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ON THE GEHRING TYPE CONDITION AND PROPERTIES OF MAPPINGS[#]

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Abstract. The goal of this work is to obtain an analytical description of mappings satisfying some capacity inequality (so called G_p-condition): we study mappings for which the G_p-condition holds for a cubical ring. In other words, we replace rings with concentric spheres in the G_p-condition by rings with concentric cubes. We obtain new analytic properties of homeomophisms in $\mathbb{R}^n$ meeting Gehring type capacity inequality. In this paper the capacity inequality means that the capacity of the image of a cubical ring is controlled by the capacity of the given ring. From the analytic properties we conclude some geometric properties of mappings under consideration. The method is new and is based on an equivalent analytical description of such mappings previously established by the author. Our arguments are based on assertions and methods discovered in author's recent papers [1] and [2] (see also some references inside). Then we obtain geometric properties of these mappings.

Keywords: quasiconformal analysis, Sobolev space, capacity inequality, pointwise condition.

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1. Introduction

In paper [3] F. W. Gehring studied some geometric properties of mappings in $\mathbb{R}^n$, $n \geqslant 2$, meeting so called G_p-condition. More precisely, suppose that D and D' are domains in $\mathbb{R}^n$ and $f : D \to D'$ is a homeomorphism. Then f maps each ring $U \subset D$ onto a ring $f(U) \subset D'$. Gehring says that $f \in G_p(K)$, $0 < K < \infty$, if

$$\operatorname{cap}_p(f(U)) \leqslant K \operatorname{cap}_p(U) \tag{1}$$

for all spherical rings $U \subset \mathbb{R}^n$. When $p = n$, a homeomorphism is in $G_n(K)$ for some K if and only if it is a quasiconformal mapping.

Recall that a bounded domain $U \subset D$ is said to be a *ring* if $\mathbb{R}^n \setminus U$ has exactly two components: bounded component F_1 and unbounded F_0. Then for $1 \leqslant p < \infty$ we define the p-capacity of U as

$$\operatorname{cap}_p(U) = \inf \int\limits_U |\nabla u|^p \, dx, \quad \nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right),$$

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where the infimum is taken over all functions $u \in L^1_p(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ with $u = 0$ on F_0 and $u = 1$ on F_1 (called *admissible*). A function $u : D \to \mathbb{R}$ belongs to the Sobolev class $L^1_p(D)$, if $u \in L_{1,\mathrm{loc}}(D)$ and its weak derivative $\frac{\partial u}{\partial x_i} \in L_p(D)$ for any $i = 1, \dots, n$. The seminorm of u equals $\|u \mid L^1_p(D)\| = \|\nabla u \mid L_p(D)\|$, $1 \leqslant p \leqslant \infty$.

A ring U is said to be a spherical ring if it is bounded by two concentric spheres, that is, if $U = \{x : a < |x - P| < b\}$, where $0 < a < b < \infty$ and $P \in \mathbb{R}^n$ is a center of spheres. Here and further $|P|$ is the Euclidean norm of $P \in \mathbb{R}^n$.

The purpose of paper [3] is to establish some relations between the classes $G_p(K)$ and $\mathrm{Lip}(K)$[1]. They are given in the following statements of paper [3].

Theorem 1 [3, Theorem 2]**.** *If $f, f^{-1} \in G_p(K)$, where $p \neq n$, then $f, f^{-1} \in \mathrm{Lip}(K_0)$, where K_0 depends only on K, n and p.*

Theorem 2 [3, Theorem 3]**.** *If $f \in G_p(K)$, where $n - 1 < p < n$, then $f \in \mathrm{Lip}(K_0)$. If $f \in G_p(K)$, where $n < p < \infty$, then $f^{-1} \in \mathrm{Lip}(K_0)$. In both cases K_0 depends only on K, n and p.*

The goal of this work is to obtain an analytical description of mappings satisfying some capacity inequality similar to (1): we study mappings for which (1) holds whenever U is a cubical ring. In another words we replace rings with concentric spheres in the right hand side of (1) by rings with concentric cubes. Our arguments are based on assertions and methods discovered in recent papers [1] and [2] (see also some references inside). Then we obtain geometric properties of these mappings.

There is also another approach to this subject. For instance, authors of paper [4] study properties of homeomorptisms under stronger capacity inequality:

$$\mathrm{cap}_p(\varphi(F_0), \varphi(F_1); D') \leqslant K_p \,\mathrm{cap}_p(F_0, F_1; D), \quad 1 < p < \infty,$$

for an arbitrary condenser $(F_0, F_1) \subset D$. However, method of paper (4) is not applicable to a minimal collection of rings (spherical or cubical). See [4] for more details.

$1^{\mathbf{st}}$ **STEP.** The crucial result for our study is the following theorem proved in [2]. Before formulating this theorem we give some necessary definitions.

Definition 1. A ring U in $\mathbb{R}^n$ is called, *cubical* whenever $U = Q(x, R) \setminus \overline{Q(x, r)}$, where $Q(x, R) = \{z \in \mathbb{R}^n : |z - x|_\infty < R\}$ and $0 < r < R < \infty$. Recall that $|x|_\infty = \max_{k=1,\dots,n} |x_k|$.

Definition 2. Suppose that D is an open set in $\mathbb{R}^n$. Denote by $\mathscr{O}_c(D)$ some system of open sets in D with the following properties:

(a) if the closure $\overline{Q}$ of an open cube Q lies in D, then $Q \in \mathscr{O}_c(D)$;

(b) if $U_1, \dots, U_k \in \mathscr{O}_c(D)$ is a disjoint system of open sets, then $\bigcup_{i=1}^k U_i \in \mathscr{O}_c(D)$ for arbitrary $k \in \mathbb{N}$.

(c) in the case $n = 2$, $q = 1$ we will consider an expanded family $\widetilde{\mathscr{O}}_c(D) \supset \mathscr{O}_c(D)$: we include additional rings of the following shape to this family:

$$U = ([a - r, a + r] \times [b, c]) \setminus (\{a\} \times [b + r, c - r]) \subset D, \quad 2r < c - b,$$

and

$$U = ([s, t] \times [d - \tau, d + \tau]) \setminus ([s + \tau, t - \tau] \times \{d\}) \subset D, \quad 2\tau < t - s.$$

Definition 3. A mapping $\Psi : \mathscr{O}_c(D) \to [0, \infty]$ is called a κ-*quasiadditive* set function, whenever

[1] Gehring says that $f \in \mathrm{Lip}(K)$, $0 < K < \infty$, if $L(P, f) = \overline{\lim\limits_{x \to P}} \frac{|f(x) - f(P)|}{|x - P|} \leqslant K$ whenever $P \in D$.

(a) for each point $x \in D$ there exists δ with $0 < \delta < \operatorname{dist}(x, \partial D)$, such that $0 < \Psi(Q(x,\delta)) < \infty$, and if $D = \mathbb{R}^n$, then the inequality $0 \leqslant \Psi(Q(x,\delta)) < \infty$ must hold for all $\delta \in (0, \delta(x))$, where $\delta(x) > 0$ may depend on x;

(b) for every finite disjoint collection of open sets $U_i \in \mathscr{O}_c(D)$, where $i = 1, \dots, l$, with

$$\bigcup_{i=1}^{l} U_i \subset U, \text{ where } U \in \mathscr{O}_c(D), \quad \text{we have } \sum_{i=1}^{l} \Psi(U_i) \leqslant \kappa \Psi(U). \tag{2}$$

If (2) holds with $\kappa = 1$, then we refer to Ψ as a *quasiadditive* set function instead of 1-quasi-additive. If for every finite collection $\{U_i \in \mathscr{O}_c(D)\}$ of disjoint open sets we have

$$\sum_{i=1}^{l} \Psi(U_i) = \Psi\left(\bigcup_{i=1}^{l} U_i\right),$$

then Ψ is called *finitely additive.*

A function Ψ is *monotone* whenever $\Psi(U_1) \leqslant \Psi(U_2)$ provided that $U_1 \subset U_2 \subset D$ and $U_1, U_2 \in \mathscr{O}_c(D)$. It is obvious that every quasiadditive set function is monotone. A κ-quasiadditive set function $\Psi : \mathscr{O}_c(D) \to [0,\infty]$ is called *bounded,* whenever $\sup_{U \in \mathscr{O}_c(D)} \Psi(U) < \infty$.

The Sobolev space $W^1_p(D)$ in a domain $D \subset \mathbb{R}^n$ consists of functions $u \in L^1_p(D)$ with the finite norm $\|u \mid W^1_p(D)\| = \|u \mid L_p(D)\| + \|\nabla u \mid L_p(D)\|$, $1 \leqslant p \leqslant \infty$.

Let D and D' be domains in the Euclidean space $\mathbb{R}^n$. Then a homeomorphism $\varphi : D \to D'$ belongs to the Sobolev space $W^1_{p,\mathrm{loc}}(D)$ ($L^1_p(D)$), if its coordinate functions belong to $W^1_{p,\mathrm{loc}}(D)$ ($L^1_p(D)$). Then Jacobi matrix $D\varphi(x) = \left(\frac{\partial \varphi_i}{\partial x_j}\right)_{i,j=1,\dots,n}$ and its Jacobian $\det D\varphi(x)$ are well defined at almost all points $x \in D$.

Notice that Gehring's condition $\varphi \in \operatorname{Lip}(K)$ in D, $0 < K < \infty$, is equivalent to $\varphi \in L^1_\infty(D)$ and the norm $\|\varphi \mid L^1_\infty(D)\| = \|D\varphi \mid L_\infty(D)\|$ can be taken as K.

Theorem 3 [2, Theorems 18 and 23]**.** *Given a homeomorphism $\psi : D' \to D$ of domains $D', D \subset \mathbb{R}^n$, where $n \geqslant 2$, the following statements are equivalent*:

(1) *Every cubical ring $U = Q(y,R) \setminus \overline{Q(y,r)} \subset D$ with the preimage $\psi^{-1}(U) = \psi^{-1}(Q(y,R)) \setminus \psi^{-1}(\overline{Q(y,r)})$ in D' satisfies*

$$\operatorname{cap}_q^{\frac{1}{q}}(\psi^{-1}(U)) \leqslant \begin{cases} K_p \operatorname{cap}_p^{\frac{1}{p}}(U), & 1 < q = p < \infty, \\ \Psi_{q,p}^{\frac{1}{\sigma}}(U) \operatorname{cap}_p^{\frac{1}{p}}(U), & 1 < q \neq p < \infty, \end{cases} \tag{3}$$

where $K_p \in (0,\infty)$ is some constant and $\Psi_{q,p}$ is some bounded quasiadditive set function on the system $\mathscr{O}_c(D)$, and σ is determined from $\frac{1}{\sigma} = \frac{1}{q} - \frac{1}{p}$, if $1 < q < p < \infty$ and $\sigma = \infty$, if $1 < q = p < \infty$.

(2) *The homeomorphism $\psi : D' \to D$ belongs to $W^1_{q,\mathrm{loc}}(D')$, has finite distortion: $D\psi(y) = 0$ holds almost everywhere on $Z = \{y \in D' \mid \det D\psi(y) = 0\}$, and the operator distortion function*

$$D' \ni y \to K_{q,p}(y,\psi) = \begin{cases} \frac{|D\psi(y)|}{|\det D\psi(y)|^{\frac{1}{p}}}, & \det D\psi(y) \neq 0, \\ 0, & \det D\psi(y) = 0, \end{cases} \tag{4}$$

belongs to $L_\sigma(D')$.

(3) *The composition operator $\psi^* : L^1_p(D) \cap \operatorname{Lip}_l(D) \to L^1_q(D')$, $\psi^*(f) = f \circ \psi$ if $f \in L^1_p(D) \cap \operatorname{Lip}_l(D)$, where $1 < q \leqslant p < \infty$, is bounded.*

Moreover,

$$\|\psi^*\| \leqslant \|K_{q,p}(\cdot,\psi) \mid L_\sigma(D')\| \leqslant \begin{cases} 7^{\frac{n}{p}} n K_p, & 1 < q = p < \infty, \\ 7^{\frac{n}{q}} n \Psi_{q,p}(D)^{\frac{1}{\sigma}}, & 1 < q < p < \infty. \end{cases} \tag{5}$$

(4) *Every ring U in D with the preimage $\psi^{-1}(U)$ in D' satisfies*

$$\operatorname{cap}_q^{\frac{1}{q}}(\psi^{-1}(U)) \leqslant \begin{cases} 7^{\frac{n}{p}} n K_p \operatorname{cap}_p^{\frac{1}{p}}(U), & 1 < q = p < \infty, \\ 7^{\frac{n}{q}} n \Psi_{q,p}^{\frac{1}{\sigma}}(U) \operatorname{cap}_p^{\frac{1}{p}}(U), & 1 < q \neq p < \infty, \end{cases}$$

where $K_p \in (0,\infty)$ and $\Psi_{q,p}$ are from (3) *and σ is determined from $\frac{1}{\sigma} = \frac{1}{q} - \frac{1}{p}$, if $1 < q < p < \infty$ and $\sigma = \infty$, if $1 < q = p < \infty$.*

(5) *The claims of Theorem* 3 *remain valid in the case $1 = q \leqslant p < \infty$ and $n = 2$, if* (3) *holds for $U \in \widetilde{\mathscr{O}}_c(D)$* (*see Definition* 2) *with probably different constant instead of $7^{\frac{n}{p}} n$.*

Put $K_{q,p}(\psi; D') = \|K_{q,p}(\cdot,\psi) \mid L_\sigma(D')\|$.

Remark 1. In the case $q = 1$ analytic properties of ψ are proved in [2, Theorem 23]. Unfortunately, in Statement 5 of Theorem 18 of [2] the condition $U \in \widetilde{\mathscr{O}}_c(D)$ (see Definition 2) is missing.

We will apply Theorem 3 to mappings meeting capacity inequality (3) instead of (1). In another words we study mappings in $\mathbb{R}^n$ which control changing of capacity of cubical rings instead of spherical ones.

The next statement is evident.

Proposition 1. *Given a homeomorphism $\varphi : D \to D'$ of domains $D, D' \subset \mathbb{R}^n$, where $n \geqslant 2$, the inequality*

$$\operatorname{cap}_q^{\frac{1}{q}}(\varphi(U)) \leqslant \begin{cases} K_p \operatorname{cap}_p^{\frac{1}{p}}(U), & 1 \leqslant q = p < \infty, \\ \Psi_{q,p}(U) \operatorname{cap}_p^{\frac{1}{p}}(U), & 1 \leqslant q < p < \infty, \end{cases} \tag{6}$$

holds for every cubical ring $U = Q(y,R) \setminus \overline{Q(y,r)} \subset D$, iff inequality (3) *holds for the homeomorphism $\psi = \varphi^{-1} : D' \to D$. Here $K_p \in (0,\infty)$ is some constant and $\Psi_{q,p}$ is some bounded quasiadditive set function on the system $\mathscr{O}_c(D)$.*

Definition 4. Suppose that D' and D are domains in $\mathbb{R}^n$ and that $\psi : D' \to D$ is a homeomorphism.

1) We say that $\psi \in Q_{q,p}$, if

a) in the case $q > 1$ inequality (3) holds for each cubical ring $U \subset \mathscr{O}_c(D)$;

b) in the case $n = 2$, $q = 1$ we ask for (3) to be true for an expanded family $\widetilde{\mathscr{O}}_c(D)$ (see Definition 2).

2) We say that $\varphi = \psi^{-1} \in G_{p,q}$, if $\psi \in Q_{q,p}$.

Theorem 3 implies Theorem 1:

Corollary 1. *Given homeomorphism $\varphi : D \to D'$ of domains $D, D' \subset \mathbb{R}^n$, where $n \geqslant 2$, meeting conditions $\varphi \in G_{p,p}$ and $\varphi^{-1} \in G_{p,p}$ with $1 < p < \infty$ the following properties hold:*

1) $\varphi, \varphi^{-1} \in \operatorname{Lip}(K_0)$, *where K_0 depends only on K_p, n, and p, $p \neq n$;*

2) *φ is quasiconformal mapping, if $p = n$.*

$\lhd$ If condition 1) holds, then the relation (4) holds for both φ and φ^{-1}. The desired results are proved in [1, Subsections 1.2, 1.3]. $\rhd$

Proposition 2. *Given a homeomorphism $\varphi : D \to D'$ of domains $D, D' \subset \mathbb{R}^n$, where $n \geqslant 2$, of class $G_{p,q}$ with $n-1 \leqslant q \leqslant p < \infty$ the following properties hold:*

1) $\varphi \in W^1_{p',\mathrm{loc}}(D)$, where $p' = \frac{p}{p-(n-1)}$, if $p > n-1$, and $\varphi \in L^1_\infty(D)$, if $p = n-1$;

2) φ has the finite distortion;

3) the codistortion function

$$D \ni x \to \mathscr{K}_{q,p}(x,\varphi) = \begin{cases} \frac{|\operatorname{adj} D\varphi(x)|}{|\det D\varphi(x)|^{1-\frac{1}{q}}}, & \det D\varphi(x) \neq 0, \\ 0, & \det D\varphi(x) = 0, \end{cases} \tag{7}$$

belongs to $L_\sigma(D)$, where σ is determined from $\frac{1}{\sigma} = \frac{1}{q} - \frac{1}{p}$, if $n-1 \leqslant q < p < \infty$, and $\sigma = \infty$, if $n-1 \leqslant q = p < \infty$;

4) the distortion function

$$D \ni x \to K_{p',q'}(x,\varphi) = \begin{cases} \frac{|D\varphi(x)|}{|\det D\varphi(x)|^{\frac{1}{q'}}}, & \det D\varphi(x) \neq 0, \\ 0, & \det D\varphi(x) = 0, \end{cases} \tag{8}$$

belongs to $L_\varrho(D)$, where $q' = \frac{q}{q-(n-1)}$ at $q > n-1$, $q' = \infty$ at $q = n-1$ and ρ is determined from $\frac{1}{\varrho} = \frac{1}{p'} - \frac{1}{q'}$, if $n-1 \leqslant q < p < \infty$, and $\varrho = \infty$, if $n-1 \leqslant q = p < \infty$.

$\lhd$ If $\varphi \in G_{p,q}$ then, according Proposition 1, $\psi = \varphi^{-1} \in Q_{q,p}$. By Theorem 3 the homeomorphism $\psi : D' \to D$

1) belongs to $W^1_{q,\mathrm{loc}}(D')$;

2) has the finite distortion: $D\psi(y) = 0$ holds almost everywhere on $Z = \{y \in D' : \det D\psi(y) = 0\}$,

3) the distortion function

$$D' \ni y \to K_{q,p}(y,\psi) = \begin{cases} \frac{|D\psi(y)|}{|\det D\psi(y)|^{\frac{1}{p}}}, & \det D\psi(y) \neq 0, \\ 0, & \det D\psi(y) = 0, \end{cases} \tag{9}$$

belongs to $L_\sigma(D')$, where σ is determined from $\frac{1}{\sigma} = \frac{1}{q} - \frac{1}{p}$, if $1 \leqslant q < p < \infty$, and $\sigma = \infty$, if $1 \leqslant q = p < \infty$.

By [1, Theorem 4] we conclude that $\varphi \in W^1_{p',\mathrm{loc}}(D)$, where $p' = \frac{p}{p-(n-1)}$ if $p > n-1$, $\varphi \in L^1_\infty(D)$, if $p = n-1$, and φ has the finite distortion.

Take into account that $D\psi(y) = \frac{\operatorname{adj} D\varphi(x)}{\det D\varphi(x)}$ and $\det D\psi(y) = (\det D\varphi(x))^{-1}$ at points $y = \varphi(x) \neq Z' \cap \Sigma'$, where $Z' = \{y \in D' : \det D\psi(y) = 0\}$ and $\Sigma' \subset D'$ is a maximal Borel null-set such that measure of $Z = \psi(\Sigma')$ is positive. Notice that up to a set of measure zero $Z = \{x \in D : \det D\varphi(x) = 0\}$, and $\Sigma = \psi(Z') \subset D$ is a null-set. The mapping φ has Luzin property N outside of Σ.

By change of variable formula in the case $q < p$ we get

$$\begin{aligned} \|\mathscr{K}_{q,p}(\cdot,\varphi) \mid L_\sigma(D)\|^\sigma &= \int\limits_{D\setminus(Z\cap\Sigma)} \left(\frac{|\operatorname{adj} D\varphi(x)|}{|\det D\varphi(x)|^{1-\frac{1}{q}}}\right)^\sigma dx \\ &= \int\limits_{D\setminus(Z\cap\Sigma)} \left(\frac{|\operatorname{adj} D\varphi(x)|}{|\det D\varphi(x)|^{1-\frac{1}{p}}}\right)^\sigma |\det D\varphi(x)|\, dx \\ &= \int\limits_{D'\setminus(Z'\cap\Sigma')} \left(\frac{|D\psi(y)|}{|\det D\psi(y)|^{\frac{1}{p}}}\right)^\sigma dy = \|K_{q,p}(\cdot,\psi) \mid D'\|^\sigma. \end{aligned} \tag{10}$$

From the left hand side of this equality it follows (7): $\mathscr{K}_{q,p}(\cdot,\varphi)\in L_\sigma(D)$.

In the case if $n-1\leqslant q=p<\infty$ we have

$$\|\mathscr{K}_{q,p}(\cdot,\varphi)\mid L_\infty(D)\|=\operatorname*{ess\,sup}_{x\in D\setminus(Z\cap\Sigma)}\frac{|\operatorname{adj}D\varphi(x)|}{|\det D\varphi(x)|^{1-\frac{1}{q}}}$$
$$=\operatorname*{ess\,sup}_{y\in D'\setminus(Z'\cap\Sigma')}\frac{|D\psi(y)|}{|\det D\psi(y)|^{\frac{1}{p}}}=\|K_{p,q}(\cdot,\psi)\mid L_\infty(D')\|.$$

Integrability $K_{p',q'}(\cdot,\varphi)\in L_\varrho(D)$ is proved in [1, Theorem 4]. $\rhd$

From Proposition 2 it follows a part of Theorem 2.

Corollary 2. *Given a homeomorphism $\varphi : D\to D'$ of domains $D, D'\subset\mathbb{R}^n$, where $n\geqslant 2$, of class $G_{p,q}$ with $n-1\leqslant q\leqslant p<\infty$ the following properties hold*:

1) $\varphi^{-1}=\psi\in L^1_\infty(D')$ *and* $\|\varphi^{-1}\mid L^1_\infty(D')\|\leqslant\|K_{p,p}(\cdot,\psi)\mid L_\infty(D')\|^{\frac{p}{p-n}}$ *in the case* $n<q=p<\infty$;

2) $\varphi\in L^1_\infty(D)$ *and*

$$\big\|\varphi\mid L^1_\infty(D)\big\|\leqslant\big\|K_{p',q'}(\cdot,\varphi)\mid L_\varrho(D)\big\|^{\frac{p'}{p'-n}} \tag{11}$$

in the case $n-1\leqslant q=p<n$.

$\lhd$ Really, taking (9) into account at $q=p>n$ and the inequality $1\leqslant\frac{|D\psi(y)|^n}{|\det D\psi(y)|}$ in points, where $\det D\psi(y)\neq 0$ we have

$$|D\psi(y)|^{\frac{p-n}{p}}\leqslant\left(\frac{|D\psi(y)|^n|D\psi(y)|^{p-n}}{|\det D\psi(y)|}\right)^{\frac{1}{p}}=\frac{|D\psi(y)|}{|\det D\psi(y)|^{\frac{1}{p}}}\leqslant\|K_{p,p}(\cdot,\psi)\mid L_\infty(D')\|.$$

It follows $\psi=\varphi^{-1}\in L^1_\infty(D')$ in the case $n<q=p<\infty$ and

$$\big\|\psi\mid L^1_\infty(D')\big\|\leqslant\big\|K_{p,p}(\cdot,\psi)\mid L_\infty(D')\big\|^{\frac{p}{p-n}}.$$

In the case $n-1<q=p<n$ we have integrability $K_{p'}(\cdot,\varphi)=L_\infty(D)$ with $p'>n$. Therefore with above-mentioned arguments applied to (8) we obtain $\varphi\in L^1_\infty(D)$ and (11) holds.

Property $\varphi\in L^1_\infty(D)$ in the case $q=p=n-1$ is just statement 1) of Proposition 2. $\rhd$

2nd STEP. Proposition 3. *Let $\varphi\in W^1_{n-1,\mathrm{loc}}(D)$, φ has the finite codistortion* ($\operatorname{adj}D\varphi(x)=0$ *almost everywhere on the set Z*) *and the codistortion function*

$$D\ni x\to\mathscr{K}_{q,p}(x,\varphi)=\begin{cases}\frac{|\operatorname{adj}D\varphi(x)|}{|\det D\varphi(x)|^{\frac{q-1}{q}}}, & \det D\varphi(x)\neq 0,\\ 0, & \det D\varphi(x)=0,\end{cases} \tag{12}$$

belongs to $L_\sigma(D')$, where σ is determined from $\frac{1}{\sigma}=\frac{1}{q}-\frac{1}{p}$, if $n-1\leqslant q<p<\infty$, and $\sigma=\infty$, if $n-1\leqslant q=p<\infty$. Then $\varphi\in G_{p,q}$.

$\lhd$ As soon as $\varphi\in W^1_{n-1,\mathrm{loc}}(D)$ and has the finite codistortion then $\psi=\varphi^{-1}\in W^1_{1,\mathrm{loc}}(D')$ and has the finite distortion (see [1, Corollary 2]). Because of this we can apply (10) again for obtaining statement 4) of Proposition 2. It left to verify that $\psi\in Q_{q,p}$. For doing this we take an arbitrary cubical ring in $U\subset D$ and an admissible function u for this ring, and evaluate the norm of the composition $u\circ\psi$. As soon as $u\circ\psi$ is an admissible function for the ring $\psi^{-1}(U)$ we have:

$$\operatorname{cap}_q(\psi^{-1}(U)) \leqslant \big\|u\circ\psi \mid L^1_q(\psi^{-1}(U))\big\|^q$$

$$\leqslant \int\limits_{\psi^{-1}(U)\setminus Z'} |\nabla u(\psi(y))|^q |\det D\psi(y)|^{\frac{q}{p}} \cdot \frac{|D\psi(y)|^q}{|\det D\psi(y)|^{\frac{q}{p}}}\,dy$$

$$\leqslant \Bigg(\int\limits_{\psi^{-1}(U)\setminus(Z'\cup\Sigma')} |\nabla u(\psi(y))|^p |\det D\psi(y)|\,dy\Bigg)^{\frac{q}{p}} \cdot \Bigg(\int\limits_{\psi^{-1}(U)\setminus(Z'\cup\Sigma')} \bigg(\frac{|D\psi(y)|}{|\det D\psi(y)|^{\frac{1}{p}}}\bigg)^{\sigma} dy\Bigg)^{\frac{q}{\sigma}}$$

$$\leqslant \begin{cases} \big\|K_{q,p}(\cdot,\psi) \mid L_\sigma(\psi^{-1}(U))\big\|^q \Big(\int\limits_U |\nabla u(x)|^p\,dx\Big)^{\frac{q}{p}}, & q<p,\\ \big\|K_{p,p}(\cdot,\psi) \mid L_\infty(\psi^{-1}(U))\big\|^p \Big(\int\limits_U |\nabla u(x)|^p\,dx\Big), & q=p. \end{cases}$$

It follows (3) with bounded quasiadditive set function $\Psi_{q,p}$ equal to

$$D\supset U\mapsto \Psi_{q,p}(U) = \|\mathscr{K}_{q,p}(\cdot,\varphi) \mid L_\sigma(U)\|^\sigma = \|K_{q,p}(\cdot,\psi) \mid L_\sigma(\psi^{-1}(U))\|^\sigma,$$

and $K_p = \|K_{p,p}(\cdot,\psi) \mid L_\infty(\psi^{-1}(U))\|$.

Hence we proved $\varphi=\psi^{-1}\in G_{p,q}$. $\rhd$

3^{rd} **STEP.** From Proposition 2 and 3 it follows the following criterium.

Theorem 4. *A homeomorphism $\varphi : D\to D'$ of domains $D, D'\subset\mathbb{R}^n$, where $n\geqslant 2$, belongs to class $G_{p,q}$ with $n-1\leqslant q\leqslant p<\frac{(n-1)^2}{n-2}$, iff the following properties hold*:

1) $\varphi\in W^1_{n-1,\mathrm{loc}}(D)$;

2) *φ has the finite codistortion*;

3) *the codistortion function*

$$D\ni x\to \mathscr{K}_{q,p}(x,\varphi) = \begin{cases} \frac{|\operatorname{adj} D\varphi(x)|}{|\det D\varphi(x)|^{\frac{q-1}{q}}}, & \det D\varphi(x)\neq 0,\\ 0, & \det D\varphi(x)=0, \end{cases} \tag{13}$$

belongs to $L_\sigma(D')$, where σ is determined from $\frac{1}{\sigma}=\frac{1}{q}-\frac{1}{p}$, if $n-1\leqslant q<p<\frac{(n-1)^2}{n-2}$, and $\sigma=\infty$, if $n-1\leqslant q=p<\frac{(n-1^2)}{n-2}$.

$\lhd$ If a homeomorphism $\varphi : D\to D'$ of domains $D, D'\subset\mathbb{R}^n$, where $n\geqslant 2$, belongs to class $G_{p,q}$ with $n-1\leqslant q\leqslant p<\frac{(n-1)^2}{n-2}$, then we apply Proposition 2 for obtaining that

1) $\varphi\in W^1_{p',\mathrm{loc}}(D)$, where $p'=\frac{p}{p-(n-1)}$, if $p\geqslant n-1$. Since $p'>n-1$ it follows that $\varphi\in W^1_{n-1,\mathrm{loc}}(D)$.

2) φ has the finite distortion.

3) the codistortion function (13) is in $L_\sigma(D')$.

Thus the necessity is proved. The sufficiency is proved in Proposition 3. $\rhd$

References

1. *Vodopyanov, S. K.* Regularity of Mappings Inverse to Sobolev Mappings, *Sbornik: Mathematics*, 2012, vol. 203, no. 10, pp. 1383–1410. DOI: 10.1070/SM2012v203n10ABEH004269.
2. *Vodopyanov, S. K.* The Regularity of Inverses to Sobolev Mappings and the Theory of $Q_{q,p}$-Homeomorphisms, *Siberian Mathematical Journal*, 2020, vol. 61, no. 6, pp. 1002–1038. DOI: 10.1134/S00374466 20060051.
3. *Gehring, F. W.* Lipschitz Mappings and the p-Capacity of Rings in n-Space, *Advances in the theory of Riemann surfaces* (Proc. Conf., Stony Brook, N.Y., 1969), 175–193, Annals of Mathematics Studies, vol. 66, Princeton, N. J., Princeton Univ. Press, 1971. DOI: 10.1515/9781400822492-013.
4. *Salimov, R., Sevost'yanov, E. and Ukhlov, A.* Capacity Inequalities and Lipschitz Continuity of Mappings, arXiv:2302.13302v1, 26 Feb 2023.

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ОБ УСЛОВИИ ТИПА ГЕРИНГА И СВОЙСТВАХ ОТОБРАЖЕНИЙ

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Аннотация. Целью данной работы является получение аналитического описания отображений, удовлетворяющих некоторому емкостному неравенству (так называемому G_p-условию); точнее, мы изучаем отображения, для которых выполнено G_p-условие для кубического кольца. Другими словами, мы заменяем кольца с концентрическими сферами в условии G_p кольцами с концентрическими кубами. Изучаются новые аналитические свойства гомеомофизмов в $\mathbb{R}^n$, удовлетворяющих емкостному неравенству типа Геринга. В этой статье емкостное неравенство означает, что емкость образа кубического конденсатора контролируется емкостью исходного конденсатора. Из полученных аналитических свойств в качестве следствия получаем некоторые геометрические свойства рассматриваемых отображений. Метод является новым и основан на эквивалентном аналитическом описании таких отображений.

Ключевые слова: квазиконформный анализ, пространство Соболева, емкостное неравенство, поточечное условие.

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