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TOSHA-DEGREE EQUIVALENCE SIGNED GRAPHS

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Abstract. The Tosha-degree of an edge α in a graph Γ without multiple edges, denoted by $T(\alpha)$, is the number of edges adjacent to α in Γ, with self-loops counted twice. A signed graph (marked graph) is an ordered pair $\Sigma = (\Gamma, \sigma)$ $(\Sigma = (\Gamma, \mu))$, where $\Gamma = (V, E)$ is a graph called the underlying graph of Σ and $\sigma : E \to \{+, -\}$ $(\mu : V \to \{+, -\})$ is a function. In this paper, we define the Tosha-degree equivalence signed graph of a given signed graph and offer a switching equivalence characterization of signed graphs that are switching equivalent to Tosha-degree equivalence signed graphs and k^{th} iterated Tosha-degree equivalence signed graphs. It is shown that for any signed graph Σ, its Tosha-degree equivalence signed graph $T(\Sigma)$ is balanced and we offer a structural characterization of Tosha-degree equivalence signed graphs.

Key words: signed graphs, balance, switching, Tosha-degree of an edge, Tosha-degree equivalence signed graph, negation.

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1. Introduction

A graph is an ordered pair $\Gamma = (V, E)$, where V is a set of vertices of Γ and E is a collection of pairs of vertices of Γ, called edges of Γ. For standard terminology and notion in graph theory, we refer the reader to the text-book of Harary [1]. All graphs considered in the paper are finite, simple and connected. The non-standard will be given in this paper as and when required.

In [2], we defined the Tosha-degree of an edge in a graph and Tosha-degree equivalence graph of a graph as follows:

Let α be an edge in a graph Γ. The Tosha-degree of α, denoted by $T(\alpha)$, is the number of edges adjacent to α in Γ, with self-loops counted twice. For any edge α in a graph Γ, $T(\alpha) \geqslant 0$.

Let $\Gamma = (V, E)$ be a graph and $|E| = m$. We define a relation $\approx$ on E as follows: for $\alpha, \beta \in E$,

$$\alpha \approx \beta \Leftrightarrow T(\alpha) = T(\beta).$$

It is easy to see that $\approx$ is an equivalence relation on E. Let $E_1, E_2, \ldots, E_k$ be the partition of E in to disjoint classes by the relation $\approx$. Let $|E_i| = m_i$, $1 \leqslant i \leqslant k$ so that $m_1 + m_2 + \ldots + m_k = m$.

The equivalence class graph on E defined by $\approxeq$ is called *Tosha-degree equivalence graph* of Γ and is denoted by $T(\Gamma)$.

A *signed graph* is an ordered pair $\Sigma = (\Gamma, \sigma)$, where $\Gamma = (V, E)$ is a graph called the *underlying graph* of Σ and $\sigma : E \to \{+, -\}$ is a function. A *marking* of Σ is a function $\mu : V(\Gamma) \to \{+, -\}$.

A signed graph $\Sigma = (\Gamma, \sigma)$ is *balanced* if every cycle in Σ has an even number of negative edges (see [3]). Equivalently, a signed graph is balanced if product of signs of the edges on every cycle of Σ is positive.

The following are the fundamental results about balance, the second being a more advanced form of the first. Note that in a bipartition of a set, $V = V_1 \cup V_2$, the disjoint subsets may be empty.

Theorem 1.1. *A signed graph Σ is balanced if and only if either of the following equivalent conditions is satisfied:*

(i) *Its vertex set has a bipartition $V = V_1 \cup V_2$ such that every positive edge joins vertices in V_1 or in V_2, and every negative edge joins a vertex in V_1 and a vertex in V_2 (Harary [3]).*

(ii) *There exists a marking μ of its vertices such that each edge uv in Γ satisfies $\sigma(uv) = \mu(u)\mu(v)$. (Sampathkumar [4]).*

Two signed graphs Σ_1 and Σ_2 are signed isomorphic (written $\Sigma_1 \cong \Sigma_2$) if there is a one-to-one correspondence between their vertex sets which preserve adjacency as well as sign.

Given a marking μ of a signed graph $\Sigma = (\Gamma, \sigma)$, *switching* Σ with respect to μ is the operation of changing the sign of every edge uv of Σ by $\mu(u)\sigma(uv)\mu(v)$. The signed graph obtained in this way is denoted by $\Sigma_\mu(\Sigma)$ and is called the *μ-switched signed graph* or just *switched signed graph*.

A signed graph $\Sigma_1 = (\Gamma, \sigma)$ *switches* to a signed graph $\Sigma_2 = (\Gamma', \sigma')$ (or that Σ_1 and Σ_2 are *switching equivalent*) written $\Sigma_1 \sim \Sigma_2$, whenever there exists a marking μ of Σ_1 such that $\Sigma_\mu(\Sigma_1) \cong \Sigma_2$. Note that $\Sigma_1 \sim \Sigma_2$ implies that $\Gamma \cong \Gamma'$, since the definition of switching does not involve change of adjacencies in the underlying graphs of the respective signed graphs. Infact, the idea of switching a signed graph was introduced by Abelson and Rosenberg [5] in connection with structural analysis of marking μ of a signed graph Σ.

Two signed graphs $\Sigma_1 = (\Gamma, \sigma)$ and $\Sigma_2 = (\Gamma', \sigma')$ are said to be *cycle isomorphic* (see [6]) if there exists an isomorphism $\phi : \Gamma \to \Gamma'$ such that the sign of every cycle Z in Σ_1 equals to the sign of $\phi(Z)$ in Σ_2. The following result is known [6]:

Theorem 1.2 (T. Zaslavsky [6]). *Two signed graphs Σ_1 and Σ_2 with the same underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

One of the important operations on signed graphs involves changing signs of their edges. The negation of a signed graph Σ, denoted $\eta(\Sigma)$, is obtained by negating the sign of every edge of Σ, i. e., by changing the sign of every edge to its opposite [7].

2. Tosha-Degree Equivalence Signed Graph of a Graph

In [2], we have defined the Tosha-degree equivalence graph of a graph which is motivated to extend this notion to signed graphs as follows: The *Tosha-degree equivalence signed graph* of a signed graph $\Sigma = (\Gamma, \sigma)$ as a signed graph $T(\Sigma) = (T(\Gamma), \sigma')$, where $T(\Gamma)$ is the underlying graph of $T(\Sigma)$ is the Tosha-degree equivalence graph of Γ, where for any edge e_1e_2 in $T(\Sigma)$, $\sigma'(e_1e_2) = \sigma(e_1)\sigma(e_2)$. Hence, we shall call a given signed graph Σ as *Tosha-degree equivalence signed graph* if it is isomorphic to the Tosha-degree equivalence signed graph $T(\Sigma')$ of some sigraph Σ'.

The following result indicates the limitations of the notion of Tosha-degree equivalence signed graphs as introduced above, since the entire class of unbalanced signed graphs is forbidden to be Tosha-degree equivalence signed graphs.

Theorem 2.1. *For any signed graph $\Sigma = (\Gamma, \sigma)$, its Tosha-degree equivalence signed graph $T(\Sigma) = (T(\Gamma), \sigma')$ is balanced.*

◁ Let E_j^+ be the set of vertices of Tosha-degree equivalence signed graph $T(\Sigma)$ each of which corresponds to a positive edge in Σ and E_j^- be the set of vertices of Tosha-degree equivalence signed graph $T(\Sigma)$ each of which corresponds to a negative edge in Σ. Let e_ie_j be any negative edge in $T(\Sigma)$. By the definition of $T(\Sigma)$, the edges e_i and e_j ofΣ are not of the same sign and hence as vertices of $T(\Sigma)$ they cannot lie in the same part of the partition $\{E_j^+, E_j^-\}$. On the other hand, if the edge e_ie_j is any positive edge of $T(\Sigma)$ then, by the definition of $T(\Sigma)$ the edges e_i and e_j of Σ are of the same sign and hence as vertices of $T(\Sigma)$ they must both lie in exactly one of the parts of the partition $\{E_j^+, E_j^-\}$ of the vertex set of $T(\Sigma)$. Thus, every negative edge of $T(\Sigma)$ has its ends in different parts of this partition whereas no positive edge of $T(\Sigma)$ has this property. Therefore, by the well known Partition Criterion for Balance of by Theorem 1.1, it follows that $T(\Sigma)$ must be balanced. ▷

For any positive integer k, the k^{th} iterated Tosha-degree equivalence signed graph, $T^k(\Sigma)$ of Σ is defined as follows:

$$T^0(\Sigma) = \Sigma, \quad T^k(\Sigma) = T(T^{k-1}(\Sigma)).$$

Corollary 2.2. *For any signed graph $S = (G, \sigma)$ and for any positive integer k, $T^k(\Sigma)$ is balanced.*

Theorem 2.3. *For any two signed graphs Σ_1 and Σ_2 with the same underlying graph, their Tosha-degree equivalence signed graphs are switching equivalent.*

◁ Suppose $\Sigma_1 = (\Gamma, \sigma)$ and $\Sigma_2 = (\Gamma', \sigma')$ be two signed graphs with $\Gamma \cong \Gamma'$. By Theorem 2.1, $T(\Sigma_1)$ and $T(\Sigma_2)$ are balanced and hence, the result follows from Theorem 1.2. ▷

In [2], we have characterize the graphs such that $\Gamma \cong T(\Gamma)$.

Theorem 2.4. *Let Γ be a connected graph with m edges. Then $\Gamma \cong T(\Gamma)$ if and only if $\Gamma \cong K_3$.*

In view of the above result, we now characterize those signed graphs that are switching equivalent to their Tosha-degree equivalence signed graphs.

Theorem 2.5. *For any connceted signed graph $\Sigma = (\Gamma, \sigma)$ with m edges. Then $\Sigma \sim T(\Sigma)$ if and only if Σ is balanced signed graph and $\Gamma \cong K_3$.*

◁ Suppose $\Sigma \sim T(\Sigma)$. This implies, $T(\Gamma) \cong \Gamma$ and hence by Theorem 2.4 we see that Γ is isomorphic to complete graph K_3. Now, if Σ is signed graph in which underlying graph Γ is isomorphic to K_3, Theorem 2.1 implies that $T(\Sigma)$ is balanced and hence if Σ is unbalanced its Tosha-degree equivalence signed graph $T(\Sigma)$ being balanced cannot be switching equivalent to S in accordance with Theorem 1.2. Therefore, Σ must be balanced.

Conversely, suppose that Σ is balanced and Γ is isomorphic to K_3. Then, by Theorem 2.1, $T(\Sigma)$ is balanced, the result follows from Theorem 1.2. ▷

By the definition of Tosha-degree of an edge in a graph, Tosha-degree equivalence graph of a graph and Theorem 2.4, we have the following result:

Theorem 2.6. *Let Γ be a connected graph with m edges. Then $\Gamma \cong T^k(\Gamma)$ if and only if $\Gamma \cong K_3$.*

In view of the above result, we now characterize those signed graphs that are switching equivalent to their k^{th} iterated Tosha-degree equivalence signed graphs.

Theorem 2.7. *For any connceted signed graph* $\Sigma = (\Gamma, \sigma)$ *with* m *edges. Then* $\Sigma \sim T^k(\Sigma)$ *if and only if* Σ *is balanced signed graph and* $\Gamma \cong K_3$.

For a signed graph $\Sigma = (\Gamma, \sigma)$, the $T(\Sigma)$ is balanced (Theorem 2.1). We now examine, the conditions under which negation η of $T(\Sigma)$ is balanced.

Theorem 2.8. *Let* $\Sigma = (\Gamma, \sigma)$ *be a signed graph. If* $T(\Gamma)$ *is bipartite then* $\eta(T(\Sigma))$ *is balanced.*

◁ Since, by Theorem 2.1, $T(\Sigma)$ is balanced, if each cycle C in $T(\Sigma)$ contains even number of negative edges. Also, since $T(\Gamma)$ is bipartite, all cycles have even length; thus, the number of positive edges on any cycle C in $T(\Sigma)$ is also even. Hence $\eta(T(\Sigma))$ is balanced. ▷

Theorem 2.5 and 2.7 provides easy solutions to two other signed graph switching equivalence relations, which are given in the following results:

Corollary 2.9. *For any signed graph* $\Sigma = (\Gamma, \sigma)$, $\eta(\Sigma) \sim T(\Sigma)$ *if and only if* Σ *is an unbalanced signed graph and* $\Gamma = K_3$.

Corollary 2.10. *For any signed graph* $\Sigma = (\Gamma, \sigma)$, $\eta(\Sigma) \sim T(\eta(\Sigma))$ *if and only if* Σ *is an unbalanced signed graph and* $\Gamma = K_3$.

Corollary 2.11. *For any signed graph* $\Sigma = (\Gamma, \sigma)$, $\eta(\Sigma) \sim T^k(\Sigma)$ *if and only if* Σ *is an unbalanced signed graph and* $\Gamma = K_3$.

Corollary 2.12. *For any signed graph* $\Sigma = (\Gamma, \sigma)$, $\eta(\Sigma) \sim T^k(\eta(\Sigma))$ *if and only if* Σ *is an unbalanced signed graph and* $\Gamma = K_3$.

2.1. Characterization of Tosha-Degree Equivalence Signed Graphs. The following result characterize signed graphs which are Tosha-degree equivalence signed graphs.

Theorem 2.13. *A signed graph* $\Sigma = (\Gamma, \sigma)$ *is a Tosha-degree equivalence signed graph if and only if* Σ *is balanced signed graph and its underlying graph* Γ *is a Tosha-degree equivalence graph.*

◁ Suppose that Σ is balanced and Γ is a Tosha-degree equivalence graph. Then there exists a graph Γ' such that $T(\Gamma') \cong \Gamma$. Since Σ is balanced, by Theorem 1.1, there exists a marking μ of Γ such that each edge uv in Σ satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the signed graph $\Sigma' = (\Gamma', \sigma')$, where for any edge e in Γ', $\sigma'(e)$ is the marking of the corresponding vertex in Γ. Then clearly, $T(\Sigma') \cong \Sigma$. Hence Σ is a Tosha-degree equivalence signed graph.

Conversely, suppose that $\Sigma = (\Gamma, \sigma)$ is a Tosha-degree equivalence signed graph. Then there exists a signed graph $\Sigma' = (\Gamma', \sigma')$ such that $T(\Sigma') \cong \Sigma$. Hence Γ is the Tosha-degree equivalence graph and by Theorem 2.1, Σ is balanced. ▷

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ЗНАКОВЫЕ ГРАФЫ ЭКВИВАЛЕНТНОСТИ СТЕПЕНИ ТОША

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Аннотация. Степень Тоша ребра α в графе Γ без кратных ребер, обозначаемая $T(\alpha)$, — это число ребер, смежных с α в Γ, причем петли считаются дважды. Знаковый граф (помеченный граф) — это упорядоченная пара $\Sigma = (\Gamma, \sigma)$ ($\Sigma = (\Gamma, \mu)$), где $\Gamma = (V, E)$ — граф, называемый базовым графом Σ и $\sigma : E \to \{+, -\}$ ($\mu : V \to \{+, -\}$), является функцией. В данной статье определяется знаковый граф эквивалентности степени Тоша заданного знакового графа и предлагается характеристика эквивалентности по переключению знаковых графов, которые переключаются эквивалентно знаковым графам эквивалентности степени Тоша и k-ой итерации знаковых графов эквивалентности степени Тоша. Также была изучена структурная характеристика знаковых графов эквивалентности степени Тоша.

Ключевые слова: знаковый граф, баланс, ребро степени Тоша, знаковый граф эквивалентности степени Тоша, отрицание.

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