^\beta\, dx\, dt \leqslant c \iint\limits_{Q_T} |\nabla(|u|^s u)|^m dx\, dt \left(\sup_{t\in[0,T]} \int\limits_\Omega |u|^q dx\right)^{\frac{m}{n}} \tag{9.5}$$

where β is the same as in (9.4) *while constant c depends only on n, m and l.*

The proof of Proposition 9.2 is given, for example, in [3], p. 88–90.

To prove estimate (2.1) of Theorem 2.1 it suffices to establish the following

Theorem 9.1. *Let*

$$Q_\rho := B_\rho(x_0) \times [t_0 - \rho^m, t_0], \quad \bar{Q}_\rho \subset Q_T.$$

Assume that u be a generalized solution of equation (1.1) *with $(m,l) \in D \setminus \omega$ (see* (2.4)*). Then $u \in L_\infty(Q_{\rho/2})$ and moreover*

$$\sup(\widehat{u}, Q_{\rho/2}) \leqslant c, \qquad \widehat{u} = u^{\sigma+1}, \quad \sigma = \frac{l}{m-1} \tag{9.6}$$

where constant c depends only on n, m, l, ρ^{-1} and $\|\widehat{u}\|_{L_\delta(Q_\rho)}$ with

$$\delta = \frac{\sigma+2}{\sigma+1} \quad \text{if} \quad m+l<1 \quad \text{and} \quad \delta = m \quad \text{if} \quad m+l \geqslant 2. \tag{9.7}$$

Proof. First of all we note that $\|u\|_{L_\delta(Q_\rho)}$ is finite for any generalized solution u of equation (1.1) with $(m,l) \in D \setminus \omega$. Really using Proposition 9.2 for function $\widehat{u} = u^{\sigma+1}$ with $q = \frac{\sigma+2}{\sigma+1} > 1$, $s = 0$ we derive from (9.4) inequality

$$\left.\begin{aligned} \iint\limits_{Q_\rho} \widehat{u}^{\beta_0}\, dx\, dt \leqslant c \left(\sup_{t \in [t_\delta - \rho^m, t_0]} \int\limits_{B_\rho} \widehat{u}^{\frac{\sigma+2}{\sigma+1}} dx, \ \iint\limits_{Q_\rho} |\nabla \widehat{u}|^m dx\, dt \right), \\ \beta_0 := m + \frac{m}{n}\frac{\sigma+2}{\sigma+1}. \end{aligned}\right\}$$

It is easy to see that in view of condition (2.4) we have

$$\beta_0 > \max\left(m, \frac{\sigma+2}{\sigma+1}\right).$$

In particular, from here it follows that for function $\widehat{u} = u^{\sigma+1}$, $\sigma = \frac{l}{m-1}$, where u is generalized solution of equation (1.1) with parameters (m,l) satisfying condition (2.4) we have

$$\widehat{u} \in L^{\mathrm{loc}}_{\frac{\sigma+2}{\sigma+1}}(Q_T) \cap L^{\mathrm{loc}}_m(Q_T). \tag{9.8}$$

Set for some $p \geqslant 2$

$$g(\widehat{u}) = \begin{cases} [(\sigma+1)(p-1)+1]\widehat{u}^{p-1} & \text{for} \quad \widehat{u} \leqslant N \\ [(\sigma+1)(p-1)+1]N^{p-2}\widehat{u} & \text{for} \quad \widehat{u} \geqslant N. \end{cases} \tag{9.9}$$

Then

$$g'(\widehat{u}) = \begin{cases} (p-1)[(\sigma+1)(p-1)+1]\widehat{u}^{p-2} & \text{for} \quad \widehat{u} < N \\ [(\sigma+1)(p-1)+1]N^{p-2} & \text{for} \quad \widehat{u} > N, \end{cases} \tag{9.10}$$

while primitive $G(u) = \int g(\widehat{u})du$ can be taken as

$$G(u) = \begin{cases} \widehat{u}^{p-1+\frac{1}{\sigma+1}} \text{ for } \widehat{u} \leqslant N \\ \frac{\sigma+1}{\sigma+2}[(\sigma+1)(p-1)+1]\left(N^{p-2}\widehat{u}^{\frac{\sigma+2}{\sigma+1}} - N^{p-1+\frac{1}{\sigma+1}}\right) + N^{p-1+\frac{1}{\sigma+1}} \\ \text{for } \widehat{u} \geqslant N. \end{cases} \tag{9.11}$$

Let $\zeta = \zeta(x,t)$ is a cutoff function in Q_ρ, i.e.,

$$\left.\begin{array}{l} \zeta \quad \text{vanishes on the parabolic boundary of} \quad Q_\rho, \\ \zeta = 1 \quad \text{in} \quad Q_{\rho/2},\ 0 \leqslant \zeta \leqslant 1 \quad \text{in} \quad Q_\rho,\ \zeta \in C^1(\bar{Q}_\rho). \end{array}\right\}$$

Because function (9.9) satisfies all conditions of Proposition 9.1 we derive from (9.2) inequality

$$\begin{aligned} &\sup_{t\in[t_0-\rho^m,t_0]} \int\limits_{B_\rho} G(u)\zeta^m dx + \iint\limits_{Q_\rho} g'(\widehat{u})|\nabla\widehat{u}|^m \zeta^m dx\,dt \leqslant \\ &\leqslant \iint\limits_{Q_\rho} [G(u)m\zeta^{m-1}\zeta_t + g(\widehat{u})|\nabla\widehat{u}|^{m-1}m\zeta^{m-1}|\nabla\zeta|]dx\,dt \end{aligned} \tag{9.12}$$

where $B_\rho = B_\rho(x_0)$, ζ is like above while $g(\widehat{u})$, $g'(\widehat{u})$, and $G(u)$ are defined by (9.9)–(9.11). It is easy to check that

$$g(\widehat{u})|\nabla\widehat{u}|^{m-1}\zeta^{m-1}|\nabla\zeta| \leqslant \frac{1}{2}g'(\widehat{u})|\nabla\widehat{u}|^m\zeta^m + cpg(\widehat{u})\widehat{u}^m|\nabla\zeta|^m \tag{9.13}$$

with some constant c depending only on m and l.

Assume that

$$\widehat{u}^{p-1+\frac{1}{\sigma+1}} \in L_1(Q_\rho), \quad \widehat{u}^{p-1+m} \in L_1(Q_\rho). \tag{9.14}$$

We denote

$$\widehat{u}_N = \max(\widehat{u}, N). \tag{9.15}$$

Taking into account that

$$\widehat{u}_N^{p-1+\frac{1}{\sigma+1}} \leqslant G(u) \leqslant cp\widehat{u}^{p-1+\frac{1}{\sigma+1}}, \quad c = c(m,l), \tag{9.16}$$

$$\begin{gathered}\iint\limits_{Q_\rho} g'(\widehat{u})|\nabla\widehat{u}|^m\zeta^m\,dx\,dt = \\ = \iint\limits_{Q_\rho} g'(\widehat{u}_N)|\nabla\widehat{u}_N|^m\zeta^m\,dx\,dt \geqslant \iint\limits_{Q_\rho}\widehat{u}_N^{p-2}|\nabla\widehat{u}_N|^m\zeta^m\,dx\,dt,\end{gathered} \tag{9.17}$$

$$g(\widehat{u}) \leqslant c\widehat{u}^{p-1}, \quad c = c(m,l), \tag{9.18}$$

we derive from (9.12)–(9.18) inequality

$$\begin{gathered}\sup_{t\in[t_0-\rho^n,t_0]}\int\limits_{B_\rho} u_N^{p-1+\frac{1}{\sigma+1}}\zeta^m\,dx + \iint\limits_{Q_\rho}\widehat{u}_N^{p-2}|\nabla\widehat{u}_N|^m\zeta^m\,dx\,dt \leqslant \\ \leqslant cp\iint\limits_{Q_\rho}[\widehat{u}^{p-1+\frac{1}{\sigma+1}}\zeta_t + \widehat{u}^{p-2+m}|\nabla\zeta|^m]dx\,dt.\end{gathered} \tag{9.19}$$

Moreover in view of (9.14) we can realize a passage to the limit on the left-hand side in (9.19) as $N\to\infty$.

Set

$$p = q + \frac{\sigma}{\sigma+1}, \quad q \geqslant \frac{\sigma+2}{\sigma+1}. \tag{9.20}$$

Then we have

$$p-1+\frac{1}{\sigma+1} = q, \quad p-2 = q-\frac{\sigma+2}{\sigma+1}, \quad p-2+m = q+m-\frac{\sigma+2}{\sigma+1}, \tag{9.21}$$

and from (9.19)–(9.21) it follows that for any $q \geqslant \frac{\sigma+2}{\sigma+1}$

$$\begin{gathered}\sup_{t\in[t_0-\rho^m,t_0]}\int\limits_{B_\rho}\widehat{u}^q\zeta^m dx + \iint\limits_{Q_\rho}\widehat{u}^{q-\frac{\sigma+2}{\sigma+1}}|\nabla\widehat{u}|^m\zeta^m\,dx\,dt \leqslant \\ \leqslant cq\iint\limits_{Q_\rho}[\widehat{u}^q\zeta_t + \widehat{u}^{q+\varkappa}|\nabla\zeta|^m]dx\,dt\end{gathered} \tag{9.22}$$

where

$$\varkappa := m - \frac{\sigma+2}{\sigma+1} = \frac{(m+l-2)(m-1)}{m+l-1}. \tag{9.23}$$

Moreover we have proved that existence of the integrals on the right-hand side in (9.22) implies that the integrals on the left-hand side in (9.22) also exist.

Obviously from (9.23) it follows that

$$\varkappa \geqslant 0 \quad \text{if} \quad m+l \geqslant 2, \qquad \varkappa < 0 \quad \text{if} \quad m+l<2. \tag{9.24}$$

In the remainder of this paper we consider the cases $m+l \geqslant 2$ and $m+l<2$ separately.

(I) Case $m+l \leqslant 2$.

In this case $q+\varkappa<q$. Obviously for all $q \geqslant \frac{\sigma+2}{\sigma+1}$ we have

$$v^{q+\varkappa} \leqslant c(q^c v^q + q^{-q}) \tag{9.25}$$

where constant c is independent of q. Then in the case under consideration inequalities (9.22) and (9.25) imply

$$\begin{aligned} \sup_{t\in[t_0-\rho^m,t_0]} \int_{B_\rho} \widehat{u}^q \zeta^m dx + \iint_{Q_\rho} \widehat{u}^{q-\frac{\sigma+2}{\sigma+1}} |\nabla \widehat{u}|^m \zeta^m dx\, dt \leqslant \\ \leqslant cq^c \iint_{Q_\rho} (\widehat{u}^q + q^{-q})(\zeta_t + |\nabla\zeta|^m dx\, dt \end{aligned} \tag{9.26}$$

if

$$\widehat{u}^q \in L_1(Q_\rho). \tag{9.27}$$

In view of (9.23), (9.24) we have

$$q \geqslant \frac{\sigma+2}{\sigma+1} = 1 + \frac{m-1}{m+l-1} > m. \tag{9.28}$$

Using (9.27), (9.28), and (9.25) and taking into account that $0 \leqslant \zeta \leqslant 1$ we have

$$\begin{aligned} \sup_{t\in[t_0-\rho^m,t_0]} \int_{B_\rho} (\widehat{u}\zeta)^q \zeta^m dx + \iint_{Q_\rho} (\widehat{u}\zeta)^{q-\frac{\sigma+2}{\sigma+1}} |\nabla (\widehat{u}\zeta)|^m dx\, dt \leqslant \\ \leqslant \sup_{t\in[t_0-\rho^m,t_0]} \int_{B_\rho} \widehat{u}^q \zeta^m dx + \iint_{Q_\rho} \widehat{u}^{q-\frac{\sigma+2}{\sigma+1}} |\nabla \widehat{u}|^m \zeta^m dx\, dt + \\ + cq^c \iint_{Q_\rho} (\widehat{u}^q + q^{-q}) |\nabla\zeta|^m dx\, dt. \end{aligned} \tag{9.29}$$

On the other hand from (9.26), (9.27) it follows that

$$(\widehat{u}\zeta)^q \in L_\infty([t_0-\rho^m,t_0],B_\rho), \quad (\widehat{u}\zeta)^{s+1} \in \overset{\circ}{W}{}^{1,0}_m(Q_\rho) \tag{9.30}$$

for $s = \left(q - \frac{\sigma+2}{\sigma+1}\right)/m$. Then applying Proposition 9.2 we can derive (see (9.5)) inequality

$$\begin{gathered}\iint\limits_{Q_\rho} (\widehat{u}\zeta)^\beta \, dx\, dt \leqslant \\ \leqslant c \iint\limits_{Q_\rho} (\widehat{u}\zeta)^{q-\frac{\sigma+2}{\sigma+1}} |\nabla(\widehat{u}\zeta)|^m \, dx\, dt \left(\sup_{t\in[t_0-\rho^m,t_0]} \int\limits_{B_\rho} (\widehat{u}\zeta)^q \, dx \right)^{\frac{m}{n}}\end{gathered} \tag{9.31}$$

where

$$\beta = \left(1+\frac{m}{n}\right) q + m - \frac{\sigma+2}{\sigma+1} = \left(1+\frac{m}{n}\right) q + \varkappa. \tag{9.32}$$

Using inequality

$$\begin{gathered}\left(\sup_{t\in[t_0-\rho^m,t_0]} \int\limits_{B_\rho} (\widehat{u}\zeta)^q \, dx \right)^{\frac{m}{n}} \iint\limits_{Q_\rho} (\widehat{u}\zeta)^{q-\frac{\sigma+2}{\sigma+1}} |\nabla(\widehat{u}\zeta)|^m \, dx\, dt \leqslant \\ \leqslant \left[\sup_{t\in[t_0-\rho^m,t_0]} \int\limits_{B_\rho} (\widehat{u}\zeta)^q \, dx + \iint\limits_{Q_\rho} (\widehat{u}\zeta)^{q-\frac{\sigma+2}{\sigma+1}} |\nabla(\widehat{u}\zeta)|^m \, dx\, dt \right]^{1+\frac{m}{n}}\end{gathered} \tag{9.33}$$

we derive from (9.26)–(9.33)

$$\iint\limits_{Q_\rho} (\widehat{u}\zeta)^{bq+\varkappa} \, dx\, dt \leqslant c q^c \left[\iint\limits_{Q_\rho} (\widehat{u}^q + q^{-q})(\zeta_t + |\nabla\zeta|^m) dx\, dt \right]^b \tag{9.34}$$

where $b := 1 + \frac{m}{n}$ and constant c is independent of $q \geqslant \frac{\sigma+2}{\sigma+1}$ and ρ. Using notation

$$\fint\!\!\fint\limits_{Q_\rho} f(x,t) dx\, dt = |Q_\rho|^{-1} \iint\limits_{Q_\rho} f(x,t) dx\, dt \tag{9.35}$$

we can write (9.34) as

$$\iint\limits_{Q_\rho} (\widehat{u}\zeta)^{bq+\varkappa} dx\, dt \leqslant cq^c \left[\iint\limits_{Q_\rho} (\widehat{u}^q + q^{-q}) h(\rho) dx\, dt \right]^b \tag{9.36}$$

where

$$h(\rho) := \rho^{-m} \sup(\zeta_t + |\nabla\zeta|^m, Q_\rho). \tag{9.37}$$

In the proof of (9.37) we have used in particular that $|Q_\rho| = c\rho^{m+n}$ with some constant $c = c(n)$.

Obviously for any $q \geqslant \frac{\sigma+2}{\sigma+1}$ we have

$$\frac{bq+\varkappa}{q} \geqslant b + \left(m - \frac{\sigma+2}{\sigma+1}\right)\frac{\sigma+1}{\sigma+2} = 1 + m\left[\frac{1}{n} + \frac{\sigma+1}{\sigma+2} - \frac{1}{m}\right] := k > 1 \tag{9.38}$$

because the expression in the square braces is strictly positive in view of condition (2.4) which can be written as

$$\frac{\sigma+1}{\sigma+2} > \frac{1}{m} - \frac{1}{n}, \quad \sigma = \frac{l}{m-1}, \quad m > 1, \quad l > 1-m.$$

In particular from (9.38) it follows that

$$bq + \varkappa \geqslant kq, \quad k > 1, \quad q \geqslant \frac{\sigma+2}{\sigma+1}. \tag{9.39}$$

Denote for any $p \geqslant 1$

$$\|v\|_{p,Q_\rho} = \left(\iint\limits_{Q_\rho} |v|^p dx\, dt \right)^{1/p}. \tag{9.40}$$

Then form (9.36), (9.39), (9.40) we derive

$$\begin{aligned} &\|\widehat{u}\zeta\|_{kq,Q_\rho}^{\frac{bq+\varkappa}{bq}} \leqslant \|\widehat{u}\zeta\|_{bq+\varkappa,Q_\rho}^{\frac{bq+\varkappa}{bq}} \leqslant \\ &\leqslant c^{1/q} h^{1/q}(\rho) q^{c/q} \left(\|\widehat{u}\zeta\|_{q,Q_\rho} + q^{-1}\right), \quad q \geqslant \frac{\sigma+2}{\sigma+1}. \end{aligned} \tag{9.41}$$

Set

$$\rho_n = \frac{\rho}{2} + \frac{\rho}{2^{n+1}}, \quad B_{(n)} = B_{\rho_n} = B_{\rho_n}(x_0), \quad Q_{(n)} = Q_{\rho_n}, \quad n = 0, 1, \ldots. \tag{9.42}$$

Obviously $Q_{\rho/2} \subset Q_{(n+1)} \subset Q_{(n)} \subset Q_{(0)} = Q_\rho$, $n = 0, 1, \ldots$.

Let the cutoff function $\zeta = \zeta_n$ is taken to satisfy

$$\left.\begin{array}{l} \zeta_n \quad \text{vanishes on the parabolic boundary of} \quad Q_\rho, \\ \zeta_n \equiv 1 \quad \text{in} \quad Q_{(n+1)},\ 0 \leqslant \zeta_n \leqslant 1 \quad \text{in} \quad Q_{(n)}, \\ |\nabla \zeta_n| \leqslant c_0 2^n \rho^{-1}, \quad 0 \leqslant (\zeta_n)_t \leqslant c_0 2^n \rho^{-m} \end{array}\right\} \tag{9.43}$$

with some absolute constant $c_0 > 1$.

We choose

$$q = q_n = \frac{\sigma+2}{\sigma+1}\, k^n, \quad n = 0, 1, \ldots \tag{9.44}$$

and denote

$$y_n = \|\widehat{u}\|_{q_n, Q_{(n)}}, \quad n = 0, 1, \ldots. \tag{9.45}$$

Obviously (see (9.37))

$$h(\rho_n) \leqslant c_0^m 2^{n(m+1)}, \quad n = 0, 1, \ldots. \tag{9.46}$$

In view of (9.41)–(9.46) we have

$$y_{n+1} \leqslant [c^{1/k^n} c^{n/k^n} (y_n + 1/k^n)]^{1 + \frac{|\varkappa|}{bq_n + \varkappa}}, \quad n = 0, 1, \ldots \tag{9.47}$$

where constant $c > 1$ is independent of number n. More precisely, (9.47) means that if y_0 has sense (i.e., integral $y_0 = \iint\limits_{Q_\rho} \widehat{u}^{q_0}\, dx\, dt$, $q_0 = \frac{\sigma+2}{\sigma+1}$, exists) then all y_n, $n = 1, 2, \ldots$, also exist and can be estimated as in (9.47). But existence of y_0 follows from the definition of generalized solution (see (9.8)). Therefore for proving estimate (9.6) it suffices to derive from (9.47) an uniform estimate for y_n, $n = 1, 2, \ldots$, in terms of y_0.

We note that from (9.38) it follows

$$0 < \frac{|\varkappa|}{bq_n + \varkappa} \leqslant \frac{|\varkappa|}{kq_n} \leqslant \frac{c}{k^n}, \quad q \geqslant \frac{\sigma+2}{\sigma+1}, \tag{9.48}$$

with some constant c depending only on n, m, and l.

Set

$$z_n = \max(y_n, e), \quad n = 0, 1, \ldots. \tag{9.49}$$

Taking into account that

$$e \leqslant [c^{1/k^n} c^{n/k^n} (z_n + 1/k^n)]^{1 + \frac{|\varkappa|}{bq_n + \varkappa}}, \quad n = 0, 1, \ldots, \tag{9.50}$$

we derive from (9.47)–(9.50) inequalities

$$z_{n+1} \leqslant c^{(n+1)/h^n}(z_n + c/k^n)^{1+c/k^n}, \quad n = 0, 1, \dots . \tag{9.51}$$

Using the Lagrange formula

$$z_n + c/k^n \leqslant z_n^{1+c/k^n} = z_n + z_n^{1+\theta(c/k^n)}(\ln z_n)c/k^n, \quad 0 < \theta < 1, \tag{9.52}$$

because $z^{1+\theta(c/k^n)} \geqslant 1$, $\ln z_n \geqslant 1$. Then we can derive from (9.50)–(9.52) inequalities

$$z_{n+1} \leqslant c^{(n+1)/k^n} z_n^{1+c/k^n}, \quad n = 0, 1, \dots , \tag{9.53}$$

taking into account that $(1+c/k^n)(1+c/k^n) \leqslant 1+c_1/k^n$ for appropriate constant c_1.

We prove that

$$\sup_{n\in\mathbb{N}} y_n \leqslant \sup_{n\in\mathbb{N}} z_n \leqslant c^{\sum\limits_{\nu=0}^{\infty}((\nu+1)/k^\nu)\prod\limits_{\nu=0}^{\infty}(1+c/k^\nu)} z_0^{\prod\limits_{\nu=0}^{\infty}(1+c/k^\nu)} \tag{9.54}$$

where the infinite product $\prod\limits_{\nu=0}^{\infty}(1 + c/k^\nu)$ and series $\sum\limits_{\nu=0}^{\infty}(\nu + 1)/k^\nu$ are obviously convergent.

To prove (9.54) it suffices to convice ourselves that

$$z_n \leqslant c^{\sum\limits_{\nu=0}^{n}((\nu+1)/k^\nu)\prod\limits_{\nu=0}^{n}(1+c/k^\nu)} z_0^{\prod\limits_{\nu=0}^{n}(1+c/k^\nu)}, \quad n = 1, 2, \dots . \tag{9.55}$$

Assertion (9.55) can be proved by mathematical induction. This proof can be found in [3], p. 95.

Taking into account that

$$\sup(\widehat{u}, Q_{\rho/2}) = \lim_{n\to\infty} y_n \tag{9.56}$$

we derive from (9.54) and (9.56) estimate (9.6) with some constant c depending on n, m, l, and $\rho^{-(m+n)} \iint\limits_{Q_\rho} \widehat{u}^{\frac{\sigma+2}{\sigma+1}dx\,dt}$. So, Theorem 9.1 is proved in the case $m + l < 2$.

(II) Case $m + l \geqslant 2$.

Because the patological set ω (see (2.4)) contains in the subset $\mathcal{F} \subset \mathcal{D}$ which is correspondent to equations of the type of fast diffusion, we can state that condition (2.4) plays no role in the case under consideration. Moreover using (9.23), (9.24) it is easy to see that in this case

$$\varkappa \geqslant 0, \quad q + \varkappa \geqslant q. \tag{9.57}$$

In view of (9.57) we have for all $q \geqslant \frac{\sigma+2}{\sigma+1}$

$$v^q \leqslant c(q^c v^{q+\varkappa} + q^{-q})$$

while from (9.22), (9.58) we can conclude that for all $q \geqslant \frac{\sigma+2}{\sigma+1}$

$$\begin{aligned} &\sup_{t\in[t_0-\rho^m,t_0]} \int_{B_\rho} \widehat{u}^q \zeta^m dx + \iint_{Q_\rho} \widehat{u}^{q-\frac{\sigma+2}{\sigma+1}} |\nabla \widehat{u}|^m \zeta^m dx\, dt \leqslant \\ &\leqslant cq^c \iint_{Q_\rho} (\widehat{u}^{q+\varkappa} + q^{-q})(\zeta_t + |\nabla\zeta|^m) dx\, dt \end{aligned} \tag{9.59}$$

assuming that $\widehat{u}^q \in L_1(Q_\rho)$. The constants c in (9.58) and (9.59) are independent of q.

Denote

$$\zeta_1 = \zeta^m \tag{9.60}$$

where ζ is defined above. Using (9.57), (9.58) we can obtain inequality

$$\begin{aligned} &\sup_{t\in[t_0-\rho^m,t_0]} \int_{B_\rho} (\widehat{u}\zeta_1)^q dx + \iint_{Q_\rho} (\widehat{u}\zeta_1)^{q-\frac{\sigma+2}{\sigma+1}} |\nabla(\widehat{u}\zeta_1)|^m dx\, dt \leqslant \\ &\leqslant \sup_{t\in[t_0-\rho^m,t_0]} \int_{B_\rho} \widehat{u}^q \zeta^m dx + \iint_{Q_\rho} \widehat{u}^{q-\frac{\sigma+2}{\sigma+1}} |\nabla \widehat{u}|^m \zeta^m dx\, dt + \\ &\quad + cq^c \iint_{Q_\rho} (\widehat{u}^{q+\varkappa} + q^{-q}) |\nabla\zeta|^m dx\, dt \end{aligned} \tag{9.61}$$

where ζ and ζ_1 are connected by (9.60). Then from (9.59) and (9.61) we derive that inclusions (9.30) hold for $s = \left(q - \frac{\sigma+2}{\sigma+1}\right)/m$, $q \geqslant \frac{\sigma+2}{\sigma+1}$, and applying Proposition 9.2 (see (9.5)) we obtain inequality

$$\begin{aligned} &\iint_{Q_\rho} (\widehat{u}\zeta_1)^\beta dx\, dt \leqslant \\ &\leqslant c \iint_{Q_\rho} (\widehat{u}\zeta_1)^{q-\frac{\sigma+2}{\sigma+1}} |\nabla(\widehat{u}\zeta_1)|^m dx\, dt \left(\sup_{t\in[t_0-\rho^m,t_0]} \int_{B_\rho} (\widehat{u}\zeta_1)^q dx \right)^{\frac{m}{n}} \end{aligned} \tag{9.62}$$

where $\beta = bq + \varkappa$, $b = 1 + \frac{m}{n}$, $\varkappa = m - \frac{\sigma+2}{\sigma+1} \geqslant 0$.

Taking into account that the right-hand side in (9.62) can be estimated from above by the left-hand side in (9.61) we derive from (9.59)–(9.62) inequality

$$\iint\limits_{Q_\rho} (\widehat{u}\zeta_1)^{bq+\varkappa} dx\,dt \leqslant cq^c \iint\limits_{Q_\rho} (\widehat{u}^{q+\varkappa} + q^{-q})(\zeta_t + |\nabla\zeta|^m) dx\,dt, \quad q \geqslant \frac{\sigma+2}{\sigma+1}, \tag{9.63}$$

where $b = 1 + \frac{m}{n}$ and constant c is independent of $q \geqslant \frac{\sigma+2}{\sigma+1}$ and ρ. Using notation (9.35) we can write (9.63) as

$$\iint\limits_{Q_\rho} (\widehat{u}\zeta_1)^{bq+\varkappa} dx\,dt \leqslant c^{1/q} h^{1/q}(\rho) q^{c/q} \left[\left(\iint\limits_{Q_\rho} \widehat{u}^{q+\varkappa} dx\,dt \right)^{1/q} + q^{-1} \right], \quad q \geqslant \frac{\sigma+2}{\sigma+1}, \tag{9.64}$$

where $h(\rho)$ is defined by (9.37).

Introduce ρ_n, $B_{(n)}$, $Q_{(n)}$, and ζ_n like in (9.42), (9.43) and denote

$$q_n = \frac{\sigma+2}{\sigma+1} b^n, \quad b = 1 + \frac{m}{n}, \quad n = 0, 1, \ldots \tag{9.65}$$

and

$$\alpha_n = \left(\iint\limits_{Q_{(n)}} \widehat{u}^{q_n+\varkappa} dx\,dt \right)^{1/q_n}, \quad n = 0, 1 \ldots. \tag{9.66}$$

Then from (9.65), (9.66) it follows that

$$\alpha_{n+1} \leqslant c^{(n+1)/b^n} (\alpha_n + 1/b^n), \quad n = 0, 1, \ldots. \tag{9.67}$$

Iterating (9.67) we obtain

$$\alpha_{n+1} \leqslant c^{\sum\limits_{\nu=0}^{\infty} (\nu+1)/b^n} \left(\alpha_0 + \frac{b}{b-1} \right), \quad n = 0, 1, \ldots. \tag{9.68}$$

Obviously

$$\sup(\widehat{u}, Q_{\rho/2}) = \lim_{n\to\infty} \alpha_n. \tag{9.69}$$

Then from (9.68), (9.69) we derive estimate

$$\sup(\widehat{u}, Q_{\rho/2}) \leqslant c \left[\left(\iint\limits_{Q_\rho} \widehat{u}^m \, dx \, dt \right)^{\frac{\sigma+1}{\sigma+2}} + 1 \right] \tag{9.70}$$

because $q_0 + \varkappa = \frac{\sigma+2}{\sigma+1} + \varkappa = m$ (see (9.23)). The constant c in (9.70) depends only on n, m, and l. Thus, estimate (9.6) is proved in the case $m + l \geqslant 2$. Theorem 9.1 is proved.

Remark 9.2. Obviously from Theorem 9.1 it follows that the first part of Theorem 2.1 is proved. The second part of Theorem 2.1 concerning the case $(m, l) \in \omega$ was proved in [3]. So, we can conclude that Theorem 2.1 is proved completely.

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