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EXISTENCE OF CLASSICAL SOLUTIONS FOR A CLASS OF THE KHOKHLOV–ZABOLOTSKAYA–KUZNETSOV TYPE EQUATIONS

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Abstract. In medical sciences, during medical exploration and diagnosis of tissues or in medical imaging, we often use mathematical models to answer questions related to these examinations. Among these models, the nonlinear partial differential equation of the Khokhlov–Zabolotskaya–Kuznetsov type (abbreviated as the KZK equation) is of proven interest in ultrasound acoustics problems. This mathematical model describes the nonlinear propagation of a sound pulse of finite amplitude in a thermo-viscous medium. The equation is obtained by combining the conservation of mass equation, the conservation of momentum equation and the equations of state. It should be noted that for this equation little mathematical analysis is reserved. This equation takes into account three combined effects: the diffraction of the wave, the absorption of energy and the nonlinearity of the medium in which the wave propagates. KZK-type equation introduced in this paper is a modified version of the KZK model known in acoustics. We study a class of the Khokhlov–Zabolotskaya–Kuznetsov type equations for the existence of global classical solutions. We give conditions under which the considered equations have at least one and at least two classical solutions. To prove our main results, we propose a new approach based on recent theoretical results.

Keywords: KZK equation, global classical solution, fixed point, sum of operators, initial value problem.

AMS Subject Classification: 35Q35, 35G20, 35A09, 35E15, 47H10, 35G25.

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1. Introduction and Statement of the Main Results

In this paper, we investigate the following class of the Khokhlov–Zabolotskaya–Kuznetsov type equations

$$\begin{gathered} u_{x_3t} = (f(u_t))_t + \beta(t,x)u_{ttt} + \gamma(t,x)u_t + u_{x_1x_1} + u_{x_2x_2}, \\ t > 0, \quad x = (x_1,x_2,x_3) \in \mathbb{R}^3, \\ u(0,x) = u_0(x), \quad x = (x_1,x_2,x_3) \in \mathbb{R}^3, \\ u_t(0,x) = u_1(x), \quad x = (x_1,x_2,x_3) \in \mathbb{R}^3, \\ u_{tt}(0,x) = u_2(x), \quad x = (x_1,x_2,x_3) \in \mathbb{R}^3, \end{gathered} \tag{1.1}$$

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where

(H1) $u_0, u_1, u_2 \in \mathscr{C}^2(\mathbb{R}^3)$, $0 \leqslant u_0, u_1, u_2 \leqslant B$ on $[0,\infty) \times \mathbb{R}^3$ for some positive constant B, $\beta, \gamma \in \mathscr{C}([0,\infty) \times \mathbb{R}^3)$, $\beta > 0$ on $[0,\infty) \times \mathbb{R}^3$, $\beta, |\gamma| \leqslant B$ on $[0,\infty) \times \mathbb{R}^3$, $f \in \mathscr{C}^1(\mathbb{R})$ and

$$\left|f'(v)\right| \leqslant a_1(t,x) + a_2(t,x)|v|^p, \quad (t,x) \in [0,\infty) \times \mathbb{R}^3,$$

$a_1, a_2 \in \mathscr{C}([0,\infty) \times \mathbb{R}^3)$, $0 \leqslant a_1, a_2 \leqslant B$ on $[0,\infty) \times \mathbb{R}^3$, $p \geqslant 0$.

Description of nonlinear acoustics was the origin of the derivation of the Khokhlov–Zabolotskaya–Kuznetsov (KZK) equation, see [1] and [2]. Later, its fields of application expanded considerably. The KZK equation is the mathematical model of phenomena with effects of diffraction and of absorption, which can provide shock formation. It contains a nonlocal diffraction term, an absorption term and a nonlinear term, it can be used as a model that describes the propagation of the ultrasound beams in the thermo-viscous fluid [3, 4]. The KZK parabolic nonlinear wave equation is one of the most widely employed nonlinear models for propagation of $3D$ diffraction sound beams in dissipative media. It is also used for modeling of an electrohydraulic lithotripter [5]. A derivation of the KZK scalar equation for incompressible materials is provided in [6]. Mathematical and numerical analysis of the KZK equation can be found in several papers, see for example [7] and [8]. In [9], the exact analytical solutions of (3+1)-dimensional time fractional the KZK equation have been constructed in the sense of modified Riemann–Liouville derivative. In [10], invariant solutions for the modified the KZK equation are obtained by using classical Lie symmetries. Accurate numerical methods to simulate the KZK equation are important to its broad applications in medical ultrasound simulations [11].

The aim of this paper is to investigate the initial value problem (IVP) (1.1) for existence of global classical solutions. Here, by a classical solution u to the first equation of (1.1) we mean a solution at least twice times continuously differentiable in x and three times continuously differentiable in t for any t $\geqslant 0$. In other words, u belongs to the space $\mathscr{C}^3([0,\infty), \mathscr{C}^2(\mathbb{R}^3))$ of three times continuously differentiable functions on $[0,\infty)$ with values in the Banach space $\mathscr{C}^2(\mathbb{R}^3)$. So, suppose that

(H2) there exist a positive constant A and a function $g \in \mathscr{C}([0,\infty) \times \mathbb{R}^3)$ such that $g > 0$ on $(0,\infty) \times \left(\mathbb{R}^3 \backslash \left(\bigcup_{j=1}^3 \{x_j = 0\}\right)\right)$ with

$$g(0,x) = g(t,0,x_2,x_3) = g(t,x_1,0,x_3) = g(t,x_1,x_2,0) = 0, \quad t \in [0,\infty), \; x \in \mathbb{R}^3,$$

and

$$6 \cdot 2^9 \left(1 + t + t^2 + t^3 + t^4 + t^5 + t^6\right) \prod_{j=1}^{3} \left(1 + |x_j| + x_j^2\right) \int_0^t \left| \int_0^x g(t_1, s)\, ds \right| dt_1 \leqslant A,$$

$(t,x) \in [0,\infty) \times \mathbb{R}^3$, where $\int_0^x = \int_0^{x_1} \int_0^{x_2} \int_0^{x_3}$, $ds = ds_3 ds_2 ds_1$. In the last section, we will give an example for a function g that satisfies $(H2)$. Assume that the constants B and A which appear in the conditions $(H1)$ and $(H2)$, respectively, satisfy the following inequalities:

(H3) $AB_1 < B$, where $B_1 = \left(B + B^{p+1}\right) B + 2B^2 + 4B$,

and

(H4) $AB_1 < \frac{L}{5}$, where $B_1 = \left(B + B^{p+1}\right) B + 2B^2 + 4B$ and L is a positive constant that satisfies the following conditions:

$$r < L < R_1 \leqslant B, \quad R_1 + \frac{A}{m} B_1 > \left(\frac{1}{5m} + 1\right) L,$$

with positive constants r and R_1 and $m > 0$ is large enough.

Our main results are as follows.

Theorem 1.1. *Under the hypotheses* $(H1)$, $(H2)$ *and* $(H3)$, *the IVP* (1.1) *has at least one solution belonging to* $\mathscr{C}^3([0,\infty),\mathscr{C}^2(\mathbb{R}^3))$.

Theorem 1.2. *Under the hypotheses* $(H1)$, $(H2)$ *and* $(H4)$, *the IVP* (1.1) *has at least two solutions belonging to* $u\in\mathscr{C}^3([0,\infty),\mathscr{C}^2(\mathbb{R}^3))$.

The paper is organized as follows. In the next section, we give some auxiliary results. In Section 3 we give some properties of the solutions of the problem (1.1). In Section 4, we prove Theorem 1.1 and Theorem 1.2. In Section 5, we give an example to illustrate our main results.

2. On Fixed Points for the Sum of Two Operators

In this section, we will recall two results which concern the existence and multiplicity of fixed points for the sum of two operators. The proof of the following theorem can be found in [12].

Theorem 2.1. *Let E be a Banach space and $E_1=\{x\in E:\|x\|\leqslant R\}$, with $R>0$. Consider two operators T and S, where $Tx=-\epsilon x$, $x\in E_1$, with $\epsilon>0$ and $S:E_1\to E$ be continuous and such that*

(i) $(I-S)(E_1)$ resides in a compact subset of E and

(ii) $\{x\in E: x=\lambda(I-S)x, \|x\|=R\}=\varnothing$, for any $\lambda\in\left(0,\frac{1}{\epsilon}\right)$.

Then there exists $x^*\in E_1$ such that

$$Tx^*+Sx^*=x^*.$$

In the sequel, E is a real Banach space.

Definition 2.1. A closed, convex set $\mathscr{P}$ in E is said to be cone if

1) $\alpha x\in\mathscr{P}$ for any $\alpha\geqslant 0$ and for any $x\in\mathscr{P}$,

2) $x,-x\in\mathscr{P}$ implies $x=0$.

Definition 2.2. A mapping $K:E\to E$ is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.

Definition 2.3. Let X and Y be real Banach spaces. A mapping $K:X\to Y$ is said to be expansive if there exists a constant $h>1$ such that

$$\left\|Kx-Ky\right\|_Y\geqslant h\left\|x-y\right\|_X$$

for any $x,y\in X$.

The details of the proof of the following result can be found in [13].

Theorem 2.2. *Let $\mathscr{P}$ be a cone of a Banach space E; Ω a subset of $\mathscr{P}$ and U_1, U_2 and U_3 three open bounded subsets of $\mathscr{P}$ such that $\overline{U}_1\subset\overline{U}_2\subset U_3$ and $0\in U_1$. Assume that $T:\Omega\to\mathscr{P}$ is an expansive mapping, $S:\overline{U}_3\to E$ is a completely continuous and $S(\overline{U}_3)\subset(I-T)(\Omega)$. Suppose that $(U_2\backslash\overline{U}_1)\cap\Omega\neq\varnothing$, $(U_3\backslash\overline{U}_2)\cap\Omega\neq\varnothing$, and there exists $w_0\in\mathscr{P}\backslash\{0\}$ such that the following conditions hold:*

(i) $Sx\neq(I-T)(x-\lambda w_0)$, for all $\lambda>0$ and $x\in\partial U_1\cap(\Omega+\lambda w_0)$;

(ii) there exists $\varepsilon>0$ such that $Sx\neq(I-T)(\lambda x)$, for all $\lambda\geqslant 1+\varepsilon$, $x\in\partial U_2$ and $\lambda x\in\Omega$;

(iii) $Sx\neq(I-T)(x-\lambda w_0)$, for all $\lambda>0$ and $x\in\partial U_3\cap(\Omega+\lambda w_0)$.

Then $T+S$ has at least two non-zero fixed points $x_1,x_2\in\mathscr{P}$ such that

$$x_1\in\partial U_2\cap\Omega \quad \text{and} \quad x_2\in\left(\overline{U}_3\backslash\overline{U}_2\right)\cap\Omega$$

or

$$x_1\in\left(U_2\backslash U_1\right)\cap\Omega \quad \text{and} \quad x_2\in\left(\overline{U}_3\backslash\overline{U}_2\right)\cap\Omega.$$

3. Some Properties of the Solutions of the Problem (1.1)

Let $X = \mathscr{C}^3([0,\infty), \mathscr{C}^2(\mathbb{R}^3))$ be endowed with the norm

$$\|u\| = \max\Big\{ \sup_{(t,x)\in[0,\infty)\times\mathbb{R}^3} |u(t,x)|, \sup_{(t,x)\in[0,\infty)\times\mathbb{R}^3} |u_t(t,x)|, \sup_{(t,x)\in[0,\infty)\times\mathbb{R}^3} |u_{x_3t}(t,x)|, \sup_{(t,x)\in[0,\infty)\times\mathbb{R}^3} |u_{tt}(t,x)|, \sup_{(t,x)\in[0,\infty)\times\mathbb{R}^3} |u_{ttt}(t,x)|, \sup_{(t,x)\in[0,\infty)\times\mathbb{R}^3} |u_{x_i}(t,x)|, \sup_{(t,x)\in[0,\infty)\times\mathbb{R}^3} |u_{x_ix_i}(t,x)|,\ i\in\{1,2,3\}\Big\},$$

provided it exists. For $u \in X$, define the operator S_1 as follows:

$$S_1u(t,x) = u(t,x) - u_0(x) - \int_0^t u_1(s)\,ds - \int_0^t (t-s)u_2(s)\,ds - \frac{1}{2}\int_0^t (t-s)^2\Big(f'(u_t(s,x))u_{tt}(s,x) + \beta(s,x)u_{ttt}(s,x) - u_{x_3t}(s,x) + \gamma(s,x)u_t(s,x) + u_{x_1x_1}(s,x) + u_{x_2x_2}(s,x) + u_{ttt}(s,x)\Big)\,ds, \quad (t,x \in [0,\infty)\times\mathbb{R}^3.$$

Lemma 3.1. *Suppose that* $(H1)$ *holds. If* $u \in X$ *satisfies the equation*

$$S_1u(t,x) = 0, \quad (t,x)\in[0,\infty)\times\mathbb{R}^3, \tag{3.1}$$

then u *is a solution to the IVP* (1.1).

$\triangleleft$ By (3.1) and the definition of the operator S_1, we find

$$\begin{aligned} 0 = u(t,x) - u_0(x) - \int_0^t u_1(s)\,ds - \int_0^t (t-s)u_2(s)\,ds \\ - \frac{1}{2}\int_0^t (t-s)^2\Big(f'(u_t(s,x))u_{tt}(s,x) + \beta(s,x)u_{ttt}(s,x) - u_{x_3t}(s,x) \\ + \gamma(s,x)u_t(s,x) + u_{x_1x_1}(s,x) + u_{x_2x_2}(s,x) + u_{ttt}(s,x)\Big)\,ds, \end{aligned} \tag{3.2}$$

$(t,x)\in[0,\infty)\times\mathbb{R}^3$. We differentiate with respect to t the equation (3.2) and we find

$$\begin{aligned} 0 = u_t(t,x) - u_1(x) - \int_0^t u_2(s)\,ds - \int_0^t (t-s)\Big(f'(u_t(s,x))u_{tt}(s,x) + \beta(s,x)u_{ttt}(s,x) \\ - u_{x_3t}(s,x) + \gamma(s,x)u_t(s,x) + u_{x_1x_1}(s,x) + u_{x_2x_2}(s,x) + u_{ttt}(s,x)\Big)\,ds, \end{aligned} \tag{3.3}$$

$(t,x)\in[0,\infty)\times\mathbb{R}^3$, i. e., u satisfies the first equation of (1.1). Now, we differentiate with respect to t the equation (3.3) and we get

$$\begin{aligned} 0 = u_{tt}(t,x) - u_2(x) - \int_0^t \Big(f'(u_t(s,x))u_{tt}(s,x) + \beta(s,x)u_{ttt}(s,x) - u_{x_3t}(s,x) \\ + \gamma(s,x)u_t(s,x) + u_{x_1x_1}(s,x) + u_{x_2x_2}(s,x) + u_{ttt}(s,x)\Big)\,ds, \end{aligned} \tag{3.4}$$

$(t,x)\in[0,\infty)\times\mathbb{R}^3$. We differentiate with respect to t the last equation and we find

$$
\begin{aligned}
0 = u_{ttt}(t,x) - f'(u_t(t,x))u_{tt}(t,x) - \beta(t,x)u_{ttt}(t,x) + u_{x_3t}(t,x) - \gamma(t,x)u_t(t,x)\\
- u_{x_1x_1}(t,x) - u_{x_2x_2}(t,x) - u_{ttt}(t,x) = -f'(u_t(t,x))u_{tt}(t,x) - \beta(t,x)u_{ttt}(t,x)\\
+ u_{x_3t}(t,x) - \gamma(t,x)u_t(t,x) - u_{x_1x_1}(t,x) - u_{x_2x_2}(t,x),
\end{aligned}
$$

$(t,x)\in[0,\infty)\times\mathbb{R}^3$. Now, we put $t=0$ in (3.2), (3.3), (3.4) and we get

$$u(0,x)=u_0(x),\quad u_t(0,x)=u_1(x),\quad u_{tt}(0,x)=u_2(x),\quad x\in\mathbb{R}^3,$$

respectively. Consequently u satisfies the second, third and fourth equations of (1.1). This completes the proof. $\triangleright$

Lemma 3.2. *Suppose that $(H1)$ holds. If $u\in X$, $\|u\|\leqslant B$, then*

$$\big|S_1u(t,x)\big|\leqslant B_1\big(1+t+t^2+t^3\big),\quad (t,x)\in[0,\infty)\times\mathbb{R}^3,$$

where B_1 is the constant defined in $(H3)$.

$\triangleleft$ Suppose that $(H1)$ is satisfied and let $u\in X$, with $\|u\|\leqslant B$. Then, for $(t,x)\in[0,\infty)\times\mathbb{R}^3$, we have

$$
\begin{aligned}
\big|S_1u(t,x)\big| = \Big|u(t,x)-u_0(x)-\int_0^t u_1(s)\,ds-\int_0^t (t-s)u_2(s)\,ds\\
-\frac{1}{2}\int_0^t (t-s)^2\Big(f'(u_t(s,x))u_{tt}(s,x)+\beta(s,x)u_{ttt}(s,x)-u_{x_3t}(s,x)+\gamma(s,x)u_t(s,x)+u_{x_1x_1}(s,x)\\
+u_{x_2x_2}(s,x)+u_{ttt}(s,x)\Big)\,ds\Big| \leqslant \big|u(t,x)\big|+u_0(x)+\int_0^t u_1(s)\,ds+\int_0^t (t-s)u_2(s)\,ds\\
+\frac{1}{2}\int_0^t (t-s)^2\Big(\big|f'(u_t(s,x))\big|\big|u_{tt}(s,x)\big|+\beta(s,x)\big|u_{ttt}(s,x)\big|+\big|u_{x_3t}(s,x)\big|\\
+\big|\gamma(s,x)\big|\big|u_t(s,x)\big|+\big|u_{x_1x_1}(s,x)\big|+\big|u_{x_2x_2}(s,x)\big|+\big|u_{ttt}(s,x)\big|\Big)\,ds\\
\leqslant 2B+tB+t^2B+\int_0^t (t-s)^2\,\Big(\big(a_1(s,x)+a_2(s,x)|u_t(s,x)|^p\big)B+2B^2+4B\Big)\,ds\\
\leqslant 2B+tB+t^2B+t^3\,\Big(\big(B+B^{p+1}\big)B+2B^2+4B\Big)\leqslant B_1\big(1+t+t^2+t^3\big).
\end{aligned}
$$

This completes the proof. $\triangleright$

For $u\in X=\mathscr{C}^3([0,\infty),\mathscr{C}^2(\mathbb{R}^3))$, define the operator S_2 as follows:

$$S_2u(t,x)=\int_0^t\int_0^x (t-t_1)^3\prod_{j=1}^3(x_j-s_j)^2g(t_1,s)S_1u(t_1,s)\,ds dt_1,\quad (t,x)\in[0,\infty)\times\mathbb{R}^3,\quad (3.5)$$

where g is the function which appears in the condition $(H2)$.

Lemma 3.3. *Under hypothesis $(H1)$ and $(H2)$ and for $u\in X$, with $\|u\|\leqslant B$, the following estimate holds:*

$$\|S_2u\|\leqslant AB_1,$$

where B_1 is the constant defined in $(H3)$.

$\lhd$ Suppose that $(H1)$ and $(H2)$ are satisfied and let $u \in X$, with $\|u\| \leqslant B$.

(i) The estimation of $|S_2u(t,x)|$, $(t,x) \in [0,\infty) \times \mathbb{R}^3$:

$$\begin{aligned}
|S_2u(t,x)| &= \Bigg| \int\limits_0^t \int\limits_0^x (t-t_1)^3 \prod_{j=1}^3 (x_j - s_j)^2 g(t_1,s) S_1u(t_1,s)\, ds dt_1 \Bigg| \\
&\leqslant \int\limits_0^t \Bigg| \int\limits_0^x (t-t_1)^3 \prod_{j=1}^3 (x_j - s_j)^2 g(t_1,s) \big| S_1u(t_1,s) \big|\, ds \Bigg|\, dt_1 \\
&\leqslant B_1 \int\limits_0^t \Bigg| \int\limits_0^x (t-t_1)^3 \big(1 + t_1 + t_1^2 + t_1^3\big) \prod_{j=1}^3 (x_j - s_j)^2 g(t_1,s)\, ds \Bigg|\, dt_1 \\
&\leqslant B_1 2^9 \prod_{j=1}^3 x_j^2 t^3 \big(1 + t + t^2 + t^3\big) \int\limits_0^t \Bigg| \int\limits_0^x g(t_1,s)\, ds \Bigg|\, dt_1 \leqslant AB_1.
\end{aligned}$$

(ii) The estimation of $\big|\frac{\partial}{\partial t} S_2u(t,x)\big|$, $(t,x) \in [0,\infty) \times \mathbb{R}^3$:

$$\begin{aligned}
\left|\frac{\partial}{\partial t} S_2u(t,x)\right| &= 3 \Bigg| \int\limits_0^t \int\limits_0^x (t-t_1)^2 \prod_{j=1}^3 (x_j - s_j)^2 g(t_1,s) S_1u(t_1,s)\, ds dt_1 \Bigg| \\
&\leqslant 3 \int\limits_0^t \Bigg| \int\limits_0^x (t-t_1)^2 \prod_{j=1}^3 (x_j - s_j)^2 g(t_1,s) \big| S_1u(t_1,s) \big|\, ds \Bigg|\, dt_1 \\
&\leqslant 3B_1 \int\limits_0^t \Bigg| \int\limits_0^x (t-t_1)^2 \big(1 + t_1 + t_1^2 + t_1^3\big) \prod_{j=1}^3 (x_j - s_j)^2 g(t_1,s)\, ds \Bigg|\, dt_1 \\
&\leqslant 3B_1 2^9 \prod_{j=1}^3 x_j^2 t^2 \big(1 + t + t^2 + t^3\big) \int\limits_0^t \Bigg| \int\limits_0^x g(t_1,s)\, ds \Bigg|\, dt_1 \leqslant AB_1.
\end{aligned}$$

(iii) The estimation of $\big|\frac{\partial^2}{\partial t^2} S_2u(t,x)\big|$, $(t,x) \in [0,\infty) \times \mathbb{R}^3$:

$$\begin{aligned}
\left|\frac{\partial^2}{\partial t^2} S_2u(t,x)\right| &= 6 \Bigg| \int\limits_0^t \int\limits_0^x (t-t_1) \prod_{j=1}^3 (x_j - s_j)^2 g(t_1,s) S_1u(t_1,s)\, ds dt_1 \Bigg| \\
&\leqslant 6 \int\limits_0^t \Bigg| \int\limits_0^x (t-t_1) \prod_{j=1}^3 (x_j - s_j)^2 g(t_1,s) \big| S_1u(t_1,s) \big|\, ds \Bigg|\, dt_1 \\
&\leqslant 6B_1 \int\limits_0^t \Bigg| \int\limits_0^x (t-t_1) \big(1 + t_1 + t_1^2 + t_1^3\big) \prod_{j=1}^3 (x_j - s_j)^2 g(t_1,s)\, ds \Bigg|\, dt_1 \\
&\leqslant 6B_1 2^9 \prod_{j=1}^3 x_j^2 t \big(1 + t + t^2 + t^3\big) \int\limits_0^t \Bigg| \int\limits_0^x g(t_1,s)\, ds \Bigg|\, dt_1 \leqslant AB_1.
\end{aligned}$$

(iv) The estimation of $\left|\frac{\partial^3}{\partial t^3} S_2u(t,x)\right|$, $(t,x)\in[0,\infty)\times\mathbb{R}^3$:

$$\begin{aligned}\left|\frac{\partial^3}{\partial t^3}\, S_2u(t,x)\right| &= 6\left|\int_0^t\int_0^x \prod_{j=1}^3 (x_j-s_j)^2 g(t_1,s) S_1u(t_1,s)\,dsdt_1\right|\\ &\leqslant 6\int_0^t\left|\int_0^x \prod_{j=1}^3 (x_j-s_j)^2 g(t_1,s)\left|S_1u(t_1,s)\right| ds\right| dt_1\\ &\leqslant 6B_1\int_0^t\left|\int_0^x \left(1+t_1+t_1^2+t_1^3\right)\prod_{j=1}^3 (x_j-s_j)^2 g(t_1,s)\,ds\right| dt_1\\ &\leqslant 6B_1 2^9\prod_{j=1}^3 x_j^2\left(1+t+t^2+t^3\right)\int_0^t\left|\int_0^x g(t_1,s)\,ds\right| dt_1 \leqslant AB_1.\end{aligned}$$

(v) The estimation of $\left|\frac{\partial^2}{\partial x_3\partial t}\, S_2u(t,x)\right|$, $(t,x)\in[0,\infty)\times\mathbb{R}^3$:

$$\begin{aligned}\left|\frac{\partial^2}{\partial x_3\partial t}\, S_2u(t,x)\right| &= 6\left|\int_0^t\int_0^x (t-t_1)^2\prod_{j=1}^2 (x_j-s_j)^2 (x_3-s_3) g(t_1,s) S_1u(t_1,s)\,dsdt_1\right|\\ &\leqslant 6\int_0^t\left|\int_0^x (t-t_1)^2\prod_{j=1}^2 (x_j-s_j)^2 (x_3-s_3) g(t_1,s)\left|S_1u(t_1,s)\right| ds\right| dt_1\\ &\leqslant 6B_1\int_0^t\left|\int_0^x (t-t_1)^2\left(1+t_1+t_1^2+t_1^3\right)\prod_{j=1}^2 (x_j-s_j)^2 (x_3-s_3) g(t_1,s)\,ds\right| dt_1\\ &\leqslant 6B_1 2^8\prod_{j=1}^2 x_j^2\,|x_3|\,t^2\left(1+t+t^2+t^3\right)\int_0^t\left|\int_0^x g(t_1,s)\,ds\right| dt_1 \leqslant AB_1.\end{aligned}$$

(vi) The estimation of $\left|\frac{\partial}{\partial x_3}\, S_2u(t,x)\right|$, $(t,x)\in[0,\infty)\times\mathbb{R}^3$:

$$\begin{aligned}\left|\frac{\partial}{\partial x_3}\, S_2u(t,x)\right| &= 3\left|\int_0^t\int_0^x (t-t_1)^3\prod_{j=1}^2 (x_j-s_j)^2 (x_3-s_3) g(t_1,s) S_1u(t_1,s)\,dsdt_1\right|\\ &\leqslant 3\int_0^t\left|\int_0^x (t-t_1)^3\prod_{j=1}^2 (x_j-s_j)^2 (x_3-s_3) g(t_1,s)\left|S_1u(t_1,s)\right| ds\right| dt_1\\ &\leqslant 3B_1\int_0^t\left|\int_0^x (t-t_1)^3\left(1+t_1+t_1^2+t_1^3\right)\prod_{j=1}^2 (x_j-s_j)^2 (x_3-s_3) g(t_1,s)\,ds\right| dt_1\\ &\leqslant 3B_1 2^8\prod_{j=1}^2 x_j^2\,|x_3|\,t^2\left(1+t+t^2+t^3\right)\int_0^t\left|\int_0^x g(t_1,s)\,ds\right| dt_1 \leqslant AB_1.\end{aligned}$$

(vii) The estimation of $\left|\frac{\partial^2}{\partial x_3^2} S_2u(t,x)\right|$, $(t,x) \in [0,\infty) \times \mathbb{R}^3$:

$$\begin{aligned}
\left|\frac{\partial^2}{\partial x_3^2} S_2u(t,x)\right| &= 3\left|\int_0^t \int_0^x (t-t_1)^3 \prod_{j=1}^2 (x_j-s_j)^2 g(t_1,s) S_1u(t_1,s)\,ds dt_1\right| \\
&\leqslant 3\int_0^t \left|\int_0^x (t-t_1)^3 \prod_{j=1}^2 (x_j-s_j)^2 g(t_1,s)\left|S_1u(t_1,s)\right| ds\right| dt_1 \\
&\leqslant 3B_1\int_0^t \left|\int_0^x (t-t_1)^3\left(1+t_1+t_1^2+t_1^3\right) \prod_{j=1}^2 (x_j-s_j)^2 g(t_1,s)\,ds\right| dt_1 \\
&\leqslant 3B_1 2^6 \prod_{j=1}^2 x_j^2 t^2\left(1+t+t^2+t^3\right)\int_0^t \left|\int_0^x g(t_1,s)\,ds\right| dt_1 \leqslant AB_1.
\end{aligned}$$

As above,

$$\left|\frac{\partial}{\partial x_j} S_2u(t,x)\right|, \quad \left|\frac{\partial^2}{\partial x_j^2} S_2u(t,x)\right| \leqslant AB_1, \quad (t,x) \in [0,\infty) \times \mathbb{R}^3,\ j=1,2.$$

Thus, $\|S_2u\| \leqslant B$. This completes the proof. ▷

Lemma 3.4. *Suppose* $(H1)$ *and* $(H2)$. *If* $u \in X$ *satisfies the equation*

$$S_2u(t,x) = C, \quad (t,x) \in [0,\infty) \times \mathbb{R}^3, \tag{3.6}$$

where C *is an arbitrary constant, then* u *is a solution to the IVP* (1.1).

◁ We differentiate four times with respect to t and three times with respect to x_l, $l \in \{1,2,3\}$, the equation (3.6) and we find

$$g(t,x)S_1u(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}^3,$$

whereupon

$$S_1u(t,x) = 0, \quad (t,x) \in (0,\infty) \times \left(\mathbb{R}^3 \Big\backslash \left(\bigcup_{j=1}^3 \{x_j = 0\}\right)\right).$$

Since $S_1u(\cdot,\cdot) \in \mathscr{C}([0,\infty) \times \mathbb{R}^3)$, we get

$$\begin{aligned}
0 &= \lim_{t\to 0} S_1u(t,x) = S_1u(0,x) = \lim_{x_1\to 0} S_1u(t,x) = S_1u(t,0,x_2,x_3) = \lim_{x_2\to 0} S_1u(t,x) \\
&= S_1u(t,x_1,0,x_3) = \lim_{x_3\to 0} S_1u(t,x) = S_1u(t,x_1,x_2,0), \quad (t,x) \in [0,\infty) \times \mathbb{R}^3.
\end{aligned}$$

Thus,

$$S_1u(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}^3.$$

Hence and Lemma 3.1, we conclude that u is a solution to the IVP (1.1). This completes the proof. ▷

4. Proof of Theorem 1.1 and Theorem 1.2

Proof of Theorem 1.1. Assume that the hypotheses $(H1)$, $(H2)$ and $(H3)$ are satisfied. Choose $\epsilon \in (0,1)$, such that $\epsilon B_1(1+A) < B$. Let $\widetilde{\widetilde{\widetilde{Y}}}$ denote the set of all equi-continuous families in X with respect to the norm $\|\cdot\|$. Let also, $\widetilde{\widetilde{Y}} = \overline{\widetilde{\widetilde{\widetilde{Y}}}}$ be the closure of $\widetilde{\widetilde{\widetilde{Y}}}$, $\widetilde{Y} = \widetilde{\widetilde{Y}} \cup \{u_0, u_1, u_2\}$,

$$Y = \{u \in \widetilde{Y} : \|u\| \leqslant B\}.$$

Note that Y is a compact set in X. For $u \in X$, define the operators T and S as follows:

$$Tu(t,x) = -\epsilon u(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}^3,$$

$$Su(t,x) = u(t,x) + \epsilon u(t,x) + \epsilon S_2 u(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}^3,$$

where S_2 is the operator given by formula (3.5). For $u \in Y$, using Lemma 3.3, we have

$$\|(I-S)u\| = \|\epsilon u - \epsilon S_2 u\| \leqslant \epsilon\|u\| + \epsilon\|S_2 u\| \leqslant \epsilon B_1 + \epsilon A B_1 = \epsilon B_1(1+A) < B.$$

Thus, $S : Y \to X$ is continuous and $(I-S)(Y)$ resides in a compact subset of X. Now, suppose that there is a $u \in X$ so that $\|u\| = B$ and

$$u = \lambda(I-S)u \quad \text{or} \quad \frac{1}{\lambda} u = (I-S)u = -\epsilon u - \epsilon S_2 u \quad \text{or} \quad \left(\frac{1}{\lambda} + \epsilon\right) u = -\epsilon S_2 u$$

for some $\lambda \in \left(0, \frac{1}{\epsilon}\right)$. Hence, $\|S_2 u\| \leqslant AB_1 < B$,

$$\epsilon B < \left(\frac{1}{\lambda} + \epsilon\right) B = \left(\frac{1}{\lambda} + \epsilon\right) \|u\| = \epsilon \|S_2 u\| < \epsilon B,$$

which is a contradiction. Hence and Theorem 2.1, it follows that the operator $T+S$ has a fixed point $u^* \in Y$. Therefore

$$u^*(t,x) = Tu^*(t,x) + Su^*(t,x) = -\epsilon u^*(t,x) + u^*(t,x) + \epsilon u^*(t,x) + \epsilon S_2 u^*(t,x),$$
$$(t,x) \in [0,\infty) \times \mathbb{R}^3,$$

whereupon

$$0 = S_2 u^*(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}^3.$$

From here and from Lemma 3.4, it follows that u is a solution to the IVP (1.1). This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. Assume that the hypotheses $(H1)$, $(H2)$ and $(H4)$ are satisfied. Let

$$\widetilde{P} = \{u \in X : u \geqslant 0 \text{ on } [0,\infty) \times \mathbb{R}^3\}.$$

With $\mathscr{P}$ we will denote the set of all equi-continuous families in $\widetilde{P}$. For $v \in X$, define the operators T_1 and S_3 as follows:

$$T_1 v(t,x) = (1 + m\epsilon) v(t,x) - \epsilon \frac{L}{10}, \quad (t,x) \in [0,\infty) \times \mathbb{R}^3,$$

$$S_3 v(t,x) = -\epsilon S_2 v(t,x) - m\epsilon v(t,x) - \epsilon \frac{L}{10}, \quad (t,x) \in [0,\infty) \times \mathbb{R}^3,$$

where $\epsilon > 0$ and S_2 is the operator given by formula (3.5). Note that, by Lemma 3.4, it follows that any fixed point $v \in X$ of the operator $T_1 + S_3$ is a solution to the IVP (1.1). Let us define the following sets:

$$U_1 = \mathscr{P}_r = \{v \in \mathscr{P} : \|v\| < r\}, \quad U_2 = \mathscr{P}_L = \{v \in \mathscr{P} : \|v\| < L\},$$

$$U_3 = \mathscr{P}_{R_1} = \{v \in \mathscr{P} : \|v\| < R_1\}, \quad \Omega = \overline{\mathscr{P}_{R_2}} = \{v \in \mathscr{P} : \|v\| \leqslant R_2\},$$

with $R_2 = R_1 + \frac{A}{m} B_1 + \frac{L}{5m}$.

1. For $v_1, v_2 \in \Omega$, we have

$$\|T_1 v_1 - T_1 v_2\| = (1 + m\epsilon)\|v_1 - v_2\|,$$

whereupon $T_1 : \Omega \to X$ is an expansive operator with a constant $h = 1 + m\epsilon > 1$.

2. For $v \in \overline{\mathscr{P}_{R_1}}$, we get

$$\|S_3 v\| \leqslant \epsilon \,\|S_2 v\| + m\epsilon \,\|v\| + \epsilon \,\frac{L}{10} \leqslant \epsilon\Big(AB_1 + mR_1 + \frac{L}{10}\Big).$$

Therefore $S_3(\overline{\mathscr{P}_{R_1}})$ is uniformly bounded. Since $S_3 : \overline{\mathscr{P}_{R_1}} \to X$ is continuous, we have that $S_3(\overline{\mathscr{P}_{R_1}})$ is equi-continuous. Consequently $S_3 : \overline{\mathscr{P}_{R_1}} \to X$ is completely continuous.

3. Let $v_1 \in \overline{\mathscr{P}_{R_1}}$. Set

$$v_2 = v_1 + \frac{1}{m}\, S_2 v_1 + \frac{L}{5m}.$$

Note that $S_2 v_1 + \frac{L}{5} \geqslant 0$ on $[0, \infty) \times \mathbb{R}^3$. We have $v_2 \geqslant 0$ on $[0, \infty) \times \mathbb{R}^3$ and

$$\|v_2\| \leqslant \|v_1\| + \frac{1}{m}\,\|S_2 v_1\| + \frac{L}{5m} \leqslant R_1 + \frac{A}{m}\, B_1 + \frac{L}{5m} = R_2.$$

Therefore $v_2 \in \Omega$ and

$$-\epsilon m v_2 = -\epsilon m v_1 - \epsilon S_2 v_1 - \epsilon\,\frac{L}{10} - \epsilon\,\frac{L}{10}$$

or

$$(I - T_1) v_2 = -\epsilon m v_2 + \epsilon\,\frac{L}{10} = S_3 v_1.$$

Consequently $S_3(\overline{\mathscr{P}_{R_1}}) \subset (I - T_1)(\Omega)$.

4. Assume that for any $v_0 \in \mathscr{P}^* = \mathscr{P} \backslash \{0\}$ there exist $\lambda > 0$ and $v \in \partial \mathscr{P}_r \cap (\Omega + \lambda v_0)$ or $v \in \partial \mathscr{P}_{R_1} \cap (\Omega + \lambda v_0)$ such that

$$S_3 v = (I - T_1)(v - \lambda v_0).$$

Then

$$-\epsilon S_2 v - m\epsilon v - \epsilon\,\frac{L}{10} = -m\epsilon(v - \lambda v_0) + \epsilon\,\frac{L}{10}$$

or

$$-S_2 v = \lambda m v_0 + \frac{L}{5}.$$

Hence,

$$\|S_2 v\| = \left\|\lambda m v_0 + \frac{L}{5}\right\| > \frac{L}{5}.$$

This is a contradiction.

5. Let $\varepsilon_1 = \frac{2}{5m}$. Assume that there exist $w \in \partial\mathscr{P}_L$ and $\lambda_1 \geqslant 1 + \varepsilon_1$ such that $\lambda_1 w \in \overline{\mathscr{P}_{R_2}}$ and

$$S_3 w = (I - T_1)(\lambda_1 w).$$

Since $w \in \partial\mathscr{P}_L$ and $\lambda_1 w \in \overline{\mathscr{P}_{R_2}}$, it follows that

$$\left(\frac{2}{5m} + 1\right) L < \lambda_1 L = \lambda_1 \|w\| \leqslant R_1 + \frac{A}{m} B_1 + \frac{L}{5m}.$$

Moreover,

$$-\epsilon S_2 w - m\epsilon w - \epsilon \frac{L}{10} = -\lambda_1 m\epsilon w + \epsilon \frac{L}{10},$$

or

$$S_2 w + \frac{L}{5} = (\lambda_1 - 1)mw.$$

From here,

$$2\frac{L}{5} > \left\| S_2 w + \frac{L}{5} \right\| = (\lambda_1 - 1)m\|w\| = (\lambda_1 - 1)mL$$

and

$$\frac{2}{5m} + 1 > \lambda_1,$$

which is a contradiction.

Therefore all conditions of Theorem 2.2 hold. Hence, the problem (1.1) has at least two solutions u_1 and u_2 so that

$$\|u_1\| = L < \|u_2\| \leqslant R_1$$

or

$$r \leqslant \|u_1\| < L < \|u_2\| \leqslant R_1.$$

This completes the proof of Theorem 1.2.

5. An Example

Take

$$h(s) = \log \frac{1 + s^{11}\sqrt{2} + s^{22}}{1 - s^{11}\sqrt{2} + s^{22}}, \quad l(s) = \arctan \frac{s^{11}\sqrt{2}}{1 - s^{22}}, \quad s \in \mathbb{R}, \ s \neq \pm 1.$$

Then

$$h'(s) = \frac{22\sqrt{2}s^{10}(1 - s^{22})}{(1 - s^{11}\sqrt{2} + s^{22})(1 + s^{11}\sqrt{2} + s^{22})}, \quad l'(s) = \frac{11\sqrt{2}s^{10}(1 + s^{22})}{1 + s^{44}}, \quad s \in \mathbb{R}, \ s \neq \pm 1.$$

Therefore

$$-\infty < \lim_{s \to \pm\infty} \left(1 + s + s^2\right)^3 h(s) < \infty, \quad -\infty < \lim_{s \to \pm\infty} \left(1 + s + s^2\right)^3 l(s) < \infty.$$

Hence, there exists a positive constant C_1 so that

$$\left(1 + s + s^2\right)^3 \left(\frac{1}{44\sqrt{2}} \log \frac{1 + s^{11}\sqrt{2} + s^{22}}{1 - s^{11}\sqrt{2} + s^{22}} + \frac{1}{22\sqrt{2}} \arctan \frac{s^{11}\sqrt{2}}{1 - s^{22}} \right) \leqslant C_1, \quad s \in \mathbb{R}.$$

Note that $\lim_{s\to\pm 1} l(s) = \frac{\pi}{2}$ and by [14, p. 707, Integral 79], we have

$$\int \frac{dz}{1+z^4} = \frac{1}{4\sqrt{2}} \log \frac{1+z\sqrt{2}+z^2}{1-z\sqrt{2}+z^2} + \frac{1}{2\sqrt{2}} \arctan \frac{z\sqrt{2}}{1-z^2}.$$

Let

$$Q(s) = \frac{s^{10}}{(1+s^{44})(1+s+s^2)^2}, \quad s \in \mathbb{R},$$

and

$$g_1(t,x) = Q(t)Q(x_1)Q(x_2)Q(x_3), \quad t \in [0,\infty),\ x = (x_1,x_2,x_3) \in \mathbb{R}^3.$$

Then there exists a constant $C > 0$ such that

$$6 \cdot 2^9 \big(1+t+t^2+t^3+t^4+t^5+t^6\big) \prod_{j=1}^{3} \big(1+|x_j|+x_j^2\big) \int_0^t \left| \int_0^x g_1(\tau,z)\,dz \right| d\tau \leqslant C,$$
$$(t,x) \in [0,\infty) \times \mathbb{R}^3.$$

Let

$$g(t,x) = \frac{A}{C}\, g_1(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}^3.$$

Then

$$6 \cdot 2^9 \big(1+t+t^2+t^3+t^4+t^5+t^6\big) \prod_{j=1}^{3} \big(1+|x_j|+x_j^2\big) \int_0^t \left| \int_0^x g(\tau,z)\,dz \right| d\tau \leqslant A,$$
$$(t,x) \in [0,\infty) \times \mathbb{R}^3,$$

i. e., $(H2)$ holds. Now, consider the following initial value problem

$$\begin{aligned} u_{x_3 t} &= \frac{u_t}{(1+u_t^2)^2}\, u_{tt} + \frac{1}{1+t^2+x_1^2+x_2^2+x_3^4}\, u_{ttt} \\ &\quad + u_t + u_{x_1x_1} + u_{x_2x_2}, \quad t>0,\ x = (x_1,x_2,x_3) \in \mathbb{R}^3, \end{aligned} \tag{5.1}$$
$$u(0,x) = \frac{1}{1+x_2^4+x_3^2}, \quad u_t(0,x) = \frac{1}{1+3x_1^6+x_2^2}, \quad u_{tt}(0,x) = \frac{1}{1+x_1^8+x_3^4}, \quad x \in \mathbb{R}^3,$$

so that $(H1)$ holds, with $B = 1$, $p = 1$. Take

$$B = p = 1, \quad \text{and} \quad A = \frac{1}{80}.$$

Then

$$B_1 = \big(B + B^{p+1}\big) B + 2B^2 + 4B = 2 + 2 + 4 = 8$$

and

$$AB_1 = \frac{1}{10} < B.$$

So, the hypothesis $(H3)$ is fulfilled. Thus, the hypotheses $(H1)$, $(H2)$ and $(H3)$ are satisfied. Hence, by Theorem 1.1, it follows that IVP (5.1) has at least one solution $u \in \mathscr{C}^3([0,\infty),\mathscr{C}^2(\mathbb{R}^3))$.

In the sequel, take

$$R_1 = 10, \quad L = 5, \quad r = 4, \quad m = 10^{50}, \quad A = \frac{1}{80}, \quad \epsilon = \frac{1}{5B_1(1+A)}.$$

Clearly,

$$r < L < R_1 \leqslant B, \quad \epsilon > 0, \quad R_1 + \frac{A}{m} B_1 > \left(\frac{1}{5m} + 1\right) L, \quad AB_1 < \frac{L}{5},$$

i. e., $(H4)$ holds. Hence, by Theorem 1.2, it follows that Problem (5.1) has at least two nonnegative solutions $u, v \in \mathscr{C}^3([0,\infty), \mathscr{C}^2(\mathbb{R}^3))$.

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СУЩЕСТВОВАНИЕ КЛАССИЧЕСКИХ РЕШЕНИЙ ДЛЯ КЛАССА УРАВНЕНИЙ ТИПА ХОХЛОВА — ЗАБОЛОЦКОЙ — КУЗНЕЦОВА

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Аннотация. В медицинских науках, во время медицинского исследования и диагностики тканей или при медицинской визуализации, мы часто используем математические модели для ответа на вопросы, связанные с этими исследованиями. Среди этих моделей значительный интерес представляет нелинейное уравнение в частных производных типа Хохлова — Заболоцкой — Кузнецова (сокращенно — уравнение ХЗК) в задачах ультразвуковой акустики. Эта математическая модель описывает нелинейное распространение звукового импульса конечной амплитуды в термовязкой среде. Уравнение получается путем объединения уравнения сохранения массы, уравнения сохранения импульса и уравнений состояния. Следует отметить, что для этого уравнения мало математического анализа. Это уравнение учитывает три комбинированных эффекта: дифракцию волны, поглощение энергии и нелинейность среды, в которой распространяется волна. Уравнение типа ХЗК, представленное в данной работе, представляет собой модифицированную версию модели ХЗК, известной в акустике. Изучается класс уравнений типа Хохлова — Заболоцкой — Кузнецова на предмет существования глобальных классических решений. Приведены условия, при которых рассматриваемые уравнения имеют хотя бы одно или хотя бы два классических решения. Для доказательства основных результатов мы предлагаем новый подход, основанный на недавних теоретических результатах.

Ключевые слова: уравнение типа Хохлова — Заболоцкой — Кузнецова, глобальное классическое решение, неподвижная точка, сумма операторов, начальная задача.

AMS Subject Classification: 35Q35, 35G20, 35A09, 35E15, 47H10, 35G25.

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