Opportunities for the Use of Computer Algebra Systems in Middle Secondary Mathematics in England and Wales

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Abstract: This paper presents and discusses a number of ways in which mathematical attainment targets specified for pupils aged 11–16 in the revised National Curriculum can be achieved when teachers deploy computer algebra systems.

Kurzreferat: *Möglichkeiten zur Nutzung von Computeralgebrasystemen in der oberen Sekundarstufe I in England und Wales.* Eine Reihe von Möglichkeiten werden präsentiert und diskutiert, wie Lernziele für 11–16jährige, die im reformierten britischen Nationalcurriculum festgeschrieben sind, durch die Nutzung von Computeralgebrasystemen erreicht werden können.

ZDM-Classification: D10, R20

1. Introduction

In England and Wales – the system is slightly different in other parts of the United Kingdom – schooling is compulsory to the age of 16. Thereafter, pupils may follow a two-year course of further academic study (A-level) in preparation for university, pursue vocationally-oriented education and training, or seek employment.

The way in which computer algebra systems (CAS) might influence the teaching, learning and assessment of post-16 mathematics, in particular at A-level, has been widely discussed, among others by the Association of Teachers of Mathematics (1995), Bowers (1995a), Hunter (1993), Taylor (1995) and Williamson (1992). Comparatively little has been written on the use of CAS in pre-16 mathematics. One reason for this is that those areas of mathematics which suddenly become immediately accessible using symbolic manipulation software, such as the routine performance of techniques in algebra and calculus, are not normally met in any great detail until A-level or equivalent courses.

This paper will focus on middle secondary education, and highlight those areas of mathematics where CAS might be deployed meaningfully. This will be strictly within the context of the National Curriculum. Reference will be made to published studies, and some novel examples will also be introduced and discussed.

2. The National Curriculum for mathematics in England and Wales

The revised National Curriculum (Department for Education 1995) applies to pupils of compulsory school age and is organised on the basis of four Key Stages (age 5–7, 7–11, 11–14 and 14–16 respectively). Thus KS3 and KS4 correspond to secondary education. Mathematics is compulsory at all Key Stages.

For each subject and at each Key Stage there are programmes of study which set out what pupils should be taught. In mathematics at KS3/4 the programme of study consists of five sections: Number; Algebra; Shape, Space and Measures; Handling Data; Using and Applying Mathematics. There is an additional section, Further Materials, for more able pupils at KS4.

The expected standards of pupils' performance in each section of the programme of study are set out explicitly in attainment targets, described in eight levels of increasing difficulty. At KS3/4 it is expected that pupils will be able to record achievement in the attainment targets at levels 5 to 8. There is an additional level description above level 8 to help teachers in differentiating exceptional performance.

An example may help to clarify the above. The attainment targets for the programme area Shape, Space and Measures include "pupils use everyday language to describe properties and positions ... when working with 2-D and 3-D shapes" (level 1), "pupils identify all the symmetries of 2-D shapes" (level 5) and "pupils use sine, cosine and tangent in right-angled triangles when solving problems in two dimensions" (level 8). The reader will be correct in concluding that teachers in England and Wales have a considerable task to plan their teaching and assessment, and to record the results, to ensure that each pupil has the opportunity to demonstrate competence in the various skills at the appropriate level in each of the attainment targets for all of the sections of the programme of study.

At the end of their final year of compulsory schooling, at age 16, pupils take externally set and marked examinations (General Certificate of Secondary Education, or GCSE) in each subject, which provide a formal summative assessment of their attainment in the National Curriculum. In mathematics, the GCSE usually consists of two written examination papers and a piece of assessed coursework.

3. Scope for using computer algebra systems at Key Stages 3 and 4

Each section of the mathematics programme at KS3/4 will now be considered, and areas which are amenable to the use of CAS will be highlighted. Particular reference will be made to the computer programme Derive, since a recent survey (Bowers 1995b) indicates that this is by far the most common choice of software in those schools where CAS are available for staff or student use. However, the comments could be modified to suit other symbolic manipulators.

3.1 Number

This section of the National Curriculum is encouragingly prefaced with the requirement that "pupils should be given opportunities to use calculators and computer software, eg spreadsheets". Indeed, simple scientific calculators will be found in every mathematics classroom, and it may be assumed that the majority of pupils will experience using a spreadsheet during their secondary education, not only in mathematics but also in other subjects. However, there are some aspects of Number for which a CAS is the most appropriate tool.

One advantage of a CAS is its ability to display large numbers in full, and decimal numbers to a high level of precision. This should allow greater confidence and understanding in the topic "decimals, ratios, fractions and percentages, and the inter-relationships between them", especially with regard to the decimal representation of fractions. With the display set to 50 digits, say, recurring cycles of digits can be easily identified. There is no longer any doubt about the fact that the decimal representation of $1/7$ does recur, with a cycle of 6 digits. Furthermore, $1/17$ appears to have a recurring cycle of 16 digits, and $1/19$ appears to have a recurring cycle of 18 digits. Is there any pattern here? For what values of n is $1/n$ an infinite decimal? Do all such infinite decimals recur?

Winter (1996) relates her experiences with a similar investigation into recurring decimals. She concludes that a CAS can generate large quantities of data very quickly, although this is no substitution for actually grappling with the underlying mathematics. Using the CAS frees up time to concentrate on the problem. She recommends that at KS3, an investigation of the pattern in the decimal representation of $1/n, 2/n, \ldots, (n-1)/n$ for $n = 7$ and 11 would be an adequate starting point to appreciate the "fascination" of recurring decimals.

The further material for more able pupils at KS4 includes "[pupils] distinguish between rational and irrational numbers". The algorithm for converting any recurring decimal into a fraction can be introduced, as in the following example:

Suppose $u = 0.307692307692\ldots$

The recurring cycle is 6 digits long. Multiply by 10^6

$1000000u = 307692.307692307692\ldots$

Subtract to "clear the tail"

$999999u = 307692$

Thus $u = 307692/999999$

A CAS can quickly simplify the final line to give $u = 4/13$, and the division of 4 by 13 using a large number of decimal places reproduces the original recurring decimal. The conclusion is soon drawn that any recurring decimal can be expressed as a fraction and is therefore rational. Conversely, an irrational decimal will exhibit no recurring pattern, and pupils can confirm using a CAS that π, $\sqrt{2}$ and such like do indeed appear to be irrational. A formal proof of irrationality is, of course, beyond the scope of KS4.

The ability of CAS to display large numbers in full also gives it an advantage over other classroom tools such as calculators or spreadsheets when dealing with Number. At KS3/4 pupils are required to "use some common properties of numbers, including multiples, factors and primes", and a common exercise is for an integer to be expressed as the product of its prime factors. This is normally carried out with little enthusiasm, by repeatedly dividing by $2, 3, 5, \ldots$ until unity is reached. A CAS such as Derive will perform factorisation at the press of a button, meaning that pupils will now need to be motivated in other ways to look into this topic.

A nice example concerns a kind of Gödel numbering, which can be used to encrypt messages. Using $A = 1$, $B = 2$, $C = 3$, etc., an n-letter word can be expressed uniquely as a product of the first n primes each raised to a power corresponding to that letter. For example,

$$\text{CAT} = 2^3.3^1.5^{20} = 2288818359375000$$

Decoding messages relies on the ability to factorise large numbers, a task which can now be delegated to the computer. Pupils are initially keen to decode messages such as

1836660096 126252462880302696304524717 6133890

and send similar messages to their friends. More able pupils can be encouraged to consider the underlying mathematics. How could we tell in advance that the first word in the example above has just two letters? Does the second word contain an A? When pupils tire of typing in large numbers, the challenge can be set to devise a more efficient coding method (Bowers 1997).

3.2 Algebra

It has been pointed out in a variety of studies (for example Sutherland 1990, Sutherland and Pozzi 1995) that the amount of algebra taught pre-16 has diminished in recent years, and that it is possible to gain a reasonably good grade in the GCSE examination without mastering the algebra techniques specified in the higher levels of attainment. However, there is scope for using CAS to support most of the algebra topics in the National Curriculum, which may help to make them more accessible to pupils.

At KS3 (level 5), students are expected to "construct, express in symbolic form, and use simple formulae involving one or two operations", having previously used "simple formulae expressed in words". The word input mode of a CAS such as Derive can aid this shift to generality (see also Pozzi 1993). Thus the bill from a plumber who charges £12 per hour for labour plus a call-out fee of £20 could be expressed by

$$cost = 20 + hours{*}12$$

and entered as such into the CAS in word input mode. The cost of a job lasting 3 hours can be found by using the "substitute" command to replace *hours* by 3, and then using the "solve" command. This is a valuable check for the pupil, since the CAS will definitely perform the operations in the correct order! Also, when using a CAS it is no more difficult to substitute a figure for *cost* and solve to find the number of hours worked. Doing this by hand is deemed by the National Curriculum to be a higher level skill (level 6), yet within a CAS environment it would appear both easy and obvious that a functional relationship such as this can work both ways, paving the way for a more formal algebraic treatment of the solution of linear equations.

The use of Derive's vector function

$$\text{vector([hours, cost], hours, } 1, 8)$$

with the cost function declared as above, generates a table of values which can be plotted directly on suitable axes (Fig. 1). Admittedly the syntax of the vector function may not be immediately obvious to pupils at KS3/4, but it can be argued that this function is so powerful and versatile that it is worth spending time to become familiar with it (Bowers 1996). Although a spreadsheet may be able to generate a number series more efficiently, it does not allow direct equation solving or manipulation. Carefully structured examples which take into account the limitations of the CAS (for example, Derive's limited vertical display length) can meet the National Curriculum requirements of enabling pupils to "express simple functions initially in words and then symbolically, representing them in graphical or tabular form ... construct, interpret and evaluate formulae and expressions, given in words or symbols, ... using computers and calculators where appropriate ... form and manipulate equations in order to solve problems" within an integrated computer environment.

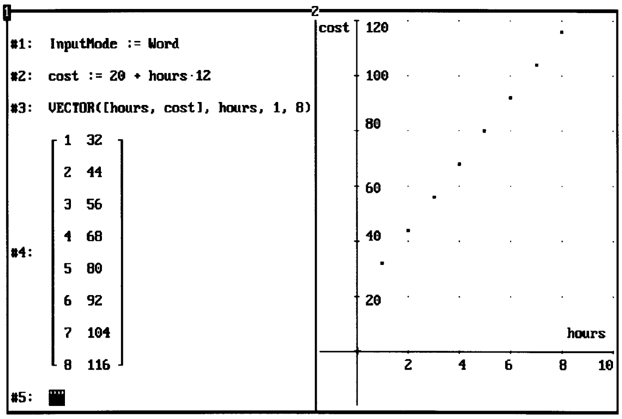

Fig. 1: A Derive session showing the inter-relationship between function, data points and graph

CAS can be used to support the teaching of the more abstract algebraic operations at KS4 such as the rules for indices or the factorisation of simple expressions. The fact that CAS can perform these operations at the press of a button can be exploited to allow pupils to discover these rules for themselves. The procedure follows the "black-box/white-box" principle, a variant of the didactic model first proposed by Buchberger (1990) and discussed widely in the international literature (for example Kutzler 1994, Drijvers 1995). An example of the approach would be as follows:

(i) the pupil is instructed to use the CAS to simplify expressions such as $x^2 \times x^4$ or $t^3 \times t^7 \times t$ and note the results;
(ii) the pupil observes the results and hypothesises a possible rule;
(iii) the pupil uses the hypothesised rule to give the answers to a set of similar questions without using the CAS;
(iv) the pupil checks the answers to the second set of questions using the CAS;
(v) if the pupil answered the second set correctly, the hypothesised rule is formulated clearly and a possible proof or justification sought; otherwise the pupil returns to step (ii).

It has been claimed that encouraging pupils to learn by discovery in this manner can motivate pupils to develop a greater ownership of the mathematics (Bowers 1995c).

A similar approach can be used for pupils to discover the concept of factorisation of algebraic expressions, which at KS3/4 would be of the form

$$4x + 10y, \quad x^2 - 3x, \quad 5ab + 10ab^2 + 20a^3b, \text{ etc.}$$

This technique is more difficult to put into words, relying on the visual identification of the structure of each term, but the CAS proves itself to be a patient provider of results for the pupils to consider at their own pace.

Under the National Curriculum, the solution of quadratic equations is only dealt with by the more able pupils working at level 8 or beyond. It is a topic which can be considered from an algebraic, numerical or graphical point of view, and a CAS is able to demonstrate the inter-relationship between these aspects within a single unified environment. By using the CAS as a black box to factorise and then solve a quadratic equation (initially of the form $x^2 + px + q = 0$), the pupil can be guided to appreciate the connection between the coefficients and the factors, and between the factors and the solutions. A graph plot illustrates the connection between the solutions and the position of the parabola on the axes, and the cycle is completed by deducing the coefficients of the quadratic from the shape of the graph. It is perhaps significant that this rich area of mathematics was used by Hunter et al. (1993) in one of the first main studies of the use of CAS with school pupils in England.

3.3 Space, Shape and Measures

This section of the National Curriculum at KS3/4 is prefaced with the recommendation that "pupils should be given opportunities to ... use computers to generate and transform graphic images and to solve problems". There exist several pieces of software specifically for this, and many schools will have one or more of Cabri Geometre, Geometer's Sketchpad or Logo. However, there are two advantages of using a CAS, where appropriate, in this curriculum area. Firstly the consistency of the computer environment should increase the pupils' confidence when working with the software in other areas, and secondly using a CAS should highlight and reinforce some of the underlying mathematics which is often hidden when using the dedicated tools of certain geometry packages.

At KS3/4 pupils are expected to use appropriate formulae to "calculate lengths, areas and volumes in plane shapes and right prisms". The formulae are generally given and are sometimes required to be transposed, for example to find the radius of a circle with known area. A CAS can be used by the pupil as a convenient checking tool here. The formula is entered, the substitution command replaces known letters by their values, and the equation is solved for the unknown quantity. A formula can be transposed algebraically simply by solving it for the required new subject. Pupils can also check the validity of their own transposition step by step. With Derive, this is achieved by applying each operation to the whole equation, and simplifying at each stage. The left part of Fig. 2 shows the stages of a typical Derive session to make b the subject of the formula $A = bh/2$, where the operations are performed in the correct sequence. Students will soon recognise if they have applied an inappropriate operation. For example, a student who thinks that the first step in the above is to transpose h by subtracting h from both sides will see something similar to that on the right of Fig. 2. Thus subtracting h from both sides has not had the desired effect, and should be reconsidered.

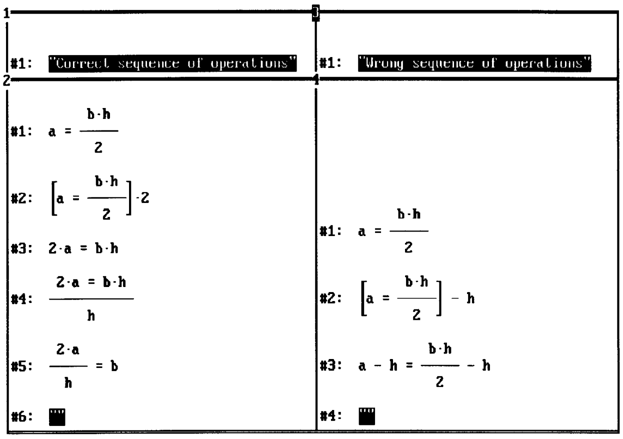

Fig. 2 Transposition of formulae using Derive

3.4 Data Handling

This is the area where there is least scope for using CAS. It covers questionnaire design, data collection, data tabulation, graphical representation, simple measures of location and dispersion, scatter diagrams and simple probability.

Derive has two functions

$$\text{AVERAGE}(x_1, x_2, \dots) \text{ and } \text{STDEV}(x_1, x_2, \dots)$$

which return the arithmetic mean and standard deviation of the arguments, although this is nowadays also available on most calculators. More interestingly, Derive will solve equations such as

$$\text{AVERAGE}(2, 3, 6, x) = 5$$

which can lead pupils to think more clearly about the definition of the mean. It is noted in passing that using Derive to solve equations such as

$$\text{STDEV}(4, 7, 8, x) = 3$$

will in general return two solutions which are either real or a complex conjugate pair. Unfortunately the mathematics involved here is well beyond the scope of pupils at KS4.

3.5 Using and Applying Mathematics

This aspect permeates all of the areas of the National Curriculum outlined above. It is concerned with teaching pupils to use effective problem solving strategies, communicate mathematically and develop mathematical reasoning. It is primarily assessed through coursework rather that by examination.

Pupils are expected to "use diagrams, graphs and symbols appropriately to convey meaning" and "use mathematical forms of communication, including diagrams, tables, graphs and computer print-outs". It has been shown above how a CAS such as Derive provides an environment for doing mathematics which brings together symbolic, numerical and graphical features, and demonstrates their inter-relationship. If pupils can be guided to observe these inter-relationships, they should gain the confidence and experience to meet the attainment target of "interpreting, discussing and synthesising information presented in a variety of mathematical forms".

This section of the programme of study stresses the importance of reviewing and checking work systematically. CAS now give the opportunity to check algebra as well as number work, and an example of reviewing methodically the steps involved in transposing a formula is given above. It has often been observed that pupils can become confused when the algebra output of a CAS is in an unusual or unexpected form. However, this is an opportunity for development rather than a reason to avoid using the CAS. Pupils can be shown ways of demonstrating the equivalence of two seemingly different expressions. This can include "justification ... by checking particular cases" (level 6), for example by evaluating both expressions for various numerical values of the letters, "showing insight into the mathematical structure" (level 7), for example by factorising or expanding one of the expressions to obtain the other, or "following alternative approaches" (level 8), such as showing that the difference of the two expressions is identical to zero.

Pupils should be taught to "make conjectures and hypotheses", and test them. This skill is fundamental to the black-box/white-box approach commended above, which encourages pupils to use the CAS to identify patterns in algebra results, and hypothesise and verify the underlying rule.

4. Discussion

The previous section describes some ways in which the facilities of a CAS can support the learning of mathematics to meet the requirements at KS3/4 of the National Curriculum. The examples given highlight five main categories of use.

Firstly, CAS can add variety to the classroom and increase motivation. In one of the earliest reports where a CAS was used with pupils as young as 14, Marshall (1992) noted an "overwhelmingly favourable" response. Even though the novelty factor can soon wear off, and it is debatable whether pupils are more inspired by the computer or by their teacher's own enthusiasm for the software, interest can be maintained by introducing problems which could not otherwise be solved, such as cracking codes using the prime factors of large numbers.

Secondly, a CAS can be a passive teacher. The pupil can generate as many examples of an algebraic process (such as multiplying two powers of x, or taking out a common factor) as are necessary to identify the underlying rule. This is in principle not different from a teacher writing some examples on the board and asking the class what they notice. However, at the computer each pupil has the chance to come up with an answer, instead of being intimidated by quicker or louder classmates.

Thirdly, a CAS is a powerful checking tool. At the simplest level, for example by just pressing the "solve" key, this is not different from looking up the answer in the back of a textbook. More usefully, a CAS can check equivalences. A common cause of confusion for pupils at this level is when their answer appears different from somebody else's answer (for example, -2 + 0.5x versus (x - 4)/2). As shown above, a CAS can offer various levels of sophistication in performing such a check. There is also a way of using CAS to check each step in the algebraic transposition of simple equations and formulae, although care must be taken that pupils are in a position to make sense of the computer output (for example, the appearance of modulus signs when square-rooting).

Fourthly, CAS can provide an environment to observe the interplay between algebraic and graphical properties of functions. The relationship between the coefficients, factors and roots of a quadratic is a classic example in the literature.

Fifthly, CAS can allow experimentation and investigation. Patterns in recurring decimals, the effect on a shape of altering the co-ordinates of its vertices, the behaviour of a family of curves or the repeated application of a function are examples of where the computer can perform the routine tasks to allow the pupil more freedom to concentrate on the results and their interpretation.

Notwithstanding the areas of scope outlined above and elsewhere, the use of CAS in mathematics classes at KS3/4 is still rare. There are no projects in the UK which can compare with those in Austria or France. CAS are still seen to be primarily relevant for A-level and university work, and this is reflected in the preponderance of case studies and research reports which concern post-16 mathematics in the published literature. Three reasons are put forward to account for this.

Firstly, the revised National Curriculum of 1995 has placed great demands on teachers in terms of developing new schemes of work and assessment plans. Graphics calculators and spreadsheets are already reasonably well established in mathematics classrooms, and teachers currently have little time or inclination to become involved in learning to use a CAS as well, especially since algebra as such is a relatively small part of the syllabus for the majority of pupils at KS3/4.

Secondly, there are virtually no CAS-based textbooks or coherent sets of worksheets readily available which have been written specifically for the needs of pupils aged 11-16.

Thirdly, there is currently a certain amount of pressure from some political quarters for education in general to return to more “traditional” methods. In mathematics this goes as far as a proposal to ban calculators from examinations. Thus there is an understandable hesitancy to embrace a new technology which might, for a while, be out of favour.

The future is there for all to see, but who knows what the future will bring?

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