Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 18 (1999), No. 2, 307–330

Initial Dirichlet Problem for Half-Plane Diffraction: Global Formulae for its Generalized Eigenfunctions, Explicit Solution by the Cagniard-de Hoop Method

E. Meister and K. Rottbrand

Dedicated to Prof. L. von Wolfersdorf on the occasion of his 65th birthday

Abstract. This paper deals with time-dependent plane wave diffraction by a soft/soft Sommerfeld half-plane $\Sigma: \ x > 0, \ y = \pm 0$. The explicit solution is obtained as a time-convolution in two ways: The first is directly applying the Cagniard de Hoop method to the generalized Wiener-Hopf solution of the corresponding stationary problem due to Meister & Speck (1989). The second way makes use of the Laplace integral representation of the generalized eigenfunctions with respect to the spatial (Cartesian) variables derived by Ali Mehmeti in his habilitation thesis (1995) from formulae of Meister (1983). After deforming the path of integration into the semi-infinte branch cut lines of the characteristic square root $\sqrt{\xi^2 - k^2}$ of the Helmholtzian, he obtains representations where real wave numbers may appear. But for convergence of the integrals one has to distinguish the cases $x \geq 0$ and $x < 0$, where the obstacle is present or not. We set the time Laplace variable $s = -ik$ and recover the time domain functions for the diffracted field from the eigenfunctions of the stationary problem. There follows a global formula representation with polar coordinates having the diffracting edge of Σ as its center. The solution of the initial boundary value problem is seen to coincide with that obtained in the first way, indeed.

Keywords: *Diffraction, half-plane, initial boundary value problems, generalized eigenfunctions, Cagniard-de Hoop method, explicit solution*

AMS subject classification: Primary 73 D 25, secondary 35 C 05, 35 L 05, 45 E 10, 78 A 45

1. Introduction

At the time $t = 0$ a plane incident wave front defined with an arbitrary integrable function $g(\cdot)$ and the Heaviside step $1_+(\cdot)\cdot$ as

$$v_{\text{inc}}(x, y, t; \theta) = G(t - x\cos\theta - y\sin\theta) \tag{1.1}$$

$$G(t) = 1_+(t) \cdot \int_0^t g(\tau)\, d\tau \tag{1.2}$$

Both authors: Techn. Univ., Dept. Math., Schloßgartenstraße 7, D-64289 Darmstadt
Email: rottbrand@mathematik.tu-darmstadt.de and meister@mathematik.tu-darmstadt.de

ISSN 0232-2064 / $ 2.50

governed by the two-dimensional wave equation (for short $\partial_{tt} = \frac{\partial^2}{\partial t^2}$, $\partial_{xx} + \partial_{yy} = \Delta_{xy}$: Laplacian)

$$(\partial_{tt} - \partial_{xx} - \partial_{yy}) v(x, y, t; \theta) = 0 \tag{1.3}$$

strikes the edge $(0,0)$ of the Sommerfeld half–plane $\Sigma :\ x > 0,\ y = \pm 0$, modelling a soft wall in acoustics for the pressure $(P = v)$, or a thin sharp-edged and perfectly conducting metallic sheet in electrodynamics for E-polarized electromagnetic waves $(E = v)$, where the covering unbounded medium is assumed to be linear homogeneous and isotropic. The travel speed of a wave is equal to one. The parameter θ denotes the incident angle between the x-axis and the normal of the wave front. See Fig. 1 drawn below.

Figure 1: Plane wave strikes edge (0,0) with incident angle θ at time $t = 0$

Before $t = 0$ the system is at rest. This implies zero initial conditions for the total wave field. And homogeneous Dirichlet (soft) boundary conditions are to be satisfied. We solve this initial value problem $\mathcal{P}_{\mathcal{I}}$ in two ways in Sections 4 and 5. It represents a reference model for general considerations of exterior domain problems with unbounded boundary by means of spectral theory. The latter still remains to be developed. In his habilitation thesis Ali Mehmeti [2] started first investigations for the half-plane problem described above. Basing on the diploma thesis of Mihalinčić recently a 3-dimensional half-space problem for the wave equation with two adjacent wedge-shaped materials is solved by spectral theoretic means [3].

We suggest that there are cases where one can make use of half-plane problem solutions in order to solve time-dependent wedge problems and even problems with a (finite) polygonial geometry with explicit formulae. This is planned for future papers. Note that procedures with Wiener-Hopf techniques run with Cartesian coordinates. As an example for rectangular wedges we refer to the paper [11] of Meister, Speck and Teixeira, where operators of the Wiener-Hopf-Hankel type appear. As another direction Budaev [5] for instance works on wedge problems with function theoretic Maliuzhinetz methods using polar coordinates for the diffracted field. For a study of inverse problems for the wave equation, in a (semi-)infinite dispersive slab, for instance, see v. Wolfersdorf [16].

In section 2 the corresponding stationary reduced inhomogeneous Dirichlet problem $\mathcal{P}$ for the scattered (= total − incident) field to the Helmholtz equation (k – wave number)

$$(\Delta_{xy} + k^2)u(x,y,k;\theta) = 0 \qquad (k = k_1 + \mathrm{i}k_2,\ k_2 > 0) \tag{1.4}$$

$$u(x,y,k;\theta) = -\exp(\mathrm{i}kx\cos\theta) \qquad (x > 0,\ y = \pm 0) \tag{1.5}$$

is solved in a rigorous way by modern Wiener-Hopf methods in Sobolev spaces, more precisely the energy norm space $H^\nu,\ \nu = 1$, where u is represented by a Fourier integral. We refer to the paper of Meister and Speck [10]. In our special case u must be a generalized eigenfunction, but real wave numbers seem not to be permitted. Suitably chosen paths of integration will permit them a posteriori in Section 3. Confer Meister [8, 9]. Ali Mehmeti [2] observed the Fourier integral to be of the inner product form $u = (\mathsf{f}, \mathcal{P}_+\mathsf{g})_{-\nu,\nu} = (\mathcal{P}_+\mathsf{f}, \mathsf{g})_{-\nu,\nu}$ (see Lemma 2 and Corollary 1). This is due to symmetry in ν (duality) of the involved Hardy spaces. The (fixed set of determined functions given as) projections to $\mathbb{R}_+$ in the Cauchy principal value sense

$$\mathcal{P}_+ f = \mathcal{P}_+\left[\exp(-\mathrm{i}\xi x)\,\frac{\exp(-|y|\sqrt{\xi^2 - k^2})}{\sqrt{\xi - k}}\right] \qquad (\xi = \eta^- = -k\cos\theta) \tag{1.6}$$

represent generalized eigenfunctions for general Dirichlet boundary data g. In our special case it is even the solution u of problem $\mathcal{P}$ (up to the factor $\sqrt{\eta^- - k}$). Moreover, the solution u corresponds to the time Laplace transformed scattered field of problem $\mathcal{P}_I$, that is

$$u(x,y,k;\theta) = u(x,y,\mathrm{i}s;\theta) = \tilde{u}_{\mathrm{scatt}}(x,y,s;\theta) = \int_0^\infty \mathrm{e}^{-st}\, u_{\mathrm{scatt}}(x,y,t;\theta)\,dt \tag{1.7}$$

$$v_{\mathrm{scatt}}(x,y,t;\theta) = \int_0^t G(t-\tau)\, u_{\mathrm{scatt}}(x,y,\tau;\theta)\,d\tau, \qquad (s = -\mathrm{i}k > 0). \tag{1.8}$$

In Section 4 we recover the time domain function from the Fourier integral solution u, and in Section 5 (as a new contribution) from the one-sided Laplace integral representation of the generalized eigenfunctions $\mathcal{P}_+\mathsf{f}$ with respect to the parametrizations of the branch cut half-lines of $\sqrt{\xi \pm k}$. There appear single poles corresponding to incident and reflected wave fields as physically expected. After splitting off the residues there remains to consider the unknown diffracted field part. Both procedures work with the Cagniard de Hoop method [7] introducing a positive-valued time variable t. After substitution $\xi = k\gamma = \mathrm{i}s\gamma$ as an example $t = |y|\sqrt{1-\gamma^2} - \gamma x \geq 0$ for some properly deformed contour $\gamma = \gamma^+$ of integration in the complex plane. This is in order to achieve the formula for time Laplace transformation and to read off the time domain function from the integrand to be evaluated along that contour. We note that in many diffraction problems appear wave numbers $k = k(s)$ of more general s-dependence.

The solution formula in Theorem 3 was first derived by Sommerfeld [15] with the aid of his theory of multi-valued functions, for the latter see also the monograph of Baker & Copson [4], and note that the Maliuzhinetz method is based on the theory of multi-valued functions. The solution coincides with that obtained by de Hoop in his Ph.-D.-thesis [7: Formula (9.34].

We emphasize that our main objective is to deal with the stationary formulae of Meister [6: p. 267/Formulas 6.1.32 and 6.1.35] given in Corollary 4, and *not* to derive Theorem 3.

2. Wiener-Hopf solution of the Dirichlet problem

2.1 Formulation of the problem ($\mathcal{P}$). Find a solution $u = (u^{(1)}, u^{(2)}) \in H^1(\Omega) = H^1(\Omega_1) \times H^1(\Omega_2)$ of the scalar Helmholtz equation

$$(\Delta_{xy} + k^2)\, u = 0 \qquad (k \in \mathbb{C}^{++}) \tag{2.1}$$

in $\Omega = \{(x,y) \in \mathbb{R}^2 : y \neq 0\}$ such that the traces of u and $\frac{\partial u}{\partial y}$ on $\mathbb{R}_x \times \{y = \pm 0\}$ denoted by

$$u_{01} = u(x,-0), \qquad u_{02} = u(x,+0) \tag{2.2}$$

$$u_{11} = \frac{\partial u}{\partial y}(x,-0), \qquad u_{12} = \frac{\partial u}{\partial y}(x,+0) \tag{2.3}$$

satisfy boundary conditions on the banks of $\Sigma = \Sigma^+ \oplus \Sigma^-$ given as

$$\left.\begin{aligned} u_{02} &= g_{02}^{\Sigma} \quad (2.4) \\ u_{01} &= g_{01}^{\Sigma} \quad (2.5) \end{aligned}\right\} \quad \text{on } \Sigma^{\pm} :\ x > 0,\ y = \pm 0$$

and fulfil zero-jump (=compatibility) conditions across $\Sigma' :\ x < 0,\ y = 0$:

$$f_\alpha := [u]_\alpha = u_{\alpha 2} - u_{\alpha 1} = 0, \qquad (\alpha \in \{0,1\}). \tag{2.6}$$

Figure 2: Dirichlet problem for a half-plane

2.2 Equivalent Wiener-Hopf formulation and solution of problem ($\mathcal{P}$). By the use of the 1-dimensional Fourier transformation parallel to Σ (to be understood in the sense of tempered distributions $\mathcal{S}'(\mathbb{R})$)

$$\widehat{\varphi}(\xi) := F_{x\mapsto\xi}\,[\varphi(x)] := \int_{-\infty}^{+\infty} \varphi(x)\;\exp(\mathrm{i}\xi x)\,dx$$

with its inverse transform

$$\varphi(x) = F^{-1}_{\xi\mapsto x}\,[\widehat{\varphi}(\xi)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \widehat{\varphi}(\xi)\;\exp(-\mathrm{i}\xi x)\,d\xi$$

there follows a representation formula for the solution with two half-space parts

$$u(x,y) = F^{-1}\Big\{1_-(y)\,\widehat{\varphi}_1(\xi)\;\exp\left(y\,t(\xi)\right) + 1_+(y)\,\widehat{\varphi}_2(\xi)\;\exp\left(-y\,t(\xi)\right)\Big\} \tag{2.7}$$

with $\mathcal{S}'$-ansatz data φ_1 and φ_2 situated on the banks $y = \mp 0$, respectively, and a characteristic square root $t(\xi) = \sqrt{\xi^2 - k^2}$ to be defined with semi-infinite branch cuts of the factors $t_\mp = \sqrt{\xi \mp k}$ taken parallel to the imaginary axis in order to have a positive real part of $t(\xi)$ in a whole strip region of the complex ξ-plane containing the real ξ-axis (see Figure 3).

Figure 3: Branch cuts of $t = \sqrt{\xi^2 - k^2} = t_- t_+ = \sqrt{\xi - k}\,\sqrt{\xi + k}$

Note that $\frac{\partial}{\partial x}$ corresponds to a multiplication by $-\mathrm{i}\xi$ in the Fourier image. The Helmholtz equation becomes the ordinary homogeneous linear differential equation

$$\frac{\partial^2 \widehat{u}}{\partial y^2}(\xi, y) = t^2(\xi; k)\, \widehat{u}(\xi; y) \tag{2.8}$$

with a real parameter ξ. This implies an ansatz of exponential type and finally leads to the representation formula given above. From this formula there follow bijective relations between the (Fourier-transformed) trace and ansatz data:

$$\left.\begin{aligned} \widehat{u}(\xi,-0) &= \widehat{\varphi}_1(\xi) = \widehat{u}_{01}(\xi) \\ \widehat{u}(\xi,+0) &= \widehat{\varphi}_2(\xi) = \widehat{u}_{02}(\xi) \\ t(\xi)\,\widehat{u}(\xi,-0) &= \widehat{u}_{11}(\xi) \\ -t(\xi)\,\widehat{u}(\xi,+0) &= \widehat{u}_{12}(\xi). \end{aligned}\right\}$$

Collecting the jumps $f_\alpha = [u]_\alpha$ to the vector $f := (f_0, f_1)^T$ one arrives at

$$\widehat{\varphi} = \widehat{u}_0 = \frac{1}{2}\begin{pmatrix} -1 & -\frac{1}{t} \\ 1 & -\frac{1}{t} \end{pmatrix} \widehat{f} \tag{2.9}$$

to be inserted into the representation formula. Hence a solution can be expressed by data jumps having support $\Sigma = \mathbb{R}_+ \times \{0\} \cong \mathbb{R}_+$: the positive axis with respect to the spatial variable x.

$$\begin{aligned} u(x,y) = F^{-1}_{\xi\mapsto x}\Big\{ & 1_-(y)\exp(y\,t(\xi))\Big(-\frac{1}{2}\Big)\Big[\widehat{f}_0 + \frac{1}{t(\xi)}\widehat{f}_1\Big] \\ & + 1_+(y)\exp(-y\,t(\xi))\Big(\frac{1}{2}\Big)\Big[\widehat{f}_0 - \frac{1}{t(\xi)}\widehat{f}_1\Big]\Big\}, \end{aligned}$$

or in a more convenient form

$$u(x,y) = \frac{1}{2}F^{-1}_{\xi\mapsto x}\left\{\exp\left(|y|\,t(\xi)\right)\left[\mathrm{sign}(y)\widehat{f}_0 - \frac{1}{t(\xi)}\widehat{f}_1\right]\right\} \tag{2.10}$$

where

$$f = \ell_0 f^\Sigma \tag{2.11}$$

defines the extension of the (partially unknown) jump vector on Σ by zero and $1_\pm(y) = \frac{1\pm\mathrm{sign}(y)}{2}$ with the usual signum function $\mathrm{sign}(\cdot)$. Let

$$(g_{01}, g_{02})^T := \left(\ell^{\mathrm{even}} g_{01}^\Sigma, \ell^{\mathrm{even}} g_{02}^\Sigma\right)^T \tag{2.12}$$

be any suitable even extension of the given Dirichlet data (e.g. by reflection) and note that

$$\mathcal{P}_\pm \widehat{\psi}(\xi) := \pm\frac{1}{2\pi\,\mathrm{i}} \lim_{\delta\mapsto+0} \int_{-\infty}^{+\infty} \widehat{\psi}(\eta)\,\frac{d\eta}{\eta - (\xi \pm \mathrm{i}\delta)} \tag{2.13}$$

$$= \frac{1}{2}\,(\mathcal{I} \pm \$_{\mathbf{R}})\widehat{\psi}(\xi) = F\,\ell_o \chi_{\mathbf{R}_\pm}\, F^{-1}_{\xi\mapsto x}\widehat{\psi}(\xi) \tag{2.14}$$

are bounded projections for any function $\widehat{\psi}$ in the Hardy spaces

$$\widehat{\psi} \in \widehat{H}^s(\mathbf{R}) := FH^s(\mathbf{R}) \qquad (-\tfrac{1}{2} < s < +\tfrac{1}{2}),$$

then having a unique direct sum representation

$$\widehat{\psi} = \mathcal{P}_-\widehat{\psi} + \mathcal{P}_+\widehat{\psi} =: \widehat{\psi}^- + \widehat{\psi}^+,$$

$$\widehat{\psi}^\pm \in \widehat{H}^s_\pm(\mathbf{R}) := F\ell_o\tilde{H}^s(\mathbf{R}_\pm), \qquad \widehat{H}^s_- \cap \widehat{H}^s_+ = \{0\}$$

(these facts follow from [6: Lemmas 5.3 and 5.4 and Theorem 5.2]). Here ℓ_o denotes the extension by zero, $\chi_{\mathbb{R}_\pm}$ the restriction to $\mathbb{R}_\pm$, $\mathcal{I}$ the identity operator and

$$\$_\Gamma\widehat{\psi}(\xi) := \frac{1}{\pi\,\mathrm{i}}\int_\Gamma \frac{\widehat{\psi}(\eta)}{\xi-\eta}\,d\eta \qquad (\xi\in\Gamma) \tag{2.15}$$

denotes the Hilbert transform to be understood as Cauchy integral in the principal value sense with respect to the unbounded curve $\Gamma = \mathbb{R}$.

Lemma 1. *For Fourier symbols* $\widehat{\psi}$ *of order* $\operatorname{ord}\widehat{\psi}$ *in* ξ

$$\widehat{\psi} \in \widehat{H}^s \qquad \big(s = -(\operatorname{ord}\widehat{\psi} + \tfrac{1}{2} + \varepsilon)\ \forall\varepsilon > 0\big)$$

holds.

Proof. This is a direct consequence from the definition of Hardy spaces ∎

Theorem 1. *The explicit Wiener-Hopf solution of the Dirichlet problem* $(\mathcal{P})$ *is given by the Fourier integral representation*

$$\begin{aligned} u(x,y) = F^{-1}_{\xi\mapsto x}\bigg\{ \mathrm{e}^{-|y|\sqrt{\xi^2-k^2}}\bigg[&\operatorname{sign}(y)\,F\ell_o\left(\frac{g^\Sigma_{02}-g^\Sigma_{01}}{2}\right) \\ &+\frac{1}{\sqrt{\xi-k}}\,\mathcal{P}_+\left[\sqrt{\xi-k}\,F\ell^{\text{even}}\left(\frac{g^\Sigma_{02}+g^\Sigma_{01}}{2}\right)\right]\bigg]\bigg\}. \end{aligned} \tag{2.16}$$

Remark 1. This formula corresponds to that given by Meister and Speck in [10: Corollary 2.6].

Proof of Theorem 1. It remains to determine the unknown Neumann jump f_1 (with support $\mathbb{R}_+$ to be inserted into the representation formula. Adding both equations of the system (1.14) yields

$$\widehat{u}_{01} + \widehat{u}_{02} = \widehat{g}_{01} + \widehat{g}_{02} + \widehat{h}_- = -\frac{1}{t(\xi)}\,\widehat{f}_1.$$

with some minus function $\widehat{h}_-$ which is caused by an unknown function to live on $\mathbb{R}_-$ due to the traces u_{01}, u_{02} both unknown on $\mathbb{R}_-$ for this half-plane problem. We write the last equation in the L^2- lifted form [10]

$$\sqrt{\xi-k}\big(\widehat{g}_{01} + \widehat{g}_{02} + \widehat{h}_-\big) = -\frac{\widehat{f}_1}{\sqrt{\xi+k}},$$

then (may!) apply the projection to $\mathbb{R}_+$

$$\begin{aligned}\mathcal{P}_+\big[\sqrt{\xi-k}\,(\widehat{g}_{01}+\widehat{g}_{02}+\widehat{h}_-)\big] &= \mathcal{P}_+\big[\sqrt{\xi-k}\,(\widehat{g}_{01}+\widehat{g}_{02})\big]\\ &= \mathcal{P}_+\Big[-\frac{\widehat{f}_1}{\sqrt{\xi+k}}\Big] = -\frac{\widehat{f}_1}{\sqrt{\xi+k}}\end{aligned}$$

and finally obtain

$$\frac{\widehat{f}_1}{t(\xi)} = -\frac{1}{\sqrt{\xi-k}}\,\mathcal{P}_+\big[\sqrt{\xi-k}\,F_{x\mapsto\xi}\,\ell^{\mathrm{even}}\left(g_{01}^{\Sigma}+g_{02}^{\Sigma}\right)\big].$$

This completes the proof ∎

In the sequel we confine ourselves to the physically most relevant case of coinciding boundary values on the screen Σ

$$g_{01}^{\Sigma} = g_{02}^{\Sigma} =: g^{\Sigma}. \tag{2.17}$$

For plane waves u_{inc} incident on Σ the function g^{Σ} describes the boundary values of the scattered field u_{scatt} when approaching the banks $x > 0,\ y = \pm 0$ of Σ.

3. On the resolvent of problem $(\mathcal{P})$ in terms of generalized eigenfunctions

3.1 Formulae by projection operator shifting. The subject of this section is to derive formulae for the eigenfunctions with real wave numbers. The Wiener-Hopf procedure presented above requires imaginary parts of k (small but) different from zero but one can manipulate the solution formula given in Theorem 1 in a suitable manner. The first step is due to the following Lemma of Ali Mehmeti [2].

Lemma 2. *Let* $\mathcal{P}_\pm := F\,\ell_o\chi_{\mathbb{R}_\pm}F^{-1}$, $\mathsf{f} \in \widehat{H}^{-s}$, $\mathsf{g} \in \widehat{H}^{s}$ $\ (-\frac{1}{2} < s < \frac{1}{2})$ *with* $(\mathsf{f},\mathsf{g})_{-s,s} := (\langle\cdot\rangle^{-s}f, \langle\cdot\rangle^{s}g)_{L^2(\mathbb{R})}$, $\langle\xi\rangle := (\xi^2+1)^{\frac{1}{2}}$ *and* $\|f\|_s := \|f\|_{L^2(\mathbb{R})}$. *Then*

$$(\mathsf{f},\ \mathcal{P}_\pm\ \mathsf{g})_{-s,s} = (\mathcal{P}_\pm\ \mathsf{f},\ \mathsf{g})_{-s,s}$$

holds.

Proof. This follows from the boundedness of the operators $\mathcal{P}_\pm$ for $-\frac{1}{2} < s < \frac{1}{2}$ and the special case $L^2:\ s = 0$, which is due to Plancherel's theorem ∎

Corollary 1. *The solution of the Dirichlet problem* $(\mathcal{P})$ *with* $u_{01} = u_{02} = g^{\Sigma}$ *on* $\Sigma = \mathbb{R}_+$ *is given by*

$$u(x,y) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\mathcal{P}_+\left[\mathrm{e}^{-\mathrm{i}\xi x}\,\frac{\mathrm{e}^{-|y|\sqrt{\xi^2-k^2}}}{\sqrt{\xi-k}}\right]\sqrt{\xi-k}\,\left(F\ell^{\mathrm{even}}g^{\Sigma}\right)(\xi)\,d\xi. \tag{3.1}$$

Remark 2. We have just reproduced [2: Formula (91)].

The advantage of this solution formula is that one can make further computations with a fixed set of known functions for general functions g^{Σ} given on the boundary. The second step is to compute the projection in Corollary 1 by means of the residue theorem for splitting off possible singularities and then deforming the integral path (real line) into the semi-infinite branch cut lines (the right "+" banks) of the square roots $\sqrt{\xi \pm k}$. As result a Laplace-type integral representation will turn out. The function

$$d_{x,y,k}(\xi) = \mathrm{e}^{-\mathrm{i}\xi x} \left[\frac{\mathrm{e}^{-|y|\sqrt{\xi^2-k^2}}}{\sqrt{\xi - k}} \right] \tag{3.2}$$

is of the general form

$$d_{x,y,k}(\xi) = c_o \, \mathrm{e}^{-\mathrm{i}\xi x} \, b_{y,k}(\xi),$$

where c_o is constant $(= 1)$ and b is a function of negative order $(-\frac{1}{2})$ in ξ without rational zeroes and poles and containing the square roots $\sqrt{\xi \pm k}$ to be defined from the right of their branch cuts when tending to $\mp (k + \mathrm{i}\infty)$ (see Figure 4). Thus $b^{\pm}_{y,k}(\xi)$ denote functions to be defined in this limiting sense.

Remark 3. Theorem 2 given below also holds for

$$e_{x,y,k}(\xi) := c_o \, \mathrm{e}^{-\mathrm{i}\xi x} \left[\frac{\mathrm{e}^{-|y|\sqrt{\xi^2-k^2}}}{\sqrt{\xi^2 - k^2}} \right]$$

with estimates of a similar form we made use of in the proof there. Note that in the second case $x \geq 0$ the term $\sqrt{\xi + k}$ appears as a square root type singularity.

The next theorem was derived in [13]. It gives Laplace integral representation formulae in a systematical manner and presents the final form of the projection terms needed for Corollary 1. Compare the calculations in [2, 8, 9] for to obtain such representations.

Theorem 2 (Laplace integral representation formulae). *Let $k = k_1 + \mathrm{i}\, k_2$ with $k_1, k_2 > 0$ and $\eta^{\pm} \notin \Gamma_{+k} \cup \Gamma_{-k}$. Consider the projections (extended to the complex halves)*

$$\mathcal{P}_+ \, d_{x,y,k}(\eta^{\pm}) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{+\infty} c_o \, e^{-\mathrm{i}\xi x} \, b_{y,k}(\xi) \, \frac{d\xi}{\xi - \eta^{\pm}} \qquad (\pm \operatorname{Im} \eta^{\pm} > 0). \tag{3.3}$$

(A) *For $x < 0$ there hold*

$$\begin{aligned} \mathcal{P}_+ d_{x,y,k}(\eta^+) &= d_{x,y,k}(\eta^+) \\ &\quad + \frac{c_o \mathrm{e}^{-\mathrm{i}kx}}{2\pi \mathrm{i}} \int_0^{+\infty} \mathrm{e}^{\varrho x} \left[b^+_{y,k}(k + \mathrm{i}\varrho) - b^-_{y,k}(k + \mathrm{i}\varrho) \right] \frac{d\varrho}{\varrho - \mathrm{i}(k - \eta^+)} \end{aligned}$$

and

$$\mathcal{P}_+ d_{x,y,k}(\eta^-) = \frac{c_o \mathrm{e}^{-\mathrm{i}kx}}{2\pi\mathrm{i}} \int_0^{+\infty} \mathrm{e}^{\varrho x} \left[b^+_{y,k}(k+\mathrm{i}\varrho) - b^-_{y,k}(k+\mathrm{i}\varrho) \right] \frac{d\varrho}{\varrho - \mathrm{i}(k-\eta^-)}$$

hold.

(B) *For $x \geq 0$ there hold*

$$\mathcal{P}_+ d_{x,y,k}(\eta^+) = -\frac{c_o \mathrm{e}^{+\mathrm{i}kx}}{2\pi\mathrm{i}} \int_0^{+\infty} \mathrm{e}^{-\varrho x} \left[b^+_{y,k}(-k-\mathrm{i}\varrho) - b^-_{y,k}(-k-\mathrm{i}\varrho) \right] \frac{d\varrho}{\varrho - \mathrm{i}(k+\eta^+)}$$

and

$$\begin{aligned}\mathcal{P}_+ d_{x,y,k}(\eta^-) &= -d_{x,y,k}(\eta^-) \\ &\quad - \frac{c_o \mathrm{e}^{+\mathrm{i}kx}}{2\pi\mathrm{i}} \int_0^{+\infty} \mathrm{e}^{-\varrho x} \left[b^+_{y,k}(-k-\mathrm{i}\varrho) - b^-_{y,k}(-k-\mathrm{i}\varrho) \right] \frac{d\varrho}{\varrho - \mathrm{i}(k+\eta^-)}\end{aligned}$$

hold.

Proof. The integrand is of order less than -1 in ξ. It remains to show that the contributions to the integrals due to the singularities of the integrands at the branch points $\pm k$ do vanish when approaching the branch cuts. Let us assume some ε small enough so that $0 < \varepsilon < |k|$ and $0 < \varepsilon < |\pm k - \eta^\pm|$ hold for fixed $\eta^\pm \notin \Gamma_{\pm k}$. Further we define lower (upper) half circles of the radius ε with their centers at $\pm k$ as

$$\begin{aligned} \partial\mathcal{C}_{+k,\varepsilon} &: \ \xi - k = \varepsilon\, \mathrm{e}^{\mathrm{i}\varphi} \qquad (\varphi \in [-\pi, 0] \\ \partial\mathcal{C}_{-k,\varepsilon} &: \ \xi + k = \varepsilon\, \mathrm{e}^{\mathrm{i}\varphi} \qquad (\varphi \in [0, +\pi]. \end{aligned}$$

They and their interiors (moreover, the ε-vicinity covering the branch cuts) do not contain $\eta^\pm$.

Statement (A) The case $x < 0$: We want to deform the integral path (= real line with positive orientation) to the plus(+) bank of the upper branch cut Γ_{+k}. The function $d_{x,y,k}$ has a singularity of squareroot-type at $\xi = k$. We obtain by simple triangle inequalities that the integral

$$\left| \int_{\partial\mathcal{C}_{+k,\varepsilon}} \frac{d_{x,y,k}}{\xi - \eta^\pm} \right| \leq |c_o| \frac{\mathrm{e}^{k_2\, x}}{\varepsilon^{\frac{1}{2}} \cdot (|+k-\eta^\pm| - \varepsilon)} \cdot 2\pi\varepsilon\,,$$

with $k_2 > 0$ denoting the imaginary part of k, tends to zero for $\varepsilon \to +0$.

Statement (B). The case $x \geq 0$ with the lower branch cut Γ_{-k}: We get the estimate

$$\left| \int_{\partial\mathcal{C}_{-k,\varepsilon}} \frac{d_{x,y,k}}{\xi - \eta^\pm} \right| \leq |c_o| \frac{\mathrm{e}^{-k_2\, x} \cdot 1}{\sqrt{|\varepsilon - |2k||} \cdot (|-k-\eta^\pm| - \varepsilon)} \cdot 2\pi\varepsilon \to 0$$

for $\varepsilon \to +0$. Hence the residue theorem will become available:

The case $x < 0$ with the upper branch cut $z = +k + \mathrm{i}\rho \quad (\rho \geq 0)$ leads to

$$\frac{1}{2\pi\mathrm{i}} \int\limits_{-\infty}^{+\infty} \frac{d_{x,y,k}(\xi)}{\xi - \eta^{\pm}}\, d\xi = 2\pi\mathrm{i}\, \mathsf{Res}_{\xi \to \eta^{\pm}} \left[\frac{1}{2\pi\mathrm{i}} \frac{d_{x,y,k}(\xi)}{\xi - \eta^{\pm}} \right] + \frac{1}{2\pi\mathrm{i}} \left(\int\limits_{+k+i\infty}^{+k} \frac{d^{-}_{x,y,k}(z)}{z - \eta^{\pm}} + \int\limits_{+k}^{+k+i\infty} \frac{d^{+}_{x,y,k}(z)}{z - \eta^{\pm}} \right) dz.$$

The case $x \geq 0$ with the lower branch cut $z = -k - \mathrm{i}\rho \quad (\rho \geq 0)$ leads to

$$\frac{1}{2\pi\mathrm{i}} \int\limits_{+\infty}^{-\infty} \frac{-d_{x,y,k}(\xi)}{\xi - \eta^{\pm}}\, d\xi = 2\pi\mathrm{i}\, \mathsf{Res}_{\xi \to \eta^{\pm}} \left[\frac{1}{2\pi\mathrm{i}} \frac{-d_{x,y,k}(\xi)}{\xi - \eta^{\pm}} \right] + \frac{1}{2\pi\mathrm{i}} \left(\int\limits_{-k-\mathrm{i}\infty}^{-k} \frac{-d^{-}_{x,y,k}(z)}{z - \eta^{\pm}} + \int\limits_{-k}^{-k-\mathrm{i}\infty} \frac{-d^{+}_{x,y,k}(z)}{z - \eta^{\pm}} \right) dz.$$

This completes the proof of Theorem 2 ∎

Figure 4: Deformed integral paths enclose the branch cuts of $\sqrt{\xi^2 - k^2}$

Remark 4. The Laplace-type integrals in Theorem 2 make sense for real wave numbers k. They are absolutely convergent if

$$\mathrm{Re}\,(k+\eta^{\pm}) \neq 0 \qquad \text{and} \qquad \mathrm{Re}\,(k-\eta^{\pm}) \neq 0$$

hold. Otherwise, if the real parts vanish,

$$\mathrm{Im}\,(k+\eta^{\pm}) > 0 \qquad \text{and} \qquad \mathrm{Im}\,(k-\eta^{\pm}) > 0$$

must hold.

Next we obtain explicit formulas for the integrands. We assume the square roots $+\sqrt{}$ to be defined from the right of the branch cut lines drawn from $\pm k$ to $\pm(k+\mathrm{i}\infty)$.

(A) Upper branch cut $\Gamma_{+k}:\ \xi = k+\mathrm{i}\rho\ \ (\rho>0)$ for the case $x<0$:

$$\begin{aligned}
\sqrt{\xi-k} &= \sqrt{\mathrm{i}\,\rho} = \mathrm{e}^{+\mathrm{i}\,\frac{\pi}{4}}\,\sqrt{\rho}.\\
\sqrt{\xi+k} &= \sqrt{\mathrm{i}\,\rho+2k} = \mathrm{e}^{+\mathrm{i}\,\frac{\pi}{4}}\,\sqrt{\rho-2\mathrm{i}k}.\\
\sqrt{\xi^2-k^2} &= +\mathrm{i}\,\sqrt{\rho(\rho-2\mathrm{i}k)}\\
\cosh\left(|y|\,\sqrt{\xi^2-k^2}\right) &= \cos\left(|y|\,\sqrt{\rho(\rho-2\mathrm{i}k)}\right).\\
\sinh\left(|y|\,\sqrt{\xi^2-k^2}\right) &= +\mathrm{i}\sin\left(|y|\,\sqrt{\rho(\rho-2\mathrm{i}k)}\right).
\end{aligned}$$

(B) Lower branch cut $\Gamma_{-k}:\ \xi = -k-\mathrm{i}\rho\ \ (\rho>0)$ for the case $x\geq 0$:

$$\begin{aligned}
\sqrt{\xi-k} &= \sqrt{-\mathrm{i}\,\rho-2k} = \mathrm{e}^{-\mathrm{i}\,\frac{\pi}{4}}\,\sqrt{\rho-2\mathrm{i}k}.\\
\sqrt{\xi+k} &= \sqrt{-\mathrm{i}\,\rho} = \mathrm{e}^{-\mathrm{i}\,\frac{\pi}{4}}\,\sqrt{\rho}.\\
\sqrt{\xi^2-k^2} &= -\mathrm{i}\,\sqrt{\rho(\rho-2\mathrm{i}k)}\\
\cosh\left(|y|\,\sqrt{\xi^2-k^2}\right) &= \cos\left(|y|\,\sqrt{\rho(\rho-2\mathrm{i}k)}\right).\\
\sinh\left(|y|\,\sqrt{\xi^2-k^2}\right) &= -\mathrm{i}\sin\left(|y|\,\sqrt{\rho(\rho-2\mathrm{i}k)}\right).
\end{aligned}$$

Remark 5. Meister [8, 9] has used this method in order to construct formulae of solutions for the Sommerfeld diffraction problem for special oscillating boundary data. Ali Mehmeti [2] recognized that these formulae appear automatically.

Corollary 2. *For $d_{x,y,k}(\cdot)$ in Theorem 2, on the "+" banks of $\Gamma_{\pm k}$,*

$$\left[b^{+}_{y,k}-b^{-}_{y,k}\right](k+\mathrm{i}\rho) = 2\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\,\frac{\cos\left(|y|\,\sqrt{\rho(\rho-2\mathrm{i}k)}\right)}{\sqrt{\rho}}$$

$$\left[b^{+}_{y,k}-b^{-}_{y,k}\right](-k-\mathrm{i}\rho) = 2\,\mathrm{i}\,\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\,\frac{\sin\left(|y|\,\sqrt{\rho(\rho-2\mathrm{i}k)}\right)}{\sqrt{\rho-2\mathrm{i}k}}$$

hold.

Corollary 3. *Let* $u(x,\pm 0) = g^{\Sigma}(x) := -\mathrm{e}^{-\mathrm{i}\eta x}$ *on* $\Sigma : x > 0,\ y = \pm 0,\ \eta = \eta^{+}$ $\mathrm{Im}(\eta^{+}) > 0$ *or* $\eta = \eta^{-},\ \mathrm{Im}(\eta^{-}) < 0$. *The solution of problem* $(\mathcal{P})$ *then reads*

$$u(x,y) = \mp \mathcal{P}_{+}\left[d_{x,y,k}(\eta^{\pm})\right]\sqrt{\eta^{\pm} - k}.$$

Note that

$$F_{x\mapsto\xi}\left[\ell^{\mathrm{even}}\, g^{\Sigma}(x)\right] = \frac{1}{\mathrm{i}\,(\xi - \eta)} - \frac{1}{\mathrm{i}\,(\xi + \eta)}. \tag{3.4}$$

Consider the special case

$$\eta = \eta^{-} := -k\cos\theta \qquad (\cos\theta > 0). \tag{3.5}$$

In Theorem 2 and Corollary 3 one has to take

$$\begin{aligned} \eta^{-} - k &= -k(\cos\theta + 1) \\ \eta^{-} + k &= k(1 - \cos\theta) = 2k\sin^2\frac{\theta}{2} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \sqrt{-k\cos\theta - k} &= -\mathrm{i}\sqrt{2k}\cos\frac{\theta}{2} \\ \sqrt{(-k\cos\theta)^2 - k^2} &= -\mathrm{i}k\sin\theta. \end{aligned}$$

The boundary value problem $(\mathcal{P})$ with its solution given above corresponds to diffraction of a time-harmonic plane incident wavefield

$$v_{\mathrm{inc}}(x,y,t) = \mathrm{Re}\left[\mathrm{e}^{\mathrm{i}k(x\cos\theta + y\sin\theta) - \mathrm{i}\omega t}\right] \tag{3.6}$$

at the edge of a half-plane $\Sigma : x > 0,\ y = 0$. The total field $v_{\mathrm{tot}}(t,x,y) = \mathrm{e}^{-\mathrm{i}\omega t}\, u_{\mathrm{tot}}(x,y)$ is a superposition of scattered (diffracted and reflected) and incident parts and obeys the wave equation, that is the stationary spatial parts (amplitudes u_{inc}, u_{scatt}, u_{tot}) obey the Helmholtzian $\Delta u + k^2 u = 0$, where here the total field is zero in a limiting sense when approaching the boundary of the obstacle (the banks $y = \pm 0$ of Σ) non-tangentially. After splitting off the time-harmonic factor $\mathrm{e}^{-\mathrm{i}\omega t}$ and making use of the previous formulae in Theorem 2 we can state

Corollary 4 (Time-harmonic diffraction problem). *For the (permanent) incident wavefield* $u_{\mathrm{inc}}(x,y) = \mathrm{e}^{\mathrm{i}k(x\,\cos\theta + y\,\sin\theta)}$ *the amplitudes of the total field read as follows:*

(A) *For* $x < 0,\ y \in \mathbb{R}$: *incident + diffracted parts*

$$\begin{aligned} u_{\mathrm{tot}}(x,y) = {} & \mathrm{e}^{\mathrm{i}k(x\,\cos\theta + y\,\sin\theta)} \\ & - \frac{\sqrt{2k}}{\pi}\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\cos\frac{\theta}{2}\,\mathrm{e}^{-\mathrm{i}kx}\int_0^{+\infty}\mathrm{e}^{\rho x}\,\frac{\cos\left(y\sqrt{\rho(\rho - 2\mathrm{i}k)}\right)}{\sqrt{\rho}}\,\frac{d\rho}{\rho - 2\mathrm{i}k\,\cos^2\frac{\theta}{2}}. \end{aligned}$$

(B) *For* $x > 0,\ y \in \mathbb{R}$: *incident + reflected + diffracted parts*

$$\begin{aligned} u_{\mathrm{tot}}(x,y) = {} & \mathrm{e}^{\mathrm{i}k(x\,\cos\theta + y\,\sin\theta)} - \mathrm{e}^{\mathrm{i}k(x\,\cos\theta + |y|\,\sin\theta)} \\ & + \frac{\sqrt{2k}}{\pi}\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\cos\frac{\theta}{2}\,\mathrm{e}^{+\mathrm{i}kx}\int_0^{+\infty}\mathrm{e}^{-\rho x}\,\frac{\sin\left(|y|\sqrt{\rho(\rho - 2\mathrm{i}k)}\right)}{\sqrt{\rho - 2\mathrm{i}k}}\,\frac{d\rho}{\rho - 2\mathrm{i}k\,\sin^2\frac{\theta}{2}}. \end{aligned}$$

4. Explicit solution of a time-dependent plane wave diffraction problem for the half-plane Σ

We have just derived the basic formulae for the treatise of a more general time-dependent half-plane diffraction problem which shall serve as a reference model for later spectral theoretical considerations. It is modelled as follows: Consider a plane incident wave field (source line) after the switch on at time $t = t_o$ travelling with speed $c = 1$ parallel to the normal direction of its wavefront

$$\mathbf{n} := (\cos\theta, \sin\theta)^T \tag{4.1}$$

where θ denotes some arbitrary (fixed) angle taken with the positive x-axis. The wavefront shall strike the edge $(x, y) = (0, 0)$ of the half-plane-shaped obstacle $\Sigma : x > 0,\ y = 0$ at time $t = t_s \geq t_o$. The incident field is of the form

$$v_{\text{inc}}(x, y, t) = G\Big(t - t_o - \frac{1}{c}(x\cos\theta + y\sin\theta)\Big) \tag{4.2}$$

where

$$G(t) = 1_+(t) \cdot \int_0^t g(\tau)\, d\tau, \tag{4.3}$$

and $1_+(t)\cdot$ denotes the Heaviside step function. Compare with Achenbach [1] who solved a time-dependent Neumann-case $(x > 0)$ with zero Dirichlet conditions along $x < 0$ by the Cagniard de Hoop method [7] there. We formulate the initial boundary value

Problem $(\mathcal{P}_I)$ (see Figure 1). Find a solution of the 2-dimensional wave equation (M) to hold for $t > 0$ in the exterior of the half-plane $\bar{\Sigma} : x \geq 0,\ y = 0$ for the following homogeneous initial conditions (I0) and (I1) and homogeneous boundary conditions (B0):

$$\left.\begin{array}{lr} \text{(M)} & (\partial_{tt} - c^2\Delta_{xy})\, v(x, y, t) = 0 \\ \text{(IO)} & v(x, y, t_0 + 0) = v_o(x, y) = 0 \\ \text{(I1)} & \partial_t v(x, y, t_0 + 0) = v_1(x, y) = 0 \\ \text{(BO)} & v(x > 0, \pm 0, t \geq t_s \geq t_0 \geq 0) = 0 \end{array}\right\} \tag{4.4}$$

where v corresponds to a total wave field $v = v_{\text{tot}} = v_{\text{scatt}} + v_{\text{inc}}$ generated by scattering of the incident wave

$$v_{\text{inc}}(x, y, t) = G\Big(t - t_o - \frac{1}{c}(x\cos\theta + y\sin\theta)\Big)$$

after striking (the edge $(0,0)$ of) the half-plane $\Sigma : x > 0, y = 0$ at time $t = t_s$. For simplicity let $c = 1$ and $t_s = t_0 = 0$.

After making use of the one-sided time Laplace transform defined as

$$\varphi := \mathcal{L}_{t\mapsto s}\,[\varphi(t)] = \int_0^{+\infty} \mathrm{e}^{-s\,t}\varphi(t)\, dt \qquad (s > 0) \tag{4.5}$$

one obtains

$$[s^2 - \Delta_{xy}]\, \tilde{v}(x,y,s) = s\, v_o(x,y) + v_1(x,y) = 0 \tag{4.6}$$

$$\gamma_+(x)\,[\tilde{v}(x,0,s)] = 0 \tag{4.7}$$

where γ_+ denotes the restriction to $\mathbb{R}_+$. The scattered field v_{scatt} obeys the same partial differential equation, but the boundary condition then reads as

$$v_{\mathrm{scatt}}(x>0,0,t) = -\gamma_+(x)\,[v_{\mathrm{inc}}(x,0,t)] \tag{4.8}$$

and its Laplace transform as

$$\tilde{v}_{\mathrm{scatt}}(x,0,s) = -\gamma_+(x)\Big[\mathrm{e}^{-s\,x\cos\theta}\,\frac{\tilde{g}(s)}{s}\Big].$$

After dropping the factor $\frac{\tilde{g}(s)}{s}$ and setting $s = -\mathrm{i}k$, that is $k = \mathrm{i}\,k_2$ $(k_2 > 0)$ one arrives at the boundary value problem $(\mathcal{P})$ we have just solved before

$$\left.\begin{aligned} &(\Delta_{xy} + k^2)\,\tilde{v}_{\mathrm{scatt}}(x,y,s) = 0 \\ &\quad \tilde{v}_{\mathrm{scatt}}(x \ge 0, 0, s) = -\gamma_+(x)\Big[\mathrm{e}^{\mathrm{i}kx\cos\theta}\,\frac{\tilde{g}(s)}{s}\Big] \end{aligned}\right\}.$$

So we are in position to write down a solution formula for the initial boundary value problem $(\mathcal{P}_I)$:

Corollary 5. *Let without loss of generality $0 < \theta \le \frac{\pi}{2}$. The solution of problem $(\mathcal{P}_I)$ is given as follows:*

(A) *For $x < 0$, $y \in \mathbb{R}$: incident + diffracted parts*

$$\begin{aligned} v(x,y,t) &= G\big(t - (x\cos\theta + y\sin\theta)\big) \\ &\quad - \mathcal{L}^{-1}_{s\mapsto t}\left[\frac{\tilde{g}(s)}{s}\,\frac{\sqrt{2s}}{\pi}\cos\frac{\theta}{2}\,\mathrm{e}^{sx}\int_0^{+\infty}\mathrm{e}^{\rho x}\,\frac{\cos\big(y\sqrt{\rho(\rho+2s)}\big)}{\sqrt{\rho}}\,\frac{d\rho}{\rho + 2s\cos^2\frac{\theta}{2}}\right]. \end{aligned}$$

(B) *For $x \ge 0$, $y \in \mathbb{R}$: incident + reflected + diffracted parts*

$$\begin{aligned} v(x,y,t) &= G\big(t - (x\cos\theta + y\sin\theta)\big) - G\big(t - (x\cos\theta + |y|\sin\theta)\big) \\ &\quad + \mathcal{L}^{-1}_{s\mapsto t}\left[\frac{\tilde{g}(s)}{s}\,\frac{\sqrt{2s}}{\pi}\cos\frac{\theta}{2}\,\mathrm{e}^{-sx}\int_0^{+\infty}\mathrm{e}^{-\rho x}\,\frac{\sin\big(|y|\sqrt{\rho(\rho+2s)}\big)}{\sqrt{\rho+2s}}\,\frac{d\rho}{\rho + 2s\sin^2\frac{\theta}{2}}\right] \end{aligned}$$

where the inverse Laplace transforms write as convolutions of

$$\mathcal{L}^{-1}_{s\mapsto t}\Big[\frac{\tilde{g}(s)}{s}\Big] = G(t) = 1_+(t)\cdot\int_0^t g(\alpha)\,d\alpha$$

and the functions corresponding to the integral terms in the original time domain.

Remark 6. After the substitution $\rho = (R-1)s$ in the integral terms given inside the brackets, the formulae for the *diffracted wave parts* take the form

$$\tilde{v}_{\text{diff}}(x<0,y,s) = -\frac{\tilde{g}(s)}{s}\frac{\sqrt{2}}{\pi}\cos\frac{\theta}{2}\int_1^{+\infty} \mathrm{e}^{sRx}\frac{\cos\left(sy\sqrt{R^2-1}\right)}{\sqrt{R-1}}\frac{dR}{R+\cos\theta} \tag{4.9}$$

$$\tilde{v}_{\text{diff}}(x\geq 0,y,s) = \frac{\tilde{g}(s)}{s}\frac{\sqrt{2}}{\pi}\cos\frac{\theta}{2}\int_1^{+\infty} \mathrm{e}^{-sRx}\frac{\sin\left(s|y|\sqrt{R^2-1}\right)}{\sqrt{R+1}}\frac{dR}{R-\cos\theta}. \tag{4.10}$$

The remaining question is how to find the Laplace inverses of the integral terms: From Theorem 1 and equation (3.4) the scattered field is easily checked to be represented by the following formula:

Corollary 6. *Making use of the two-sided Laplace transform with respect to the spatial variable x the time Laplace transformed scattered field writes*

$$\tilde{v}_{\text{scatt}}(x,y,s) = \frac{\tilde{g}(s)}{s}\,\tilde{u}_{\text{scatt}}(x,y,s)$$

where in the strip $-s < \beta_1 < +s$

$$\tilde{u}_{\text{scatt}}(x,y,s) = -\frac{\sqrt{2s}\cos\frac{\theta}{2}}{2\pi\,\mathrm{i}}\int_{\beta_1-i\infty}^{\beta_1+\mathrm{i}\infty}\frac{\mathrm{e}^{-(|y|\sqrt{s^2-\beta^2}-\beta x)}}{\sqrt{s-\beta}\,(\beta+s\cos\theta)}d\beta$$

holds.

After splitting off the residuum $-\frac{\tilde{g}(s)}{s}\,\mathrm{e}^{-s(x\cos\theta+|y|\sin\theta)}$ which is due to the pole $\beta=-s\cos\theta$ and setting $\beta = s\gamma$ the time Laplace transformed diffracted part writes in the strip $-\cos\theta < \gamma_1 < +1$ as

$$\tilde{u}_{\text{diff}}(x,y,s) = -\frac{\sqrt{2}\cos\frac{\theta}{2}}{2\pi\,\mathrm{i}}\int_{\gamma_1-\mathrm{i}\infty}^{\gamma_1+\mathrm{i}\infty}\frac{\mathrm{e}^{-s(|y|\sqrt{1-\gamma^2}-\gamma x)}}{\sqrt{1-\gamma}\,(\gamma+\cos\theta)}\,d\gamma. \tag{4.11}$$

From the physical point of view the diffracted wave field should be radial symmetric with respect to the diffracting edge (0,0). Therefore let us introduce the polar coordinates (upper half-circles)

$$\left.\begin{aligned} &r=\sqrt{x^2+y^2}\\ &\cos\alpha=\frac{x}{r},\quad \sin\alpha=\frac{y}{r}\geq 0\end{aligned}\right\}\qquad (+0\leq\alpha\leq\pi). \tag{4.12}$$

We invert the last integral given above by the Cagniard de Hoop method (see Achenbach [1]) to yield

Theorem 3. *For problem $(\mathcal{P}_I)$ the diffracted wave field is given by*

$$v_{\text{diff}}(r,\alpha,t) = \int_0^t G(t-p)\, u_{\text{diff}}(r,\alpha,p)\, dp \tag{4.13}$$

where

$$u_{\text{diff}}(r,\alpha,t) = 1_+(t-r)\cdot \frac{\sqrt{2r}}{2\pi\sqrt{t-r}} \left[\frac{\sin\frac{\theta-\alpha}{2}}{t-r\,\cos(\theta-\alpha)} - \frac{\sin\frac{\theta+\alpha}{2}}{t-r\,\cos(\theta+\alpha)} \right], \tag{4.14}$$

with $1_+(t-r)\cdot$ denoting the Heaviside step. For the scattered parts we then have

$$v_{\text{scatt}}(r,\alpha,t) = \begin{cases} v_{\text{diff}}(r,\alpha,t) - G(t-r\,\cos(\theta-\alpha)) & \text{if } +0 \le \alpha < \theta \le \frac{\pi}{2} \\ v_{\text{diff}}(r,\alpha,t) & \text{if } \theta < \alpha \le \pi. \end{cases} \tag{4.15}$$

Proof. It remains to verify the representation given for u_{diff}. The idea of the Cagniard de Hoop method is to deform the (spatially) two-sided Laplace integral such that it takes a one-sided Laplace integral with respect to the new (time) variable. So we introduce the new variable

$$t = |y|\,\sqrt{1-\gamma^2} - \gamma x \tag{4.16}$$

to be positive. The suitable Cagniard de Hoop contour is thus given by

$$\gamma^{\pm} = -\frac{t}{r}\,\cos\alpha \pm \mathrm{i}\,\sin\alpha\sqrt{\left(\frac{t}{r}\right)^2 - 1} \qquad (\sin\alpha \ge 0) \tag{4.17}$$

which describes a hyperbola in the $(\operatorname{Re}\gamma^{\pm}, \operatorname{Im}\gamma^{\pm})$-plane with its vertex at $-\cos\alpha$. For short

$$F(\gamma) := \frac{1}{\sqrt{1-\gamma}\,(\gamma+\cos\theta)} \qquad (\gamma = \gamma^+ \ \text{ or } \ \gamma = \gamma^-). \tag{4.18}$$

Note the symmetry due to complex conjugation $F(\gamma^-) = F(\overline{\gamma^+}) = \overline{F(\gamma^+)}$. One immediately obtains

$$\begin{aligned} \tilde{u}_{\text{diff}}(r,\alpha,s) &= -\frac{\sqrt{2}\,\cos\frac{\theta}{2}}{2\pi\mathrm{i}} \int_r^{+\infty} \mathrm{e}^{-st} \left[F(\gamma^+)\,\frac{\partial\gamma^+}{\partial t} - F(\gamma^-)\,\frac{\partial\gamma^-}{\partial t} \right] dt \\ &= -\frac{\sqrt{2}\,\cos\frac{\theta}{2}}{2\pi\mathrm{i}}\, 2\mathrm{i} \int_r^{+\infty} \mathrm{e}^{-st}\, \operatorname{Im}\left[F(\gamma^+)\,\frac{\partial\gamma^+}{\partial t} \right] dt \end{aligned} \tag{4.19}$$

where

$$\frac{\partial\gamma^+}{\partial t} = -\frac{\cos\alpha}{r} + \mathrm{i}\,\frac{t\,\sin\alpha}{r\,\sqrt{t^2-r^2}}$$

$$F(\gamma^+)\frac{\partial\gamma^+}{\partial t} = \frac{\frac{\partial\gamma^+}{\partial t}}{\sqrt{1-\gamma^{+2}}}\,\frac{\sqrt{1+\gamma^+}}{\gamma^+ + \cos\theta} = \mathrm{i}\,\frac{1}{\sqrt{t^2-r^2}}\,\frac{\sqrt{1+\gamma^+}}{\gamma^+ + \cos\theta}.$$

Hence, with $\operatorname{Im}(\mathrm{i}z) = \operatorname{Re}(z)$,

$$\tilde{u}_{\text{diff}}(r,\alpha,s) = -\frac{1}{\pi}\sqrt{2}\,\cos\frac{\theta}{2}\int_r^{+\infty}\frac{\mathrm{e}^{-st}}{\sqrt{t^2-r^2}}\operatorname{Re}\left[\frac{\sqrt{1+\gamma^+}}{\gamma^+ + \cos\theta}\right]dt$$

follows and it is immediately seen that

$$u_{\text{diff}}(r,\alpha,t) = -1_+(t-r)\cdot\frac{\sqrt{2}\cos\frac{\theta}{2}}{\pi\sqrt{t^2-r^2}}\operatorname{Re}\left[\frac{\sqrt{1+\gamma^+}}{\gamma^+ + \cos\theta}\right]. \tag{4.20}$$

It remains to compute the real part of

$$\mathcal{Q}(r,\alpha,t) := \frac{(-\sqrt{2}\,\cos\frac{\theta}{2})\sqrt{1+\gamma^+(r,\alpha,t)}}{\gamma^+(r,\alpha,t)+\cos\theta}. \tag{4.21}$$

For convenience let

$$\left.\begin{aligned}
&\frac{t}{r} =: \cosh\varphi \quad \big(\text{then } \gamma^+ = -\cos(\alpha+\mathrm{i}\varphi)\big)\\
&\cosh\frac{\varphi}{2} = \sqrt{\frac{\frac{t}{r}+1}{2}}, \quad \sinh\frac{\varphi}{2} = \sqrt{\frac{\frac{t}{r}-1}{2}}\\
&\sqrt{1+\gamma^+} = \sqrt{1-\cos(\alpha+\mathrm{i}\varphi)} = \sqrt{2}\,\sin\left(\frac{\alpha+\mathrm{i}\varphi}{2}\right)\\
&\sqrt{1-\gamma^+} = \sqrt{1+\cos(\alpha+\mathrm{i}\varphi)} = \sqrt{2}\,\cos\left(\frac{\alpha+\mathrm{i}\varphi}{2}\right).
\end{aligned}\right\}$$

We get after rewriting $-(\gamma^+ + \cos\theta) = 1-\gamma^+ - (1+\cos\theta)$ that

$$\begin{aligned}
\mathcal{Q} &= \frac{\sqrt{2}\,\cos\frac{\theta}{2}\,\sqrt{1+\gamma^+}}{\left(\sqrt{1-\gamma^+}-\sqrt{2}\,\cos\frac{\theta}{2}\right)\left(\sqrt{1-\gamma^+}+\sqrt{2}\,\cos\frac{\theta}{2}\right)}\\
&= \frac{2\cos\frac{\theta}{2}\sin\left(\frac{\alpha+\mathrm{i}\varphi}{2}\right)}{\sqrt{2}\left(\cos\left(\frac{\alpha+\mathrm{i}\varphi}{2}\right)-\cos\frac{\theta}{2}\right)\sqrt{2}\left(\cos\left(\frac{\alpha+\mathrm{i}\varphi}{2}\right)+\cos\frac{\theta}{2}\right)}.
\end{aligned} \tag{4.22}$$

With the trigonometrical relations

$$\left.\begin{aligned}
2\,\cos x\,\sin y &= \sin(x+y)-\sin(x-y)\\
\cos x - \cos y &= -2\,\sin\left(\frac{x+y}{2}\right)\,\sin\left(\frac{x-y}{2}\right)\\
\cos x + \cos y &= +2\,\cos\left(\frac{x+y}{2}\right)\,\cos\left(\frac{x-y}{2}\right)
\end{aligned}\right\}$$

one finally arrives at

$$\begin{aligned}
\mathcal{Q} &= \frac{1}{2}\left[\frac{1}{\sin\left(\frac{\theta-\alpha-\mathrm{i}\varphi}{2}\right)} - \frac{1}{\sin\left(\frac{\theta+\alpha+\mathrm{i}\varphi}{2}\right)}\right]\\
&= \frac{1}{2}\,\frac{\sin\left(\frac{\theta-\alpha}{2}\right)\,\cosh\frac{\varphi}{2} + \mathrm{i}\cos\left(\frac{\theta-\alpha}{2}\right)\,\sinh\frac{\varphi}{2}}{\sin^2\left(\frac{\theta-\alpha}{2}\right)\,\cosh^2\frac{\varphi}{2} + \cos^2\left(\frac{\theta-\alpha}{2}\right)\,\sinh^2\frac{\varphi}{2}}\\
&\quad - \frac{1}{2}\,\frac{\sin\left(\frac{\theta+\alpha}{2}\right)\,\cosh\frac{\varphi}{2} - \mathrm{i}\cos\left(\frac{\theta+\alpha}{2}\right)\,\sinh\frac{\varphi}{2}}{\sin^2\left(\frac{\theta+\alpha}{2}\right)\,\cosh^2\frac{\varphi}{2} + \cos^2\left(\frac{\theta+\alpha}{2}\right)\,\sinh^2\frac{\varphi}{2}}
\end{aligned} \tag{4.23}$$

and obtains

$$\operatorname{Re}(\mathcal{Q}) = \frac{1}{2}\left[\frac{\sqrt{2r}\sqrt{t+r}\,\sin\left(\frac{\theta-\alpha}{2}\right)}{t-r\,\cos(\theta-\alpha)} - \frac{\sqrt{2r}\sqrt{t+r}\,\sin\left(\frac{\theta+\alpha}{2}\right)}{t-r\,\cos(\theta+\alpha)}\right].$$

This completes the proof ∎

Remark 7. The region $+0 \leq \alpha < \theta$ is the so-called *shadow zone*; the incident wave is cancelled there.

Remark 8. The case $\pi \leq \alpha \leq 2\pi - 0$ can be done in the same manner: $\beta = \alpha - \pi$ leads to the *Cagniard contour*

$$\gamma^{\pm} = \frac{t}{r}\cos\beta \pm \mathrm{i}\sin\beta\sqrt{\left(\frac{t}{r}\right)^2 - 1} \qquad (\sin\beta \geq 0) \tag{4.24}$$

and the formulae

$$\left.\begin{aligned}
&\gamma^{+} = \cos(\beta - \mathrm{i}\varphi), \qquad \frac{\frac{\partial\gamma^{+}}{\partial t}}{\sqrt{1-\gamma^{+2}}} = \mathrm{i}\,\frac{1}{\sqrt{t^2-r^2}}\\
&\sqrt{1+\gamma^{+}} = \sqrt{2}\,\cos\frac{\beta-\mathrm{i}\varphi}{2}, \qquad \sqrt{1-\gamma^{+}} = \sqrt{2}\,\sin\frac{\beta-\mathrm{i}\varphi}{2}\\
&\mathcal{Q} = \frac{2\cos\frac{\theta}{2}\,\cos\left(\frac{\beta-\mathrm{i}\varphi}{2}\right)}{\sqrt{2}\left(\sin\left(\frac{\beta-\mathrm{i}\varphi}{2}\right) - \cos\frac{\theta}{2}\right)\sqrt{2}\left(\sin\left(\frac{\beta-\mathrm{i}\varphi}{2}\right) + \cos\frac{\theta}{2}\right)}
\end{aligned}\right\} \tag{4.25}$$

where after rewriting with $\beta = \alpha - \pi$ and complex conjugation the last expression $\mathcal{Q}$ becomes exactly that appeared before in Theorem 3 for $+0 \leq \alpha \leq \pi$. That is, the formula for the diffracted field holds in the full sector $(+0 \leq \alpha \leq 2\pi - 0)$, the whole exterior of the obstacle as physically expected. In this way it represents a *global solution* of the wave equation which satisfies homogeneous boundary conditions of the first kind for $\alpha = +0$ and $\alpha = 2\pi - 0$. Further the region $2\pi - \theta < \alpha \leq 2\pi$ is the zone of reflections. There one has to take into account poles similar as above: Note that the cosines of α coincide with those in the shadow zone. Hence Theorem 3 can be supplemented by

Corollary 7. *For $\pi \leq \alpha \leq 2\pi - 0$ the scattered wave field in Theorem 3 reads*

$$v_{\text{scatt}}(r,\alpha,t) = \begin{cases} v_{\text{diff}}(r,\alpha,t) & \textit{if } \pi \leq \alpha < 2\pi - \theta \\ v_{\text{diff}}(r,\alpha,t) - G(t - r\,\cos(\theta+\alpha)) & \textit{if } 2\pi - \theta < \alpha \leq 2\pi - 0. \end{cases}$$

Note that the incident wave can be written in the form

$$v_{\text{inc}}(r,\alpha,t) = G\big(t - r\cos(\alpha-\theta)\big) = G\Big(t - r + 2r\sin^2\frac{\alpha-\theta}{2}\Big). \tag{4.26}$$

5. Comments on the solution formulae

In this section we will see that the time functions (4.14) we derived above for the diffracted field by the Cagniard de Hoop method can also be recovered from the generalized eigenfunctions of the stationary problem given in Corollary 4 as Laplace integrals with respect to the positive-valued variable ρ for parametrization of the semi-infinite branch cut lines of the characteristic square root $\sqrt{\xi^2 - k^2}$ of the Helmholtzian. In other words: We show that we have two equivalent representations of the eigenfunctions of problem $(\mathcal{P})$. The one is with respect to polar coordinates, the other with respect to Cartesian coordinates. Let us start to compare them by regarding the boundary values.

Making use of the Laplace transforms

$$\mathcal{L}_{t\mapsto s}\left[\frac{\sqrt{b}}{\pi\sqrt{t}\,(t+b)}\right] = \mathrm{e}^{bs}\mathrm{erfc}\,(\sqrt{bs}) \qquad (b>0)$$

$$\begin{aligned}\mathrm{erfc}\,(x>0) &= \frac{2}{\sqrt{\pi}}\int\limits_{x}^{+\infty}\mathrm{e}^{-t^2}\,dt = \frac{1}{\sqrt{\pi}}\int\limits_{x^2}^{+\infty}\frac{\mathrm{e}^{-m}}{\sqrt{m}}\,dm\\ &= 1-\mathrm{erf}\,(x>0) = 1-\frac{1}{\sqrt{\pi}}\int\limits_{0}^{x^2}\frac{\mathrm{e}^{-m}}{\sqrt{m}}\,dm\end{aligned}$$

and

$$\mathcal{L}_{t\mapsto s}[1_+(t-r)\cdot f(t)] = \mathrm{e}^{-rs}\,\mathcal{L}_{t\mapsto s}[f(t)]$$

the Laplace transform of the diffracted field part in Theorem 3 can be written as

$$\begin{aligned}\tilde{u}_{\mathrm{diff}}(r,\alpha,s) &= \frac{1}{2}\,\mathrm{e}^{-sr\,\cos(\theta-\alpha)}\,\mathrm{erfc}\left(\sqrt{2sr}\,\left|\sin\frac{\theta-\alpha}{2}\right|\right)\,\mathrm{sign}\left[\sin\frac{\theta-\alpha}{2}\right]\\ &\quad -\frac{1}{2}\,\mathrm{e}^{-sr\,\cos(\theta+\alpha)}\,\mathrm{erfc}\left(\sqrt{2sr}\,\left|\sin\frac{\theta+\alpha}{2}\right|\right)\,\mathrm{sign}\left[\sin\frac{\theta+\alpha}{2}\right].\end{aligned} \tag{5.1}$$

Hint: Rewrite equation (4.14) as

$$u_{\mathrm{diff}}(r,\alpha,t) = \frac{1}{2}\,\frac{1_+(t-r)}{\sqrt{t-r}}\cdot\frac{1}{\pi}\left[\frac{\sqrt{2r}\,\sin\frac{\theta-\alpha}{2}}{t-r+2r\,\sin^2\frac{\theta-\alpha}{2}} - \frac{\sqrt{2r}\,\sin\frac{\theta+\alpha}{2}}{t-r+2r\,\sin^2\frac{\theta+\alpha}{2}}\right]. \tag{5.2}$$

For the special cases

(i) $\alpha = 0:\ x = +r > 0,\ y = 0$
(ii) $\alpha = \pi:\ x = -r < 0,\ y = 0$

the corresponding equations (4.10) and (4.9) are easily seen to yield coinciding expressions. After splitting off the common factor $\frac{\tilde{g}(s)}{s}$, that is

(i) $\tilde{u}_{\mathrm{diff}} = 0$
(ii) $\tilde{u}_{\mathrm{diff}} = -\mathrm{e}^{sr\,\cos\theta}\mathrm{erfc}\big(\sqrt{2sr}\,\cos\frac{\theta}{2}\big)$, when $\cos\frac{\theta}{2} \geq 0$.

Moreover, one observes the symmetry

$$u_{\text{diff}}(r,\theta,t) = u_{\text{diff}}(r,2\pi-\theta,t) = \frac{1}{2}\,\frac{1_{+}(t-r)}{\pi\,\sqrt{t-r}} \cdot \Big[-\frac{\sqrt{2r}\,\sin\theta}{t-r+2r\,\sin^2\theta}\Big]. \tag{5.3}$$

For $\sin\theta \geq 0$,

$$\tilde{u}_{\text{diff}}(r,\theta,s) = -\frac{1}{2}\,\mathrm{e}^{-sr\,\cos(2\theta)}\,\mathrm{erfc}(\sqrt{2sr}\,\sin\theta)$$

holds. Let us recall from Remark 6 and Corollary 5, respectively,

$$\tilde{u}_{\text{diff}}(x<0,y,s) = -\frac{\sqrt{2}}{\pi}\cos\frac{\theta}{2}\int\limits_{1}^{+\infty}\mathrm{e}^{sRx}\frac{\cos\big(sy\sqrt{R^2-1}\big)}{\sqrt{R-1}}\,\frac{dR}{R+\cos\theta} \tag{5.4}$$

This formula was derived by deforming the integral path into a branch cut line. It is of interest to recover $u_{\text{diff}}(r,\alpha,t)$ where the result is expected to coincide with that given by Theorem 3. After some simple rearrangements the Cagniard de Hoop method which means deforming of the integral path into a hyperbolic curve (instead of a straight line) will help again to succeed in this question. We want to consider the case $x<0$, so that

$$\left.\begin{aligned} x &= r\,\cos\alpha \\ y &= r\,\sin\alpha \end{aligned}\right\} \qquad (\tfrac{\pi}{2} < \alpha < \tfrac{3\pi}{2}). \tag{5.5}$$

After substitution $\xi = \sqrt{R^2-1} \geq 0$ the integral reads

$$\tilde{u}_{\text{diff}}(x,y,s) = -\frac{\sqrt{2}}{\pi}\,\cos\frac{\theta}{2}\int\limits_{0}^{\infty}\frac{\mathrm{e}^{-s|x|\sqrt{1+\xi^2}}}{\sqrt{1+\xi^2}}\,\cos(sy\xi)\,\frac{\sqrt{1+\sqrt{1+\xi^2}}}{\sqrt{1+\xi^2}+\cos\theta}\,d\xi. \tag{5.6}$$

This can be rewritten as a two-sided Laplace transform

$$\tilde{u}_{\text{diff}}(x,y,s) = -\frac{\sqrt{2}}{\pi}\,\frac{\cos\frac{\theta}{2}}{2\mathrm{i}}\int\limits_{\gamma_1-\mathrm{i}\infty}^{\gamma_1+\mathrm{i}\infty}\frac{\mathrm{e}^{-s(|x|\sqrt{1-\gamma^2}-y\gamma)}}{\sqrt{1-\gamma^2}}\,\frac{\sqrt{1+\sqrt{1-\gamma^2}}}{\sqrt{1-\gamma^2}+\cos\theta}\,d\gamma \tag{5.7}$$

which holds in a strip of $\mathbb{C}$ containing $\gamma_1 = 0$. We set (the time variable)

$$t := |x|\sqrt{1-\gamma^2} - y\gamma \tag{5.8}$$

and obtain in order to have $t \geq 0$ the Cagniard contour (hyperbola)

$$\gamma^{\pm} = -\frac{t}{r}\,\sin\alpha \mp \mathrm{i}\,\cos\alpha\,\sqrt{\frac{t^2}{r^2}-1} \qquad (\cos\alpha < 0).$$

We write this contour of integration in the more convenient form

$$\gamma^{\pm} = -\sin(\alpha \pm \mathrm{i}\varphi), \qquad \frac{t}{r} =: \cosh\varphi. \tag{5.9}$$

It follows

$$\frac{\frac{\partial\gamma^+}{\partial t}}{\sqrt{1-\gamma^{+2}}} = -\mathrm{i}\frac{\partial\varphi}{\partial t}, \qquad \frac{1}{r} = \sinh\varphi\,\frac{\partial\varphi}{\partial t}, \qquad \frac{\partial\varphi}{\partial t} = \frac{1}{\sqrt{t^2-r^2}}.$$

Choose $\gamma = \gamma^+$ with positive imaginary part. After calculating (cmp. (4.18))

$$F(\gamma)\,\frac{\partial\gamma}{\partial t} = \frac{\sqrt{1+\sqrt{1-\gamma^2}}}{\left(\sqrt{1-\gamma^2}+\cos\theta\right)\sqrt{1-\gamma^2}}\,\frac{\partial\gamma}{\partial t} = \mathrm{i}\,\frac{1}{\sqrt{t^2-r^2}}\,\frac{\sqrt{2}\,\sin\left(\frac{\alpha+\mathrm{i}\varphi}{2}\right)}{\cos\theta-\cos(\alpha+\mathrm{i}\varphi)}$$

one ends up with

$$\begin{aligned}\tilde{u}_{\text{diff}}(r,\alpha,s) &= -\frac{1}{\pi}\int\limits_r^{+\infty} \frac{\mathrm{e}^{-st}}{\sqrt{t^2-r^2}}\,\mathrm{Re}\left[\frac{\cos\frac{\theta}{2}\,\sin\frac{\alpha+i\varphi}{2}}{\sin^2\frac{\alpha+i\varphi}{2}-\sin^2\frac{\theta}{2}}\right] dt \\ &= \mathcal{L}_{t\mapsto s}\left[\frac{1_+(t-r)}{\pi\sqrt{t^2-r^2}}\cdot\,\mathrm{Re}\left[\frac{\cos\frac{\theta}{2}\,\sin\frac{\alpha+i\varphi}{2}}{\cos^2\frac{\alpha+i\varphi}{2}-\cos^2\frac{\theta}{2}}\right]\right],\end{aligned} \tag{5.10}$$

where the real-part term coincides with that of the expression given in (4.22).

Next we want to recover $u_{\text{diff}}(r,\alpha,t)$ as a representation with polar coordinates in the case $x \geq 0$. The corresponding Laplace integral to be under consideration (compare (4.10)) reads

$$\tilde{u}_{\text{diff}}(x\geq 0,y,s) = +\frac{\sqrt{2}}{\pi}\cos\frac{\theta}{2}\int\limits_1^{+\infty}\mathrm{e}^{-sRx}\,\frac{\sin\left(s|y|\sqrt{R^2-1}\right)}{\sqrt{R+1}}\,\frac{dR}{R-\,\cos\theta}. \tag{5.11}$$

We confine the considerations on $y \geq 0$ with $+0 \leq \alpha < \theta < \frac{\pi}{2}$. The case $0 < \theta < \alpha \leq \frac{\pi}{2}$ can be treated in the same manner. The case $y < 0$ with $\frac{3\pi}{2} < \alpha \leq 2\pi - 0$ can be done analogously by symmetry arguments.

The substitution $R := \cosh\beta$ leads to the integral

$$\begin{aligned}\tilde{u}_{\text{diff}}(x\geq 0,y,s) &= \frac{1}{\pi}\int\limits_0^{+\infty}\mathrm{e}^{-sx\cosh\beta}\,\sin(sy\sinh\beta)\,\frac{2\cos\frac{\theta}{2}\,\sinh\frac{\beta}{2}}{\cosh\beta-\cos\theta}\,d\beta \\ &= \mathrm{Im}\left[\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\mathrm{e}^{-s(x\cosh\beta-\mathrm{i}y\sinh\beta)}\,\frac{2\cos\frac{\theta}{2}\,\sinh\frac{\beta}{2}}{\cosh\beta-\cos\theta}\,d\beta\right].\end{aligned}$$

This reads with polar coordinates $x = r\cos\alpha$, $y = r\sin\alpha$ as

$$\begin{aligned}\tilde{u}_{\text{diff}}(r,\alpha,s) &= \mathrm{Re}\left[-\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\mathrm{e}^{-sr(\cos(\alpha+\mathrm{i}\beta))}\,\frac{2\cos\frac{\theta}{2}\,\sin\mathrm{i}\frac{\beta}{2}}{\cos\mathrm{i}\,\beta-\cos\theta}\,d\beta\right] \\ &= \mathrm{Re}\left[\frac{1}{2\pi}\int\limits_{-\infty-\mathrm{i}\alpha}^{+\infty-\mathrm{i}\alpha}\mathrm{e}^{-sr\cosh\varphi}\,\frac{2\cos\frac{\theta}{2}\,\sin\left(\frac{\alpha-\mathrm{i}\varphi}{2}\right)}{\cos(\alpha-\mathrm{i}\varphi)-\cos\theta}\,d\varphi\right].\end{aligned}$$

The last formula was obtained by the substitution $\varphi = \beta - \mathrm{i}\,\alpha$ and describes integrating parallel to the real line. Shifting the path of integration onto the real line yields

$$\begin{aligned}\tilde{u}_{\text{diff}}(r,\alpha,s) &= \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} \mathrm{e}^{-sr\cosh\varphi}\ \mathrm{Re}\left[\frac{2\cos\frac{\theta}{2}\,\sin\left(\frac{\alpha-\mathrm{i}\varphi}{2}\right)}{\cos(\alpha-\mathrm{i}\varphi)-\cos\theta}\right]d\varphi \\ &= \frac{1}{\pi}\int\limits_{0}^{+\infty} \mathrm{e}^{-sr\cosh\varphi}\ \mathrm{Re}\left[\frac{2\cos\frac{\theta}{2}\,\sin\left(\frac{\alpha-\mathrm{i}\varphi}{2}\right)}{\cos(\alpha-\mathrm{i}\varphi)-\cos\theta}\right]d\varphi.\end{aligned}$$

After introducing the real (time-) variable $t = r\cosh\varphi$ the integral takes the form

$$\tilde{u}_{\text{diff}}(r,\alpha,s) = \frac{1}{\pi}\int\limits_{r}^{+\infty} \frac{\mathrm{e}^{-st}}{\sqrt{t^2-r^2}}\,\mathrm{Re}\left[\frac{2\cos\frac{\theta}{2}\,\sin\left(\frac{\alpha-\mathrm{i}\varphi}{2}\right)}{\cos(\alpha-\mathrm{i}\varphi)-\cos\theta}\right]dt$$

to yield the original function in the time domain

$$\begin{aligned}u_{\text{diff}}(r,\alpha,t) &= \frac{1_+(t-r)}{\pi\sqrt{t^2-r^2}}\cdot\mathrm{Re}\left[\frac{2\cos\frac{\theta}{2}\,\sin\left(\frac{\alpha-\mathrm{i}\varphi}{2}\right)}{\cos(\alpha-\mathrm{i}\varphi)-\cos\theta}\right] \\ &= \frac{1_+(t-r)}{\pi\sqrt{t^2-r^2}}\cdot\mathrm{Re}\left[\frac{\cos\frac{\theta}{2}\,\sin\left(\frac{\alpha-\mathrm{i}\varphi}{2}\right)}{\cos^2\left(\frac{\alpha-\mathrm{i}\varphi}{2}\right)-\cos^2\frac{\theta}{2}}\right],\end{aligned} \tag{5.12}$$

the last term representing the complex conjugate of the expression we faced already in equation (4.22).

Acknowledgement. K. Rottbrand acknowledges support through the *Deutsche Forschungsgemeinschaft* (DFG) under grant number ME 261/13-1.

References

[1] Achenbach, J. D.: *Wave propagation in elastic solids.* Amsterdam: North-Holland Publ. Comp. 1984.

[2] Ali Mehmeti, F.: *Transient Tunnel Effect and Sommerfeld Problem: Waves in Semi-Infinite Structures.* Habilitation Thesis. Darmstadt: Techn. Univ. 1995. Published as: Mathematical Research: Vol. 91. Berlin: Akademie-Verlag 1996.

[3] Ali Mehmeti, F., Meister, E. and K. Mihalinčić: *Spectral theory for the wave equation in two adjacent wedges.* Math. Meth. Appl. Sci. 20 (1997), 1015 – 1044.

[4] Baker, B. B. and E. T. Copson: *The Mathematical Theory of Huygens' Principle.* Oxford: Clarendon Press 1950.

[5] Budaev, B.: *Diffraction by Wedges* (Pitman Research Notes in Mathematics Series). New York: Longman Sci. & Techn., John Wiley & Sons 1995.

[6] Èskin, G. I.: *Boundary Value Problems for Elliptic Pseudodifferential Equations.* Providence (R.I.): Amer. Math. Soc. 1981.

[7] Hoop, A. T. de: *Representation Theorems for the Displacement in an Elastic Solid and their Application to Elastodynamic Diffraction Theory.* Ph.D. Thesis. Delft (The Netherlands): Univ. Techn. 1958.

[8] Meister, E.: *Randwertaufgaben der Funktionentheorie.* Stuttgart: Teubner 1983.

[9] Meister, E.: *Integraltransformationen mit Anwendungen auf Probleme der mathematischen Physik.* Frankfurt/Main: Verlag Peter Lang 1983.

[10] Meister, E. and F.-O. Speck: *Modern Wiener-Hopf methods in diffraction theory.* In: Ordinary and Partial Differential Equations, Vol. 2. Proceedings of a Conference in Dundee (eds.: B. D. Sleeman and R. J. Jarvis). London: Longman 1989, pp. 130 – 172.

[11] Meister, E., Speck, F.-O. and F. S. Teixeira: *Wiener-Hopf-Hankel operators for some wedge diffraction problems with mixed boundary conditions.* J. Int. Eq. Appl. 4 (1992), 229 – 255.

[12] Rottbrand, K.: *A Canonical diffraction problem with two media.* Math. Meth. Appl. Sci. 19 (1996), 1217 – 1224.

[13] Rottbrand, K.: *Rawlins' problem for half-plane diffraction: its generalized eigenfunctions with real wave numbers.* Math. Meth. Appl. Sci. 20 (1997), 989 – 1014.

[14] Rottbrand, K.: *Time-dependent plane wave diffraction by a half-plane: explicit solution for Rawlins' mixed initial boundary value problem.* Z. Ang. Math. Mech. 78 (1998), 321 – 334.

[15] Sommerfeld, A.: *Theoretisches über die Beugung der Röntgenstrahlen.* Z. Math. Phys. 46 (1901), 11 - 97.

[16] v. Wolfersdorf, L.: *On an electromagnetic inverse problem for dispersive media.* Quart. Appl. Math. 49 (1991), 237 – 246.

Received 17.03.1998