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A POINTWISE CONDITION FOR THE ABSOLUTE CONTINUITY OF A FUNCTION OF ONE VARIABLE AND ITS APPLICATIONS[#]

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Abstract. An absolutely continuous function in calculus is precisely such a function that, within the framework of Lebesgue integration, can be restored from its derivative, that is, the Newton–Leibniz theorem on the relationship between integration and differentiation is fulfilled for it. An equivalent definition is that the the sum of the moduli of the increments of the function with respect to arbitrary pair-wise disjoint intervals is less than any positive number if the sum of the lengths of the intervals is small enough. Certain sufficient conditions for absolute continuity are known, for example, the Banach–Zaretsky theorem. In this paper we prove a new sufficient condition for the absolute continuity of a function of one variable and give some of its applications to problems in the theory of function spaces. The proved condition makes it possible to significantly simplify the proof of the theorems on the pointwise description of functions of the Sobolev classes defined on Euclidean spaces and Carnot groups.

Key words: absolutely continuous function, Sobolev space, pointwise description.

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1. Introduction

The function $f : [a, b] \to \mathbb{R}$ is called absolutely continuous, if for any $\varepsilon > 0$ there is $\delta > 0$ such that for any disjoint set of intervals $(\alpha_j, \beta_j) \subset [a, b]$, which has the property:

$$\sum_j (\beta_j - \alpha_j) < \delta \quad \text{implies} \quad \sum_j |f(\beta_j) - f(\alpha_j)| < \varepsilon.$$

Below we formulate two classical criteria for absolute continuity. The first of them is the well-known Banach–Zaretsky theorem.

Theorem 1 [1]. *If a continuous function $f : [a, b] \to \mathbb{R}$ has bounded variation and possesses Luzin's property $\mathscr{N}$, i. e., $|f(E)|_1 = 0$* for any set $E \subset [a, b]$ of measure zero, then this function is absolutely continuous.*

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* Here and below, $|A|_1$ is the Lebesgue measure of a measurable set $A \subset \mathbb{R}$.

In the following statement a local condition of absolute continuity of a function is given. It was proved in [2].

Lemma 1 [2, Lemma 8.3]**.** *Let the function $f : [a,b] \to \mathbb{R}$ be continuous. Let $f : [a,b] \to \mathbb{R}$ also have Luzin's property $\mathscr{N}$, and the upper derivative $Df(x) = \overline{\lim}_{h\to 0} \frac{f(x+h)-f(x)}{h}$ is integrable, i. e., $\int_{[a,b]} |Df(x)|\,dx < \infty$. Then $f : [a,b] \to \mathbb{R}$ is absolutely continuous.*

In [3], a more general local condition for the absolute continuity of a function was obtained, which applied in [3] for describing regularity properties of mappings inverse to Sobolev.

Lemma 2 [3, Lemma 1]**.** *Let $\psi : [a,b] \to \mathbb{R}$ be a continuous function, $[a,b] = A \cup B$, $A \cap B = \varnothing$, where A and B are Borel sets such that*

1) $|\psi(B)|_1 = 0$, and the function $\psi : A \to \mathbb{R}$ has the Luzin's property $\mathscr{N}$ on the set A: $|\psi(E)|_1 = 0$ for each subset E of A of zero measure;

*2) $\psi(t)$ has an approximate derivative*** $\operatorname{app}\psi'(t)$ *almost everywhere on A;*

3) $\operatorname{app}\psi' \in L_1(A)$.

Then the function $\psi : [a,b] \to \mathbb{R}$ is absolutely continuous and its ordinary derivative

$$\psi'(t) = \begin{cases} \operatorname{app}\psi'(t), & \text{for almost every } t \in A, \\ 0, & \text{for almost every } t \in B. \end{cases}$$

It can be verified that from Lemma 2 one can deduce Banach–Zaretsky theorem. Indeed, let a function $f : [a,b] \to \mathbb{R}$ meet the conditions of Banach–Zaretsky theorem. Since the function $f(x)$ has the bounded variation, $f(x)$ is differentiable on the segment $[a,b]$ for almost all points $x \in [a,b]$, and $\int_{[a,b]} |f'(x)|\,dx < \infty$. We define the set

$$A = \{x \in [a,b] : \text{there is the derivative } f'(x)\}.$$

Complement $B = [a,b] \setminus A$ has zero measure. Moreover, we have $|f(B)|_1 = 0$, and the function $f : A \to \mathbb{R}$ possesses the Luzin's property $\mathscr{N}$ on the set A. By Lemma 2 the function $f : [a,b] \to \mathbb{R}$ is absolutely continuous.

Obviously, from Lemma 2 one can also deduce Lemma 1.

2. Pointwise Absolute Continuity Condition

In the next statement, we will establish a new pointwise criterion for the absolute continuity of a function defined on the real line.

Theorem 2. *Let $\mathbb{I} = (a,b)$ be an arbitrary interval in $\mathbb{R}$. Let a function $f : \mathbb{I} \to \mathbb{R}$ and a function $g : \mathbb{I} \to \mathbb{R}$ of the class $L_1(\mathbb{I})$ satisfy the pointwise inequality*

$$|f(\tau) - f(t)| \leqslant |\tau - t|\,(g(\tau) + g(t)) \tag{1}$$

for almost all $\tau, t \in \mathbb{I} \setminus S$ where $S \subset \mathbb{I}$ is some set of mesuare zero. Then the function f is measurable, and it can be changed on a set of measure zero so that it becomes absolutely continuous on $\mathbb{I}$, and its derivative enjoys the estimate

$$|f'(t)| \leqslant 2g(t) \quad \text{for almost all} \quad t \in \mathbb{I}. \tag{2}$$

** Recall that a number a is the approximative derivative of a function $\psi : A \to \mathbb{R}$ at a point x if the point x is the density point the set $\{y \in A : \left|\frac{\psi(y)-\psi(x)}{y-x} - a\right| < \varepsilon\}$ for any $\varepsilon > 0$.

◁ For any $k \in \mathbb{N}$, define the measurable set

$$A_k = \{t \in \mathbb{I} \setminus S : g(t) \leqslant k\}.$$

Obviously, $A_k \subset A_l$ for all $k < l$, and $|\mathbb{I} \setminus \bigcup_{k=1}^{\infty} A_k|_1 = 0$.

For all points $\tau, t \in A_k$ we have

$$|f(\tau) - f(t)| \leqslant 2k|\tau - t| \tag{3}$$

(here it is assumed that k is big enough so that the set A_k has positive measure). Thus, on the set A_k the function f satisfies the Lipschitz condition. Therefore, the function f is uniformly continuous, is extended by continuity to the closure $\overline{A_k}$, and the inequality (3) holds for all points $\tau, t \in \overline{A_k}$.

The complement $\mathbb{R} \setminus \overline{A_k}$ is an open set. It is known that an open set on $\mathbb{R}$ is the union of an at most countable collection of intervals: $\mathbb{R} \setminus \overline{A_k} = \bigcup_{i=1}^{\infty} (\alpha_i, \beta_i)$. In view of the above, we can assume that the function f is defined at the endpoints of a finite interval (α_i, β_i). We extend it to the segment $[\alpha_i, \beta_i] \Subset \mathbb{I}$ so that it is linear and takes in the boundary point α_i (β_i) the value $f(\alpha_i)$ ($f(\beta_i)$):

$$(\alpha_i, \beta_i) \ni t \mapsto \tilde{f}_k(t) = f(\alpha_i) + \frac{f(\beta_i) - f(\alpha_i)}{\beta_i - \alpha_i}\,(t - \alpha_i); \tag{4}$$

in the case of unbounded intervals $(\alpha_i, \beta_i) = (\alpha_i, \infty)$ or (and) $(\alpha_i, \beta_i) = (-\infty, \beta_i)$, we put

$$(\alpha_i, \beta_i) \ni t \mapsto \tilde{f}_k(t) = \begin{cases} f(\beta_i), & \text{if } \alpha_i = -\infty, \\ f(\alpha_i), & \text{if } \beta_i = +\infty. \end{cases} \tag{5}$$

The function extended in this way will be denoted by the symbol $\tilde{f}_k \colon \mathbb{I} \to \mathbb{R}$. The function $\tilde{f}_k$ has the following properties:

5) $\tilde{f}_k \colon \mathbb{I} \to \mathbb{R}$ satisfies the Lipschitz condition with the same Lipschitz constant as the function $f \colon A_k \to \mathbb{R}$ (see (3) and (4));

6) $\tilde{f}_k|_{A_k} = f|_{A_k}$;

It is evident that $f(x) = \lim_{k\to\infty} \tilde{f}_k(x) \cdot \chi_{A_k}(x)$ for almost all $x \in \mathbb{I}$. As soon as functions

$$\mathbb{I} \ni x \mapsto \tilde{f}_k(x) \cdot \chi_{A_k}(x) = \begin{cases} f_k(x), & \text{if } x \in A_k, \\ 0, & \text{otherwise}, \end{cases}$$

are measurable, $k \in \mathbb{N}$, the limits $f(x)$ is measurable too.

7) the function $\tilde{f}_k \colon \mathbb{I} \to \mathbb{R}$ is bounded on $\mathbb{I}$ and, for almost $t \in \mathbb{I}$, there is a derivative

$$\frac{d\tilde{f}_k}{dt}(t) \quad \text{for which the estimate} \quad \Big|\frac{d\tilde{f}_k}{dt}(t)\Big| \leqslant 2k \;\; \text{is valid;} \tag{6}$$

8) if $A_{k,l} \subset A_k$, $l \geqslant k$, is the collection of all points of differentiability of the function $\tilde{f}_l \colon \mathbb{I} \to \mathbb{R}$ on the set A_k, then $\bigcap_{l \geqslant k} A_{k,l}$ is a full measure set*** in A_k, and the equality holds

$$\frac{d\tilde{f}_l}{dt}(t) = \frac{d\tilde{f}_k}{dt}(t) \quad \text{for all } l \geqslant k \text{ and all } t \in \bigcap_{l \geqslant k} A_{k,l}. \tag{7}$$

*** Since $A_{k,l}$ is a set of full measure in A_k for any $l \geqslant k$.

It is known that the following properties are fulfilled:

9) almost every point $t \in A_k$ has density 1 with respect to the set A_k:

$$\lim_{\delta\to 0} \frac{|\Delta \cap A_k|_1}{|\Delta|} = 1, \tag{8}$$

where $\Delta = (t-\delta, t+\delta)$;

10) almost every point $t \in A_k$ is a Lebesgue point for the function $g\colon \mathbb{I} \to \mathbb{R}$.

Next, we will establish an estimate for the derivative of the function $\tilde{f}_k\colon \mathbb{I} \to \mathbb{R}$:

$$\left|\frac{d\tilde{f}_k}{dt}\right|(t) \leqslant 2g(t) \quad \text{for almost all } t \in \mathbb{I}. \tag{9}$$

Let $t \in \bigcap_{l\geqslant k} A_{k,l}$ be such a point that the above properties 8)–10) hold. From (1), for function

$$A_k \setminus S \ni \tau \to \tilde{f}_k(\tau) = f(\tau),$$

we have an estimate for the difference ratio:

$$\left|\frac{\tilde{f}_k(\tau) - \tilde{f}_k(t)}{\tau - t}\right| = \left|\frac{f(\tau) - f(t)}{\tau - t}\right| \leqslant g(t) + g(\tau) = 2g(t) + g(\tau) - g(t). \tag{10}$$

From relations (10) it is seen that the estimate for the derivative $\tilde{f}_k'(t)$ depends on the behavior of the difference $g(\tau) - g(t)$. Since t is a Lebesgue point for $g\colon \mathbb{I} \to \mathbb{R}$ then

$$\frac{1}{2\delta} \int\limits_{t-\delta}^{t+\delta} |g(\tau) - g(t)|\, d\tau = o(1), \quad \text{if } \delta \to 0. \tag{11}$$

Put $\Delta = (t-\delta, t+\delta)$. From (10), (11) we have

$$\begin{aligned} \frac{1}{2\delta} \int\limits_{\Delta\cap A_k} \left|\frac{f(\tau) - f(t)}{\tau - t}\right| d\tau &\leqslant 2g(t) + \frac{1}{2\delta} \int\limits_{\Delta\cap A_k} |g(\tau) - g(t)|\, d\tau \\ &\leqslant 2g(t) + \frac{1}{2\delta} \int\limits_{t-\delta}^{t+\delta} |g(\tau) - g(t)|\, d\tau = 2g(t) + o(1), \end{aligned} \tag{12}$$

if $\delta \to 0$.

We will show that the limit of the left-hand side of (12) at $\delta \to 0$ equals $\bigl|\tilde{f}_k'(t)\bigr| = \bigl|\frac{d\tilde{f}_k}{dt}\bigr|(t)$. Indeed, the left-hand side of (12) is equal to

$$\begin{aligned} \frac{1}{2\delta} \int\limits_{\Delta\cap A_k} \left|\frac{f(\tau) - f(t)}{\tau - t}\right| d\tau &= \frac{1}{2\delta} \int\limits_{\Delta\cap A_k} \left|\frac{\tilde{f}_k(\tau) - \tilde{f}_k(t) - \tilde{f}_k'(t)(\tau - t) + \tilde{f}_k'(t)(\tau - t)}{(\tau - t)}\right| d\tau \\ &= \frac{1}{2\delta} \int\limits_{\Delta\cap A_k} \left|\frac{o(1)(\tau - t) + \tilde{f}_k'(\tau - t)}{(\tau - t)}\right| d\tau = \frac{1}{2\delta} \int\limits_{\Delta\cap A_k} \bigl|\tilde{f}_k'(t) + o(1)\bigr|\, d\tau \end{aligned}$$

$$= \frac{|\Delta \cap A_k|_1}{|\Delta|} \bigl|\tilde{f}_k'(t) + o(1)\bigr| \to \bigl|\tilde{f}_k'(t)\bigr|, \tag{13}$$

if $\delta \to 0$ (see (8) for the limit of the fraction on line (13)). Therefore, the inequality (9) is proved in almost all $t \in A_k$. Note that at all points of $t \notin \overline{A_k}$ the following relations hold:

$$\left|\frac{d\tilde{f}_k}{dt}\right|(t) \leqslant 2k \leqslant 2g(t), \tag{14}$$

so (9) is proved.

Since $A_k \subset A_{k+1}$, $k = 1, 2, \dots$, and $|\mathbb{I} \setminus \bigcup_{k=1}^{\infty} A_k|_1 = 0$, for almost all $t \in \mathbb{I}$ there exists a limit

$$\lim_{k\to\infty} \tilde{f}_k(t) = f(t),$$

and taking into account (7), we have

$$\bigcup_{k=1}^{\infty} A_k \ni t \mapsto w(t) \stackrel{\text{def}}{=} \lim_{l\to\infty} \frac{d\tilde{f}_l}{dt}(t) = \begin{cases} \dfrac{d\tilde{f}_1}{dt}(t), & \text{if } t \in A_1, \\ \dfrac{d\tilde{f}_k}{dt}(t), & \text{if } t \in A_k \setminus A_{k-1},\ k = 2, 3, \dots, n, \dots \end{cases}$$

By virtue of (9), the inequality

$$|w(t)| \leqslant 2g(t) \quad \text{holds for almost all } t \in \mathbb{I}. \tag{15}$$

As noted above, the set $\bigcup_{k=1}^{\infty} A_k$ is a set of full measure in $\mathbb{I}$. Therefore, for arbitrary points $\alpha, \beta \in A_k$ and for $l \geqslant k$, by the Lebesgue dominated convergence theorem (see (9) and (15)), we have

$$f(\beta) = f(\alpha) + \int_{\alpha}^{\beta} \frac{d\tilde{f}_l}{dt}(t)\,dt \to f(\alpha) + \int_{\alpha}^{\beta} w(t)\,dt.$$

From the obtained equality, we deduce that the function

$$\mathbb{I} \ni \beta \mapsto f(\alpha) + \int_{\alpha}^{\beta} w(t)\,dt$$

is absolutely continuous and coincides with the function $f(\beta)$ for almost all $\beta \in \mathbb{I}$. From this we get $f'(\beta) = w(\beta)$ for almost all $\beta \in \mathbb{I}$. From (15) we get the inequality (9). Theorem 2 is proved. $\rhd$

3. A Short Proof of Some Pointwise Estimates for Sobolev Functions

We say that a function $f : \Omega \to \mathbb{R}^n \to \mathbb{R}$ where $\Omega \subset \mathbb{R}^n$ is an open set, belongs to $L^1_q(\Omega)$ ($W^1_q(\Omega)$), $q \in [1, \infty)$, if f is locally integrable in Ω and its gradient ∇f in the sense of distribution belongs to $L_q(\Omega)$ (both f and its gradient ∇f belong to $L_q(\Omega)$); one more notation: $f \in W^1_{q,\mathrm{loc}}(\Omega)$ if $f \in W^1_q(U)$ for any $U \Subset \Omega$, that is $\overline{U} \subset \Omega$ and $\overline{U}$ is a compact.

We apply Theorem 2 for proving the following statement.

Theorem 3. 1) *A function $f : \Omega \to \mathbb{R}$ belongs to $W^1_{1,\mathrm{loc}}(\Omega)$ if there exists a non-negative function $g \in L_{1,\mathrm{loc}}(\Omega)$ such that the inequality*

$$|f(x) - f(y)| \leqslant |x - y|(g(x) + g(y)) \tag{16}$$

holds for all x, y outside of some set $\Sigma \subset \Omega$ of measure zero. Mareover, the estimate $|\nabla f(x)| \leqslant \sqrt{n} \cdot g(x)$ holds a. e. in Ω.

2) *If $g \in L_1(\Omega)$ then $f \in L_1^1(\Omega)$.*

3) *If $f \in L_1(\Omega)$ and $g \in L_1(\Omega)$ then $f \in W_1^1(\Omega)$.*

$\lhd$ Fix a cube $Q(a,r) = \{x = (x_1, x_2, \dots, x_n) : |(x-a)_i| < r,\ i = 1, \dots, n\}$ such that $Q(a,r) \Subset \Omega$. Every point $x \in Q(a,r)$ can be represented as $x = (\bar{x}_i, x_i)$ where $\bar{x}_i$ is a projection of x on the hyperplane $P_i = (x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)$ and x_i is a projection of x on the line $L_i = \{x = te_i : t \in \mathbb{R}\}$ (here $e_i = (0, \dots, \underset{i}{1}, \dots, 0)$ is ith vector of the canonical basis in $\mathbb{R}^n$).

It follows from the conditions of Theorem 3 and (16) that $g \in L_1(Q(a,r))$, f is measurable in $Q(a,r)$ and $f \in L_1(Q(a,r))$. The first property is evident. For proving the second one we define a measurable set

$$A_k = \{x \in Q(a,r) : g(x) \leqslant k\}, \quad k \in \mathbb{N}.$$

Then the restriction f_k of f to A_k is Lipschitz and therefore is measurable. Moreover, by Kirszbraun theorem $f_k : A_k \to \mathbb{R}$ can by extended to $Q(a,r)$ to be a Lipschitz function $\tilde{f}_k : Q(a,r) \to \mathbb{R}$ on $Q(a,r)$.

Further, $\bigcup_{k\in\mathbb{N}} A_k$ is a set of full measure in $Q(a,r)$. Hence,

$$f(x) = \lim_{k\to\infty} \chi_{A_k}(x) \cdot \tilde{f}_k(x)$$

a. e. in $Q(a,r)$. Since $\chi_{A_k}(x) \cdot \tilde{f}_k(x)$ is a measurable function on $Q(a,r)$, $k \in \mathbb{N}$, $f(x)$ coincides with the limit of measurable functions a. e. By this reason $f(x)$ is measurable function in $Q(a,r)$. Finally, (16) gives the inequality

$$|f(x)| \leqslant |f(x) - f(y)| + |f(y)| \leqslant \sqrt{n} \cdot rg(x) + \sqrt{n} \cdot rg(y)$$

for all $x \in Q(a,r) \setminus \Sigma$ with a fixed point $y \in Q(a,r) \setminus \Sigma$. It follows immediately that $f \in L_1(Q(a,r))$ for any $Q(a,r) \Subset \Omega$ and consequently $f \in L_{1,\mathrm{loc}}(\Omega)$.

Then we apply Fubini theorem to integrable functions f and g on $Q(a,r)$ for coming to the conclusion that both f and g are integrable on an interval

$$L_i(r) = \bar{x}_i + \{x = x_i e_i : |x_i - a_i| < r\} \quad \text{for almost all} \quad \bar{x}_i \in P_i(Q(a,r)).$$

Thus f and g meets the conditions ot Theorem 2 on $L_i(r)$ including inequality (1).

By conclusion of Theorem 2, for almost all $\bar{x}_i \in P_i(Q(a,r))$, the function $f : L_i(r) \to \mathbb{R}$ can be redefined on a set of measure zero to be absolutely continuous on $L_i(r)$; moreover, the estimate $\left|\frac{\partial f'(x)}{\partial x_i}(x)\right| \leqslant 2g(x)$ holds for almost all points $x \in L_i(r)$. As soon as $i = 1, \dots, n$ and $Q(a,r) \Subset \Omega$ are arbitrary we have proved that $f \in W_{1,\mathrm{loc}}^1(\Omega)$, and the inequality $|\nabla f(x)| \leqslant 2\sqrt{n} \cdot g(x)$ is valid a. e. in Ω.

The asserions 2) and 3) of the Theorem 3 follows immediately from the saying above. $\rhd$

Now we are ready to give a short proof of the known statement [4, 5].

Theorem 4 [4, Theorem 1]; [5, Theorem 3]**.** *Let $1 < q < \infty$. A function $f \in L_q(\mathbb{R}^n)$ ($f : \mathbb{R}^n \to \mathbb{R}$) belongs to $W_q^1(\mathbb{R}^n)$ ($L_q^1(\mathbb{R}^n)$) if and only if there exists a non negative $g \in L_q(\mathbb{R}^n)$ such that the inequality*

$$|f(x) - f(y)| \leqslant |x-y|(g(x) + g(y)) \tag{17}$$

holds for all x, y outside of some set $\Sigma \subset \mathbb{R}^n$ of measure zero.

$\lhd$ The sufficiency of conditions is proved in [6]. We prove the necessity of conditions below.

1) Let $f \in L_q(\mathbb{R}^n)$. Fix a cube $Q(0,r) = \{x = (x_1, x_2, \dots, x_n) : |x_i| < r,\ i = 1, \dots, n\}$. By conditions of Theorem 4 we have $f, g \in L_1(Q(0,r))$. Therefore conditions of Theorem 3 hold with $\Omega = Q(0,r)$. By its conclusion we have that $f \in W_1^1(Q(0,r))$ and the estimate $|\nabla f(x)| \leqslant 2\sqrt{n} \cdot g(x)$ holds a. e. in $Q(0,r)$. By this reason, $f \in W^1_{q,\mathrm{loc}}(Q(0,r))$ for any $r \in (0,\infty)$. The properties $f \in L_q(\mathbb{R}^n)$ and $g \in L_q(\mathbb{R}^n)$ provide also $f \in W_q^1(\mathbb{R}^n)$.

2) If for a function $f : \mathbb{R}^n \to \mathbb{R}$ the inequality (17) holds with $g \in L_q(\mathbb{R}^n)$ then applying above mentioned arguments we come to conclusion that $f \in L_{1,\mathrm{loc}}(\mathbb{R}^n)$. The property $g \in L_q(\mathbb{R}^n)$ provides $\nabla f \in L_q(\mathbb{R}^n)$. Hence, $f \in L_q^1(\mathbb{R}^n)$. $\rhd$

4. A Short Proof of Some Pointwise Estimates for Banach Function Spaces

Definition 1. Let (T,μ) be a σ-finite measure space and let $\mathscr{M}$ denote the set of all measurable functions on (T,μ). We say that a function $\|\cdot\| \colon \mathscr{M} \to [0,\infty]$ is a *Banach function norm* if for all f_n, f, $g \in \mathscr{M}$ and $\alpha \in \mathbb{R}$:

(i) $\|f\| = 0$ if and only if $f = 0$ a. e., $\|\alpha f\| = |\alpha|\|f\|$ and $\|f+g\| \leqslant \|f\| + \|g\|$;

(ii) if $|f| \leqslant |g|$ a.e. then $\|f\| \leqslant \|g\|$;

(iii) if $0 \leqslant f_n \nearrow f$ then $\|f_n\| \nearrow \|f\|$;

(iv) for every measurable $E \subset T$, $\mu(E) < \infty$: $\|\chi_E\| < \infty$;

(v) for every measurable $E \subset T$, there exists a constant $C_E > 0$ (independent of f), such that $\int_E |f|\,d\mu \leqslant C_E\|f\|$.

The space $X(T) = \{f \in \mathscr{M} : \|f\| < \infty\}$ with norm $\|\cdot\|$ is called a *Banach function space.*

Let $\Omega \subset \mathbb{R}^n$ be an open set and $X(\Omega)$ be a Banach function space w.r.t. the Lebesgue measure. The *Sobolev space* $WX(\Omega)$ denotes the space of weakly differentiable mappings f with f, $\nabla f \in X(\Omega)$. This space is equipped with a norm

$$\|f\|_{WX(\Omega)} := \|f\|_{X(\Omega)} + \|\nabla f\|_{X(\Omega)}\,.$$

In [7] the following statement is proved.

Theorem 5 [7, Theorem 2.2]**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and $X(\Omega)$ be a Banach function space such that the Hardy–Littlewood maximal operator is bounded in $X(\Omega)$. Then a function f belongs to $WX(\Omega)$ if and only if $f \in X(\Omega)$ and there exists a non negative function $g \in X(\Omega)$ such that the inequality*

$$|f(x) - f(y)| \leqslant |x-y|(g(x) + g(y)) \tag{18}$$

holds for almost all x, $y \in \Omega$ with $B(x, 3|x-y|) \subset \Omega$.

Note, that the proof of necessity in Theorem 4 is based on the following facts:

1) f, ∇f, $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$;

2) the space $L_q(\mathbb{R}^n)$ satisfies the lattice property, i. e. if $|f| \leqslant |g|$ a. e. then $\|f\| \leqslant \|g\|$.

Hence, the proof of Theorem 4 can be applied almost verbatim for proving a similar result for function spaces meeting conditions of Theorem 5. Notice that Theorem 5 can be applied for many various spaces, for example, weighted Lebesgue (with Muckenhaupt's weight), grand Lebesgue, Musielak–Orlicz, Lorentz and Marcinkiewicz spaces, as well as Lebesgue spaces with variable exponents. In particular, it includes the general concept of Banach function spaces. So the method of proving Theorem 4 simpifies the proof of necessity in Theorem 5.

5. A Short Proof of Some Pointwise Estimates for Sobolev Functions on Carnot Groups

Similar arguments can be applied for simplifying the proof of pointwise description of Sobolev functions of Carnot groups given in [5]. For doing this it is enough to generalize Theorem 2 and its proof in Carnot groups,

Theorem 6. *Let* $\mathbb{I} = (a, b)$ *be an arbitrary interval in* $\mathbb{R}$. *Let* $\mathbb{G}$ *be a Carnot group and* X_i *is some horizontal vector field. Let a function* $f : \exp \mathbb{I} X_i \to \mathbb{R}$ *and a function* $g : \exp \mathbb{I} X_i \to \mathbb{R}$ *of the class* $L_1(\exp \mathbb{I})$ *satisfy the pointwise inequality*

$$|f(\tau) - f(t)| \leqslant |\tau - t|(g(\tau) + g(t))$$

for almost all $\tau, t \in \exp \mathbb{I} \setminus S$ *where* $S \subset \exp \mathbb{I}$ *is some set of mesuare zero.*

Then the function f *is measurable, and it can be changed on a set of measure zero so that it becomes absolutely continuous on* $\exp \mathbb{I}$, *and its derivative* $X_i f(\exp tX_i)$, $t \in \mathbb{I}$, *enjoys the estimate*

$$|X_i f(\exp tX_i)| \leqslant 2g(\exp tX_i) \quad \text{for almost all} \quad t \in \mathbb{I}.$$

The proof of Theorem 6 can be obtained from the proof of Theorem 2 almost verbatim.

By means of Theorem 6 all previous results of the paper can be generalized to Carnot groups.

We formulate here a statement proved in [5].

Theorem 7 [5, Theorem 3]**.** *Let* $1 < q < \infty$ *and* $\mathbb{G}$ *be a Carnot group. A function* $f \in L_q(\mathbb{G})$ $(f : \mathbb{G} \to \mathbb{R})$ *belongs to* $W^1_q(\mathbb{G})$ $(L^1_q(\mathbb{G}))$ *if and only if there exists a non negative* $g \in L_q(\mathbb{G})$ *such that the inequality*

$$|f(x) - f(y)| \leqslant d_{cc}(x, y)(g(x) + g(y))$$

holds for all x, y *outside of some set* $\Sigma \subset \mathbb{G}$ *of measure zero.*

Here $d_{cc}(x, y)$ is the Carnot–Carathéodory metric [8] between points x and y in G.

As in Euclidean spaces Theorem 6 allows us to significantly simplify the necessity in the proof of Theorem 7.

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ПОТОЧЕЧНОЕ УСЛОВИЕ АБСОЛЮТНОЙ НЕПРЕРЫВНОСТИ ФУНКЦИИ ОДНОЙ ПЕРЕМЕННОЙ И ЕГО ПРИМЕНЕНИЯ

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Аннотация. Абсолютно непрерывная функция в математическом анализе это в точности такая функция, которая в рамках интегрирования по Лебегу может быть восстановлена по своей производной, то есть для нее выполнена теорема Ньютона — Лейбница о связи между интегрированием и дифференцированием. Эквивалентное определение состоит в том, что сумма модулей приращений функции по произвольному дизьюнктому набору интервалов меньше любого положительного числа, если сумма длин интервалов достаточно мала. Известны некоторые достаточные условия вбсолютной непрерывности, например теорема Банаха — Зарецкого. В этой статье мы доказываем новое достаточное условие абсолютной непрерывности функции одной переменной и приводим некоторые его применения к задачам теории функциональных пространств. Доказанное условие дает возможность значительно упростить доказательство теорем о поточечном описании функций классов Соболева, определенных на евклидовых пространствах и группах Карно.

Ключевые слова: абсолютно непрерывная функция, пространство Соболева, поточечное описание.

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