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AN ANALYSIS OF CONVEXITY AND STARLIKENESS ATTRIBUTES FOR BREAZ INTEGRO-DIFFERENTIAL OPERATOR[#]

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Abstract. The Geometric Theory of Analytic Functions (GTAF) is the attractive part of complex analysis, which correlates with the rest of the themes in mathematics. Its essential purpose is to formulate numerous classes of geometric analytic functions and to discuss their geometric attributes. In continuation, the association between operator theory and the GTAF area started to take shape and has remained a topic of wide attention today. In the previous century, operator theory was extended to the complex open unit disk and has been applied to propose diverse sorts of generalizations of normalized analytic functions. As a result, the operator theory appears to be a good way to look for things in the GTAF area. Since then, the acquisition of geometric attributes by employing operators has become a significant theme of research studies. The current study centers on and investigates, in the classes of ℓ-uniformly convex and starlike functions of order β, the convexity attribute by utilizing a modified Breaz integro-differential operator in the unit disk. Furthermore, in the class of analytic functions, some conditions that make the Breaz operator look like a star are looked into.

Key words: analytic function, uniformly convex function, uniformly starlike function, Breaz operator.

AMS Subject Classification: 30C45, 30C50, 30C10.

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1. Overviews

In the territory of complex analysis, the Geometric Theory of Analytic Functions (GTRF) studies the connection between the analytic structure of given functions and the geometrical behavior of their image domain on the unit disc. The geometrical and analytical interaction is the most catchy aspect of GTRF, therefore, it has remained remarkably one of the vivid themes in current seeking. In this discipline, special functions (SFs) also play a vast role in GTRF due to the solution of the renowned problem "Bieberbach conjecture" in GTRF by the interested researcher L. de-Branges [1]. Subsequently, the research dealing with assorted geometric aspects of the analytic functions correlating with numerous SFs has been condensed [2–6]. Actually, SFs have contributed significantly to the development of complex analysis [7–12].

The base catalyst for founding the GTRF is a significant outcome called "Riemann Mapping theorem" (RMT) dating back to 1851. Five decades later, in 1907, Koebe [13] posed

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an immensely important paper which led to the evolution of GTRF. Koebe's paper presents that analytic and univalent (one-to-one) functions on the unit disk have a "conformality" attribute in view of RMT. Since then, the pivotal attribute of such functions is, that their ranges will describe diverse geometrics, such as convex, star-shaped (starlike), close-to-convex and others. This means that studies on certain subclasses of analytic functions such as convex, starlike, and close-to-convex functions are defined in the unit disk. The term "convexity attribute" is one of the delightful attributes in GTRF, which was first originated in 1913, due to E. Study [14]. In 1915, J. W. Alexander [15] detected this attribute and noticed a beautiful outcome that yields a bridge between these attributes, namely, Alexander's theorem. Then, in 1921, R. Nevanlinna [16] gave the analytic description for starlikeness. Following Nevanlinna's notation, M. S. Robertson [17] in 1954 imposed the concepts of the order $\rho \in [0,1)$ of convexity and starlikeness attributes of analytic functions and also studied their analytic descriptions. These are new subclasses of the classes of convex and starlike functions, consecutively. Subsequent studies about some adequate stipulations for univalentness, convexity, starlikeness were conducted by H. Shiraishi and S. Owa [18], M. Nunokawa *et al.* [19], J. Sokól and Nunokawa [20] and M. Nunokawa and Sokól [21] and others.

The corresponding studies "uniform attribute", in 1991 by A. W. Goodman [22, 23] first presented the notions of uniform convexity and uniform starlikeness and investigated two-variable analytic descriptions of such attributes. The following year, F. Rønning [24] and W. C. Ma and D. Minda [25] independently considered one-variable analytic description of uniform convexity attribute. The usage of this stipulation leads to achieving numerous important properties. However, a one-variable analytic description of uniform starlikeness attribute is not yet available. In [22], A. W. Goodman displayed that the acclaimed Alexander's theorem does not materialize to uniform convexity and starlikeness attributes. In exploring the possibility of this analogous outcome, F. Rønning [26] in 1993 presented the corresponding uniform starlikeness attribute which is called the parabolic starlikeness attribute related to uniform convexity attribute. Thereafter, in 1997, R. Bharati *et al* [27] studied and examined more general attributes of the order ρ of uniform σ-convexity and the order ρ of parabolic σ-starlikeness attributes, $\rho \in (-1,1]$, $\sigma \in [0,\infty)$ consecutively. Besides, several outcomes emerged that deal with the study of uniformly convex and starlike functions, for instance, [28–32].

On the other trend, complex linear and non-linear operators are remarkable themes in Operation Theory, which contributed splendidly to the evolution of GTRF. Due to fruitful implementations of complex operators, researchers are interested in examining geometric attributes of functions by utilizing complex operators. The Alexander operator is the first integral complex operator. It was coined by J. W. Alexander [15] in 1915. Later, in 1965, R. J. Libera [33] began studying another integral complex operator, namely Libera operator, and discussed starlikeness attribute under this operator. This operator was circulated in [34, 35] and others. After that, in 1975, S. Ruscheweyh [36] utilized the convolution technique to impose linear operator on the class of analytic functions, the so-called Ruscheweyh operator. Subsequent, in 1983, G. S. Sălăgean [37] posed a differential and integral operators formulate called the Sălăgean differential operator and Sălăgean integral operator, consecutively. Following, in 1984, B. C. Carlson and D. B. Shaffer [38] provided a linear operator by employing the convolution tool between the class of analytic functions with an impressive class of special functions called incomplete beta function. Then, several complex analysts have taken an interest in contributing to the creation of highlighted (linear and non-linear) operators on the complex domain, see [39, 40].

In this context, in 1990, N. N. Pascu and V. Pescar [41] first posed integro-differential operator. Since then, attention has been devoted to the study further generalizations and extensions of the integro-differential (integral) operators. In 2002, D. Breaz and N. Breaz [42] studied a new general integro-differential operator and investigated several geometric attributes for this operators on the unit disk. In 2008, D. Breaz *et al* [43] considered a generalized integro-differential operator and discussed some attributes of univalent function associated with this operator. Afterward, in 2011, B. A. Frasin [44] defined a more general integro-differential operators based on Breaz study. Further recent investigations were made on Breaz integral operators, see [45, 46] and [32].

This effort, in terms of modified Breaz integro-differential operator, highlights the study of the convexity attribute on the classes of ℓ-uniformly convex and starlike functions of order β. Besides, some adequate stipulations for this Breaz operator to be starlike in the unit disk are introduced and discussed.

2. Introduction

Let $\mathfrak{D} = \{z : |z| < 1\}$ be the open unit disc in the z-plane $\mathscr{C}$ and let $\mathscr{H}(\mathfrak{D})$ represent the class of all analytic functions in $\mathfrak{D}$. Denoted by $\mathfrak{A}$, the class of analytic functions f are given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

are normalized in $\mathfrak{D}$. Consider $\mathfrak{S}$ to be the subclass of $\mathfrak{A}$ including univalent functions. For $0 \leqslant \beta < 1$, let $\mathscr{C}_{\mathscr{V}}(\beta)$ and $\mathscr{S}^*(\beta)$ denote the subclasses of $\mathfrak{A}$ that involve, consecutively, convex and starlike functions of order β. These functions are noted to be univalent and are defined analytically by

$$\beta < \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right)$$

and

$$\beta < \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right),$$

consecutively [13]. Obviously, if $\beta = 0$, then $\mathscr{C}_{\mathscr{V}}(\beta)$ and $\mathscr{S}^*(\beta)$ coincide with $\mathscr{C}_{\mathscr{V}}$ and $\mathscr{S}^*$, consecutively. Furthermore, the subclass $\ell - \mathscr{C}_{\mathscr{V}}(\beta)$ of $\mathfrak{A}$ consists of ℓ-uniformly convex functions f of order β formulated as:

$$\ell \left| \frac{zf''(z)}{f'(z)} \right| \leqslant \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)} - \beta\right), \tag{2}$$

for $z \in \mathfrak{D}$, $\beta \in [-1, 1)$, $0 \leqslant \ell$ [27]. Whilst, the subclass $\ell - \mathscr{S}^*(\beta)$ of $\mathfrak{A}$ includes ℓ-uniformly starlike functions f of order β defined as:

$$\ell \left| \frac{zf'(z)}{f(z)} - 1 \right| \leqslant \operatorname{Re}\left(\frac{zf'(z)}{f(z)} - \beta\right), \tag{3}$$

for $z \in \mathfrak{D}$, $\beta \in [-1, 1)$, $0 \leqslant \ell$ [27]. Moreover, the following lemmas will be advantageous tools in acquiring the main outcomes.

Lemma 2.1 [18]. *If $f \in \mathfrak{A}$ achieves*

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{\mu + 1}{2(\mu - 1)}, \quad z \in \mathfrak{D},\ 2 \leqslant \mu < 3,$$

or

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) < \frac{5\mu-1}{2\,(\mu+1)}, \quad z\in\mathfrak{D},\ 1<\mu\leqslant 2,$$

then $f\in\mathscr{S}^*$.

Lemma 2.2 [18]**.** *If* $f\in\mathfrak{A}$ *achieves*

$$-\frac{\mu+1}{2\mu\,(\mu-1)} < \operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right), \quad z\in\mathfrak{D},\ \mu\leqslant -1,$$

or

$$\frac{3\mu+1}{2\mu\,(\mu+1)} < \operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right), \quad z\in\mathfrak{D},\ 1<\mu,$$

then $f\in\mathscr{S}^*\big(\frac{\mu+1}{2\mu}\big)$.

Based on the study in [44], consider the following modified Breaz integro-differential operator $\Omega_\aleph(f_\kappa;\tau_\kappa):\mathfrak{A}^\aleph\longrightarrow\mathfrak{A}$ as:

$$\Omega_\aleph(f_\kappa;\tau_\kappa)(z)=\int_0^z\prod_{\kappa=1}^{\aleph}\big(f'_\kappa(\omega)\big)^{\frac{1}{\rho_\kappa}}\left(\frac{\tau_\kappa(\omega)}{\omega}\right)^{\frac{1}{\sigma_\kappa}}d\omega, \tag{4}$$

where $f_\kappa,\tau_\kappa\in\mathfrak{A}$, $0<\rho_\kappa$, σ_κ for $1\leqslant\kappa\leqslant\aleph$.

3. Convexity of Breaz Operator

This section studies the convexity attribute for the modified Breaz integro-differential operator given by (23) on the classes $\ell-\mathscr{C}_\mathscr{V}(\beta)$ and $\ell-\mathscr{S}^*(\beta)$.

Theorem 3.1. *If* $f_\kappa\in\ell_\kappa-\mathscr{C}_\mathscr{V}(\beta_\kappa)$, $\tau_\kappa\in\ell_\kappa-\mathscr{S}^*(\beta_\kappa)$, *with* $\beta_\kappa\in[-1,1)$, $0\leqslant\ell_\kappa$ *for* $1\leqslant\kappa\leqslant\aleph$, $\sum_{\kappa=1}^{\aleph}\frac{1}{\rho_\kappa}\leqslant\frac14$, *and* $\sum_{\kappa=1}^{\aleph}\frac{1}{\sigma_\kappa}\leqslant\frac14$, *then* $\Omega_\aleph(f_\kappa;\tau_\kappa)\in\mathscr{C}_\mathscr{V}(\zeta)$, *where* $\zeta=1+\sum_{\kappa=1}^{\aleph}\big(\frac{1}{\rho_\kappa}+\frac{1}{\sigma_\kappa}\big)(\beta_\kappa-1)$.

$\lhd$ In view of (23), we have

$$\Omega'_\aleph(f_\kappa;\tau_\kappa)(z)=\prod_{\kappa=1}^{\aleph}\big(f'_\kappa(z)\big)^{\frac{1}{\rho_\kappa}}\left(\frac{\tau_\kappa(z)}{z}\right)^{\frac{1}{\sigma_\kappa}}. \tag{5}$$

By utilizing natural logarithm for equation (), we derive

$$\ln\Omega'_\aleph(f_\kappa;\tau_\kappa)(z)=\sum_{\kappa=1}^{\aleph}\left[\frac{\ln\left(f'_\kappa(z)\right)}{\rho_\kappa}+\frac{1}{\sigma_\kappa}\left(\ln\left(\tau_\kappa(z)\right)-\ln(z)\right)\right]. \tag{6}$$

Differentiating (24), we deduce

$$\frac{z\Omega''_\aleph(f_\kappa;\tau_\kappa)(z)}{\Omega'_\aleph(f_\kappa;\tau_\kappa)(z)}=\sum_{\kappa=1}^{\aleph}\frac{1}{\rho_\kappa}\left(\frac{zf''_\kappa(z)}{f'_\kappa(z)}\right)+\sum_{\kappa=1}^{\aleph}\frac{1}{\sigma_\kappa}\left[\left(\frac{z\tau'_\kappa(z)}{\tau_\kappa(z)}\right)-1\right]. \tag{7}$$

Using the real part of equation (25), we obtain

$$\operatorname{Re}\left(\frac{z\Omega''_\aleph(f_\kappa;\tau'_\kappa(z)}{\Omega'_\aleph(f_\kappa;\tau_\kappa)(z)}\right)=\sum_{\kappa=1}^{\aleph}\frac{1}{\rho_\kappa}\operatorname{Re}\left(\frac{zf''_\kappa(z)}{f'_\kappa(z)}\right)+\sum_{\kappa=1}^{\aleph}\frac{1}{\sigma_\kappa}\operatorname{Re}\left(\frac{z\tau'_\kappa(z)}{\tau_\kappa(z)}\right)-\sum_{\kappa=1}^{\aleph}\frac{1}{\sigma_\kappa}.$$

Therefore,

$$\operatorname{Re}\left(\frac{z\Omega''_{\aleph}(f_\kappa;\tau_\kappa)(z)}{\Omega'_{\aleph}(f_\kappa;\tau_\kappa)(z)}\right)=\sum_{\kappa=1}^{\aleph}\frac{1}{\rho_\kappa}\operatorname{Re}\left(1+\frac{zf''_\kappa(z)}{f'_\kappa(z)}-\beta_\kappa\right)+\sum_{\kappa=1}^{\aleph}\left(\frac{\beta_\kappa}{\rho_\kappa}-\frac{1}{\rho_\kappa}\right)$$
$$+\sum_{\kappa=1}^{\aleph}\frac{1}{\sigma_\kappa}\operatorname{Re}\left(\frac{z\tau'_\kappa(z)}{\tau_\kappa(z)}-\beta_\kappa\right)+\sum_{\kappa=1}^{\aleph}\left(\frac{\beta_\kappa}{\sigma_\kappa}-\frac{1}{\sigma_\kappa}\right).$$

Since $f_\kappa \in \ell_\kappa - \mathscr{C}_{\mathscr{V}}(\beta_\kappa)$ and $\tau_\kappa \in \ell_\kappa - \mathscr{S}^*(\beta_\kappa)$ for $1 \leqslant \kappa \leqslant \aleph$ lead to the application of (2) and (7) in the latter equation and imply that

$$\operatorname{Re}\left(\frac{z\Omega''_{\aleph}(f_\kappa;\tau_\kappa)(z)}{\Omega'_{\aleph}(f_\kappa;\tau_\kappa)(z)}\right)=\sum_{\kappa=1}^{\aleph}\frac{1}{\rho_\kappa}\,\ell_\kappa\left|\frac{zf''_\kappa(z)}{f'_\kappa(z)}\right|+\sum_{\kappa=1}^{\aleph}\frac{1}{\sigma_\kappa}\,\ell_\kappa\left|\frac{zf'_\kappa(z)}{f_\kappa(z)}-1\right|$$
$$+\sum_{\kappa=1}^{\aleph}\left(\frac{1}{\rho_\kappa}+\frac{1}{\sigma_\kappa}\right)(\beta_\kappa-1)\geqslant\sum_{\kappa=1}^{\aleph}\left(\frac{1}{\rho_\kappa}+\frac{1}{\sigma_\kappa}\right)(\beta_\kappa-1). \quad (8)$$

The equation (29) yields

$$\operatorname{Re}\left(1+\frac{z\Omega''_{\aleph}(f_\kappa;\tau_\kappa)(z)}{\Omega'_{\aleph}(f_\kappa;\tau_\kappa)(z)}\right)\geqslant 1+\sum_{\kappa=1}^{\aleph}\left(\frac{1}{\rho_\kappa}+\frac{1}{\sigma_\kappa}\right)(\beta_\kappa-1).$$

Thus, $\Omega_{\aleph}(f_\kappa;\tau_\kappa) \in \mathscr{C}_{\mathscr{V}}(\zeta)$, where $\zeta = 1+\sum_{\kappa=1}^{\aleph}\left(\frac{1}{\rho_\kappa}+\frac{1}{\sigma_\kappa}\right)(\beta_\kappa-1)$. Since $\beta_\kappa \in [-1,1)$ for $1 \leqslant \kappa \leqslant \aleph$, $\sum_{\kappa=1}^{\aleph}\frac{1}{\rho_\kappa} \leqslant \frac{1}{4}$, and $\sum_{\kappa=1}^{\aleph}\frac{1}{\sigma_\kappa} \leqslant \frac{1}{4}$ lead to $0 \leqslant \zeta < 1$. This completes the proof. ▷

By setting $\ell = \ell_1 = \ell_2 = \ldots = \ell_{\aleph}$ and $\beta = \beta_1 = \beta_2 = \ldots = \beta_{\aleph}$ in Theorem 3.1, the following convexity attribute is obtained.

Corollary 3.1. *If $f_\kappa \in \ell - \mathscr{C}_{\mathscr{V}}(\beta)$, $\tau_\kappa \in \ell - \mathscr{S}^*(\beta)$ for $1 \leqslant \kappa \leqslant \aleph$, with $\beta \in [-1,1)$, $0 \leqslant \ell$, $\sum_{\kappa=1}^{\aleph}\frac{1}{\rho_\kappa} \leqslant \frac{1}{4}$, and $\sum_{\kappa=1}^{\aleph}\frac{1}{\sigma_\kappa} \leqslant \frac{1}{4}$, then $\Omega_{\aleph}(f_\kappa;\tau_\kappa) \in \mathscr{C}_{\mathscr{V}}(\zeta)$, where $\zeta = 1+(\beta-1)\sum_{\kappa=1}^{\aleph}\left(\frac{1}{\rho_\kappa}+\frac{1}{\sigma_\kappa}\right)$.*

The special case $\aleph = 1$ above yields the following outcome.

Corollary 3.2. *If $f \in \ell - \mathscr{C}_{\mathscr{V}}(\beta)$, $\tau \in \ell - \mathscr{S}^*(\beta)$, with $\beta \in [-1,1)$, $0 \leqslant \ell$, $\frac{1}{\rho} \leqslant \frac{1}{4}$, and $\frac{1}{\sigma} \leqslant \frac{1}{4}$, then $\Omega(f;\tau) \in \mathscr{C}_{\mathscr{V}}(\zeta)$, where $\zeta = 1+(\beta-1)\left(\frac{1}{\rho}+\frac{1}{\sigma}\right)$ and $\Omega(f;\tau) = \int_0^z (f'(\omega))^{\frac{1}{\rho}}\left(\frac{\tau(\omega)}{\omega}\right)^{\frac{1}{\sigma}} d\omega$.*

4. Starlikeness of Breaz Operator

This section examines some adequate stipulations for starlikeness of Breaz operator on class $\mathfrak{A}$.

An implementation of Lemma 2.1 to Breaz operator (23) yields the following outcome.

Theorem 4.1. *Let $0 < \rho_\kappa$, σ_κ for $1 \leqslant \kappa \leqslant \aleph$. If f_κ, $\tau_\kappa \in \mathfrak{A}$ for $1 \leqslant \kappa \leqslant \aleph$ achieves*

$$\operatorname{Re}\left(\frac{zf''_\kappa(z)}{f'_\kappa(z)}\right)<\frac{\rho_\kappa(3-\mu)}{4\aleph(\mu-1)} \quad \text{and} \quad \operatorname{Re}\left(\frac{z\tau'_\kappa(z)}{\tau_\kappa(z)}\right)<1+\frac{\sigma_\kappa(3-\mu)}{4\aleph(\mu-1)}, \quad (9)$$

for some $2 \leqslant \mu < 3$, or

$$\operatorname{Re}\left(\frac{zf''_\kappa(z)}{f'_\kappa(z)}\right)<\frac{\rho_\kappa(\mu-1)}{\aleph(\mu+1)} \quad \text{and} \quad \operatorname{Re}\left(\frac{z\tau'_\kappa(z)}{\tau_\kappa(z)}\right)<1+\frac{\sigma_\kappa(\mu-1)}{2\aleph(\mu+1)}, \quad (10)$$

for some $1 < \mu \leqslant 2$, then $\Omega_{\aleph}(f_\kappa;\tau_\kappa) \in \mathscr{S}^*$.

◁ We have

$$1+\frac{z\Omega_{\aleph}(f_\kappa;\tau_\kappa)''(z)}{\Omega_{\aleph}(f_\kappa;\tau_\kappa)'(z)}=\sum_{\kappa=1}^{\aleph}\frac{1}{\rho_\kappa}\left(\frac{zf_\kappa''(z)}{f_\kappa'(z)}\right)+\sum_{\kappa=1}^{\aleph}\frac{1}{\sigma_\kappa}\left(\frac{z\tau_\kappa'(z)}{\tau_\kappa(z)}\right)+1-\sum_{\kappa=1}^{\aleph}\frac{1}{\sigma_\kappa}. \tag{11}$$

Taking the real part of equation (11), we obtain

$$\operatorname{Re}\left(1+\frac{z\Omega_{\aleph}(f_\kappa;\tau_\kappa)''(z)}{\Omega_{\aleph}(f_\kappa;\tau_\kappa)'(z)}\right)=\sum_{\kappa=1}^{\aleph}\frac{1}{\rho_\kappa}\operatorname{Re}\left(\frac{zf_\kappa''(z)}{f_\kappa'(z)}\right)+\sum_{\kappa=1}^{\aleph}\frac{1}{\sigma_\kappa}\operatorname{Re}\left(\frac{z\tau_\kappa'(z)}{\tau_\kappa(z)}\right)+1-\sum_{\kappa=1}^{\aleph}\frac{1}{\sigma_\kappa}. \tag{12}$$

From equations (9) and (12), we can acquire the following:

$$\begin{aligned}\operatorname{Re}\left(1+\frac{z\Omega_{\aleph}(f_\kappa;\tau_\kappa)''(z)}{\Omega_{\aleph}(f_\kappa;\tau_\kappa)'(z)}\right)&<\frac{1}{\rho_1}\left(\frac{\rho_1(3-\mu)}{4\aleph(\mu-1)}\right)+\frac{1}{\rho_2}\left(\frac{\rho_2(3-\mu)}{4\aleph(\mu-1)}\right)+\dots\\&+\frac{1}{\rho_\aleph}\left(\frac{\alpha_\aleph(3-\mu)}{4\aleph(\mu-1)}\right)+\frac{1}{\sigma_1}\left(1+\frac{\sigma_1(3-\mu)}{4\aleph(\mu-1)}\right)+\frac{1}{\sigma_2}\left(1+\frac{\sigma_2(3-\mu)}{4\aleph(\mu-1)}\right)+\dots\\&+\frac{1}{\sigma_\aleph}\left(1+\frac{\sigma_\aleph(3-\mu)}{4\aleph(\mu-1)}\right)+1-\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}+\dots+\frac{1}{\sigma_\aleph}\right).\end{aligned}$$

A computation of the above gains

$$\operatorname{Re}\left(1+\frac{z\Omega_{\aleph}(f_\kappa;\tau_\kappa)''(z)}{\Omega_{\aleph}(f_\kappa;\tau_\kappa)'(z)}\right)<\frac{3-\mu}{2(\mu-1)}+1=\frac{\mu+1}{2(\mu-1)}$$

for some $2\leqslant\mu<3$. Also from Equations (10) and (12), we derive

$$\begin{aligned}\operatorname{Re}\left(1+\frac{z\Omega_{\aleph}(f_\kappa;\tau_\kappa)''(z)}{\Omega_{\aleph}(f_\kappa;\tau_\kappa)'(z)}\right)&<\frac{1}{\rho_1}\left(\frac{\rho_1(\mu-1)}{\aleph(\mu+1)}\right)+\frac{1}{\rho_2}\left(\frac{\rho_2(\mu-1)}{\aleph(\mu+1)}\right)+\dots\\&+\frac{1}{\rho_\aleph}\left(\frac{\rho_\aleph(\mu-1)}{\aleph(\mu+1)}\right)+\frac{1}{\sigma_1}\left(1+\frac{\sigma_1(\mu-1)}{2\aleph(\mu+1)}\right)+\frac{1}{\sigma_2}\left(1+\frac{\sigma_2(\mu-1)}{2\aleph(\mu+1)}\right)+\dots\\&+\frac{1}{\sigma_\aleph}\left(1+\frac{\sigma_\aleph(\mu-1)}{2\aleph(\mu+1)}\right)+1-\left(\frac{1}{\sigma_1}+\frac{1}{\sigma_2}+\dots+\frac{1}{\sigma_\aleph}\right).\end{aligned}$$

Hence

$$\operatorname{Re}\left(1+\frac{z\Omega_{\aleph}(f_\kappa;\tau_\kappa)''(z)}{\Omega_{\aleph}(f_\kappa;\tau_\kappa)'(z)}\right)<\frac{3(\mu-1)}{2(\mu+1)}+1=\frac{5\mu-1}{2(\mu+1)}.$$

for some $1<\mu\leqslant 2$. Thus by Lemma 2.1, we yield $\Omega_{\aleph}(f_\kappa;\tau_\kappa)(z)\in\mathscr{S}^*$. This completes the proof. ▷

For $\aleph=1$, $\rho=\rho_1$, $\sigma=\sigma_1$ and $f_1=f$, $\tau_1=\tau$ in Theorem 4.1, we deduce the following outcome:

Corollary 4.1. *Let $\rho,\sigma>0$. If $f,\tau\in\mathfrak{A}$ achieves*

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right)<\frac{\rho(3-\mu)}{4(\mu-1)}\quad\text{and}\quad\operatorname{Re}\left(\frac{z\tau'(z)}{\tau(z)}\right)<1+\frac{\sigma(3-\mu)}{4(\mu-1)},$$

for some $2\leqslant\mu<3$, or

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right)<\frac{\rho(\mu-1)}{(\mu+1)}\quad\text{and}\quad\operatorname{Re}\left(\frac{z\tau'(z)}{\tau(z)}\right)<1+\frac{\sigma(\mu-1)}{2(\mu+1)},$$

for some $1 < \mu \leqslant 2$, *then*

$$\int_0^z \left(f'(\omega)\right)^{\frac{1}{\rho}} \left(\frac{\tau(\omega)}{\omega}\right)^{\frac{1}{\sigma}} d\omega \in \mathscr{S}^*.$$

Adequate stipulations for Breaz operator (23) to be starlike is given in the following outcome.

Theorem 4.2. *Let* $0 < \rho_\kappa$, σ_κ *for* $1 \leqslant \kappa \leqslant \aleph$. *If* $f_\kappa, \tau_\kappa \in \mathfrak{A}$ *for* $1 \leqslant \kappa \leqslant \aleph$ *achieves*

$$\operatorname{Re}\left(\frac{zf_\kappa''(z)}{f_\kappa'(z)}\right) > \frac{\rho_i\left(\mu - 2\mu^2 - 1\right)}{4\aleph\mu\left(\mu - 1\right)} \quad \text{and} \quad \operatorname{Re}\left(\frac{z\tau_\kappa'(z)}{\tau_\kappa(z)}\right) > 1 + \frac{\sigma_\kappa\left(\mu - 2\mu^2 - 1\right)}{4\aleph\mu\left(\mu - 1\right)}, \qquad (13)$$

for some $\mu \leqslant -1$, *or*

$$\operatorname{Re}\left(\frac{zf_\kappa''(z)}{f_\kappa'(z)}\right) > \frac{\rho_\kappa}{2\aleph\mu\left(\mu + 1\right)} \quad \text{and} \quad \operatorname{Re}\left(\frac{z\tau_\kappa'(z)}{\tau_\kappa(z)}\right) > 1 + \frac{\sigma_\kappa\left(1 - 2\mu\right)}{2\aleph\left(\mu + 1\right)}, \qquad (14)$$

for some $1 < \mu$, *then* $\Omega_\aleph(f_\kappa; \tau_\kappa)(z) \in \mathscr{S}^*\left(\frac{\mu+1}{2\mu}\right)$.

$\lhd$ Using equations (12) and (13), we achieve

$$\operatorname{Re}\left(1 + \frac{z\Omega_\aleph(f_\kappa;\tau_\kappa)''(z)}{\Omega_\aleph(f_\kappa;\tau_\kappa)'(z)}\right) > \frac{1}{\rho_1}\left(\frac{\rho_1\left(\mu - 2\mu^2 - 1\right)}{4\aleph\mu\left(\mu - 1\right)}\right) + \frac{1}{\rho_2}\left(\frac{\rho_2\left(\mu - 2\mu^2 - 1\right)}{4\aleph\mu\left(\mu - 1\right)}\right) + \dots$$
$$+ \frac{1}{\alpha_\aleph}\left(\frac{\rho_\aleph\left(\mu - 2\mu^2 - 1\right)}{4\aleph\mu\left(\mu - 1\right)}\right) + \frac{1}{\sigma_1}\left(1 + \frac{\sigma_1\left(\mu - 2\mu^2 - 1\right)}{4\aleph\mu\left(\mu - 1\right)}\right) + \frac{1}{\sigma_2}\left(1 + \frac{\sigma_2\left(\mu - 2\mu^2 - 1\right)}{4\aleph\mu\left(\mu - 1\right)}\right) + \dots$$
$$+ \frac{1}{\sigma_\aleph}\left(1 + \frac{\sigma_\aleph\left(\mu - 2\mu^2 - 1\right)}{4\aleph\mu\left(\mu - 1\right)}\right) + 1 - \left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2} + \dots + \frac{1}{\sigma_\aleph}\right) = \frac{-\left(\mu + 1\right)}{2\mu\left(\mu - 1\right)}.$$

for some $\mu \leqslant -1$. Also from equations (14) and (12), we acquire

$$\operatorname{Re}\left(1 + \frac{z\Omega_\aleph(f_\kappa;\tau_\kappa)''(z)}{\Omega_\aleph(f_\kappa;\tau_\kappa)(z)}\right) > \frac{1}{\rho_1}\left(\frac{\rho_1}{2\aleph\mu\left(\mu + 1\right)}\right) + \frac{1}{\rho_2}\left(\frac{\rho_2}{2\aleph\mu\left(\mu + 1\right)}\right) + \dots$$
$$+ \frac{1}{\rho_\aleph}\left(\frac{\rho_\aleph}{2\aleph\mu\left(\mu + 1\right)}\right) + \frac{1}{\sigma_1}\left(1 + \frac{\sigma_1\left(1 - 2\mu\right)}{2\aleph\left(\mu + 1\right)}\right) + \frac{1}{\sigma_2}\left(1 + \frac{\sigma_2\left(1 - 2\mu\right)}{2\aleph\left(\mu + 1\right)}\right) + \dots$$
$$+ \frac{1}{\sigma_\aleph}\left(1 + \frac{\sigma_\aleph\left(1 - 2\mu\right)}{2\aleph\left(\mu + 1\right)}\right) + 1 - \left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2} + \dots + \frac{1}{\sigma_\aleph}\right) = \frac{3\mu + 1}{2\mu\left(\mu + 1\right)}.$$

for some $1 < \mu$. Thus by Lemma 2.2, we gain $\Omega_\aleph(f_\kappa; \tau_\kappa)(z) \in \mathscr{S}^*\left(\frac{\mu+1}{2\mu}\right)$. This completes the proof.

If $\aleph = 1$, $\rho = \rho_1$, $\sigma = \sigma_1$ and $f_1 = f$, $\tau_1 = \tau$. In this instance, Theorem 4.2 gives the following outcome.

Corollary 4.2. *Let* $\rho, \sigma > 0$ *be real numbers. If* $f, \tau \in \mathfrak{A}$ *achieves*

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right) > \frac{\rho\left(\mu - 2\mu^2 - 1\right)}{4\aleph\mu\left(\mu - 1\right)} \quad \text{and} \quad \operatorname{Re}\left(\frac{z\tau'(z)}{\tau(z)}\right) > 1 + \frac{\sigma\left(\mu - 2\mu^2 - 1\right)}{4\aleph\mu\left(\mu - 1\right)},$$

for some $\mu \leqslant -1$, *or*

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right) > \frac{\rho}{2\aleph\mu\left(\mu + 1\right)} \quad \text{and} \quad \operatorname{Re}\left(\frac{z\tau'(z)}{\tau(z)}\right) > 1 + \frac{\sigma\left(1 - 2\mu\right)}{2\aleph\left(\mu + 1\right)},$$

for some $1 < \mu$, *then*

$$\int_0^z \left(f'(t)\right)^{\frac{1}{\rho}} \left(\frac{\tau(t)}{t}\right)^{\frac{1}{\sigma}} dt \in \mathscr{S}^* \left(\frac{\mu+1}{2\mu}\right).$$

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АНАЛИЗ ПРИЗНАКОВ ВЫПУКЛОСТИ И ЗВЕЗДНОСТИ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНОГО ОПЕРАТОРА БРИЗА

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Аннотация. Геометрическая теория аналитических функций (ГТАФ) является привлекательной частью комплексного анализа, взаимосвязанная с другими разделами математики. Его основная цель состоит в том, чтобы определить различные классы геометрических аналитических функций и обсудить их геометрические свойства. В дальнейшем появилась взаимосвязь между теорией операторов и ГТАФ, которая до сих пор привлекает широкое внимание. В прошлом столетии теория операторов была распространена на открытый единичный круг комплексной плоскости и применялась для предложения разнообразных обобщений нормализованных аналитических функций. В результате теория операторов оказалась хорошим способом исследования в области ГТАФ. С тех пор изучение геометрических свойств с помощью операторов стало важной темой исследований. Настоящее исследование сосредоточено на изучении свойства выпуклости в классах ℓ-равномерно выпуклых и звездообразных функций порядка β с использованием модифицированного интегро-дифференциального оператора Бриза в единичном круге. Кроме того, в классе аналитических функций рассматриваются некоторые условия, обеспечивающие звездообразность оператора Бриза.

Ключевые слова: аналитическая функция, равномерно выпуклая функция, равномерно звездообразная функция, оператор Бриза.

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