

Oscillations of Certain Delay Integro-Differential Equations

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Abstract. In this paper, the oscillation of solutions of certain delay integro-differential systems are considered. Sufficient conditions for the non-existence of non-oscillatory solutions for these systems are obtained. Comparison results are obtained also.

Keywords: *Integro-differential systems, oscillation, non-oscillation, comparison theorems*

AMS subject classification: 34 A 15

1. Introduction

Recently the oscillation in systems of functional-differential equations has received attention in the literature [1 - 6]. In this paper we consider the delay integro-differential system

$$x'_i(t) + \sum_{j=1}^n \int_0^t D_{i,j}(t-s)x_j(s-\tau) ds = 0 \quad (i = 1, 2, \dots, n) \quad (1)$$

and

$$x'_i(t) + \sum_{j=1}^n \int_0^t D_{i,j}(t-s)x_j(s-\tau) ds = f_i(t) \quad (i = 1, 2, \dots, n) \quad (2)$$

with initial condition

$$x(s) = \phi(s) \quad (s \in [-\tau, 0]) \quad (3)$$

where $\tau > 0$, $D_{i,j} \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $\phi \in C([-\tau, 0])$. System (1) with $n = 1$ has been considered in [7].

By a *solution* of (1) - (3) we mean a vector $x = (x_1, \dots, x_n)^T$, which is continuous on $[-\tau, \infty)$ and continuously differentiable and satisfies (1) for $t \geq 0$ and such that (3) holds. By a *solution* of (1) on $[T, \infty)$ we mean a vector $x \in \mathbb{R}^n$ which is continuous on $[-\tau, \infty)$ and continuously differentiable and satisfies (1) for $t \geq T$. A vector $x \in \mathbb{R}^n$ is said to be *non-oscillatory*, if $x_i(t) \neq 0$ on $[-\tau, \infty)$ ($i = 1, 2, \dots, n$), and to be *positive* if every component of x is positive on $[-\tau, \infty)$.

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We will reduce the non-existence of non-oscillatory solutions of (1) to the same for the scalar integro-differential inequality

$$y'(t) + \int_0^t D(t-s)y(s-\tau) ds \leq 0 \tag{4}$$

where

$$D(t) = \begin{cases} D_{11}(t) & \text{for } n = 1 \\ \min_{1 \leq j \leq n} [D_{jj}(t) - \sum_{i=1, i \neq j}^n |D_{i,j}(t)|] & \text{for } n > 1 \end{cases} \quad (t \geq 0).$$

By a *solution* of (4) on $[T, \infty)$ we mean a function $y \in C([-\tau, \infty), \mathbb{R}) \cap C^1([T, \infty), \mathbb{R})$ which satisfies (4) on $[T, \infty)$ where $T \geq 0$. A solution y of (4) is called *positive*, if $y > 0$ on $[-\tau, +\infty)$.

2. Main results

Let us start with the following theorem.

Theorem 1. *If (4) has no positive solutions, then (1) has no non-oscillatory solutions.*

Proof. Suppose the contrary, let x be a non-oscillatory solution of (1) with $x_i(t) \neq 0$ ($t \geq -\tau; i = 1, 2, \dots, n$) and set $\delta_i = \frac{x_i(t)}{|x_i(t)|}$. Then $w = (w_1, w_2, \dots, w_n)^T = (\delta_1 x_1, \dots, \delta_n x_n)^T$ satisfies $w_i(t) > 0$ ($t \geq -\tau$) and

$$w'_i(t) + \sum_{j=1}^n \int_0^t \bar{D}_{i,j}(t-s)w_j(s-\tau) ds = 0 \quad (t \geq 0) \tag{5}$$

where $\bar{D}_{i,j} = \frac{\delta_i}{\delta_j} D_{i,j}$. Clearly, $\bar{D}_{i,i} = D_{i,i}$ and $|\bar{D}_{i,j}| = |D_{i,j}|$ ($i \neq j$). Define $y(t) = \sum_{i=1}^n w_i(t) > 0$ ($t \geq -\tau$). Summing (5), we have

$$\begin{aligned} 0 &= \sum_{i=1}^n w'_i(t) + \sum_{i=1}^n \sum_{j=1}^n \int_0^t \bar{D}_{i,j}(t-s)w_j(s-\tau) ds \\ &= y'(t) + \sum_{j=1}^n \int_0^t \bar{D}_{j,j}(t-s)w_j(s-\tau) ds + \sum_{i,j=1, i \neq j}^n \int_0^t \bar{D}_{i,j}(t-s)w_j(s-\tau) ds \\ &\geq y'(t) + \sum_{j=1}^n \int_0^t \left(\bar{D}_{j,j}(t-s) - \sum_{i,j=1, i \neq j}^n |\bar{D}_{i,j}(t-s)| \right) w_j(s-\tau) ds \\ &\geq y'(t) + \int_0^t D(t-s)y(s-\tau) ds \end{aligned}$$

which is a contradiction ■

Theorem 2. Assume that $D_{ii}(t) \not\equiv 0$ on $[0, \tau]$. If

$$z'_i(t) + \sum_{j=1}^n \int_0^t D_{i,j}(t-s)z_j(s-\tau) ds \leq 0 \quad (i = 1, 2, \dots, n) \tag{6}$$

has a positive solution $z(t)$ on $[-\tau, \infty)$, then (1) has a positive solution $x(t)$ with $0 < x(t) \leq z(t)$ ($t \geq 0$) and $\lim_{t \rightarrow \infty} x(t) = 0$.

Proof. Define the set

$$X = \left\{ x \in C([-\tau, \infty), \mathbb{R}^n) \mid 0 \leq x(t) \leq z(t) \ (t \geq -\tau), \ x(t) > 0 \ \text{on} \ [-\tau, 0] \right\}$$

and an operator S on X by

$$(Sx_i)(t) = \begin{cases} \sum_{j=1}^n \int_t^\infty \int_0^u D_{i,j}(u-s)x_j(s-\tau) ds du & \text{if } t \geq 0 \\ \left(\sum_{j=1}^n \int_0^\infty \int_0^u D_{i,j}(u-s)x_j(s-\tau) ds du \right) \left(1 - \frac{|t|}{\tau} \right) \frac{z_i(t)}{z_i(0)} + z_i(t) \frac{|t|}{\tau} & \text{if } -\tau \leq t \leq 0 \end{cases} \tag{7}$$

for $i = 1, 2, \dots, n$. Clearly, $(Sx_i)(t) \leq z_i(t)$ ($t \geq -\tau$) and $(Sx_i)(t) > 0$ on $[-\tau, 0]$, i.e. $SX \subset X$. Define the sequences $\{y_m(t)\}, y_m \in \mathbb{R}^n$ by

$$\left. \begin{aligned} y_0 &= z \\ y_m &= Sy_{m-1} \quad (m \in \mathbb{N}). \end{aligned} \right\}$$

Then

$$y_0 \geq y_1 \geq \dots \geq y_m(t) \geq \dots \quad (t \geq -\tau)$$

and hence $\lim_{m \rightarrow \infty} y_m(t) = y(t)$ ($t \geq -\tau$) exists. It is easy to see that $y(t)$ is a solution of (1) for $t \geq 0$ and $y(t) \leq z(t)$. We claim that $y(t) > 0$ ($t \geq 0$). From (7) and the condition $D_{i,i}(t) \not\equiv 0$ on $[0, \tau]$ we have $y(t) > 0$ on $[-\tau, 0]$. If ξ is the first zero of y_i on $[0, \infty)$, i.e. $y_i(t) > 0$ ($0 \leq t < \xi$) and $y_i(\xi) = 0$, then

$$\begin{aligned} 0 &= y_i(\xi) \\ &= \sum_{j=1}^n \int_\xi^\infty \int_0^u D_{i,j}(u-s)y_j(s-\tau) ds du \\ &\geq \int_\xi^\infty \int_0^u D_{i,i}(u-s)y_i(s-\tau) ds du \end{aligned}$$

which implies that $\int_0^u D_{i,i}(u-s)y_i(s-\tau) ds = 0$, which contradicts the fact that $D_{i,i} \not\equiv 0$ on $[0, \tau]$ and $y_i(s) > 0$ ($s \in [-\tau, 0]$). This contradiction proves the Theorem ■

From Theorem 2 we can derive a comparison theorem.

Corollary 1. *If $D_{i,j}(t) \leq E_{i,j}(t)$ ($t \geq 0$) and*

$$y'_i(t) + \sum_{j=1}^n \int_0^t E_{i,j}(t-s)y_j(s-\tau) ds = 0 \quad (i = 1, 2, \dots, n)$$

has a positive solution $y(t)$, then (1) has a positive solution $x(t)$ with $x(t) \leq y(t)$.

Proof. In fact, we have

$$y'_i(t) + \sum_{j=1}^n \int_0^t D_{i,j}(t-s)y_j(t-\tau) ds \leq y'_i(t) + \sum_{j=1}^n \int_0^t E_{i,j}(t-s)y_j(t-\tau) ds = 0.$$

By Theorem 2, (1) has a positive solution ■

Corollary 2. *Let $d(t) = \max_{1 \leq i \leq n} \sum_{j=1}^n D_{i,j}(t)$. If there exists $\lambda > 0$ such that*

$$-\lambda + e^{\lambda\tau} \int_0^\infty d(s)e^{\lambda s} ds \leq 0,$$

then system (1) has a positive solution x with

$$0 < x_i(t) \leq e^{-\lambda t} \quad (t \geq 0; i = 1, 2, \dots, n). \tag{8}$$

Proof. In fact, let $z_i(t) = e^{-\lambda t}$ ($i = 1, 2, \dots, n$). Then

$$\begin{aligned} z'_i(t) + \sum_{j=1}^n \int_0^t D_{i,j}(t-s)z_j(s-\tau) ds \\ &= z'_i(t) + \sum_{j=1}^n \int_0^t D_{i,j}(s)z_j(t-s-\tau) ds \\ &= e^{-\lambda t} \left[-\lambda + \sum_{j=1}^n e^{\lambda\tau} \int_0^t D_{i,j}(s)e^{\lambda s} ds \right] \\ &\leq e^{-\lambda t} \left[-\lambda + e^{\lambda\tau} \int_0^t d(s)e^{\lambda s} ds \right] \\ &\leq 0. \end{aligned}$$

By Theorem 2, (1) has a solution x , which satisfies (8) ■

Now we want to show a sufficient condition for the non-existence of positive solutions of (4). Let

$$D_1(t) = \int_{t-\tau}^t \int_0^u D(u-s) dsdu$$

$$D_n(t) = \int_{t-\tau}^t \int_0^u D(u-s)D_{n-1}(s) dsdu \quad (n \geq 2)$$

and

$$p_0 = \min_{0 \leq t \leq n\tau} D_n(t)$$

$$p_k = \min_{kn\tau \leq t \leq (k+1)n\tau} D_n(t) \quad (k \geq 1).$$

Lemma 1. *Assume that*

$$D_n(t) \geq \frac{1}{e^n}. \tag{9}$$

Let $y(t)$ be a positive solution of (4). Then

$$\frac{y(t-\tau)}{y(t)} \leq 4e^{2n}. \tag{10}$$

Proof. Integrating (4), we have

$$y(t) - y(t-\tau) + \int_{t-\tau}^t \int_0^u D(u-s)y(s-\tau) dsdu \leq 0. \tag{11}$$

Thus

$$y(t-\tau) > \int_{t-\tau}^t \int_0^u D(u-s)y(s-\tau) dsdu$$

$$\geq y(t-\tau) \int_{t-\tau}^t \int_0^u D(u-s) dsdu$$

$$= D_1(t)y(t-\tau).$$
(12)

Substituting (12) into (11) we obtain $y(t-\tau) > D_2(t)y(t-\tau)$. In general, we have

$$y(t-\tau) > y(t-\tau)D_n(t). \tag{13}$$

Assumption (9) implies that there exists $t^* \in [t-\tau, t]$ such that

$$\int_{t-\tau}^{t^*} \int_0^u D(u-s)D_{n-1}(s) dsdu \geq \frac{1}{2e^n} \quad \text{and} \quad \int_{t^*}^t \int_0^u D(u-s)D_{n-1}(s) dsdu \geq \frac{1}{2e^n}.$$

Integrating (4), we have

$$y(t^*) - y(t - \tau) + \int_{t-\tau}^{t^*} \int_0^u D(u - s)y(s - \tau) dsdu \leq 0. \tag{14}$$

In view of (13) and (14), we obtain

$$\begin{aligned} y(t - \tau) &> \int_{t-\tau}^{t^*} \int_0^u D(u - s)y(s - \tau) dsdu \\ &\geq \int_{t-\tau}^{t^*} \int_0^u D(u - s)D_{n-1}(s)y(s - \tau) dsdu \\ &\geq \frac{y(t^* - \tau)}{2e^n}. \end{aligned}$$

Similarly, we have $y(t^*) > \frac{y(t-\tau)}{2e^n}$. Combining the above two inequalities we have $\frac{y(t^* - \tau)}{y(t^*)} < (2e^n)^2$ and the proof is complete ■

Lemma 2. Assume that (9) holds. Let $y(t)$ be a positive solution of (4) and define

$$M = \min_{n(m-1)\tau \leq t \leq mn\tau} \frac{y(t - \tau)}{y(t)} \quad \text{and} \quad N = \min_{mn\tau \leq t \leq (m+1)n\tau} \frac{y(t - \tau)}{y(t)}.$$

Then $M > 1$ and

$$N \geq \exp(e^{n-1}Mp_m) \geq \exp\left(\frac{M}{e}\right) \geq M. \tag{15}$$

Proof. $M > 1$ is obvious. Dividing (4) by $y(t)$ and integrating it we have

$$\frac{y(t - \tau)}{y(t)} \geq \exp \int_{t-\tau}^t \frac{1}{y(s)} \int_0^s D(s - u)y(u - \tau) dud s. \tag{16}$$

Let

$$\begin{aligned} d_k &= \min_{(mn+k-1)\tau \leq t \leq (mn+k)\tau} \{D_n(t)\} \\ N_k &= \min_{(mn+k-1)\tau \leq t \leq (mn+k)\tau} \frac{y(t - \tau)}{y(t)} \\ M_l &= \min_{((m-1)n+l)\tau \leq t \leq (mn+l)\tau} \frac{y(t - \tau)}{y(t)} \end{aligned}$$

for $k, l = 1, 2, \dots, n$. By definition, $p_m \leq d_k$ ($k = 1, 2, \dots, n$) and $N = \min_{1 \leq k \leq n} N_k$.

For $t \in [mn\tau, (mn + 1)\tau]$, from (16) and the inequality $e^x \geq ex$ for $x > 0$, we have

$$\begin{aligned} \frac{y(t - \tau)}{y(t)} &\geq \exp \int_{t-\tau}^t \int_0^{s_1} D(s_1 - u) \frac{y(u - \tau)}{y(u)} du ds_1 \\ &\geq \exp \int_{t-\tau}^t \int_0^{s_1} D(s_1 - u) \exp \int_{u-\tau}^u \int_0^{s_2} D(s_2 - u_1) \frac{y(u_1 - \tau)}{y(u_1)} du_1 ds_2 du ds_1 \\ &\geq \exp \left(e \int_{t-\tau}^t \int_0^{s_1} D(s_1 - u) \int_{u-\tau}^u \int_0^{s_2} D(s_2 - u_1) \frac{y(u_1 - \tau)}{y(u_1)} du_1 ds_2 du ds_1 \right) \\ &\geq \\ &\vdots \\ &\geq \exp \left(e^{n-1} \int_{t-\tau}^t \int_0^{s_1} D(s_1 - u) \int_{u-\tau}^u \int_0^{s_2} D(s_2 - u_2) \dots \right. \\ &\quad \left. \dots \int_{u_{n-2}-\tau}^{u_{n-2}} \int_0^{s_n} D(s_n - u_{n-1}) \frac{y(u_{n-1} - \tau)}{y(u_{n-1})} du_{n-1} ds_n \dots du ds_1 \right) \\ &\geq \exp \left(e^{n-1} \int_{t-\tau}^t \int_0^{s_1} D(s_1 - u) \int_{u-\tau}^u \int_0^{s_2} D(s_2 - u_2) \dots \right. \\ &\quad \left. \dots \int_{u_{n-2}-\tau}^{u_{n-2}} \frac{y(s_n - \tau)}{y(s_n)} \int_0^{s_n} D(s_n - u_{n-1}) du_{n-1} ds_n \dots du ds_1 \right) \end{aligned}$$

where $s_n \in [(m - 1)n\tau, (mn + 1)\tau]$. In view of the inequality $\exp(\frac{x}{e}) > x$ for $x \neq e$ we have

$$\begin{aligned} N_1 &\geq \exp(e^{n-1} \min(M, N_1)d_1) \\ &\geq \exp(e^{n-1} \min(M, N_1)p_m) \\ &\geq \exp\left(\frac{\min(M, N_1)}{e}\right) \\ &> \min(M, N_1). \end{aligned}$$

Hence $\min(M, N_1) = M$. Therefore

$$N_1 \geq \exp(e^{n-1}Mp_m) \geq \exp\left(\frac{M}{e}\right) \geq M$$

and $M_1 \geq \min(M, N_1) \geq M$. Similar to the above, we can prove that

$$N_k \geq \exp(e^{n-1}Mp_m) \geq \exp\left(\frac{M}{e}\right) \geq M \quad (k = 1, 2, \dots, n).$$

Therefore

$$N = \min_{1 \leq k \leq n} N_k \geq \exp(e^{n-1}Mp_m) \geq \exp\left(\frac{M}{e}\right) \geq M$$

and the proof is complete ■

Theorem 3. Assume that (9) holds and

$$\sum_{i=1}^{\infty} \left(p_i - \frac{1}{e^n}\right) = \infty. \tag{17}$$

Then (4) has no positive solutions.

Proof. Suppose the contrary, let $y(t)$ be a positive solution of (4). Define the sequence $\{\bar{N}_i\}$ by

$$\bar{N}_i = \min_{(k+i-1)n\tau \leq t \leq (k+i)n\tau} \frac{y(t-\tau)}{y(t)} \quad (i \geq 0).$$

By Lemma 2, we have $\bar{N}_i > 1$ and

$$\begin{aligned} \bar{N}_{i+1} &\geq \exp(e^{n-1}\bar{N}_i p_{k+i}) \\ &\geq \exp\left(\frac{\bar{N}_i}{e}\right) \exp\left(\bar{N}_i\left(e^{n-1}p_{k+i} - \frac{1}{e}\right)\right) \\ &\geq \exp\left(\frac{\bar{N}_i}{e}\right) \\ &> \bar{N}_i. \end{aligned} \tag{18}$$

Therefore $\{\bar{N}_i\}$ is increasing. On the other hand, by Lemma 1, $\{\bar{N}_i\}$ is bounded. Hence $\lim_{i \rightarrow \infty} \bar{N}_i = \bar{N}$ exists. From the last inequality, we have $\bar{N} \geq \exp\left(\frac{\bar{N}}{e}\right) > \bar{N}$ if $\bar{N} \neq e$ which implies that $\bar{N} = e$. From (18),

$$\bar{N}_{i+1} \geq \bar{N}_i(1 + e^{n-1}\bar{N}_i(p_{k+i} - e^{-n})).$$

Hence

$$\bar{N}_{i+1} - \bar{N}_i \geq e^{n-1}\bar{N}_i^2(p_{k+i} - e^{-n})$$

and

$$\bar{N}_{i+2} - \bar{N}_{i+1} \geq e^{n-1}\bar{N}_{i+1}^2(p_{k+i+1} - e^{-n}) > e^{n-1}\bar{N}_i^2(p_{k+i+1} - e^{-n}).$$

Summing up the above inequality, we get

$$e - \bar{N}_i > e^{n-1}\bar{N}_i^2 \sum_{j=i}^{\infty} (p_{k+j} - e^{-n})$$

for some k , which contradicts (17) ■

Corollary 3. *If there exists a positive integer n such that*

$$\liminf_{t \rightarrow \infty} D_n(t) > \frac{1}{e^n}, \tag{19}$$

then (4) has no positive solutions.

In fact, (19) implies (17). Combining Theorem 1 and Corollary 3 we have the following result.

Corollary 4. *If (19) holds, then (1) has no non-oscillatory solutions.*

Now we consider the forced system (2).

Theorem 4. *Let $F(t) = \sum_{i=1}^n \delta_i f_i(t)$ with $\delta_i = \pm 1$ and $F(t) = h'(t)$, $h_+(t) = \max(h(t), 0) \not\equiv 0$, $h_-(t) = \max(-h(t), 0) \not\equiv 0$, and*

$$\int_0^\infty \int_0^t D(t-s)h_+(s-\tau) dsdt = \infty. \tag{20}$$

Then (2) has no non-oscillatory solutions.

Proof. Suppose the contrary, let $\{x_i(t)\}$ ($i = 1, 2, \dots, n$) be a non-oscillatory solution of (2). Then we have

$$\begin{aligned} \delta_i x_i'(t) + \sum_{j=1}^n \int_0^t D_{i,j}(t-s)\delta_i x_j(s-\tau) ds &= \delta_i f_i(t) \\ w_i'(t) + \sum_{j=1}^n \int_0^\infty D_{i,j}(t-s)\delta_i \delta_j^{-1} w_j(s-\tau) ds &= \delta_i f_i(t). \end{aligned}$$

That is,

$$w_i'(t) + \sum_{i=1}^n \int_0^t \bar{D}_{i,j}(t-s)w_j(s-\tau) ds = \delta_i f_i(t).$$

Summing the above equation, we obtain

$$\begin{aligned} \sum_{i=1}^n w_i'(t) + \sum_{j=1}^n \int_0^t \bar{D}_{i,j} \bar{D}_{j,j}(t-s)w_j(s-\tau) ds \\ + \sum_{i,j=1, i \neq j}^n \int_0^t \bar{D}_{i,j}(t-s)w_j(s-\tau) ds = \sum_{i=1}^n \delta_i f_i(t). \end{aligned}$$

Hence

$$y'(t) + \sum_{j=1}^n \int_0^t \left(\bar{D}_{j,j}(t-s) - \sum_{i,j=1, i \neq j} |\bar{D}_{i,j}(t-s)| \right) w_j(s-\tau) ds \leq \sum_{i=1}^n \delta_i f_i(t)$$

and hence

$$y'(t) + \int_0^t D(t-s)y(s-\tau) ds \leq F(t).$$

Thus,

$$(y(t) - h(t))' + \int_0^t D(t-s)y(s-\tau) ds \leq 0. \quad (21)$$

If $y(t) > 0$ eventually, then $y(t) - h(t)$ is non-increasing. There are two possible cases:

(i) $y(t) - h(t) \leq 0$ eventually

and

(ii) $y(t) - h(t) \geq 0$ eventually.

For the case (i), $y(t) \leq h(t)$ eventually, which contradicts the positivity of y . Therefore, the case (ii) holds. Hence $y(t) \geq h_+(t)$ eventually. From this and (21), we obtain

$$(y(t) - h(t))' + \int_0^t D(t-s)h_+(s-\tau) ds \leq 0. \quad (22)$$

This together with condition (20) lead to a contradiction and the proof is completed ■

Remark 1. (2) has no positive solution, if (20) holds, where $F(t) = \sum_{i=1}^n f_i(t)$.

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