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ON NORMAL SUBGROUPS OF THE GROUP REPRESENTATION OF THE CAYLEY TREE[#]

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Abstract. Gibbs measure plays an important role in statistical mechanics. On a Cayley tree, for describing periodic Gibbs measures for models in statistical mechanics we need subgroups of the group representation of the Cayley tree. A normal subgroup of the group representation of the Cayley tree keeps the invariance property which is a significant tool in finding Gibbs measures. By this occasion, a full description of normal subgroups of the group representation of the Cayley tree is a significant problem in Gibbs measure theory. For instance, in [1, 2] a full description of normal subgroups of indices four, six, eight, and ten for the group representation of a Cayley tree is given. The present paper is a generalization of these papers, i. e., in this paper, for any odd prime number p, we give a characterization of the normal subgroups of indices $2n$, $n \in \{p, 2p\}$ and $2^i, i \in \mathbb{N}$, of the group representation of the Cayley tree.

Keywords: Cayley tree, G_k-group, subgroups of finite index, abelian group, homomorphism.

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1. Introduction

In group theory, there are some significant open problems, the majority of which arise in solving of problems of sciences such as physics, biology, chemistry, etc. Especially, if the configuration of the particle and lattice system is located on a graph such as lattice, tree, etc (in our case regular tree) then the configuration can be considered as a mapping which is defined on the graph. As usual, the main configurations (mappings) are the periodic ones. It is known that if the graph has a group representation then the periodicity of a mapping can be defined by the given subgroup of the representation. Namely, if H is a given subgroup then we can define a H-periodic mapping, which has a constant value (depending only on the coset) on each (right or left) coset of H. So the periodicity is related to a partition of the group (that presents the graph on which our physical system is located). There are many research manuscripts devoted to several kinds of partitions of groups (lattices) (detail in [3–6]).

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The Gibbs measure is a probability measure, which has been an important object in many problems of probability theory and statistical mechanics. In turn, there are many papers which is devoted to periodic and weakly periodic Gibbs measures. In Ref. [5] a bijection between the set of vertices V of the Cayley tree Γ^k and the group G_k is given. Also, a full description of normal subgroups of index two is found and some normal subgroups of the group G_k are constructed. To define periodic and weakly periodic Gibbs measures we need subgroups of G_k. In [7], invariance property of subgroups of group representation of Cayley trees is given and by using this property, the description of the set of periodic or weakly periodic Gibbs measures for Hamiltonians with finite spin values on Cayley trees is reduced to solve the system of algebraic equations. In [8, 9], the problem of describing periodic or weakly periodic Gibbs measures for statistical models on Cayley trees is reduced to solve the system of algebraic equations. If the invariance property holds for any subgroup of the group G_k then we have the opportunity of finding periodic and weakly periodic Gibbs measures corresponding to an arbitrary subgroup of finite index for the group G_k. Also, for any normal subgroup of finite index for the group representation of Cayley tree, the invariance property holds but to study periodic and weakly periodic Gibbs measure we need the exact view of normal subgroups. That is why, we need the description of normal subgroups of finite index (without index two) and to the best of our knowledge there was are full description of a (not normal) subgroup of index 4 of the group representation of the Cayley tree is given in [10]. In [1] and [2] full descriptions of normal subgroups of indices $2i$, $i \in \{2, 3, 4, 5\}$, for the group G_k are given.

In this paper, we continue this investigation and construct all normal subgroups of index $2n$, $n \in \{p, 2p\}$, and 2^i, $i \in \mathbb{N}$, for the group representation of the Cayley tree, where p is an odd prime number.

2. Preliminaries

A Cayley tree (Bethe lattice) Γ^k of degree (order) $k \geqslant 1$ is a $k+1$-regular tree, i. e., a connected, non-directed, acyclic graph with degree of every vertex is $k+1$. We denote Cayley tree of degree $k+1$ by $\Gamma^k = (V, L)$ where V and L are the set of vertices and edges respectively.

Let $(G_k := \langle a_1, a_2, \ldots, a_{k+1} \rangle, *)$ be a group such that $o(a_i) = 2$, $i \in N_k := \{1, 2, \ldots, k+1\}$, the operation $*$ is a free product. It is known that there exists a bijection from the set of vertices V of the Cayley tree Γ^k to the set of element of the group G_k. To give this correspondence we fix an element $x_0 \in V$ and let it correspond to the identity element e (i. e., the length of element equals zero) of the group G_k. In positive direction, we label the nearest-neighbors of element e by $a_1, \ldots, a_{k+1}$. Let us label the neighbors of each $a_i, i = 1, \ldots, k+1$ by $a_i a_j$, $j = 1, \ldots, k+1$. Since all a_i have the common neighbor e we have $a_i a_i = a_i^2 = e$. Other neighbors are labeled starting from $a_i a_i$ in positive direction. We label the set of all the nearest-neighbors of each $a_i a_j$ by words $a_i a_j a_q$, $q = 1, \ldots, k+1$, starting from $a_i a_j a_j = a_i$ by the positive direction. We continue the process and give bijection from the set of vertices V of the Cayley tree Γ^k to the group G_k.

Any (minimal represented) element $x \in G_k$ has the following form: $x = a_{i_1} a_{i_2} \ldots a_{i_n}$, where $1 \leqslant i_m \leqslant k+1, m = 1, \ldots, n$. The number n is called the length of the word x and is denoted by $l(x)$. The number of letters $a_i, i = 1, \ldots, k+1$, that enter the non-contractible representation of the word x is denoted by $w_x(a_i)$.

The following result is well-known in group theory.

Let f be a homomorphism of a group G onto a group G_1, $H \lhd G$ such that $H \subseteq \operatorname{Ker} f$, and g be the natural homomorphism of G onto G/H. Then there exists a unique homomorphism h of G/H onto G_1 such that $f = h \circ g$. Furthermore, h is one-one if and only if $H = \operatorname{Ker} f$.

Put $H \lhd G$ and g from G to G/H by $g(a) = aH$ for all $a \in G$. From group theory (e. g., [11]), g is an epimorphism from G onto G/H with $\operatorname{Ker} g = H$.

One of our aim in this paper, we shall give a full description of normal subgroups of finite index of the group G_k.

Let $A_1, A_2, \dots, A_m$ be subsets N_k and $A_i \neq A_j$, for $i \neq j$. The intersection is said "contractible" if there exists i_0 ($1 \leqslant i_0 \leqslant m$) such that

$$\bigcap_{i=1}^{m} A_i = \left(\bigcap_{i=1}^{i_0-1} A_i \right) \cap \left(\bigcap_{i=i_0+1}^{m} A_i \right).$$

Denote

$$H_A = \left\{ x \in G_k : \sum_{i \in A} \omega_x(a_i) \ is \ even \right\}, \quad A \subset N_k. \tag{1}$$

We recall main results in [5].

Let $A \subseteq N_k$ be a non empty set then $H_A \lhd G_k$ and $|G_k : H_A| = 2$. Also, for $A_1, A_2, \dots, A_m \subseteq N_k$ if $\bigcap_{i=1}^{m} H_{A_i}$ is non-contractible, then $\bigcap_{i=1}^{m} H_{A_i} \lhd G_k$ and $|G_k : \bigcap_{i=1}^{m} H_{A_i}| = 2^m$. One of the important theorem in the book: If H is a subgroup G_k with odd index ($\neq 1$) then H is not normal subgroup.

3. Normal Subgroups of Finite Index $2n$, $n \in \{p, 2p\}$, and 2^i, $i \in \mathbb{N}$.

Definition 1 (e. g. [12]). An elementary abelian group (or elementary abelian p-group) is an abelian group in which every nontrivial element has order p. The number p must be prime, and the elementary abelian groups are a particular kind of p-group. The case where $p = 2$, i. e., an elementary abelian 2-group, is sometimes called a Boolean group.

We denote Boolean group of order 2^n by K_{2^n}. From group theory it's known that if φ be a homomorphism of the group G_k onto a finite commutative group G. Then $\varphi(G_k)$ is isomorphic to K_{2^i} for some $i \in \mathbb{N}$.

Indeed, let $(G, *)$ be a commutative group of order n and $\varphi : G_k \to G$ be an epimorphism. Then the group $G_k/\operatorname{Ker}\varphi$ has (up to isomorphism) generators and relations $\langle b_1, \dots, b_n : b_i^2 = e_1, [b_i, b_j] = e_1 \rangle$, where e_1 is an identity element of $G_k/\operatorname{Ker}\varphi$ and $[b_i, b_j]$ are commutators. This is an elementary abelian group of order 2^k. So any homomorphism of G_k into an abelian group is isomorphic to a subgroup of an elementary abelian 2-group, and this is necessarily another elementary abelian 2-group.

Let $A_1, A_2, \dots, A_m \subset N_k$, $m \in \mathbb{N}$ and $\bigcap_{i=1}^{m} H_{A_i}$ is non-contractible. Then we denote by Re the following set

$$\mathrm{Re} = \left\{ \bigcap_{i=1}^{m} H_{A_i} : A_1, A_2, \dots, A_m \subset N_k, \, m \in \mathbb{N} \right\}.$$

The next theorem gives us a family of all subgroups of index 2^i of the group G_k coincides with the set Re. For any subgroup $H \in \mathrm{Re}$, periodic and weakly periodic Gibbs measures corresponding to H are well studied in [5]. We are going to show that there is not any normal subgroup H of index 2^i of G_k such that $H \notin \mathrm{Re}$.

Theorem 1 [1]. *Let φ be a homomorphism from G_k to a finite commutative group. Then there exists an element H of* Re *such that* $\operatorname{Ker}\varphi \simeq H$ *and conversely.*

Note that, by Theorem 1 we can get easily the following results:

Corollary 1 [1, 2]. *Any normal subgroup of index 4 has the form $H_A \cap H_B$, i. e.*

$$\{H:\ |G_k : H| = 4\} = \{H_A \cap H_B:\ A, B \subseteq N_k,\ A \neq B\}.$$

Any normal subgroup of index 8 has the form $H_A \cap H_B \cap H_C$, i. e.

$$\{H:\ |G_k : H| = 8\} = \{H_A \cap H_B \cap H_C:\ A, B, C \subseteq N_k,\ A \neq B,\ B \neq C,\ A \neq C\}.$$

Let $G = \langle b_1, b_2, \ldots, b_r\rangle$ be a group with free product. If $\Xi_n = \{A_1, A_2, \ldots, A_n\}$ be a partition of $N_k \backslash A_0$, $0 \leqslant |A_0| \leqslant k + 1 - n$. Then we define the following homomorphism $u_n : \{a_1, a_2, \ldots, a_{k+1}\} \to \{e_1, b_1 \ldots, b_m\}$ given by

$$u_n(x) = \begin{cases} e_1, & \text{if } x = a_i,\ i \in A_0; \\ b_j, & \text{if } x = a_i,\ i \in A_j,\ j \in \{1, 2, \ldots, n\}, \end{cases} \tag{2}$$

where e_1 is the identity element of G.

Put $R_b[b_1, b_2, \ldots, b_m]$ is a minimal representation of the word b. Then we introduce another mapping $\gamma_n : G \to G$ by the following formula:

$$\gamma_n(x) = \begin{cases} e_1, & \text{if } x = e_1; \\ b_i, & \text{if } x = b_i,\ i \in \{1, 2, \ldots, r\}; \\ R_{b_i}[b_1, \ldots, b_r], & \text{if } x = b_i,\ i \notin \{1, \ldots, r\}. \end{cases} \tag{3}$$

Denote

$$H^{(p)}_{\Xi_n}(G) = \{x \in G_k:\ \ l(\gamma_n(u_n(x))) \vdots 2p\}, \quad 2 \leqslant n \leqslant k - 1. \tag{4}$$

We define the following relation on $G_k : x \sim y$ if $\tilde{x} = \tilde{y}$, where $\gamma_n(u_n(x))) = \tilde{x}$. Note that defined relation is an equivalence relation.

Proposition 1. *Let $\Im_n$ be a family of groups of order n which has 2 generators with order two. Then the following equality holds*

$$\{\operatorname{Ker}\varphi:\ \varphi : G_k \to G \in \Im_{2n} \textit{ is an epimorphism}\}$$
$$= \left\{H^{(n)}_{B_0B_1B_2}(G):\ B_1, B_2 \textit{ is a partition of the set } N_k \setminus B_0,\ 0 \leqslant |B_0| \leqslant k - 1\right\}.$$

$\lhd$ Let $G \in \Im_{2n}$. We construct a bijection between the two given sets. Note that e_1 is the unit element of the group G. For a set $B_0 \subset N_k$, $0 \leqslant |B_0| \leqslant k - 1$ we take B_1, B_2 which is a partition of $N_k \setminus B_0$. Consequently, we can give the homomorphism $\varphi_{B_0B_1B_2} : G_k \to G$ by the formula

$$\varphi_{B_0B_1B_2}(a_i) = \begin{cases} b_1, & \text{if } i \in B_1; \\ b_2, & \text{if } i \in B_2. \end{cases} \tag{5}$$

It's easy to see that for the given subsets B_0, B_1, B_2 we can construct a unique such homomorphism. Also, we have $x \in \operatorname{Ker}\varphi_{B_0B_1B_2}$ iff $\tilde{x}$ equals e_1. Therefore, it is sufficient to prove the following claim: if $y \in H^{(n)}_{B_0B_1B_2}(G)$ then $\tilde{y} = e_1$. Suppose that there exist $y \in G_k$ such that $l(\tilde{y}) \geqslant 2n$. Put

$$\tilde{y} = b_{i_1} b_{i_2} \ldots b_{i_q}, \quad q \geqslant 2n, \quad S = \{b_{i_1}, b_{i_1} b_{i_2}, \ldots, b_{i_1} b_{i_2} \ldots b_{i_q}\}.$$

Since $S \subseteq G$ there exist $x_1, x_2 \in S$ such that $x_1 = x_2$ which contradicts the fact that $\tilde{y}$ is a non-contractible. Hence, we showed the inequality $l(\tilde{y}) < 2n$. From $y \in H^{(n)}_{B_0B_1B_2}(G)$ the integer number $l(\tilde{y})$ have to be divided by $2n$. Consequently, we have $\tilde{y} = e_1$ for any $y \in H^{(n)}_{B_0B_1B_2}(G)$. For the group G we have $\operatorname{Ker}\varphi_{B_0B_1B_2} = H^{(n)}_{B_0B_1B_2}(G)$. $\rhd$

Denote

$$\aleph_n := \Big\{H^{(n)}_{B_0B_1B_2}(G):\ B_1, B_2 \ \textit{is a partition of the set}\ N_k \setminus B_0,\ 0 \leqslant |B_0| \leqslant k-1,\ |G| = 2n\Big\}$$

$$\cup \Big\{H^{(n)}_{B_0B_1B_2B_3}(G):\ B_1, B_2, B_3 \ \textit{is a partition of the set}\ N_k \setminus B_0,\ 0 \leqslant |B_0| \leqslant k-2,\ |G| = 2n\Big\}.$$

Theorem 2. *Let p be an odd prime number. Any normal subgroup of index $2n$, $n \in \{p, 2p\}$, has the form $H^{(n)}_{B_0B_1B_2}(G) \cup H^{(n)}_{B_0B_1B_2B_3}(G)$, $|G| = 2n$ i.e.,*

$$\aleph_n = \{H:\ H \lhd G_k,\ |G_k : H| = 2n\}.$$

$\lhd$ At first we prove that

$$\aleph_n \subseteq \{H:\ H \lhd G_k,\ |G_k : H| = 2n\}.$$

Let G be a finite group and the number of elements is $2n$. Also, B_1, B_2 is a partition of $N_k \setminus B_0$, $0 \leqslant |B_0| \leqslant k-1$. It is enough to show that $x^{-1} H^{(n)}_{B_0B_1B_2}(G)\ x \subseteq H^{(n)}_{B_0B_1B_2}(G)$, for all $x \in G_k$. Similar to the proof of Proposition 1, we can conclude that if $y \in H^{(n)}_{B_0B_1B_2}(G)$ then $\tilde{y} = e_1$, where e_1 is the identity element of G. If we take an element z from the coset $x^{-1} H^{(n)}_{B_0B_1B_2}(G)\ x$, then $z = x^{-1} h\ x$ for some $h \in H^{(n)}_{B_0B_1B_2}(G)$. Consequently, one gets

$$\tilde{z} = \gamma_n(v_n(z)) = \gamma_n\left(v_n(x^{-1} h\ x)\right) = \gamma_n\left(v_n(x^{-1}) v_n(h) v_n(x)\right)$$
$$= \gamma_n\left(v_n(x^{-1})\right) \gamma_n\left(v_n(h)\right) \gamma_n\left(v_n(x)\right) = \left(\gamma_n\left(v_n(x)\right)\right)^{-1} \gamma_n\left(v_n(h)\right) \gamma_n\left(v_n(x)\right).$$

Since $\gamma_n\left(v_n(h)\right) = e_1$ we have $\tilde{z} = e_1$, i.e., $z \in H^{(n)}_{B_0B_1B_2}(G)$. Namely

$$H^{(n)}_{B_0B_1B_2}(G) \in \{H:\ H \lhd G_k,\ |G_k : H| = 2n\}.$$

Now we show that $\{H:\ H \lhd G_k, |G_k : H| = 2n\} \subseteq \aleph_n$. Put $H \lhd G_k$, $|G_k : H| = 2n$. We consider a *natural homomorphism* $\phi : G_k \to G_k : H$, i.e., $\phi(x) = xH$, $x \in G_k$. We can find elements: $e, b_1, b_2, \ldots, b_{2n-1}$ such that $\phi : G_k \to \{H, b_1H, \ldots, b_{2n-1}H\}$ is an epimorphism. Let $(\{H, b_1H, \ldots, b_{2n-1}H\}, *) = \wp$, i.e., $\wp$ is the factor group. If we show that $\wp \in \Im_{2n}$ then the theorem will be proved. Assuming that $\wp \notin \Im_{2n}$, then there are at least three generators: $c_1, c_2, \ldots, c_q \in \wp$, $q \geqslant 3$, such that $\wp = \langle c_1, \ldots, c_q\rangle$. Clearly, $\langle c_1, c_2\rangle$ is a subgroup of $\wp$ and elements of the group $\langle c_1, c_2\rangle|$ are greater than 3. By Lagrange's theorem and $n \in \{p, 2p\}$, we obtain $|\langle c_1, c_2\rangle| \in \{4, 2p, 4p\}$.

Let us consider the case: $|\langle c_1, c_2\rangle| = 4$. If the number four isn't equal to one of these numbers $|\langle c_1, c_3\rangle|$ or $|\langle c_2, c_3\rangle|$ then we shall take these pairs. If $|\langle c_1, c_2\rangle| = |\langle c_1, c_3\rangle| = |\langle c_2, c_3\rangle| = 4$, then elements of the group $\langle c_1, c_2, c_3\rangle$ is 8. We use Lagrange's theorem and conclude $|\wp| = 2n$ is divided by eight, i.e., it is impossible.

For the case $n = p$, since Lagrange's theorem one gets:

$$|\langle c_1, c_2\rangle| \in \left\{m:\ \frac{2p}{m} \in \mathbb{N}\right\}.$$

If e_2 is the identity element of $\wp$, then from $c_1^2 = e_2$ we take $|\langle c_1, c_2\rangle| = 2n$. Consequently, we have $\langle c_1, c_2\rangle = \wp$, but the second handside, $c_3 \notin \langle c_1, c_2\rangle$. Hence, $\wp \in \Im_{2n}$.

Finally, we consider the case $n = 2p$. Again by Lagrange's theorem we obtain

$$|\langle c_1, c_2\rangle| \in \left\{m:\ \frac{2n}{m} \in \mathbb{N}\right\}.$$

Let e_2 be the unit element of $\wp$. Then since $c_1^2 = e_2$ one gets $|\langle c_1, c_2\rangle| = 2n$. Consequently, $\langle c_1, c_2\rangle = \wp$ which contradicts to $c_3 \notin \langle c_1, c_2\rangle$. Hence, $\wp \in \Im_{2n}$.

If $|\langle c_1, c_2\rangle| = 2p$, then

$$\langle c_1, c_2\rangle = \{e, c_1, c_2, c_1c_2, c_1c_2c_1, \ldots, \underbrace{c_1c_2\ldots c_1}_{2(p-1)}\} = A.$$

It's easy to check that

$$c_3A \cup A \subseteq \wp, \quad c_3A \cap A = \varnothing, \quad |c_3A \cup A| = |c_3A| + |A| = 2n = |\wp|.$$

We then deduce that $c_3A \cup A = \wp$. On the second hand side, we showed that $c_3c_1c_3 \in \wp$ does not belong to $c_3A \cup A$. Clearly, from c_1, c_2, c_3 are generators, our conclusion is $c_3c_1c_3 \notin A$. Thus, $c_3c_1c_3 \in c_3A \ \Rightarrow\ c_3c_1c_3 = c_3x$ with $x \in \langle c_1, c_2\rangle$. But $x = c_1c_3 \notin \langle c_1, c_2\rangle$.

If $|\langle c_1, c_2\rangle| = 4p$, then $\langle c_1, c_2\rangle = \wp$, but $c_3 \notin \langle c_1, c_2\rangle$. Hence $\wp \in \Im_{2n}$. $\triangleright$

As a corollary of Theorem 2, we give the main theorems in [1, 2], i.e., let $\Xi_n = \{A_1,\ A_2, \ldots, A_n\}$ be a partition of $\{1, 2, \ldots, k+1\}\backslash A_0$, $0 \leqslant |A_0| \leqslant k+1-n$, and it is considered function $u_n : \{a_1, a_2, \ldots, a_{k+1}\} \to \{e, a_1 \ldots, a_{k+1}\}$ as

$$u_n(x) = \begin{cases} e, & \text{if } x = a_i,\ i \in A_0; \\ a_{m_j}, & \text{if } x = a_i,\ i \in A_j,\ j = 1, 2, \ldots, n. \end{cases}$$

Define $\gamma_n : G_k \to G_k$ by the formula

$$\gamma_n(x) = \gamma_n(a_{i_1}a_{i_2}\ldots a_{i_s}) = u_n(a_{i_1})u_n(a_{i_2})\ldots u_n(a_{i_s}).$$

Put

$$H^{(i)}_{\Xi_n} = \{x:\ l(\gamma_n(x)) \vdots 2i\}, \quad n < k+1,\ i \in \{3, 5\}.$$

Note that these corollary is not so difficult in group theory but our main aim is to feel elements of these subgroups as vertices of Cayley tree. Only in this case we have a chance to study periodic and weakly periodic Gibbs measures on Cayley trees.

Corollary 2 [1, 2]**.** *Let H be a normal subgroup of the group G_k. Then*
1. $\{H^{(3)}:\ |G_k : H| = 6\} = \{H_{\Xi_2},\ H_{\Xi_3}\}$;
2. $\{H^{(4)}:\ |G_k : H| = 8\} = \{H_{\Xi_2},\ H_{\Xi_3}\}$.

Remark. In general, we can not say any normal subgroup of index $2i$, $i \in \mathbb{N}$, has the form $H^{(i)}_{\Xi_n}$, $n \in \mathbb{N}$.

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О НОРМАЛЬНЫХ ПОДГРУППАХ ГРУППОВОГО ПРЕДСТАВЛЕНИЯ ДЕРЕВА КЭЛИ

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Аннотация. Мера Гиббса играет важную роль в статистической механике. На дереве Кэли для описания периодических мер Гиббса для моделей статистической механики нам нужны подгруппы группового представления дерева Кэли. Нормальная подгруппа группового представления дерева Кэли сохраняет свойство инвариантности, которое является важным инструментом при поиске мер Гиббса. В связи с этим полное описание нормальных подгрупп группового представления дерева Кэли является важной проблемой теории меры Гиббса. Например, в [1, 2] дано полное описание нормальных подгрупп индексов четыре, шесть, восемь и десять для группового представления дерева Кэли. Настоящая работа является обобщением этих работ, т. е. в ней для любого нечетного простого числа p дается характеризация нормальных подгрупп индексов $2n$, $n \in \{p, 2p\}$ и $2^i, i \in \mathbb{N}$, группового представления дерева Кэли.

Ключевые слова: дерево Кэли, G_k-группа, подгруппы конечного индекса, абелева группа, гомоморфизм.

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