Vladikavkaz Mathematical Journal
2019, Volume 21, Issue 4, P. 11–24

УДК 517.98
DOI 10.23671/VNC.2019.21.44607

ISOMETRIES OF REAL SUBSPACES OF SELF-ADJOINT OPERATORS IN BANACH SYMMETRIC IDEALS

B. R. Aminov[1], **V. I. Chilin**[1]

[1] National University of Uzbekistan,
Vuzgorodok, Tashkent 100174, Uzbekistan

E-mail: aminovbehzod@gmail.com, vladimirchil@gmail.com, chilin@ucd.uz

Dedicated to E. I. Gordon on the occasion of his 70th birthday

Abstract. Let $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ be a Banach symmetric ideal of compact operators, acting in a complex separable infinite-dimensional Hilbert space $\mathcal{H}$. Let $\mathcal{C}_E^h = \{x \in \mathcal{C}_E : x = x^*\}$ be the real Banach subspace of self-adjoint operators in $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$. We show that in the case when $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ is a separable or perfect Banach symmetric ideal ($\mathcal{C}_E \neq \mathcal{C}_2$) any skew-Hermitian operator $H : \mathcal{C}_E^h \to \mathcal{C}_E^h$ has the following form $H(x) = i(xa - ax)$ for same $a^* = a \in \mathcal{B}(\mathcal{H})$ and for all $x \in \mathcal{C}_E^h$. Using this description of skew-Hermitian operators, we obtain the following general form of surjective linear isometries $V : \mathcal{C}_E^h \to \mathcal{C}_E^h$. Let $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ be a separable or a perfect Banach symmetric ideal with not uniform norm, that is $\|p\|_{\mathcal{C}_E} > 1$ for any finite dimensional projection $p \in \mathcal{C}_E$ with $\dim p(\mathcal{H}) > 1$, let $\mathcal{C}_E \neq \mathcal{C}_2$, and let $V : \mathcal{C}_E^h \to \mathcal{C}_E^h$ be a surjective linear isometry. Then there exists unitary or anti-unitary operator u on $\mathcal{H}$ such that $V(x) = uxu^*$ or $V(x) = -uxu^*$ for all $x \in \mathcal{C}_E^h$.

Key words: symmetric ideal of compact operators, skew-Hermitian operator, isometry.

Mathematical Subject Classification (2010): 46L52, 46B04.

For citation: *Aminov, B. R. and Chilin, V. I.* Isometries of Real Subspaces of Self-Adjoint Operators in Banach Symmetric Ideals, *Vladikavkaz Math. J.*, 2019, vol. 21, pp. 11–24. DOI: 10.23671/VNC.2019.21.44607.

1. Introduction

The study of linear isometries on classical Banach spaces was initiated by S. Banach. In [1, Ch. XI], he described all isometries on the space $L_p[0,1]$ with $p \neq 2$. In [2], J. Lamperti characterized all linear isometries on the L_p-space $L_p(\Omega, \mathcal{A}, \mu)$, where $(\Omega, \mathcal{A}, \mu)$ is a measure space with a complete σ-finite measure μ. Both S. Banach and J. Lamberti used a method for description of linear isometries on L_p-spaces that was independent of the choice of a scalar field. For studying linear isometries on the broader class of function symmetric spaces $E(\Omega, \mathcal{A}, \mu)$, different approaches are required that depend on a scalar field. If $E(\Omega, \mathcal{A}, \mu)$ is a complex symmetric space then G. Lumer's method [3] based on the theory of Hermitian operators can be effectively applied. For example, M. G. Zaidenberg [4, 5] used this method for description of all surjective linear isometries on the complex symmetric space $E(\Omega, \mathcal{A}, \mu)$, where μ is a continuous measure. For the symmetric space $E = E(0,1)$ of real-valued measurable functions on the segment $[0,1]$ with a Lebesgue measure μ, where E is a separable space or has the Fatou property, a description of surjective linear isometries on E was given by N. J. Kalton and B. Randrianantoanina [6]. They used methods of the theory of positive numerical operators. For real symmetric sequence spaces, a general form of surjective linear isometries was described by M. Sh. Braverman and E. M. Semenov [7, 8]. They used methods based on the theory of finite groups. For complex separable symmetric sequence spaces (symmetric sequence spaces with the Fatou property), a general form of surjective linear isometries was described in [9] (respectively, in [10]).

Naturally, the next step is to describe surjective linear isometries in the noncommutative situation, when symmetric sequence spaces are replaced by symmetric ideals of compact operators.

Assume $(\mathcal{H},(\cdot,\cdot))$ is an infinite-dimensional complex separable Hilbert space. Let $\mathcal{B}(\mathcal{H})$ (respectively, $\mathcal{K}(\mathcal{H})$) be the C^*-algebra of all bounded (respectively, compact) linear operators on $\mathcal{H}$. For a compact operator $x\in\mathcal{K}(\mathcal{H})$, we denote by $\mu(x):=\big\{\mu(n,x)\big\}_{n=1}^{\infty}$ the singular value sequence of x, that is, the decreasing rearrangement of the eigenvalue sequence of $|x|=(x^*x)^{\frac12}$. We let Tr denote the standard trace on $\mathcal{B}(\mathcal{H})$. For $p\in[1,\infty)$ $(p=\infty)$, we let

$$\mathcal{C}_p:=\Big\{x\in\mathcal{K}(\mathcal{H})\ :\ \mathrm{Tr}\big(|x|^p\big)<\infty\Big\}\quad(\text{respectively, } \mathcal{C}_\infty=\mathcal{K}(\mathcal{H}))$$

denote the p-th Schatten ideal of $\mathcal{B}(\mathcal{H})$, with the norm

$$\|x\|_p:=\mathrm{Tr}\big(|x|^p\big)^{\frac1p}\quad(\text{respectively, } \|x\|_\infty:=\sup_{n\geqslant1}|\mu(n,x)|).$$

In 1975, J. Arazy [11], [12, Ch. 11, § 2, Theorem 11.2.5] gave the following description of all the surjective isometries of Schatten ideals $\mathcal{C}_p$.

Theorem 1. *Let $V:\mathcal{C}_p\to\mathcal{C}_p$, $1\leqslant p\leqslant\infty$, $p\neq2$, be an surjective isometry. Then there exist unitary operators u_1 and u_2 or anti-unitary operators v_1 and v_2 on $\mathcal{H}$ such that either $Vx=u_1xu_2$ or $Vx=v_1x^*v_2$ for all $x\in\mathcal{C}_p$.*

Recall that a mapping $v:\mathcal{H}\to\mathcal{H}$ is an anti-unitary operator if

$$v(\lambda h+f)=\overline{\lambda}v(h)+v(f)\quad\text{and}\quad\|v(h)\|_{\mathcal{H}}=\|h\|_{\mathcal{H}}$$

for every complex number λ and $h,f\in\mathcal{H}$. If v is an anti-unitary operator then there exists an anti-unitary operator v^* such that $(h,v(f))=(f,v^*(h))$ for all $h,f\in\mathcal{H}$ (see, for example, [12, Ch. 11, § 2]).

The Schatten ideals $\mathcal{C}_p$ are examples of Banach symmetric ideals $(\mathcal{C}_E,\|\cdot\|_{\mathcal{C}_E})$ of compact operators associated with symmetric sequence spaces $(E,\|\cdot\|_E)$ (see Section 2.2 below). In 1981 A. Sourour [13] proved a version of Theorem 1 for separable Banach symmetric ideal $(\mathcal{C}_E,\|\cdot\|_{\mathcal{C}_E})$ such that $\mathcal{C}_E\neq\mathcal{C}_2$. Recently [14], a variant of Theorem 1 was obtained for any perfect Banach symmetric ideals $(\mathcal{C}_E,\|\cdot\|_{\mathcal{C}_E})$, $\mathcal{C}_E\neq\mathcal{C}_2$ (recall that $(\mathcal{C}_E,\|\cdot\|_{\mathcal{C}_E})$ is a perfect ideals, if $\mathcal{C}_E=\mathcal{C}_E^{\times\times}$ [15] (see Section 2.2 below)).

It is clear that for any unitary or anti-unitary operator u the linear operators $V_1(x)=uxu^*$ and $V_2(x)=-uxu^*$ acting in a real Banach space $(\mathcal{C}_E^h,\|\cdot\|_{\mathcal{C}_E})$ are surjective isometries, where $\mathcal{C}_E^h=\{x\in\mathcal{C}_E: x=x^*\}$.

Our main result states that if $(\mathcal{C}_E,\|\cdot\|_{\mathcal{C}_E})$ is a separable or a perfect Banach symmetric ideal of compact operators such that $\mathcal{C}_E\neq\mathcal{C}_2$, there are no other isometries in $(\mathcal{C}_E^h,\|\cdot\|_{\mathcal{C}_E})$:

Theorem 2. *Let $(\mathcal{C}_E,\|\cdot\|_{\mathcal{C}_E})$ be a separable or a perfect Banach symmetric ideal with not uniform norm, $\mathcal{C}_E\neq\mathcal{C}_2$, and let $V:\mathcal{C}_E^h\to\mathcal{C}_E^h$ be a surjective isometry. Then there exists unitary or anti-unitary operator u on $\mathcal{H}$ such that V can be written in the form $V(x)=uxu^*$ $(x\in\mathcal{C}_E^h)$ or in the form $V(x)=-uxu^*$ $(x\in\mathcal{C}_E^h)$.*

An analogous result for the space of self-adjoint traceless operators on a finite dimensionalal Hilbert space was obtained by G. Nagy [16].

2. Preliminaries

2.1. Symmetric Sequence Spaces. Let ℓ_∞ (respectively, c_0) be the Banach lattice of all bounded (respectively, converging to zero) sequences $\{\xi_n\}_{n=1}^{\infty}$ of real numbers with respect to the uniform norm $\|\{\xi_n\}_{n=1}^{\infty}\|_\infty = \sup_{n\in\mathbb{N}} |\xi_n|$, where $\mathbb{N}$ is the set of natural numbers. If $2^{\mathbb{N}}$ is the σ-algebra of all subsets of $\mathbb{N}$ and $\mu(\{n\}) = 1$ for each $n \in \mathbb{N}$, then $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ is a σ-finite measure space, $\mathcal{L}_\infty(\mathbb{N}, 2^{\mathbb{N}}, \mu) = \ell_\infty$,

$$\mathcal{L}_1(\mathbb{N}, 2^{\mathbb{N}}, \mu) = \ell_1 = \left\{ \{\xi_n\}_{n=1}^{\infty} \subset \mathbb{R} : \ \|\{\xi_n\}\|_1 = \sum_{n=1}^{\infty} |\xi_n| < \infty \right\},$$

where $\mathbb{R}$ is the field of real numbers. If $\xi = \{\xi_n\}_{n=1}^{\infty} \in \ell_\infty$, then the non-increasing rearrangement $\xi^* : (0,\infty) \to (0,\infty)$ of ξ is defined by

$$\xi^*(t) = \inf\{\lambda : \ \mu(\{|\xi| > \lambda\}) \leqslant t\}, \quad t > 0,$$

(see, for example, [17, Ch. 2, Definition 1.5]).

Therefore the non-increasing rearrangement ξ^* is identified with the sequence $\xi^* = \{\xi_n^*\}$, where

$$\xi_n^* = \inf_{\substack{F\subset\mathbb{N}, \\ \operatorname{card}(F)<n}} \ \sup_{n\notin F} |\xi_n|.$$

A non-zero linear subspace $E \subseteq \ell_\infty$ with a Banach norm $\|\cdot\|_E$ is called *symmetric* sequence space if conditions $\eta \in E$, $\xi \in \ell_\infty$, $\xi^* \leqslant \eta^*$ imply that $\xi \in E$ and $\|\xi\|_E \leqslant \|\eta\|_E$.

If $(E, \|\cdot\|_E)$ is a symmetric sequence space, then $\ell_1 \subset E \subset \ell_\infty$, in addition, $\|\xi\|_E \leqslant \|\xi\|_1$ for all $\xi \in \ell_1$ and $\|\xi\|_\infty \leqslant \|\xi\|_E$ for all $\xi \in E$ [17, Ch. 2, § 6, Theorem 6.6]. If there exists $\xi \in (E \setminus c_0)$ then $\xi^* \geqslant \alpha\mathbf{1}$ for some $\alpha > 0$, and therefore $\mathbf{1} \in E$, where $\mathbf{1} = \{1, 1, \ldots\}$. Consequently, for any symmetric sequence space E we have that $E \subseteq c_0$ or $E = \ell_\infty$.

2.2. Banach Symmetric Ideal of Compact Operators. Let $(\mathcal{H}, (\cdot,\cdot))$ be an infinite-dimensional complex separable Hilbert space, let $\mathcal{B}(\mathcal{H})$ (respectively, $\mathcal{K}(\mathcal{H}), \mathcal{F}(\mathcal{H})$) be the $*$-algebra of all bounded (respectively, compact, finite rank) linear operators in $\mathcal{H}$, and let $\mathcal{P}(\mathcal{H}) = \{p \in \mathcal{B}(\mathcal{H}) : p = p^* = p^2\}$. It is known that $*$-algebras $\mathcal{B}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ are C^*-algebras with respect to the uniform operator norm, which we shall denote by $\|\cdot\|_\infty$. For a subset $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$, we set $\mathcal{A}^h = \{x \in \mathcal{A} : x = x^*\}$.

It is well known that $\mathcal{F}(\mathcal{H}) \subset \mathcal{I} \subset \mathcal{K}(\mathcal{H})$ for any proper two-sided ideal $\mathcal{I}$ in $\mathcal{B}(\mathcal{H})$ (see for example, [18, Proposition 2.1]).

If $(E, \|\cdot\|_E) \subset c_0$ is a symmetric sequence space, then the set

$$\mathcal{C}_E := \{x \in \mathcal{K}(\mathcal{H}) : \{\mu(n,x)\}_{n=1}^{\infty} \in E\}$$

is a proper two-sided ideal in $\mathcal{B}(\mathcal{H})$ (see [18, Theorem 2.5]). In addition, $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ is a Banach space with respect to the norm $\|x\|_{\mathcal{C}_E} = \|\{\mu(n,x)\}\|_E$ [19] (see also [20, Ch. 3, § 3.5]), and the norm $\|\cdot\|_{\mathcal{C}_E}$ has the following properties:

1) $\|xzy\|_{\mathcal{C}_E} \leqslant \|x\|_\infty \|y\|_\infty \|z\|_{\mathcal{C}_E}$ for all $x, y \in \mathcal{B}(\mathcal{H})$ and $z \in \mathcal{C}_E$;
2) $\|x\|_{\mathcal{C}_E} = \|x\|_\infty$ if $x \in \mathcal{F}(\mathcal{H})$ is of rank 1.

In this case we say that $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ is a *Banach symmetric ideal* (cf. [18, Ch. 1, §1.7], [21, Ch. III]). It is known that $\mathcal{C}_1 \subset \mathcal{C}_E \subset \mathcal{K}(\mathcal{H})$ and $\|x\|_{\mathcal{C}_E} \leqslant \|x\|_1$, $\|y\|_\infty \leqslant \|y\|_{\mathcal{C}_E}$ for all $x \in \mathcal{C}_1$, $y \in \mathcal{C}_E$.

If $(E, \|\cdot\|_E)$ is a symmetric sequence space (respectively, $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ is a Banach symmetric ideal), then the Köthe dual $E^\times$ (respectively, $\mathcal{C}_E^\times$) is defined as

$$E^\times = \Big\{\xi = \{\xi_n\}_{n=1}^\infty \in \ell_\infty : \xi\eta = \{\xi_n\eta_n\}_{n=1}^\infty \in \ell_1 \ \text{ for all } \ \eta = \{\eta_n\}_{n=1}^\infty \in E\Big\},$$

$$\Big(\text{respectively,} \ \ \mathcal{C}_E^\times = \Big\{x \in \mathcal{B}(\mathcal{H}) : \ xy \in \mathcal{C}_1 \ \text{ for all } \ y \in \mathcal{C}_E\Big\}\Big),$$

and

$$\|\xi\|_{E^\times} = \sup\Big\{\sum_{n=1}^\infty |\xi_n\eta_n| : \ \eta = \{\eta_n\}_{n=1}^\infty \in E, \ \|\eta\|_E \leqslant 1\Big\}, \quad \xi \in E^\times,$$

$$\Big(\text{respectively,} \ \ \|x\|_{\mathcal{C}_E^\times} = \sup\Big\{\mathrm{Tr}\big(|xy|\big) : \ y \in \mathcal{C}_E, \ \|y\|_{\mathcal{C}_E} \leqslant 1\Big\}, \ x \in \mathcal{C}_E^\times\Big).$$

It is known that $(E^\times, \|\cdot\|_{E^\times})$ is a symmetric sequence space [22, Ch. II, § 4, Theorems 4.3, 4.9] and $\ell_1^\times = \ell_\infty$. In addition, if $E \neq \ell_1$ then $E^\times \subset c_0$. Therefore, if $E \neq \ell_1$, the space $(\mathcal{C}_E^\times, \|\cdot\|_{\mathcal{C}_E^\times})$ is a symmetric ideal of compact operators.

A Banach symmetric ideal $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ is said to be *perfect* if $\mathcal{C}_E = \mathcal{C}_E^{\times\times}$ (see, for example, [15]). It is clear that $\mathcal{C}_E$ is perfect if and only if $E = E^{\times\times}$.

A symmetric sequence space $(E, \|\cdot\|_E)$ (a Banach symmetric ideal $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$) is said to possess *Fatou property* if the conditions

$$0 \leqslant \xi_k \leqslant \xi_{k+1}, \ \xi_k \in E \ \ (\text{respectively, } 0 \leqslant \ x_k \leqslant x_{k+1}, \ x_k \in \mathcal{C}_E) \ \text{ for all } \ k \in \mathbb{N}$$

and $\sup_{k\geqslant 1} \|\xi_k\|_E < \infty$ (respectively, $\sup_{k\geqslant 1} \|x_k\|_{\mathcal{C}_E} < \infty$) imply that there exists an element $\xi \in E$ (respectively, $x \in \mathcal{C}_E$) such that $\xi_k \uparrow \xi$ and $\|\xi\|_E = \sup_{k\geqslant 1} \|\xi_k\|_E$ (respectively, $x_k \uparrow x$ and $\|x\|_{\mathcal{C}_E} = \sup_{k\geqslant 1} \|x_k\|_{\mathcal{C}_E}$).

It is known that $(E, \|\cdot\|_E)$ (respectively, $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$) has the Fatou property if and only if $E = E^{\times\times}$ [23, Vol. II, Ch. 1, Section a] (respectively, $\mathcal{C}_E = \mathcal{C}_E^{\times\times}$ [24, Theorem 5.14]). Therefore $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ is a perfect Banach symmetric ideal if and only if $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ has the Fatou property.

If $y \in \mathcal{C}_E^\times$, then a linear functional $f_y(x) = \mathrm{Tr}\big(x \cdot y\big)$, $x \in \mathcal{C}_E$, is continuous on $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$, in addition, $\|f_y\|_{\mathcal{C}_E^*} = \|y\|_{\mathcal{C}_E^\times}$, where $(\mathcal{C}_E^*, \|\cdot\|_{\mathcal{C}_E^*})$ is the dual of the Banach space $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ (see, for example, [15]). Identifying an element $y \in \mathcal{C}_E^\times$ and the linear functional f_y, we may assume that $\mathcal{C}_E^\times$ is a closed linear subspace in $\mathcal{C}_E^*$. Since $\mathcal{F}(\mathcal{H}) \subset \mathcal{C}_E^\times$, it follows that $\mathcal{C}_E^\times$ is a total subspace in $\mathcal{C}_E^*$, that is, the conditions $x \in \mathcal{C}_E$, $f(x) = 0$ for all $f \in \mathcal{C}_E^\times$ imply $x = 0$. Thus, the weak topology $\sigma(\mathcal{C}_E, \mathcal{C}_E^\times)$ is a Hausdorff topology, in addition $\mathcal{F}(\mathcal{H})$ (respectively, $\mathcal{F}(\mathcal{H})^h$) is $\sigma(\mathcal{C}_E, \mathcal{C}_E^\times)$-dense in $\mathcal{C}_E$ (respectively, $\mathcal{C}_E^h$).

3. Skew-Hermitian Operators in Banach Symmetric Ideals

Let X be a linear space over the field $\mathbb{K}$ of real or complex numbers. *A semi-inner product* on a space X is a $\mathbb{K}$-valued form $[\cdot,\cdot]\colon X \times X \to \mathbb{K}$ which satisfies

(i) $[\alpha x + y, z] = \alpha \cdot [x, z] + [y, z]$ for all $\alpha \in \mathbb{K}$ and $x, y, z \in X$;

(ii) $[x, \alpha y] = \overline{\alpha} \cdot [x, y]$ for all $\alpha \in \mathbb{K}$ and $x, y \in X$;

(iii) $[x, x] \geqslant 0$ for all $x \in X$ and $[x, x] = 0$ implies that $x = 0$;

(iv) $|[x,y]|^2 \leqslant [x,x]\cdot[y,y]$ for all $x,y \in X$

(see, for example, [25, Ch. 2, § 1]).

The function $\|x\| = \sqrt{[x,x]}$ is the norm on a linear space X. Conversely, if $(X, \|\cdot\|_X)$ is a normed linear space, then there exists semi-inner product $[\cdot,\cdot]$ on X *compatible with the norm* $\|\cdot\|_X$, that is, $\|x\|_X = \sqrt{[x,x]}$ [25, Ch. 2, § 1]. In particular, the semi-inner product (compatible with the norm $\|\cdot\|_X$) can be defined using the equation $[x,y] = \varphi_y(x)$, where $\varphi_y \in X^*$, $\|\varphi_y\|_{X^*} = \|y\|_X$ and $\varphi_y(y) = \|y\|_X^2$ (such functional is called a *support functional* at $y \in X$) [25, Ch. 2, §1, Theorem 10].

Let $(X, \|\cdot\|_X)$ be Banach space over field $\mathbb{K}$, and let $[\cdot,\cdot]$ be a semi-inner product on X which is compatible with the norm $\|\cdot\|_X$. A linear bounded operator $H\colon X \to X$ is said to be *skew-Hermitian*, if $\mathrm{Re}([H(x),x]) = 0$ for all $x \in X$, where $\mathrm{Re}(\alpha)$ is the real part of number $\alpha \in \mathbb{K}$ [12, Ch. 9, § 4]. In particular, if $\mathbb{K} = \mathbb{R}$ then $\varphi_x(H(x)) = [H(x),x] = 0$ for every $x \in X$.

The following Proposition is well known [12, Ch. 9, § 4, Proposition 9.4.2].

Proposition 1. *Let $(X, \|\cdot\|_X)$ be a real Banach space and let H be a skew-Hermitian operator on X. If $V\colon X \to X$ is a surjective isometry then an operator $V \cdot H \cdot V^{-1}$ is a skew-Hermitian.*

It is clear that in the case $(X, \|\cdot\|_X) = (\mathcal{C}_E^h, \|\cdot\|_{\mathcal{C}_E})$ every linear operator $H : \mathcal{C}_E^h \to \mathcal{C}_E^h$ defined by $H(x) = i(xa - ax), x \in \mathcal{C}_E^h$, where $a \in \mathcal{B}(H)^h$, $i^2 = -1$ is a skew-Hermitian operator.

The following Theorem gives a description of skew-Hermitian operators acting on $\mathcal{C}_E^h$ when $\mathcal{C}_E$ is a separable or perfect Banach symmetric ideal other than $\mathcal{C}_2$.

Theorem 3. *Let $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ be a separable or perfect Banach symmetric ideal, and let $\mathcal{C}_E \neq \mathcal{C}_2$. Then for any skew-Hermitian operator $H : \mathcal{C}_E^h \to \mathcal{C}_E^h$ there exists $a \in \mathcal{B}(H)^h$ such that $H(x) = i(xa - ax)$ for all $x \in \mathcal{C}_E^h$.*

$\lhd$ We slightly modify the original proof of Sourour [13]. For vectors $\xi, \eta \in \mathcal{H}$, denote by $\xi \otimes \eta$ the rank one operator on $\mathcal{H}$ given $(\xi\otimes\eta)(h) = (h,\eta)\xi$, $h \in \mathcal{H}$. It is easily seen $\langle x, \xi\otimes\eta\rangle := \mathrm{Tr}((\eta\otimes\xi)\cdot x) = (x(\eta),\xi)$ for any $x \in \mathcal{B}(\mathcal{H})^h$ and $\xi,\eta \in \mathcal{H}$. If $y = \xi\otimes\xi$, $\|\xi\|_{\mathcal{H}} = 1$, then y is an one dimensional projection on $\mathcal{H}$ and $\|y\|_{\mathcal{C}_E} = \|y\|_\infty = 1$. Thus for a linear functional $f_y(x) := \langle x, y\rangle = \mathrm{Tr}(y^*x), x \in \mathcal{C}_E^h$, we have that $f_y(y) = \mathrm{Tr}(y^2) = \mathrm{Tr}(y) = (\xi,\xi) = 1 = \|y\|_{\mathcal{C}_E}^2$. In addition, if $x \in \mathcal{C}_E^h$ and $\|x\|_{\mathcal{C}_E} \leqslant 1$ then $|f_y(x)| = |\mathrm{Tr}(yx)| = |(x(\xi),\xi)| \leqslant \|x\|_\infty \leqslant \|x\|_{\mathcal{C}_E} \leqslant 1$. Consequently, $\|f_y\|_{(\mathcal{C}_E^h)^*} = 1 = \|y\|_{\mathcal{C}_E}$. This means that f_y is a support functional at $y \in \mathcal{C}_E^h$, and $[x,y] = f_y(x)$ is a semi-inner product on $\mathcal{C}_E^h$ compatible with the norm $\|\cdot\|_{\mathcal{C}_E^h}$ [25, Ch. 2, § 1, Theorem 10].

Step 1. If $\xi,\eta \in \mathcal{H}$, $(\eta,\xi) = 0$, then $\langle H(\eta\otimes\eta), \xi\otimes\xi\rangle = 0$.

We can assume that $\|\eta\|_{\mathcal{H}} = \|\xi\|_{\mathcal{H}} = 1$. Since $p = \eta\otimes\eta$ is one dimensional projections and H is a skew-Hermitian operator, it follows that

$$0 = [H(p),p] = f_p(H(p)) = \langle H(p),p\rangle. \tag{1}$$

By Lemma 9.2.7 ([12, Ch. 9, §9.2], see also the proof of Lemma 11.3.2 [12, Ch. 9, §11.3]), there exists a vector $\xi = \{\xi_1,\xi_2\} \in (\mathbb{R}^2, \|\cdot\|_E)$, $\xi_1 > 0, \xi_2 > 0$, $\|\xi\|_E = 1$, such that the functional $f(\{\eta_1,\eta_2\}) = \eta_1\xi_1 + \eta_2\xi_2$, $\{\eta_1,\eta_2\} \in \mathbb{R}^2$, is a support functional at ξ for space $(\mathbb{R}^2, \|\cdot\|_E)$.

Let us show that the linear functional

$$\varphi(y) = \langle y, x\rangle, \quad y \in \mathcal{C}_E^h, \;\; x = \xi_1 p + \xi_2 q,$$

is a support functional at x for $(\mathcal{C}_E^h, \|\cdot\|_{\mathcal{C}_E})$.

Since f is support functional at ξ for $(\mathbb{R}^2, \|\cdot\|_E)$ and $\|\xi\|_E = 1$, it follows that $\xi_1^2 + \xi_2^2 = f(\{\xi_1, \xi_2\}) = f(\xi) = \|\xi\|_E^2 = 1$. Furthermore, by $\|f\| = \|\xi\|_E = 1$, we have that $|f(\{\eta_1, \eta_2\})| = |\xi_1\eta_1 + \xi_2\eta_2| \leqslant 1$ for every $\{\eta_1, \eta_2\} \in \mathbb{R}^2$ with $\|\{\eta_1, \eta_2\}\|_E \leqslant 1$.

Further, by [21, Ch. II, § 4, Lemma 4.1], we have

$$|(y(\eta), \eta)| \leqslant \mu(1, y), \quad |(y(\xi), \xi)| \leqslant \mu(1, y), \quad |(y(\eta), \eta)| + |(y(\xi), \xi)| \leqslant \mu(1, y) + \mu(2, y),$$

that is, $\{(y(\eta), \eta), (y(\xi), \xi)\} \prec\prec \{\mu(1, y), \mu(2, y)\}$. Since $(E, \|\cdot\|_E)$ is a fully symmetric sequence space, it follows that

$$\|\{(y(\eta), \eta), (y(\xi), \xi)\}\|_E \leqslant \|\{\mu(1, y), \mu(2, y)\}\|_E \leqslant \|y\|_{\mathcal{C}_E}.$$

Consequently, if $y \in \mathcal{C}_E^h$ and $\|y\|_{\mathcal{C}_E} \leqslant 1$, then

$$|\varphi(y)| = |\langle y, x\rangle| = |\xi_1 \mathrm{Tr}(py) + \xi_2 \mathrm{Tr}(qy)| = \big|f\big(\big\{(y(\eta), \eta), (y(\xi), \xi)\big\}\big)\big| \leqslant 1,$$

that is, $\|\varphi\|_{(\mathcal{C}_E^h, \|\cdot\|_E)^*} \leqslant 1$. Since $\|x\|_{\mathcal{C}_E} = \|\xi\|_E = 1$ and

$$\varphi(x) = \langle x, x\rangle = \langle \xi_1 p + \xi_2 q, \xi_1 p + \xi_2 q\rangle = \mathrm{Tr}(\xi_1 p + \xi_2 q)(\xi_1 p + \xi_2 q) = \xi_1^2 + \xi_2^2 = 1,$$

it follows that $\|\varphi\|_{(\mathcal{C}_E^h, \|\cdot\|_E)^*} = 1 = \|x\|_{\mathcal{C}_E}$ and $\varphi(x) = \|x\|_{\mathcal{C}_E}^2$. This means that φ is a support functional at x for space $(\mathcal{C}_E^h, \|\cdot\|_{\mathcal{C}_E})$.

Hence,

$$0 = [H(x), x] = \varphi(H(x)) = \big\langle H(x), x\big\rangle = \big\langle \xi_1 H(p) + \xi_2 H(q), \xi_1 p + \xi_2 q\big\rangle.$$

Since $\langle H(p), p\rangle = \langle H(q), q\rangle = 0$ (see (1)), it follows that

$$\big\langle H(p), q\big\rangle = -\big\langle H(q), p\big\rangle. \tag{2}$$

We extend $\eta_1 = \eta$, $\eta_2 = \xi$ up to an orthonormal basis $\{\eta_i\}_{i=1}^{\infty}$, and let $p_i = \eta_i \otimes \eta_i$. Now we replace our operator H with another skew-Hermitian operator H_0. Let u be a unitary operator such that $u(\eta_1) = \eta_2$, $u(\eta_2) = \eta_1$ and $u(\eta_k) = \eta_k$ if $k \neq 1, 2$. It is clear that $u^* = u^{-1} = u$, $up_1u = p_2$, $up_2u = p_1$, $up_iu = p_i$, $i \neq 1, 2$, and $V(x) = uxu^* = uxu$ is an surjective isometry on $\mathcal{C}_E^h$, in addition, $V^{-1} = V$.

By Proposition 1, a linear operator $H_1 = VHV^{-1}$ is a skew-Hermitian operator, in particular, $\langle H_1(p_k), p_k\rangle = 0$ for all $k \in \mathbb{N}$ (see (1)).

If $i, j \neq 1, 2$, then

$$\begin{aligned}\big\langle H_1(p_i), p_j\big\rangle &= \big\langle uH(p_i)u, p_j\big\rangle = \mathrm{Tr}(p_j uH(p_i)u) = (uH(p_i)u(\eta_j), \eta_j) \\ &= (H(p_i)u(\eta_j), u^*(\eta_j)) = (H(p_i)(\eta_j), \eta_j) = \mathrm{Tr}(p_j H(p_i)) = \big\langle H(p_i), p_j\big\rangle.\end{aligned}$$

If $i = 1$, $j \neq 1, 2$, then

$$\begin{aligned}\langle H_1(p_1), p_j\rangle &= \langle uH(p_2)u, p_j\rangle = \mathrm{Tr}(p_j uH(p_2)u) = (uH(p_2)u(\eta_j), \eta_j) \\ &= (H(p_2)u(\eta_j), u^*(\eta_j)) = (H(p_2)(\eta_j), \eta_j) = \mathrm{Tr}(p_j H(p_2)) = \langle H(p_2), p_j\rangle.\end{aligned}$$

Similarly, we get the following equalities

(i) $\langle H_1(p_2), p_j\rangle = \langle H(p_1), p_j\rangle$ if $i = 2$, $j \neq 1, 2$;

(ii) $\langle H_1(p_i), p_1\rangle = \langle H(p_i), p_2\rangle$ if $j = 1$, $i \neq 1, 2$;

(iii) $\langle H_1(p_1), p_2\rangle = \langle H(p_2), p_1\rangle$ if $i = 1,\ j = 2$;

(iv) $\langle H_1(p_2), p_1\rangle = \langle H(p_1), p_2\rangle$ if $i = 2,\ j = 1$.

It is clear that $H_0 = \frac{1}{2}(H - H_1)$ is a skew-Hermitian operator, and if $i, j \neq 1, 2$, then $\langle H_0(p_i), p_j\rangle = \frac{1}{2}(\langle H(p_i), p_j\rangle - \langle H_1(p_i), p_j\rangle) = 0$. Similarly, if $i = 1,\ j \neq 1, 2$ (respectively, $i = 2,\ j \neq 1, 2$) we get

$$\big\langle H_0(p_1), p_j\big\rangle = \frac{1}{2}\big(\big\langle H(p_1), p_j\big\rangle - \big\langle H(p_2), p_j\big\rangle\big)$$

$$\Big(\text{respectively, } \big\langle H_0(p_2), p_j\big\rangle = \frac{1}{2}\big(\big\langle H(p_2), p_j\big\rangle - \big\langle H(p_1), p_j\big\rangle\big)\Big),$$

that is, $\langle H_0(p_1), p_j\rangle + \langle H_0(p_2), p_j\rangle = 0$ in the case $j \neq 1, 2$.

Similarly, $\langle H_0(p_j), p_1\rangle + \langle H_0(p_j), p_2\rangle = 0$ if $j \neq 1, 2$. Since

$$\big\langle H_0(p_1), p_2\big\rangle = \frac{1}{2}\big(\big\langle H(p_1), p_2\big\rangle - \big\langle H(p_2), p_1\big\rangle\big), \quad \big\langle H(p_1), p_2\big\rangle = -\big\langle H(p_2), p_1\big\rangle$$

(see (2)), it follows that $\langle H_0(p_1), p_2\rangle = \langle H(p_1), p_2\rangle$. Similarly, we get that $\langle H_0(p_2), p_1\rangle = -\langle H(p_1), p_2\rangle$. Finally, since H_0 is a skew-Hermitian operator, we have $\langle H_0(p_k), p_k\rangle = 0$ for all $k \in \mathbb{N}$ (see (1)).

Let n be the smallest natural number such that the norm $\|\cdot\|_E$ is not Euclidian on $\mathbb{R}^n$. Then there exist (see, [10, Lemma 5.4]) linear independent vectors $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$, $\eta = (\eta_1, \eta_2, \ldots, \eta_n) \in \mathbb{R}^n$, $\|\xi\|_E = 1$, such that

$$\|\xi\|_E = \|f_\eta\|_{E^*} = f_\eta(\xi) = 1, \tag{3}$$

where $f_\eta(\zeta) = \sum_{i=1}^n \zeta_i\eta_i$, $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \in \mathbb{R}^n$. By rearranging the coordinates we may assume that $\xi_1\eta_2 \neq \xi_2\eta_1$.

Let $x = \sum_{j=1}^n \xi_j p_j$, $y = \sum_{j=1}^n \eta_j p_j$, and let $\varphi_y(z) = \langle z, y\rangle = \sum_{j=1}^n \eta_j \cdot \operatorname{Tr}(p_j z)$, $z \in \mathcal{C}_E^h$.

Let us show that φ_y is a support functional at x for $(\mathcal{C}_E^h, \|\cdot\|_E)$. Since $\|f_\eta\|_{E^*} = 1$ (see (3)), it follows that $|f_\eta(\zeta)| = |\sum_{i=1}^n \eta_i\zeta_i| \leqslant 1$ for every $\zeta = \{\zeta_i\}_{i=1}^n \in \mathbb{R}^n$ with $\|\zeta\|_E \leqslant 1$. Note that $\|x\|_{\mathcal{C}_E} = \|\xi\|_E = 1$.

We should show that $\|\varphi_y\| = \|x\|_{\mathcal{C}_E} = 1$ and $\varphi_y(x) = \|x\|^2_{\mathcal{C}_E} = 1$. Indeed,

$$\varphi_y(x) = \langle x, y\rangle = \left\langle \sum_{j=1}^n \xi_j p_j, \sum_{j=1}^n \eta_j p_j \right\rangle = \sum_{j=1}^n \xi_j\eta_j = f_\eta(\xi) = 1 = \|x\|^2_{\mathcal{C}_E}.$$

If $z \in \mathcal{C}_E^h$, $\|z\|_{\mathcal{C}_E} \leqslant 1$ then $|\varphi_y(z)| = \big|\sum_{j=1}^n \eta_j(z(\eta_j), \eta_j)\big| \leqslant 1$. The last inequality follows from

$$\big\{(z(\eta_1), \eta_1), (z(\eta_2), \eta_2), \ldots, (z(\eta_n), \eta_n)\big\} \prec \big\{\mu(1, z), \mu(2, z), \ldots, \mu(n, z)\big\}$$

(see [21, Ch. II, § 4, Lemma 4.1]). Therefore $\|\varphi_y\| = \|x\|_{\mathcal{C}_E} = 1$ and $\varphi_y(x) = \|x\|^2_{\mathcal{C}_E} = 1$. This means that φ_y is a support functional at x for $(\mathcal{C}_E^h, \|\cdot\|_E)$.

Consequently,

$$\begin{aligned} 0 = \big\langle H_0(x), y\big\rangle &= \big\langle \xi_1 H_0(p_1) + \ldots + \xi_n H_0(p_n), \eta_1 p_1 + \ldots + \eta_n p_n\big\rangle \\ &= (\xi_1\eta_2 - \xi_2\eta_1)\big\langle H_0(p_1), p_2\big\rangle + (\xi_1\eta_3 - \xi_2\eta_3)\big\langle H_0(p_1), p_3\big\rangle \\ &\quad + \ldots + (\xi_1\eta_n - \xi_2\eta_n)\big\langle H_0(p_1), p_n\big\rangle + (\xi_3\eta_1 - \xi_3\eta_2)\big\langle H_0(p_3), p_1\big\rangle \\ &\quad + \ldots + (\xi_n\eta_1 - \xi_n\eta_2)\big\langle H_0(p_n), p_1\big\rangle. \end{aligned} \tag{4}$$

Let now $\tilde{x} = \xi_1 p_1 + \xi_2 p_2 - \xi_3 p_3 - \ldots - \xi_n p_n$ and $\tilde{y} = \eta_1 p_1 + \eta_2 p_2 - \eta_3 p_3 - \ldots - \eta_n p_n$. As above, we have that $\varphi_{\tilde{y}}(\cdot) = \langle \cdot, \tilde{y} \rangle$ is a support functional at $\tilde{x}$. Consequently,

$$\begin{aligned} 0 = \big\langle H_0(\tilde{x}), \tilde{y} \big\rangle &= (\xi_1\eta_2 - \xi_2\eta_1)\big\langle H_0(p_1), p_2 \big\rangle + (-\xi_1\eta_3 + \xi_2\eta_3)\langle H_0(p_1), p_3 \rangle \\ &+ \ldots + (-\xi_1\eta_n + \xi_2\eta_n)\big\langle H_0(p_1), p_n \big\rangle + (-\xi_3\eta_1 + \xi_3\eta_2)\big\langle H_0(p_3), p_1 \big\rangle \\ &+ \ldots + (-\xi_n\eta_1 + \xi_n\eta_2)\big\langle H_0(p_n), p_1 \big\rangle. \end{aligned} \tag{5}$$

Summing (4) and (5) we obtain $2(\xi_1\eta_2 - \xi_2\eta_1)\langle H_0(p_1), p_2 \rangle = 0$, that is, $\langle H(p_1), p_2 \rangle = \langle H_0(p_1), p_2 \rangle = 0$.

Step 2. Let $\eta \in \mathcal{H}$, $\|\eta\|_{\mathcal{H}} = 1$, $p = \eta \otimes \eta$, $x \in \mathcal{K}(\mathcal{H})^h$, and let $\mathrm{Tr}(xq) = 0$ for any one dimensional projection q with $qp = 0$. Then there exists $f \in \mathcal{H}$ such that $x = \eta \otimes f + f \otimes \eta - (\eta \otimes \eta)(f \otimes \eta)$, $\|f\|_{\mathcal{H}} \leqslant \|x\|_\infty$.

Indeed, if q is an one dimensional projection with $qp = 0$ then $qxq = \alpha q$ for some $\alpha \in \mathbb{R}$, and $0 = \mathrm{Tr}(xq) = \mathrm{Tr}(qxq) = \mathrm{Tr}(\alpha q) = \alpha$, that is, $\alpha = 0$ and $qxq = 0$. Let $e \in \mathcal{P}(\mathcal{H})$, $\dim e(\mathcal{H}) = 1$, $ep = 0$, $eq = 0$, $y = (q+e)x(q+e)$. If $y \neq 0$ then there exists $r \in \mathcal{P}(\mathcal{H})$, $\dim r(\mathcal{H}) = 1$ such that $r \leqslant q + e$ and $rxr = ryr = \beta r$ for some $0 \neq \beta \in \mathbb{R}$. Since $rp = 0$, it follows that $0 = \mathrm{Tr}(xr) = \mathrm{Tr}(rxr) = \beta \neq 0$. Thus $y = 0$. Continuing this process, we construct a sequence of finite-dimensional projections $g_n \uparrow (I - p)$ such that $g_n x g_n = 0$ for all $n \in \mathbb{N}$, where $I(h) = h$, $h \in \mathcal{H}$. Consequently, $(I - p)x(I - p) = 0$.

If $f = x(\eta)$ then $xp = f \otimes \eta$ and $px = \eta \otimes f$. In addition,

$$(I - p)xp(h) = (I - p)x((h, \eta)\eta)) = (h, \eta)(I - p)f, \quad h \in \mathcal{H},$$

that is, $(I - p)xp = (I - p)f \otimes \eta$. Therefore,

$$x = px + (I - p)xp = \eta \otimes f + (I - p)f \otimes \eta \ \text{ and } \ \|f\|_{\mathcal{H}} \leqslant \|x\|_\infty.$$

Step 3. Let $\eta \in \mathcal{H}$, $\|\eta\|_{\mathcal{H}} = 1$, $p = \eta \otimes \eta$. Then there exists $f \in \mathcal{H}$ such that

$$H(\eta \otimes \eta) = \eta \otimes f + f \otimes \eta, \quad \|f\|_{\mathcal{H}} \leqslant \|H\|.$$

Indeed, if $x = H(\eta \otimes \eta)$, $\xi \in \mathcal{H}$, $(\eta, \xi) = 0$, $q = \xi \otimes \xi$, then by Step 1 we obtain that $(x(\xi), \xi) = \langle x, \xi \otimes \xi \rangle = \mathrm{Tr}(x \cdot \xi \otimes \xi) = 0$. Using Step 2, we have that there exists $f \in \mathcal{H}$ such that $H(\eta \otimes \eta) = x = \eta \otimes f + f \otimes \eta - (\eta \otimes \eta)(f \otimes \eta)$. Since H is a skew-Hermitian operator, it follows that

$$\begin{aligned} 0 = \big\langle H(\eta \otimes \eta), \eta \otimes \eta \big\rangle &= \big\langle \eta \otimes f + f \otimes \eta - (\eta \otimes \eta)(f \otimes \eta), \eta \otimes \eta \big\rangle \\ &= \mathrm{Tr}\big((\eta \otimes \eta)\big(\eta \otimes f + f \otimes \eta - (\eta \otimes \eta)(f \otimes \eta)\big)\big) \\ &= \mathrm{Tr}\big((\eta \otimes \eta)(\eta \otimes f)\big) = \big((\eta \otimes f)(\eta), \eta\big) = (\eta, f). \end{aligned}$$

Thus $(\eta, f) = 0$ and $x = \eta \otimes f + f \otimes \eta - (\eta \otimes \eta)(f \otimes \eta) = \eta \otimes f + f \otimes \eta$. In addition,

$$\|f\|_{\mathcal{H}} \leqslant \|x\|_\infty \leqslant \|x\|_{\mathcal{C}_E} = \|H(\eta \otimes \eta)\|_{\mathcal{C}_E} \leqslant \|H\| \cdot \|\eta \otimes \eta\|_{\mathcal{C}_E} = \|H\| \cdot \|\eta \otimes \eta\|_\infty = \|H\|.$$

Step 4. There exists $a \in \mathcal{B}(\mathcal{H})$ such that $H(x) = ax + xa^*$ for every $x \in \mathcal{C}_E^h$.

Let $\{p_i\}_{i=1}^\infty = \{\eta_i \otimes \eta_i\}_{i=1}^\infty$ be a basis in real linear space $\mathcal{F}(\mathcal{H})^h$, where $\{\eta_i\}_{i=1}^\infty$ is an orthonormal basis of $\mathcal{H}$. For every $\eta_i \in \mathcal{H}$ there exists $f_i \in \mathcal{H}$ such that $H(\eta_i \otimes \eta_i) = \eta_i \otimes f_i + f_i \otimes \eta_i$, and $\|f_i\|_{\mathcal{H}} \leqslant \|H\|$ for all $i \in \mathbb{N}$ (see Step 3). Define a linear operator $a \colon \mathcal{H} \to \mathcal{H}$ setting $a(\eta_i) = f_i$. Since $\|f_i\|_{\mathcal{H}} \leqslant \|H\|$ for all $i \in \mathbb{N}$, it follows that $a \in \mathcal{B}(\mathcal{H})$, in addition,

$H(p_i) = \eta_i \otimes a(\eta_i) + a(\eta_i) \otimes \eta_i$. Since $\eta_i \otimes a(\eta_i) = (\eta_i \otimes \eta_i)a^*$ and $a(\eta_i) \otimes \eta_i = a(\eta_i \otimes \eta_i)$, it follows that $H(x) = ax + xa^*$ for all $x \in \mathcal{F}(\mathcal{H})^h$.

If $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ is a separable space then $\mathcal{F}(H)^h$ is dense in $(\mathcal{C}_E^h, \|\cdot\|_{\mathcal{C}_E})$. Consequently, $H(x) = ax + xa^*$ for all $x \in \mathcal{C}_E^h$.

Let now $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ be a perfect Banach symmetric ideal. Repeating the proof of Theorem 4.4 [14] that establishes the $\sigma(\mathcal{C}_E, \mathcal{C}_E^\times)$-continuity of the Hermitian operators acting in $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$, we obtain that the skew-Hermitian operator H also $\sigma(\mathcal{C}_E^h, (C_E^\times)^h)$-continuous. Since the space $\mathcal{F}(\mathcal{H})^h$ is $\sigma(\mathcal{C}_E^h, (C_E^\times)^h)$-dense in $\mathcal{C}_E^h$, it follows that $H(x) = ax + xa^*$ for all $x \in \mathcal{C}_E^h$.

Step 5. $a = ib$ for some $b \in \mathcal{B}(\mathcal{H})^h$.

Indeed, if $a = a_1 + ia_2$, $a_1, a_2 \in \mathcal{B}(\mathcal{H})^h$, then

$$H(x) = ax + xa^* = a_1x + xa_1 + i(a_2x - xa_2) = S_1(x_1) + S_2(x),$$

where $S_1(x) = a_1x + xa_1$, $S_2(x) = i(a_2x - xa_2)$, $x \in \mathcal{C}_E^h$. Since H and S_2 are skew-Hermitian, it follows that $S_1 = H - S_2$ is also skew-Hermitian.

If $p \in \mathcal{P}(\mathcal{H})$, $\dim p(\mathcal{H}) = 1$, then the lineal functional $\varphi(y) = \langle y, p\rangle = \mathrm{Tr}(yp)$, $y \in \mathcal{C}_E^h$, is support functional at p. Thus $\mathrm{Tr}(pa_1p + pa_1) = \mathrm{Tr}(S_1(p)p) = 0$, that is, $-\mathrm{Tr}(pa_1) = \mathrm{Tr}(pa_1p) = \mathrm{Tr}(pa_1)$. This means that $\mathrm{Tr}(pa_1) = 0$ for all $p \in \mathcal{P}(\mathcal{H})$ with $\dim p(\mathcal{H}) = 1$. Consequently, $\mathrm{Tr}(xa_1) = 0$ for all $x \in \mathcal{F}(\mathcal{H})$, and by [26, Lemma 2.1] we have $a_1 = 0$. Therefore, $a = ia_2$. $\rhd$

4. The Proof of Theorem 2

Let $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ be a Banach symmetric ideal. We say that a bounded linear operator $T\colon \mathcal{C}_E^h \to \mathcal{C}_E^h$ has the property **(P)** if for any $a \in \mathcal{B}(\mathcal{H})^h$ there are operators $b \in \mathcal{B}(\mathcal{H})^h$ and $c \in \mathcal{B}(\mathcal{H})^h$ such that $T(i(bx - xb)) = i(aT(x) - T(x)a)$ and $T(i(ax - xa)) = i(cT(x) - T(x)c)$ for all $x \in \mathcal{C}_E^h$.

It is clear that a bounded linear bijection $T\colon \mathcal{C}_E^h \to \mathcal{C}_E^h$ has the property **(P)** if and only if T^{-1} has the property **(P)**.

Lemma 1. *Let $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ be a separable or a perfect Banach symmetric ideal other than $\mathcal{C}_2$, and let $V : \mathcal{C}_E^h \to \mathcal{C}_E^h$ be a surjective isometry. Then an isometry V has the property* **(P)**.

$\lhd$ If $a \in \mathcal{B}(\mathcal{H})^h$ then the linear operator $H : \mathcal{C}_E^h \to \mathcal{C}_E^h$ defined by $H(x) = i(xa - ax)$, $x \in \mathcal{C}_E^h$, is a skew-Hermitian operator. By the Proposition 1 the operator $V^{-1} \cdot H \cdot V$ is also skew-Hermitian. Using the Theorem 3 we obtain that there exists $b \in \mathcal{B}(\mathcal{H})^h$ such that $V^{-1} \cdot H \cdot V(x) = i(bx - xb)$, that is, $i(aV(x) - V(x)a) = V(i(bx - xb))$ for all $x \in \mathcal{C}_E^h$.

Similarly, $V \cdot H \cdot V^{-1}$ is a skew-Hermitian operator. Hence, there exists an operator $c \in \mathcal{B}(\mathcal{H})^h$ such that $V \cdot H \cdot V^{-1}(y) = i(cy - yc)$ for all $y \in \mathcal{C}_E^h$. If $V^{-1}(y) = x$, then $V(i(ax - xa)) = i(cV(x) - V(x)c)$ for all $x \in \mathcal{C}_E^h$. $\rhd$

Let $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ be a Banach symmetric ideal, $0 \neq x \in \mathcal{C}_E^h$, and let $Z(x) = \{x\}' \cap \mathcal{B}(\mathcal{H})^h = \{y \in \mathcal{B}(\mathcal{H})^h : xy = yx\}$. A non-zero operator $x \in \mathcal{C}_E^h$ is said to be a $\mathcal{C}_E^h$-maximal if $Z(x) = Z(y)$ for any $0 \neq y \in \mathcal{C}_E^h$ with $Z(x) \subset Z(y)$ (cf. [27, Definition 1.4]).

Lemma 2. *The following conditions are equivalent:*

(i) *$x \in \mathcal{C}_E^h$ is a $\mathcal{C}_E^h$-maximal operator;*

(ii) *$x = \alpha p$, where $0 \neq p \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$, $0 \neq \alpha \in \mathbb{R}$.*

$\lhd$ (i) $\Longrightarrow$ (ii). Since $x \in \mathcal{C}_E^h$, it follows that $x = \sum_{i=1}^t \lambda_i p_i$, $t \in \mathbb{N}$ or $t = \infty$ (the series converges with respect to the norm $\|\cdot\|_\infty$), where $0 \neq p_i \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$, $p_i p_j = 0$, $i \neq j$, $0 \neq \lambda_i \in \mathbb{R}$, for all $i, j = 1, \ldots, t$. If $y \in Z(x)$ then $y p_i = p_i y$ [28, Ch. 1, §4, p. 17], that is, $Z(x) \subset Z(p_i)$ for all $i = 1, \ldots, t$. Since, x is a $\mathcal{C}_E^h$−maximal operator, it follows that $Z(x) = Z(p_i)$, thus $Z(p_i) = Z(p_k)$ for all $i, k = 1, \ldots, t$.

Suppose that $t \geqslant 2$. As $Z(p_1) = Z(p_2)$, we have

$$\{p_1\}'' = \{p_2\}'' = \{\alpha \cdot p_2 + \beta \cdot (I - p_2) : \alpha, \beta \in \mathbb{C}\},$$

that is, $p_1 = \alpha_0 \cdot p_2 + \beta_0 \cdot (I - p_2)$ for some $\alpha_0, \beta_0 \in \mathbb{C}$. Consequently, $0 = p_1 p_2 = \alpha_0 \cdot p_2$, and $\alpha_0 = 0$. Therefore $p_1 = \beta_0 \cdot (I - p_2)$, which contradicts the inclusion $p_1 \in \mathcal{F}(\mathcal{H})$. Thus $t = 1$ and $x = \lambda_1 p_1$.

(ii) $\Longrightarrow$ (i). Let $x = \alpha p$, where $0 \neq p \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$, $0 \neq \alpha \in \mathbb{R}$. If $0 \neq y \in \mathcal{C}_E^h$ and $Z(x) \subset Z(y)$ then $Z(p) = Z(x) \subset Z(y)$, and $y \in \{y\}'' \subseteq \{p\}'' = \{\alpha \cdot p + \beta \cdot (I - p) : \alpha, \beta \in \mathbb{C}\}$, that is, $y = \alpha_0 \cdot p + \beta_0 \cdot (I - p)$ for some $\alpha_0, \beta_0 \in \mathbb{C}$. Since y is a compact operator, it follows that $\beta_0 = 0$, that is, $y = \alpha_0 \cdot p$ and $Z(x) = Z(y)$. $\rhd$

Lemma 3. *Let $T\colon \mathcal{C}_E^h \to \mathcal{C}_E^h$ be a bounded linear bijective operator with the property* **(P)**. *Then $T(x)$ is a $\mathcal{C}_E^h$-maximal operator for any $\mathcal{C}_E^h$-maximal operator $x \in \mathcal{C}_E^h$.*

$\lhd$ Suppose that $x \in \mathcal{C}_E^h$ is a $\mathcal{C}_E^h$-maximal operator, but $T(x)$ is not $\mathcal{C}_E^h$-maximal, that is, there exists $z \in \mathcal{C}_E^h$ such that $Z(T(x)) \subset Z(z)$ and $Z(T(x)) \neq Z(z)$. Since T is a bijection, $z = T(y)$ for some $y \in \mathcal{C}_E^h$. Hence, $Z(T(x)) \subset Z(T(y))$ and $Z(T(x)) \neq Z(T(y))$.

We show that $Z(x) \subset Z(y)$. Since an operator T has property **(P)**, it follows that for $a \in Z(x)$ there exists $b \in \mathcal{B}(\mathcal{H})^h$ such that

$$T(i(ac - ca)) = i(bT(c) - T(c)b) \tag{6}$$

for all $c \in \mathcal{C}_E^h$. Using equations (6) and $T(i(ax - xa)) = T(0) = 0$, and the injectivity of the mapping T, we obtain that $bT(x) = T(x)b$, that is, $b \in Z(T(x)) \subset Z(T(y))$. Consequently, $T(i(ay - ya)) = 0$ and $ay - ya = 0$ (see (6)), i.e. $a \in Z(y)$. Therefore $Z(x) \subset Z(y)$, and by the $\mathcal{C}_E^h$-maximality of the operator x we obtain that $Z(x) = Z(y)$.

Since $Z(T(x)) \neq Z(T(y))$, there exists an operator $a \in Z(T(y))$ such that $a \notin Z(T(x))$. By the property **(P)** we can choose $b \in \mathcal{B}(\mathcal{H})^h$ such that

$$T(i(bc - cb)) = i(aT(c) - T(c)a) \tag{7}$$

for all $c \in \mathcal{C}_E^h$. Thus $T(i(by - yb)) = 0$, and $by - yb = 0$, that is, $b \in Z(y)$. Besides, $aT(x) - T(x)a \neq 0$ implies that $bx - xb \neq 0$ (see (7)), that is, $b \notin Z(x)$, which contradicts the equality $Z(x) = Z(y)$. $\rhd$

Lemma 4. *Let $V : \mathcal{C}_E^h \to \mathcal{C}_E^h$ be a surjective linear isometry with the property* **(P)**. *Then for every $p \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H}))$ there exists $q_p \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ such that $V(p) = q_p$ or $V(p) = -q_p$.*

$\lhd$ Let $0 \neq p_i \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$, $i = 1, 2$, $p_1 p_2 = 0$. Since p_i is a $\mathcal{C}_E^h$-maximal operator (Lemma 2), it follows that $V(p_i)$ is a $\mathcal{C}_E^h$-maximal operator too, $i = 1, 2$ (Lemma 3). Consequently, there exist $0 \neq q_i \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$, and $0 \neq \alpha_i \in \mathbb{R}$ such that $V(p_i) = \alpha_i q_i$, $i = 1, 2$ (Lemma 2). Since $p_1 p_2 = 0$, it follows that $(p_1 + p_2) \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ and $V(p_1 + p_2) = \alpha_3 q_3$ for some non-zero projection $q_3 \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ and $0 \neq \alpha_3 \in \mathbb{R}$ (Lemma 2). Therefore $\frac{\alpha_1}{\alpha_3} q_1 + \frac{\alpha_2}{\alpha_3} q_2 = q_3$. By [29] there are four possibilities:

(i) $\frac{\alpha_1}{\alpha_3} = 1$, $\frac{\alpha_2}{\alpha_3} = 1$ if $q_1 q_2 = 0$;

(ii) $\frac{\alpha_1}{\alpha_3} = 1,\ \frac{\alpha_2}{\alpha_3} = -1$ if $q_1q_2 = q_2$;

(iii) $\frac{\alpha_1}{\alpha_3} = -1,\ \frac{\alpha_2}{\alpha_3} = 1$ and $q_1q_2 = q_1$;

(iv) $\frac{\alpha_1}{\alpha_3} + \frac{\alpha_2}{\alpha_3} = 1$ and $(q_1 - q_2)^2 = 0$ if $q_1q_2 \neq q_2q_1$.

The case (iv) is impossible because $\|(q_1 - q_2)\|_\infty^2 = \|(q_1 - q_2)^2\|_\infty = 0$, which contradicts the bijectivity of V. In other cases we have $V(p_2) = \alpha q_2$ or $V(p_2) = -\alpha q_2$, where $\alpha = \alpha_1$. Consequently, $V(p) = \alpha q_p$ or $V(p) = -\alpha q_p$ for an arbitrary $0 \neq p \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$, $p_1p = 0$.

Let now $0 \neq e \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ and $p_1e \neq 0$. Then there exists a non-zero finite dimensional projection f, such that $p_1f = 0$ and $ef = 0$. According to above, we have $\alpha_1q_1 = V(p_1) = \alpha_f q_{p_1}$ or $V(p_1) = -\alpha_f q_{p_1}$ and $V(e) = \alpha_f q_e$ or $V(e) = -\alpha_f q_e$ for some non-zero finite dimensional projections q_f, q_e and for non-zero real number α_f. Consequently, $q_1 = q_{p_1}$ and $\alpha_1 = \pm\alpha_f$. In particular, $V(e) = \alpha_1 q_e$ or $V(f) = -\alpha_1 q_e$.

If $e \in \mathcal{P}(\mathcal{H})$ and $\dim e(\mathcal{H}) = 1$, then $1 = \|e\|_{\mathcal{C}_E} = \|V(e)\|_{\mathcal{C}_E} = |\alpha|\|q_e\|_{\mathcal{C}_E} \geqslant |\alpha|\|q_e\|_\infty = |\alpha|$, that is, $|\alpha| \leqslant 1$.

Replacing the isometry V with V^{-1}, we get that $V^{-1}(p) = \beta r_p$ or $V^{-1}(p) = -\beta r_p$ for arbitrary $p \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$, where $r_p \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ and β does not depend on the projection p. In particular, if $e \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ and $\dim e(\mathcal{H}) = 1$, then $1 = \|e\|_{\mathcal{C}_E} = \|V^{-1}(e)\|_{\mathcal{C}_E} = |\beta|\|r_e\|_{\mathcal{C}_E} \geqslant |\beta|\|r_e\|_\infty = |\beta|$, i. e. $|\beta| \leqslant 1$.

Therefore, for $p \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ we obtain that $V(p) = \pm\alpha q_p$, and $p = V^{-1}(\pm\alpha q) = \pm(\alpha\beta) r_q$. Hence $|\alpha\beta| = 1$ and $|\alpha| = 1$. $\rhd$

We say that the norm $\|\cdot\|_{\mathcal{C}_E}$ is *a not uniform* if $\|p\|_{\mathcal{C}_E} > 1$ for any $p \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ with $\dim p(\mathcal{H}) > 1$.

Lemma 5. *Let $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ be a Banach symmetric ideal with not uniform norm, and let $V\colon \mathcal{C}_E^h \to \mathcal{C}_E^h$ be a surjective isometry with the property* **(P)***. Then $V(p)$ or $(-V)(p)$ is one dimensional projection for any one dimensional projection p.*

$\lhd$ Let $p \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$, $\dim p(\mathcal{H}) = 1$. By Lemma 4 we have that there exists $q_p \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ such that $V(p) = q_p$ or $V(p) = -q_p$. If $\dim q_p(\mathcal{H}) > 1$ then $1 = \|p\|_{\mathcal{C}_E} = \|V(p)\|_{\mathcal{C}_E} = \|q_p\|_{\mathcal{C}_E} > 1$, what is wrong. $\rhd$

Lemma 6. *Let $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ and an isometry V be the same as in the conditions of the Lemma* 5*. Then*

$$V(\mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})) \subseteq \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$$

or

$$(-V)(\mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})) \subseteq \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H}).$$

$\lhd$ Let $\mathcal{P}_1(\mathcal{H}) = \{p \in \mathcal{P}(\mathcal{H}) : \dim p(\mathcal{H}) = 1\}$, and let $p, e \in \mathcal{P}_1(\mathcal{H})$. By Lemma 5, there exists $q, r \in \mathcal{P}_1(\mathcal{H})$ such that $V(p) = q$ or $V(p) = -q$ and $V(e) = r$ or $V(e) = -r$. If $V(p) = q$, $V(e) = -r$ then $q - r = V(p + q) = \pm f$ for some $0 \neq f \in \mathcal{P}(\mathcal{H})$ (see Lemma 4), which is not possible because $q, r \in \mathcal{P}_1(\mathcal{H})$. Similarly, the case $V(p) = -q$, $V(e) = r$ is also impossible. Consequently, $V(\mathcal{P}_1(\mathcal{H})) \subseteq \mathcal{P}_1(\mathcal{H})$ or $(-V)(\mathcal{P}_1(\mathcal{H})) \subseteq \mathcal{P}_1(\mathcal{H})$. Since each projector $p \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ is the final sum of one-dimensional projectors, it follows that $V(\mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})) \subseteq \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ or $(-V)(\mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})) \subseteq \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$. $\rhd$

Corollary 1. *Let $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ and V be the same as in the conditions of the Lemma* 5*. Then*

(i) *$V(p)V(e) = 0$ for any $p, e \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ with $pe = 0$;*

(ii) *V is a bijection from $\mathcal{P}_1(\mathcal{H})$ onto $\mathcal{P}_1(\mathcal{H})$.*

◁ (i). By Lemma 5, $V(p) = q_p \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ for all $p \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ or $V(p) = -q_p \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ for all $p \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$. In the first case for $p, e \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ with $pe = 0$, we have that $V(p) = q_p$, $V(e) = q_p$, $q_r + q_e = V(r + e) = q_{r+e}$, that is, $V(r)V(e) = q_r q_e = 0$.

The case $V(p) = -q_p \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ for all $p \in \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ is proved similarly.

Item (ii) directly follows from Lemma 5. ▷

◁ Proof of Theorem 2. We suppose that $V(\mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})) \subseteq \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ (the case $(-V)(\mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})) \subseteq \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ is proved by replacing V with $(-V)$). Let

$$x = \sum_{n=1}^{k} \lambda_n p_n \in \mathcal{F}(\mathcal{H})^h, \quad p_n \in \mathcal{P}_1(\mathcal{H}),\ p_n p_m = 0,$$
$$n \neq m,\ \ 0 \neq \lambda_n \in \mathbb{R},\ \ n, m = 1, \ldots, k.$$

Since $V(p_n) \cdot V(p_m) = 0,\ n \neq m$ (Corollary 1 (i)), it follows that

$$V(x^2) = V\left(\sum_{n=1}^{k} \lambda_n^2 p_n\right) = \sum_{n=1}^{k} \lambda_n^2 V(p_n) = V(x)^2$$

and

$$\mathrm{Tr}(V(x)) = \sum_{n=1}^{k} \lambda_n \mathrm{Tr}(V(p_n)) = \sum_{n=1}^{k} \lambda_n = \mathrm{Tr}(x).$$

If $p, e, q, f \in \mathcal{P}_1(\mathcal{H})$, $V(p) = q$, $V(e) = f$, then

$$2\,\mathrm{Tr}(pe) = \mathrm{Tr}(pe) + \mathrm{Tr}(ep) = \mathrm{Tr}((p+e)^2 - p - e)$$
$$= \mathrm{Tr}(V((p+e)^2)) - 2 = \mathrm{Tr}(V(p+e))^2 - 2 = \mathrm{Tr}((q+f)^2) - 2 = 2\mathrm{Tr}(qf).$$

Consequently, $\mathrm{Tr}(pe) = \mathrm{Tr}(V(p)V(e))$ for all $p, e \in \mathcal{P}_1(H)$. By [30, Ch. 3, § 3.2, Theorem 3.2.8] we obtain that there exists an unitary or anti-unitary operator u such that $V(p) = upu^*$ for all $p \in \mathcal{P}_1(H)$. Thus $V(x) = u^* x u$ for all $x \in \mathcal{F}(H)^h$.

If $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ is a separable space then $\mathcal{F}(H)^h$ is dense in $(\mathcal{C}_E^h, \|\cdot\|_{\mathcal{C}_E})$. Consequently, $V(x) = u^* x u$ (respectively, $V(x) = -uxu^*$) for all $x \in \mathcal{C}_E^h$.

If $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ is a perfect Banach symmetric ideal, then V is $\sigma(\mathcal{C}_E, \mathcal{C}_E^\times)$-continuous (see proof of Step 4 in Theorem 4). Since $\mathcal{F}(H)^h$ is $\sigma(\mathcal{C}_E, \mathcal{C}_E^\times)$-dense in $(\mathcal{C}_E^h, \|\cdot\|_{\mathcal{C}_E})$, it follows that $V(x) = u^* x u$ (respectively, $V(x) = -uxu^*$) for all $x \in \mathcal{C}_E^h$.

In the case $(-V)(\mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})) \subseteq \mathcal{P}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H})$ we get that $V(x) = -uxu^*$ for all $x \in \mathcal{C}_E^h$. ▷

References

1. *Banach, S. Theorie des Operations Lineaires*, Warsaw, 1932.
2. *Lamperti, J.* On the Isometries of Some Function Spaces, *Pacific Journal of Mathematics*, 1958, vol. 8, no. 3, pp. 459–466. DOI: 10.2140/pjm.1958.8.459.
3. *Lumer, G.* On the Isometries of Reflexive Orlicz Spaces, *Annales de l'Institut Fourier*, 1963, vol. 13, no. 1, p. 99–109. DOI: 10.5802/aif.132.
4. *Zaidenberg, M. G.* On Isometric Classification of Symmetric Spaces, *Doklady Akademii Nauk SSSR*, 1977, vol. 234, pp. 283–286 (in Russian).
5. *Zaidenberg, M. G.* A Representation of Isometries of Functional Spaces, *Zhurnal Matematicheskoi Fiziki, Analiza, Geometrii* [Journal of Mathematical Physics, Analysis, Geometry], 1997, vol. 4, no. 3, pp. 339–347.
6. *Kalton, N. J. and Randrianantoanina, B.* Surjective Isometries on Rearrangment Invariant Spaces, *The Quarterly Journal of Mathematics*, 1994, vol. 45, no. 3, pp. 301–327. DOI: 10.1093/qmath/45.3.301.
7. *Braverman, M. Sh. and Semenov, E. M.* Isometries on Symmetric Spaces, *Doklady Akademii Nauk SSSR*, 1974, vol. 217, pp. 257–259 (in Russian).

8. *Braverman, M. Sh. and Semenov, E. M.* Isometries on Symmetric Spaces, *Trudy NII Matem. Voronezh. Gos. Univ.*, 1975, vol. 17, pp. 7–18 (in Russian).
9. *Arazy, J.* Isometries on Complex Symmetric Sequence Spaces, *Mathematische Zeitschrift*, 1985, vol. 188, no. 3, pp. 427–431. DOI: 10.1007/BF01159187.
10. *Aminov, B. R. and Chilin, V. I.* Isometries and Hermitian Operators on Complex Symmetric Sequence Spaces, *Siberian Advances in Mathematics*, 2017, vol. 27, no. 4, pp. 239–252. DOI: 10.3103/S105513 4417040022.
11. *Arazy, J.* The Isometries of $\mathcal{C}_p$, *Israel Journal of Mathematics*, 1975, vol. 22, no. 3–4, pp. 247–256. DOI: 10.1007/BF02761592.
12. *Fleming, R. J. and Jamison, J. E. Isometries on Banach Spaces: Vector-Valued Function Spaces*, Chapman-Hall/CRC, 2008.
13. *Sourour, A.* Isometries of Norm Ideals of Compact Operators, *Journal of Functional Analysis*, 1981, vol. 43, no. 1, pp. 69–77. DOI: 10.1016/0022-1236(81)90038-0.
14. *Aminov, B. R. and Chilin, V. I.* Isometries of Perfect Norm Ideals of Compact Operators, Studia Math., 2018, vol. 241(1), pp. 87–99. DOI: 10.4064/sm170306-19-4.
15. *Garling, D. J. H.* On Ideals of Operators in Hilbert Space, *Proceedings of the London Mathematical Society*, 1967, vol. 17, no. 1, 115–138. DOI: 10.1112/plms/s3-17.1.115.
16. *Nagy, G.* Isometries of the Spaces of Self-Adjoint Traceless Operators, *Linear Algebra and its Applications*, 2015, vol. 484, pp. 1–12. DOI: 10.1016/j.laa.2015.06.026.
17. *Bennett, C. and Sharpley, R. Interpolation of Operators*, Academic Press Inc., 1988.
18. *Simon, B. Trace Ideals and Their Applications*, Mathematical Surveys and Monographs, vol. 120, 2nd edition, Providence, R. I., Amer. Math. Soc., 2005.
19. *Kalton, N. J. and Sukochev, F. A.* Symmetric Norms and Spaces of Operators, *Journal für die Reine und Angewandte Mathematik*, 2008, vol. 621, pp. 81–121. DOI: 10.1515/CRELLE.2008.059.
20. *Lord, S., Sukochev, F. and Zanin, D. Singular Traces. Theory and Applications*, Berlin/Boston, Walter de Gruyter GmbH, 2013.
21. *Gohberg, I. C. and Krein, M. G. Introduction to the Theory of Linear Nonselfadjoint Operators*, Translations of Mathematical Monographs, vol. 18, Providence, R. I., Amer. Math. Soc., 1969.
22. *Krein, M. G., Petunin, Ju. I. and Semenov, E. M. Interpolation of Linear Operators*, Translations of Mathematical Monographs, vol. 54, Providence, R. I., Amer. Math. Soc., 1982.
23. *Lindenstrauss, J. and Tzafriri, L. Classical Banach Spaces*, Berlin and N. Y., Springer-Verlag, 1996.
24. *Dodds, P. G., Dodds, T. K. and Pagter, B.* Noncommutative Köthe Duality, *Transactions of the American Mathematical Society*, 1993, vol. 339, no. 2, pp. 717–750. DOI: 10.2307/2154295.
25. *Dragomir, S. S. Semi-Inner Products and Applications*, N. Y., Hauppauge, Nova Science Publishers Inc., 2004.
26. *Ayupov, Sh. and Kudaybergenov, K.* 2-Local Derivations and Automorphisms on $B(H)$, *Journal of Mathematical Analysis and Applications*, 2012, vol. 395, no. 1, pp. 15–18. DOI: 10.1016/j.jmaa. 2012.04.064.
27. *Dolinar, G., Guterman, A., Kuzma, B. and Oblak, P.* Extremal Matrix Centralizers, *Linear Algebra and its Applications*, 2013, vol. 438, no. 7, pp. 2904–2910. DOI: 10.1016/j.laa.2012.12.010.
28. *Schatten, R. Norm Ideals of Completely Continuous Operators*, Berlin and N. Y., Springer-Verlag, 1960.
29. *Baksalary, J. K. and Baksalary, O. M.* Idempotency of Linear Combinations of Two Idempotent Matrices, *Linear Algebra and its Applications*, 2000, vol. 321, no. 1–3, pp. 3–7. DOI: 10.1016/S0024-3795(00)00225-1.
30. *Bratteli, O. and Robinson, D. W. Operator Algebras and Quantum Statistical Mechaniks*, N. Y.–Heidelber–Berlin, Springer-Verlag, 1979.

Received 13 June, 2019

Behzod R. Aminov
National University of Uzbekistan,
Vuzgorodok, Tashkent 100174, Uzbekistan,
Teacher
E-mail: `aminovbehzod@gmail.com`

Vladimir I. Chilin
National University of Uzbekistan,
Vuzgorodok, Tashkent 100174, Uzbekistan
Professor
E-mail: `vladimirchil@gmail.com`, `chilin@ucd.uz`

Владикавказский математический журнал
2019, Том 21, Выпуск 4, С. 11–24

ИЗОМЕТРИИ ДЕЙСТВИТЕЛЬНЫХ ПОДПРОСТРАНСТВ САМОСОПРЯЖЕННЫХ ОПЕРАТОРОВ В БАНАХОВЫХ СИММЕТРИЧНЫХ ИДЕАЛАХ

Аминов Б. Р.[1], Чилин В. И.[1]

[1] Национальный университет Узбекистана,
Узбекистан, 100174, Ташкент, Вузгородок
E-mail: aminovbehzod@gmail.com,
vladimirchil@gmail.com, chilin@ucd.uz

Аннотация. Пусть $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ банахов симметричный идеал компактных операторов, действующих в комплексном сепарабельном бесконечномерном гильбертовом $\mathcal{H}$. Пусть $\mathcal{C}_E^h = \{x \in \mathcal{C}_E : x = x^*\}$ действительное банахово подпространство самосопряженных операторов в $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$. Доказывается, что в случае, когда $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ есть сепарабельный или совершенный банахов симметричный идеал $(\mathcal{C}_E \neq \mathcal{C}_2)$ каждый косоэрмитовый оператор $H : \mathcal{C}_E^h \to \mathcal{C}_E^h$ имеет следующий вид $H(x) = i(xa - ax)$ для некоторого $a^* = a \in \mathcal{B}(\mathcal{H})$ и для всех $x \in \mathcal{C}_E^h$. Используя это описание косоэрмитовых операторов мы получаем следующий общий вид сюръективных линейных изометрий $V : \mathcal{C}_E^h \to \mathcal{C}_E^h$: Пусть $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ сепарабельный или совершенный банахов симметричный идеал с неравномерной нормой, т. е. $\|p\|_{\mathcal{C}_E} > 1$ для всех конечномерных проекторов $p \in \mathcal{C}_E$ с $\dim p(\mathcal{H}) > 1$, пусть $\mathcal{C}_E \neq \mathcal{C}_2$, и пусть $V : \mathcal{C}_E^h \to \mathcal{C}_E^h$ сюръективная линейная изометрия. Тогда существует такой унитарный или антиунитарный оператор u на $\mathcal{H}$, что $V(x) = uxu^*$ или $V(x) = -uxu^*$ для всех $x \in \mathcal{C}_E^h$.

Ключевые слова: симметричный идеал компактных операторов, косоэрмитовый оператор, изометрия.

Mathematical Subject Classification (2010): 46L52, 46B04.

Образец цитирования: *Aminov B. R., Chilin V. I.* Isometries of Real Subspaces of Self-Adjoint Operators in Banach Symmetric Ideals // Владикавк. мат. журн.—2019.—Т. 21, № 4.—С. 11–24 (in English). DOI: 10.23671/VNC.2019.21.44607.