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Existence and Uniqueness of Solutions to a Class of Stochastic Functional Partial Differential Equations via Integral Contractors

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Abstract. Existence and uniqueness theorem for first order stochastic functional partial differential equations with the white noise as a coefficient is proved. In the proof the characteristics method and the concept of integral contractors are used.

Keywords: *Stochastic functional partial differential equations, integral contractors*

AMS subject classification: 60 H 15, 35 R 60, 35 R 10

1. Introduction

Let $\mathbb{R}^m$ denote the m-dimensional Euclidean space with the norm $|\cdot|$, and let $B = [-r,0] \times [-d,+d]$ where $r \geq 0$ and $d = (d_1,...,d_m) \in \mathbb{R}^m_+$ (in particular, it may be $d_i = \infty$ for some i, $1 \leq i \leq m$) and $\mathbb{R}_+ = [0,\infty)$. Put $I = [0,T] \times \mathbb{R}^m$, $I_0 = [-r,0] \times \mathbb{R}^m$ and $D = I \cup I_0$, where $T > 0$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. We assume that there is a set of sub-σ-algebras $\mathcal{F}_t$ $(t \in [0,T])$ in $\mathcal{F}$ such that $\mathcal{F}_s \subset \mathcal{F}_t$ if $s \leq t$. Let $w(t;\omega)$ be a p-dimensional standard Brownian motion process adapted to $\mathcal{F}_t$ such that $\mathcal{F}\big(w(t+h;\omega) - w(t;\omega), h > 0\big)$ is independent on $\mathcal{F}_t$ $(t \in [0,T])$, where $\mathcal{F}(w(\xi), \xi > 0)$ denotes the σ-algebra generated by the process $w(\xi)$ $(\xi > 0)$. For any function $u : D \times \Omega \to \mathbb{R}^n$ and a fixed $(t,x;\omega) \in I \times \Omega$ we define the *Hale-type operator* $u_{(t,x)}(\omega) : B \to \mathbb{R}^n$ by

$$u_{(t,x)}(\omega)(\tau,\theta) = u(t+\tau, x+\theta;\omega) \qquad \big((\tau,\theta) \in B, \omega \in \Omega\big).$$

Let L_2 be the space of all random variables $\xi : \Omega \to \mathbb{R}^n$ with finite L_2-norm $||\xi||_2 = \{E|\xi|^2\}^{\frac{1}{2}}$, where E is an expectation. Denote by $C_B = C(B, L_2)$ the space of all continuous processes $v : B \to L_2$. Let $\mathcal{L}(\mathbb{R}^p, \mathbb{R}^m)$ be the space of all linear maps from $\mathbb{R}^p$ into $\mathbb{R}^m$.

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Let us consider the functional partial differential equations of first order with a random coefficient

$$\left.\begin{aligned} &\frac{\partial u}{\partial t}(t,x;\omega) + \big\{a(t,x;\omega) + b(t,x;\omega)\dot{w}(t;\omega)\big\}\frac{\partial u}{\partial x}(t,x;\omega) \\ &\quad = f\big(t,x,u_{(t,x)}(\omega);\omega\big) + g\big(t,x,u_{(t,x)}(\omega);\omega\big)\dot{w}(t;\omega) \quad ((t,x)\in I) \\ &u(t,x;\omega) = \varphi(t,x;\omega) \quad ((t,x)\in I_0, \omega\in\Omega) \end{aligned}\right\} \tag{1}$$

where

$$\begin{aligned} &a : I\times\Omega \to \mathbb{R}^m \\ &b : I\times\Omega \to \mathcal{L}(\mathbb{R}^p,\mathbb{R}^m) \\ &f : I\times C_B\times\Omega \to \mathbb{R}^n \\ &g : I\times C_B\times\Omega \to \mathcal{L}(\mathbb{R}^p,\mathbb{R}^n) \\ &\varphi : I_0\times\Omega \to \mathbb{R}^n \end{aligned}$$

and $\dot{w}(t;\omega)$ is the formal derivative of the process $w(t;\omega)$, namely the so called *white noise*. Equation (1) contains as particular cases the equations investigated in [3, 5 - 8].

Now, we consider the stochastic functional integral equation

$$\left.\begin{aligned} u(t,x) &= \varphi(0,y(0;t,x)) \\ &\quad + \int_0^t f\big(s,y(s;t,x),u_{(s,y(s;t,x))}\big)\,ds \\ &\quad + \int_0^t g\big(s,y(s;t,x),u_{(s,y(s;t,x))}\big)\,dw(s) \quad \big((t,x)\in I\big) \\ u(t,x) &= \varphi(t,x) \quad \big((t,x)\in I_0\big) \end{aligned}\right\} \tag{2}$$

where $y(s;t,x)$ is a solution of the stochastic integral equation

$$y(s) = x + \int_s^t a(\tau,y(\tau))\,d\tau + \int_s^t b(\tau,y(\tau))\,dw(\tau). \tag{3}$$

We can notice the close analogy between our consideration and the common theory of partial differential equations of first order. In this sense we call the stochastic process $\{y(s;t,x), s\le t\}$ the characteristic line of equation (1) through (t,x) and equation (2) can be considered as equation (1) integrated along the characteristic line.

The concept of contractors by Altman [1] has been used by Constantin [2] to prove the existence and uniqueness solutions of a stochastic integral equation. A particular case of equation (2) has been studied in [4] under the condition that the functions f and g satisfy a Lipschitz condition with respect to the last variable (see also [5]). In this paper, using the characteristics and integral contractors methods, we obtain more general conditions for the existence and uniqueness of solutions to equation (2).

2. The existence of characteristic lines

Denote by $C = C([0,T], L_2)$ the space of all processes $y : [0,T] \to L_2$ which are continuous and adapted to the $\mathcal{F}_t$ $(t \in [0,T])$. We consider on C the norm $||y|| = \sup_{t\in[0,T]} ||y(t)||_2$.

Define the integral operators $\tilde{J}_1$ and $\tilde{J}_2$ on C by

$$(\tilde{J}_1 y)(t) = \int_0^t y(s)\, ds$$
$$(\tilde{J}_2 y)(t) = \int_0^t y(s)\, dw(s).$$

It is easy to seen (see [2]) that

$$\left.\begin{aligned} ||\tilde{J}_1 y|| &\le T||y|| \\ ||\tilde{J}_2 y|| &\le \sqrt{T}||y|| \end{aligned}\right\} \qquad (y \in C). \tag{4}$$

Assumption (H$_1$). Suppose the following:

(i) The functions $a : I \times \Omega \to \mathbb{R}^m$ and $b : I \times \Omega \to \mathcal{L}(\mathbb{R}^p, \mathbb{R}^m)$ are such that $a(\cdot, y(\cdot)), b(\cdot, y(\cdot)) \in C$ for all $y \in C$.

(ii) For each $t \in [0,T]$ and $y \in L_2$ there exist bounded linear operators $\tilde{\Gamma}_i(t,y)$ $(i = 1,2)$ on C such that $||\tilde{\Gamma}_i(t,y)||$ are continuous in (t,y) and there is a constant $\tilde{Q} > 0$ such that

$$\left\|\big(\tilde{\Gamma}_i(t, y(t))v\big)(t)\right\|_2 \le \tilde{Q}||v(t)||_2$$

for every $v \in C$.

(iii) There exist continuous functions $\tilde{\gamma}_i : \mathbb{R}_+ \to \mathbb{R}_+$ with $\tilde{\gamma}_i(0) = 0$ $(i = 1,2)$ such that for each $t \in [0,T]$ and $y, v \in C$ we have

$$\begin{aligned}\Big\|a\Big(t, y(t) + v(t) + \big(\tilde{J}_1\tilde{\Gamma}_1(t, y(t))v\big)(t) + \big(\tilde{J}_2\tilde{\Gamma}_2(t, y(t))v\big)(t)\Big)& \\ -a(t, y(t)) - (\tilde{\Gamma}_1\big(t, y(t))v\big)(t)\Big\|_2 &\le \tilde{\gamma}_1(||v(t)||_2)\end{aligned}$$

and

$$\begin{aligned}\Big\|b\Big(t, y(t) + v(t) + \big(\tilde{J}_1\tilde{\Gamma}_1(t, y(t))v\big)(t) + \big(\tilde{J}_2\tilde{\Gamma}_2(t, y(t))v\big)(t)\Big)& \\ -b(t, y(t)) - \big(\tilde{\Gamma}_2(t, y(t))v\big)(t)\Big\|_2 &\le \tilde{\gamma}_2(||v(t)||_2).\end{aligned}$$

The vector of functions (a, b) satisfying assumption (H$_1$) is said to have a *bounded integral vector contractor* $(\tilde{\Gamma}_1, \tilde{\Gamma}_2)$ with nonlinear majorants $(\tilde{\gamma}_1, \tilde{\gamma}_2)$ with respect to C.

Remark 1. If $\tilde{\gamma}_i = \alpha_i t$ $(t \in \mathbb{R}_+)$ where $\alpha_i > 0$ $(i = 1,2)$ are constants, we have that the vector functions (a, b) has a bounded integral vector contractor $(\tilde{\Gamma}_1, \tilde{\Gamma}_2)$. These conditions are weaker than the usual Lipschitz condition. Indeed, if $\tilde{\Gamma}_i = 0$ $(i = 1,2)$, the condition in assumption (H$_1$) reduces to Lipschitz condition on a and b.

Definition 1. Let $\mathcal{H}$ be the family of all functions $\gamma \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ satisfying $\gamma(0) = 0$ and $\gamma'(t) \in [0, 1)$ $(t \in \mathbb{R}_+)$.

Lemma 1 (see [2]). *If $\gamma \in \mathcal{H}$, we have that γ is non-decreasing on $\mathbb{R}_+$, $\gamma(t) < t$ for $t > 0$ and $\sum_{k=0}^{\infty} \gamma^{(k)}(t) < \infty$ for $t \in \mathbb{R}_+$, where $\gamma^{(k)}$ denotes the k-th iterate of γ.*

Remark 2. Examples of functions $\gamma \in \mathcal{H}$ are $\gamma(t) = \alpha t$ $(t \in \mathbb{R}_+)$, where $\alpha \in (0,1)$, $\gamma(t) = \frac{t^2}{t+1}$ $(t \in \mathbb{R}_+)$, $\gamma(t) = t - \arctan t$ $(t \in \mathbb{R}_+)$, and $\gamma(t) = t - ln(1+t)$ $(t \in \mathbb{R}_+)$.

Lemma 2 (see [2], but also [9]). *Let us suppose the following:*

(i) *The functions $a : I \times \Omega \to \mathbb{R}^m$ and $b : I \times \Omega \to \mathcal{L}(\mathbb{R}^p, \mathbb{R}^m)$ are such that if $y_n \to y$ in C, then $a(\cdot, y_n(\cdot)) \to a(\cdot, y(\cdot))$ and $b(\cdot, y_n(\cdot)) \to b(\cdot, y(\cdot))$ in C for $n \to \infty$.*

(ii) *Assumption* (H$_1$) *is satisfied with the vector of nonlinear majorants $(\tilde{\gamma}_1, \tilde{\gamma}_2)$ such that $T\tilde{\gamma}_1 + \sqrt{T}\tilde{\gamma}_2 = \tilde{\gamma} \in \mathcal{H}$.*

Then equation (3) *has a unique solution $y(s) = y(s; t, x)$ in C.*

Remark 3. Note that y satisfies the group property

$$y(s; \tau, y(\tau; t, x)) = y(s; t, x) \tag{5}$$

for $\tau \in [s, t]$ and $(t, x) \in I$, since $y(s; t, x)$ is the unique solution of equation (3).

3. Assumptions and lemma

Let $C_Y = C(Y, L_2)$ be the space of all processes $v : Y \to L_2$ which are continuous bounded and adapted to the $\mathcal{F}_t$ for each x, where $Y \subset I$ or $Y \subset D$ (let $\mathcal{F}_t = \mathcal{F}_0$ for $-r \le t \le 0$). We consider on C_Y the norm $||v||_Y = \sup_{(t,x)\in Y} ||v(t,x)||_2$. Define the integral operators J_1 and J_2 on C_I by

$$(J_1 u)(t, x) = \int_0^t u(s, x)\, ds$$
$$(J_2 u)(t, x) = \int_0^t u(s, x)\, dw(s).$$

For these operators we have analogous estimates as in (4)

$$\left.\begin{array}{l} ||J_1 u||_I \le T||u||_I \\ ||J_2 u||_I \le \sqrt{T}||u||_I \end{array}\right\} \qquad (u \in C_I).$$

Put

$$f[u](s; t, x) = f\big(s, y(s; t, x), u_{(s, y(s;t,x))}\big)$$
$$g[u](s; t, x) = g\big(s, y(s; t, x), u_{(s, y(s;t,x))}\big).$$

Assumption (H$_2$). Suppose the following:

(i) The functions $f : I \times C_B \times \Omega \to \mathbb{R}^n$ and $g : I \times C_B \times \Omega \to \mathcal{L}(\mathbb{R}^p, \mathbb{R}^n)$ are such that $f[u](\cdot) \in C_I$ for $u \in C_B$ and $g[u](\cdot) \in C_I$ for $y \in C$.

(ii) For each $s \in [0,T]$, $y \in L_2$ and $u \in C_D$ there exist bounded linear operators $\Gamma_i(s,y,u)$ $(i=1,2)$ on C_D such that $||\Gamma_i(s,y,u)||$ are continuous in (s,y,u) and there is a constant $Q>0$ such that

$$\big\|(\Gamma_i[y,u]v)(s;t,x)\big\|_2 \le Q\big\|v(s,y(s;t,x))\big\|_2$$

for every $v \in C_D$, where

$$(\Gamma_i[y,u]v)(s;t,x) = \big(\Gamma_i\big(s,y(s;t,x),u_{(s,y(s;t,x))}\big)v\big)(s,y(s;t,x)).$$

(iii) There exist continuous functions $\gamma_i : \mathbb{R}_+ \to \mathbb{R}_+$ with $\gamma_i(0)=0$ $(i=1,2)$ such that for each $u,v \in C_D$ and $y \in C$ we have

$$\Big\|f\big[u+v+J_1\Gamma_1[y,u]v+J_2\Gamma_2[y,u]v\big](s;t,x) - f[u](s;t,x) - (\Gamma_1[y,u]v)(s;t,x)\Big\|_2$$

$$\le \gamma_1\big(||v_{(s,y(s;t,x))}||_B\big)$$

and

$$\Big\|g\big[u+v+J_1\Gamma_1[y,u]v+J_2\Gamma_2[y,u]v\big](s;t,x) - g[u](s;t,x) - (\Gamma_2[y,u]v)(s;t,x)\Big\|_2$$

$$\le \gamma_2\big(||v_{(s,y(s;t,x))}||_B\big)$$

for all $s \in [0,T]$ and $(t,x) \in I$.

Lemma 3. *If assumption* $(\mathrm{H}_2)/(ii)$ is satisfied and y is a solution of equation (3) and $u(\cdot,y(\cdot;t,x)) \in C_D$, then for every $h \in C_I$ such that $h(0;x)=0$, there is a unique solution $v \in C_D$ to the stochastic integral equation

$$\left.\begin{aligned} v(t;x) &= h(t;x) \\ &\quad - \int_0^t (\Gamma_1[y,u]v)(s;t,x)\,ds \\ &\quad - \int_0^t (\Gamma_2[y,u]v)(s;t,x)\,dw(s) \quad \big((t,x)\in I\big) \\ v(t,x) &= 0 \quad \big((t,x)\in I_0\big). \end{aligned}\right\} \tag{6}$$

Proof. Define an operator K on C_D as

$$(Kv)(t;x) = \begin{cases} h(t;x) - \int_0^t (\Gamma_1[y,u]v)(s;t,x)\,ds - \int_0^t (\Gamma_2[y,u]v)(s;t,x)\,dw(s) & \text{if } (t,x)\in I \\ 0 & \text{if } (t,x)\in I_0. \end{cases}$$

It is obvious that K maps C_D into itself. Let us introduce the norm

$$||v||_* = \sup_{(t,x)\in D}\big\{e^{-\lambda t}||v(t,x)||_2\big\}$$

where $\lambda > Q^2(T+1)$. Now we prove that K is a contraction. Indeed, since $(x+y)^2 \le 2(x^2+y^2)$ we have

$$\begin{aligned}
&\big\|(Kv_1)(t,x) - (Kv_2)(t,x)\big\|_2^2 \\
&\quad \le 2Q^2(T+1)\int_0^t \big\|v_1(s,y(s;t,x)) - v_2(s,y(s;t,x))\big\|_2^2 ds \\
&\quad \le 2Q^2(T+1)\|v_1 - v_2\|_*^2 \int_0^t e^{2\lambda s} ds \\
&\quad \le \frac{Q^2(T+1)}{\lambda}(e^{2\lambda t} - 1)\|v_1 - v_2\|_*^2
\end{aligned} \tag{7}$$

for all $(t,x) \in I$. Multiplying (7) by $e^{-2\lambda t}$ we obtain

$$||Kv_1 - Kv_2||_*^2 \le \frac{Q^2(T+1)}{\lambda}||v_1 - v_2||_*^2$$

thus

$$||Kv_1 - Kv_2||_* \le q||v_1 - v_2||_*$$

where $q = \big[\frac{Q^2(T+1)}{\lambda}\big]^{\frac{1}{2}}$. The assertion of Lemma 3 now follows from the Banach fixed point theorem ∎

4. The main results

We are now in the position to prove the main results.

Theorem. *Let us suppose the following:*

(i) *$\varphi : I_0 \to L_2$ is continuous and $\mathcal{F}_0$-adapted for each x, and $\varphi(0, y(0;t,x))$ is independent on $\{w(t),\, t \in [0,T]\}$ for each x.*

(ii) *The functions $f : I \times C_B \times \Omega \to \mathbb{R}^n$ and $g : I \times C_B \times \Omega \to \mathcal{L}(\mathbb{R}^p, \mathbb{R}^n)$ are such that if $u^n \to u$ in C_B, then $f[u^n](\cdot) \to f[u](\cdot)$ and $g[u^n](\cdot) \to g[u](\cdot)$ for $n \to \infty$.*

(iii) *Assumptions of Lemma 2 are satisfied.*

(iv) *Assumption* (H_2) *is satisfied with the vector of nonlinear majorants (γ_1, γ_2) such that $T\gamma_1 + \sqrt{T}\gamma_2 = \gamma \in \mathcal{H}$.*

Then equation (2) *has a unique solution in C_D.*

Proof. Consider the sequence $\{u^n\}$ defined by

$$\left.\begin{aligned}
u^{n+1}(t,x) &= u^n(t,x) - v^n(t,x) \\
&\qquad - \int_0^t (\Gamma_1[y,u^n]v^n)(s;t,x)\,ds \\
&\qquad - \int_0^t (\Gamma_2[y,u^n]v^n)(s;t,x)\,dw(s) \quad ((t,x) \in I) \\
u^{n+1}(t,x) &= \varphi(t,x) \quad ((t,x) \in I_0)
\end{aligned}\right\} \tag{8}$$

where

$$\left.\begin{aligned} v^n(t,x) &= u^n(t,x) - \varphi(0, y(0;t,x)) \\ &\quad - \int_0^t f[u^n](s;t,x)\,ds \\ &\quad - \int_0^t g[u^n](s;t,x)\,dw(s) \quad ((t,x) \in I) \\ v^n(t,x) &= 0 \quad ((t,x) \in I_0) \end{aligned}\right\} \tag{9}$$

and $u^0 \in C_D$. We will now demonstrate that the auxiliary sequence $\{v^n\}$ is such that $||v^n||_D \to 0$ as $n \to \infty$. By (8) - (9) applying (5) we deduce that

$$\begin{aligned} v^{n+1}(t,x) &= \int_0^t f[u^n](s;t,x)\,ds + \int_0^t g[u^n](s;t,x)\,dw(s) \\ &\quad - \int_0^t (\Gamma_1[y,u^n]v^n)(s;t,x)\,ds - \int_0^t (\Gamma_2[y,u^n]v^n)(s;t,x)\,dw(s) \\ &\quad - \int_0^t f\Big[u^n - v^n - J_1\Gamma_1[y,u^n]v^n - J_2\Gamma_2[y,u^n]v^n\Big](s;t,x)\,ds \\ &\quad - \int_0^t g\Big[u^n - v^n - J_1\Gamma_1[y,u^n]v^n - J_2\Gamma_2[y,u^n]v^n\Big](s;t,x)\,dw(s) \\ &\qquad ((t,x) \in I) \\ v^{n+1}(t,x) &= 0 \quad ((t,x) \in I_0). \end{aligned}$$

Using assumption (H$_2$)/(iii) on f and g we obtain

$$\begin{aligned} ||v^{n+1}||_D &\le T\Big\| f\left[u^n - v^n - J_1\Gamma_1[y,u^n]v^n - J_2\Gamma_2[y,u^n]v^n\right](s;t,x) \\ &\qquad - f[u^n](s;t,x) - \Gamma_1[y,u^n](-v^n)(s;t,x)\Big\|_D \\ &\qquad + \sqrt{T}\Big\| g\left[u^n - v^n - J_1\Gamma_1[y,u^n]v^n - J_2\Gamma_2[y,u^n]v^n\right](s;t,x) \\ &\qquad - g[u^n](s;t,x) - \Gamma_2[y,u^n](-v^n)(s;t,x)\Big\|_D \\ &\le \gamma\big(||v^n_{(s,y(s;t,x))}||_B\big) \\ &\le \gamma(||v^n||_D) \end{aligned}$$

since $||v^n_{(t,x)}||_B \le ||v^n||_D$. Thus $||v^{n+1}||_D \le \gamma^{(n+1)}(||v^0||_D)$. Since $lim_{n\to\infty}\gamma^{(n)}(t) = 0$ for all $t \in \mathbb{R}_+$ (see Lemma 1) we get $lim_{n\to\infty}||v^n||_D = 0$.

From (8) we see that

$$\begin{aligned} ||u^{n+1} - u^n||_D &\le ||v^n||_D + (T + \sqrt{T})Q||v^n||_D \\ &\le \\ &\vdots \\ &\le \big(1 + TQ + \sqrt{T}Q\big)\gamma^{(n)}(||v^0||_D). \end{aligned}$$

Since $\gamma^{(n)}(||v^0||_D) \to 0$ as $n \to \infty$, we get that $\{u^n\}$ is a Cauchy sequence and thus there exists $u \in C_D$ such that $lim_{n\to\infty} u^n = u$.

From (9) and assumption (ii) it follows that u is a solution of equation (2). Let us now prove the uniqueness of solutions to equation (2). Let $u, \bar{u}$ be two solutions in C_D of equation (2) with $u(t,x) = \bar{u}(t,x) = \varphi(t,x)$ for $(t,x) \in I_0$. Then

$$\left.\begin{aligned} u(t,x) - \bar{u}(t,x) &= \int_0^t \big\{f[u](s;t,x) - f[\bar{u}](s;t,x)\big\}ds \\ &\quad + \int_0^t \big\{g[u](s;t,x) - g[\bar{u}](s;t,x)\big\}dw(s) \quad ((t,x) \in I) \\ u(t,x) - \bar{u}(t,x) &= 0 \quad ((t,x) \in I_0). \end{aligned}\right\} \tag{10}$$

We denote $h(t,x) = u(t,x) - \bar{u}(t,x) \quad ((t,x) \in I)$, and let $v \in C_D$ be a solution to the stochastic integral equation (see Lemma 3)

$$\left.\begin{aligned} v(t,x) &= h(t,x) \\ &\quad - \int_0^t (\Gamma_1[y,\bar{u}]v)(s;t,x)\,ds \\ &\quad - \int_0^t (\Gamma_2[y,\bar{u}]v)(s;t,x)\,dw(s) \quad ((t,x) \in I) \\ v(t,x) &= 0 \quad ((t,x) \in I_0). \end{aligned}\right\} \tag{11}$$

By (10), (11) and (5) we get

$$\begin{aligned} ||v||_D &\le \bigg\| \int_0^t \Big\{ f\big[\bar{u} + v + J_1\Gamma_1[y,\bar{u}]v + J_2\Gamma_2[y,\bar{u}]v)\big](s;t,x) \\ &\qquad - f[\bar{u}](s;t,x) - (\Gamma_1[y,\bar{u}]v)(s;t,x)\Big\}ds \bigg\|_I \\ &\qquad + \bigg\| \int_0^t \Big\{ g\big[\bar{u} + v + J_1\Gamma_1[y,\bar{u}]v + J_2\Gamma_2[y,\bar{u}]v)\big](s;t,x) \\ &\qquad - g[\bar{u}](s;t,x) - (\Gamma_2[\bar{u}]v)(s;t,x)\Big\}ds \bigg\|_I \\ &\le \gamma(||v||_D). \end{aligned}$$

Hence (see Lemma 1) $v(t,x) = 0$ a.s., $(t,x) \in D$, and so by (11) we obtain $h(t,x) = 0$, a.s., $(t,x) \in D$, i.e. $u = \bar{u}$ in C_D and the uniqueness is proved. This completes the proof of the Theorem ∎

Corollary. *Let assumptions* (i), (iii) *and* (iv) *of Theorem be satisfied. If* $3(T^2 + T)Q^2 < 1$, *then equation* (2) *has a unique solution in* C_D.

Proof. Let us prove that if $u^n \to u$ in C_B, then $f[u^n] \to f[u]$ and $g[u^n] \to g[u]$ in

C_I for $n \to \infty$. Let $v^n \in C_D$ be a solution to the stochastic integral equation

$$\left.\begin{aligned} v^n(t,x) &= u^n(t,x) - u(t,x) \\ &\quad - \int_0^t (\Gamma_1[y,u]v^n)(s;t,x)\,ds \\ &\quad - \int_0^t (\Gamma_2[y,u]v^n)(s;t,x)\,dw(s) \quad ((t,x)\in I) \\ v^n(t,x) &= 0 \quad ((t,x)\in I_0). \end{aligned}\right\}$$

The existence of a such solution follows from Lemma 3. Since $(x+y+z)^2 \leq 3(x^2+y^2+z^2)$ we have

$$\begin{aligned} ||v^n(t,x)||_2^2 &\leq 3\big\|u^n(t,x) - u(t,x)\big\|_2^2 \\ &\quad + 3TQ^2\int_0^t \big\|v^n(s,y(s;t,x))\big\|_2^2 ds + 3Q^2\int_0^t \big\|v^n(s,y(s;t,x))\big\|_2^2 ds \end{aligned}$$

therefore

$$||v^n||_D^2 \leq 3||u^n - u||_D^2 + 3(T^2+T)Q^2||v^n||_D^2.$$

Since $3(T^2+T)Q^2 < 1$ and $\lim_{n\to\infty} ||u^n - u||_D = 0$ we obtain $\lim_{n\to\infty} ||v^n||_D = 0$.

Writing relation (iii) from Assumption (H_2) with u and v^n, we get

$$\Big\|f\big[u+v^n+J_1\Gamma_1[y,u]v^n+J_2\Gamma_2[y,u]v^n\big](s;t,x) - f[u](s;t,x) - (\Gamma_1[y,u]v^n)(s;t,x)\Big\|_2$$
$$\leq \gamma_1(||v^n_{(s,y(s;t,x))}||_B)$$

and

$$\Big\|g\big[u+v^n+J_1\Gamma_1[y,u]v^n+J_2\Gamma_2[y,u]v^n\big](s;t,x) - g[u](s;t,x) - (\Gamma_2[y,u]v^n)(s;t,x)\Big\|_2$$
$$\leq \gamma_2(||v^n_{(s,y(s;t,x))}||_B).$$

Thus

$$\begin{aligned} \big\|f[u^n](s;t,x) - f[u](s;t,x)\big\|_2 &\leq \gamma_1\big(\big\|v^n_{(s,y(s;t,x))}\big\|_B\big) + Q\big\|v^n(s,y(s;t,x))\big\|_2 \\ \big\|g[u^n](s;t,x) - g[u](s;t,x)\big\|_2 &\leq \gamma_2\big(\big\|v^n_{(s,y(s;t,x))}\big\|_2\big) + Q\big\|v^n(s,y(s;t,x))\big\|_2. \end{aligned}$$

Hence we get

$$\big\|f[u^n] - f[u]||_I + ||g[u^n] - g[u]\big\|_I \leq \sup_{\tau\in[0,||v^n||_D]} \{\gamma_1(\tau)+\gamma_2(\tau)\} + 2Q||v^n||_D$$

and since $v^n \to 0$ as $n \to \infty$ in C_D and $\gamma_i \;\; (i=1,2)$ are continuous with $\gamma_i(0) = 0$ we obtain the desired property of f and g. The result now follows from Theorem ∎

5. Particular forms of functional dependence

We give now a few examples that show how the Hale-type operator defined in Introduction acts in particular forms of functional dependence such as delays, integrals and other Volterra functionals.

Example 1. Let $\alpha, \beta : I \to \mathbb{R}^{m+1}$, $\alpha = (\alpha_0, \alpha_1, ..., \alpha_m)$ and $\beta = (\beta_0, \beta_1, ..., \beta_m)$, and let $\tilde{f} : I \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ and $\tilde{g} : I \times \mathbb{R}^n \times \Omega \to \mathcal{L}(\mathbb{R}^p, \mathbb{R}^n)$ be given functions such that $-r \leq \alpha_0(t,x)$ and $\beta_0(t,x) \leq t$ for $(t,x) \in I$. If we define

$$
\begin{aligned}
f(t,x,v) &= \tilde{f}\big(t, x, v\big(\alpha(t,x) - (t,x)\big)\big) \\
g(t,x,v) &= \tilde{g}\big(t, x, v\big(\beta(t,x) - (t,x)\big)\big)
\end{aligned}
$$

then equation (1) reduces to the differential equation with retarded argument

$$
\begin{aligned}
&\frac{\partial u}{\partial t}(t,x) + \big\{a(t,x) + b(t,x)\dot{w}(t)\big\}\frac{\partial u}{\partial x}(t,x) \\
&\qquad = f\big(t, x, u(\alpha(t,x))\big) + g\big(t, x, u(\beta(t,x))\big)\dot{w}(t)
\end{aligned}
\qquad ((t,x) \in I).
$$

Example 2. Suppose that $\tilde{f} : I \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n$, $\tilde{g} : I \times \mathbb{R}^n \times \Omega \to \mathcal{L}(R^p, R^n)$, $\alpha, \beta : I \to \mathbb{R}^{m+1}$ and $k_i : I \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ $(i = 1, 2)$ are the given functions. Let

$$
\begin{aligned}
f(t,x,v) &= \tilde{f}\left(t, x, \int_{\alpha(t,x)}^{\beta(t,x)} k_1(s,\xi) v(s-t, \xi - x)\, ds d\xi\right) \\
g(t,x,v) &= \tilde{g}\left(t, x, \int_{\alpha(t,x)}^{\beta(t,x)} k_2(s,\xi) v(s-t, \xi - x)\, ds d\xi\right).
\end{aligned}
$$

Then equation (1) reduces to the differential-integral equation

$$
\begin{aligned}
&\frac{\partial u}{\partial t}(t,x) + \big\{a(t,x) + b(t,x)\dot{w}(t)\big\}\frac{\partial u}{\partial x}(t,x) \\
&\qquad = f\left(t, x, \int_{\alpha(t,x)}^{\beta(t,x)} k_1(s,\xi) u(s,\xi)\, ds d\xi\right) \\
&\qquad\quad + g\left(t, x, \int_{\alpha(t,x)}^{\beta(t,x)} k_2(s,\xi) u(s,\xi) ds d\xi\right)\dot{w}(t)
\end{aligned}
\qquad ((t,x) \in I).
$$

Example 3. Take $\tilde{f} : I \times C(D, \mathbb{R}^n) \times \Omega \to \mathbb{R}^n$ and $\tilde{g} : I \times C(D, \mathbb{R}^n) \times \Omega \to \mathcal{L}(\mathbb{R}^p, \mathbb{R}^n)$. Consider the equation

$$
\frac{\partial u}{\partial t}(t,x) + \big\{a(t,x) + b(t,x)\dot{w}(t)\big\}\frac{\partial u}{\partial x}(t,x) = f(t,x,u) + g(t,x,u)\dot{w}(t) \quad ((t,x) \in I).
$$

The dependence on the past is expressed by means of so called Volterra condition which reads as follows: if $u, \bar{u} \in C_D$ and $u(s,x) = \bar{u}(s,x)$ for $(s,x) \in [-r,t] \times \mathbb{R}^m$, then $\tilde{f}(t,x,u) = \tilde{f}(t,x,\bar{u})$. The definition of the Volterra condition for $\tilde{g}$ is analogous. There are various possibilities of extending this notation. For instance, if we want to describe the dependence of $\tilde{f}$ locally on the past and locally on the space, then we can formulate the Volterra-type condition as follows: if $u, \bar{u} \in C_D$ and $u(s,\xi) = \bar{u}(s,\xi)$ for $(s,\xi) \in B + (t,x)$, then $\tilde{f}(t,x,u) = \tilde{f}(t,x,\bar{u})$, where $B + (t,x) = \{(s+t, \xi+x) : (s,\xi) \in B\}$ is the translation of the set B. In this case we can define

$$
\begin{aligned}
f(t,x,v) &= \tilde{f}(t,x,\mathcal{I}_{t,x} v(\cdot - t, \cdot - x)) \\
g(t,x,v) &= \tilde{g}(t,x,\mathcal{I}_{t,x} v(\cdot - t, \cdot - x))
\end{aligned}
$$

where $\mathcal{I}_{t,x} : C_B \to C_{D+(-t,-x)}$ is defined by $(\mathcal{I}_{t,x} v)(s,\xi) = v(s-t, \xi-x)$.

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