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USING HOMOLOGICAL METHODS ON THE BASE OF ITERATED SPECTRA IN FUNCTIONAL ANALYSIS

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We introduce new concepts of functional analysis: Hausdorff spectrum and Hausdorff limit or H-limit of Hausdorff spectrum of locally convex spaces. Particular cases of regular H-limit are projective and inductive limits of separated locally convex spaces. The class of H-spaces contains Fréchet spaces and is stable under forming countable inductive and projective limits, closed subspaces and quotient spaces. Moreover, for H-space an unproved variant of the closed graph theorem holds true. Homological methods are used for proving of theorems of vanishing at zero for first derivative of Hausdorff limit functor: $\mathrm{Haus}^1(\boldsymbol{X}) = 0$.

Key words: topology, spectrum, closed graph theorem, differential equation, homological methods, category.

Introduction

The study which was carried out in [1–2] of the derivatives of the projective limit functor acting from the category of countable inverse spectra with values in the category of locally convex spaces made it possible to resolve universally homomorphism questions about a given mapping in terms of the exactness of a certain complex in the abelian category of vector spaces. Later in [3] a broad generalization of the concepts of direct and inverse spectra of objects of an additive semiabelian category $\boldsymbol{G}$ (in the sense V. P. Palamodov) was introduced: the concept of a Hausdorff spectrum, analogous to the δ_s-operation in descriptive set theory. This idea is characteristic even for algebraic topology, general algebra, category theory and the theory of generalized functions. The construction of Hausdorff spectra $\boldsymbol{X} = \{X_s, \boldsymbol{F}, h_{s's}\}$ is achieved by successive standard extension of a small category of indices Ω. The category $\boldsymbol{H}$ of Hausdorff spectra turns out to be additive and semiabelian under a suitable definition of spectral mapping. In particular, $\boldsymbol{H}$ contains V. P. Palamodov's category of countable inverse spectra with values in the category TLG of locally convex spaces [1]. The H-limit of a Hausdorff spectrum in the category TLG generalizes the concepts of projective and inductive limits and is defined by the action of the functor $\mathrm{Haus} : \ \boldsymbol{H} \to TLC$. The class of H-spaces is defined by the action of the functor Haus on the countable Hausdorff spectra over the category of Banach spaces; the closed graph theorem holds for its objects [8] and it contains the category of Fréchet spaces and the categories of spaces due to De Wilde [7], D. A. Rajkov [5] and Suslin [6]. The H-limit of a Hausdorff spectrum of H-spaces is an H-space [7]. There are many injective objects in the category $\boldsymbol{H}$ and the right derivatives Haus^i $(i = 1, 2, \ldots)$ are defined, while the “algebraic” functor $\mathrm{Haus} : \boldsymbol{H}(L) \to L$ over the abelian category L of vector spaces (over $\mathbb{R}$ or $\mathbb{C}$) has injective type, that is if $0 \to \boldsymbol{X} \to \boldsymbol{Y} \to \boldsymbol{Z}$ is an exact sequence of mappings of Hausdorff spectra with values in L, then the limit sequence $0 \to \mathrm{Haus}(\boldsymbol{X}) \to$

Haus($\boldsymbol{Y}$) $\to$ Haus($\boldsymbol{Z}$) is exact or acyclic in the terminology of V. P. Palamodov [2]. In particular, regularity of the Hausdorff spectrum $\boldsymbol{X}$ of the nonseparated parts of $\boldsymbol{Y}$ guarantees the exactness of the functor Haus : $\boldsymbol{H}(TLC) \to TLC$ and the condition of vanishing at zero: $\mathrm{Haus}^1(\boldsymbol{X}) = 0$. The classical results of Malgrange and Ehrenpreis on the solvability of the unhomogeneous equation $p(D)D' = D'$, where $p(D)$ is a linear differential operator with constant coefficients in $\mathbb{R}^n$ and $D' = D'(S)$ is the space of generalized functions on a convex domain $S \subset \mathbb{R}^n$, can be extended to the case of sets S which are not necessarily open or closed. Analogous theorems for Fréchet spaces were first proved by V. P. Palamodov [1–2].

1. We recall certain definitions and theorems which are used in this chapter and which were brought into the discussion in [3–6]. Let Ω be a small category. By a *directed class* in the category we mean a subcategory satisfying the following properties:

(i) no more than one morphism is defined between any two objects;

(ii) for any objects a, b there exists an object c such that there exist $a \to c$ and $b \to c$.

Let A be some category and s denotes the object of a category A (if $Q \in \Omega$ and $a, b \in Q$ we will denote the corresponding morphisms of category Ω by $a \xrightarrow{Q} b$). We shall call the category B with objects S, where S is a subcategory of A, a *standard extension of the category* A if the following conditions are satisfied:

1°. A is a complete subcategory of B;

2°. The morphism $\omega_{SS'} : S' \to S$ of the category B is defined by the collection of morphisms $\omega_{ss'} : s' \to s$ ($s' \xrightarrow{\omega_{ss'}} s$) of the category A such that

(a) for every $s' \in S'$ there exists $s \in S$ such that $s' \xrightarrow{\omega_{ss'}} s$;

(b) if $s' \xrightarrow{\omega_{ss'}} s$, $p' \xrightarrow{\omega_{pp'}} p$, $s \xrightarrow{\omega^S_{ps}} p$, then there exists a morphism $s' \xrightarrow{\omega^{S'}_{ps}} p'$ and the following diagram is commutative:

$$\begin{array}{ccc} s & \xrightarrow{S} & p \\ {\scriptstyle \omega_{ss'}}\big\uparrow & & \big\uparrow{\scriptstyle \omega_{pp'}} \\ s' & \xrightarrow{S'} & p' \end{array}$$

We will establish the successive standard extensions of categories

$$\Omega(s) \subset \boldsymbol{B}(T) \subset \Sigma(F) \to \Sigma^0(F) \subset \boldsymbol{D}(\boldsymbol{F}),$$

where $T \subset \Omega$ denotes directed classes of objects $s \in \Omega$, coincides as object of category $\boldsymbol{B}$; F, $F \in \boldsymbol{B}$, denote filter bases of sets $T \in \boldsymbol{B}$, considered as objects of category Σ, and $\boldsymbol{F}$, $\boldsymbol{F} \subset \Sigma$, denote directed classes of objects $F \in \Sigma$ of the dual category Σ^0, considered as objects of category $\boldsymbol{D}$. We shall say that such classes $\boldsymbol{F}$ are *admissible* for Ω; put $|F| = \bigcup_{T \in F} T$, $|\boldsymbol{F}| = \bigcup_{F \in \boldsymbol{F}} |F|$, so that $|F| \subset \Omega$ and $|\boldsymbol{F}| \subset \Omega$. The most characteristic constructions connected with Hausdorff spectra use in the role of the small category $\Omega = \mathrm{Ord}\, I$, where I is a partially ordered set of indices, considered as category.

Example 1 (standard extension of the category A). Let G and A be categories, $T(F)$ the category of covariant functors $F : G \to A$ with functorial morphism $\Phi : F_1 \to F_2$ defined by the rule [2] which assigns to each object $g \in G$ a morphism $\Phi(g) : F_1(g) \to F_2(g)$ of the category A such that for any morphism $\omega : g \to h$ of the category G the following diagram is commutative

$$\begin{array}{ccc} F_1(h) & \xrightarrow{\Phi(h)} & F_2(h) \\ {\scriptstyle F_1(\omega)}\big\uparrow & & \big\uparrow{\scriptstyle F_2(\omega)} \\ F_1(g) & \xrightarrow{\Phi(g)} & F_2(g) \end{array}$$

It is clear that each object $s \in A$ generates a covariant functor $F_s : g \in G \mapsto s \in A$ such that $A \subset T$. Moreover, A is a complete subcategory of T.

We will show that T provides a standard extension of the category A (by means of the category G). Let $F \in T$ and $S \subset A$ be such that $S = \cup_{g \in G} F(g)$ and for $s', s \in S$ the set of morphisms $\mathrm{Hom}(s', s) = \cup_\omega F(\omega)$, where $\omega : g \to h$ and $s' = F(q)$, $s = F(h)$. Therefore the category B is defined, where S is a subcategory of A and the morphisms $\omega_{SS'} : S' \to S$ of the category B are generated by the collection of functorial morphisms $\Phi : F' \to F$, where $F' \in T$ generates S', while F generates S according to the method indicated above.

If we take such a functorial morphism $\Phi : F' \to F$, then the morphisms $\Phi(g) : F'(g) \to F(g)$ $(g \in G)$ of the category A form a collection of morphisms $\omega_{ss'} : s' \to s$ $(s' = F'(g)$, $s = F(g))$ such that (a) is satisfied. Condition (b) follows from consideration of the definition of the functorial morphism.

Thus, B is a standard extension of the category A. If $G = \mathrm{Ord}\, I$, where I is a linearly ordered set, then $T = B(S)$.

Example 2 (Palamodov [1]). The categories of direct and inverse spectra over a semiabelian category K are standard extensions of the category K.

Example 3 (construction of an admissible class for Ω). Let T be a separated topological space and Ω a countable set. We shall call a set $A \subset T$ an s-set if

$$A = \bigcup_{B \in \mathscr{K}} \bigcap_{t \in B} T_t \,,$$

where T_t $(t \in \Omega)$ is a subset of T and $\boldsymbol{K}$ is the family of subsets B of the set Ω such that

(a) for each $B \in \boldsymbol{K}$ the set $T_B = \cap_{t \in B} T_t$ is compact in T,

(b) the sets T_B $(B \in \boldsymbol{K})$ form a fundamental system of compact subsets of A.

Proposition 1. *Every separable metric space is an s-set.*

Proposition 2. *Let A be a subset of the finite-dimensional space $\mathbb{R}^n$. Then A is an s-set and moreover*

$$A = \bigcup_{B \in \mathscr{K}} \bigcap_{t \in B} T_t, \tag{1}$$

where the T_t are compact subsets of $\mathbb{R}^n$.

Thus, s-sets are a generalization on the one hand of compact spaces (and locally compact spaces which are countable at infinity) and on the other of separable metric spaces. However, s-sets will be of interest to us in connection with the possibility of constructing the associated functor of a simple Hausdorff spectrum.

Let A be some s-set, so that

$$A = \bigcup_{B \in \mathscr{K}} \bigcap_{t \in B} T_t,$$

where $T_t \subset T$, $B \subset \Omega$. We may assume without loss of generality that the family Q of subsets T_t $(t \in \Omega)$ is closed with respect to finite intersections and unions (that is, there exist corresponding surjections $\Phi_s, \Psi_s : d(\Omega) \to \Omega$, where $d(\Omega)$ is the set of finite subsets of Ω).

The set Ω will be partially ordered if we put $t' \leqslant t$ whenever $T_t \subset T_{t'}$; let $\boldsymbol{G} = \mathrm{Ord}\, Q$. Further, we may assume that each set $B \in \boldsymbol{K}$ is directed in $(\Omega, \leqslant)$. Let I be the factor set of all possible complexes $s = [t_1, t_2, \ldots, t_n]$, where $t_i \in |\boldsymbol{K}|$, $t_i = pr_i s$ $(i = 1, 2, \ldots, n$, $n \in \mathbb{N})$, with respect to the equivalence relation on the set of ordered n-tuples of elements of $|\boldsymbol{K}| : (t_1, t_2, \ldots, t_n) \sim (t'_1, t'_2, \ldots, t'_n)$ if and only if $\{t_1, t_2, \ldots, t_n\} = \{t'_1, t'_2, \ldots, t'_n\}$. The set I becomes partially ordered if we put $s' \leqslant s$, where $s = [t_1, t_2, \ldots, t_n]$, $s' = [t'_1, t'_2, \ldots, t'_n]$, whenever for each t_i there exists t'_j such that $t'_j \leqslant t_i$; let $\Omega = \mathrm{Ord}\, I$.

By continuing the construction following the method of transformation of indices we will construct an admissible class $\boldsymbol{F}$ for Ω. For each $s = [t_1, t_2, \ldots, t_n] \in |\boldsymbol{F}|$ the subset $R_s = \bigcup_{i=1}^n T_{t_i}$ is defined and moreover if $s' \leqslant s$ then $R_s \subset R_{s'}$. Thus a contravariant functor of the simple Hausdorff spectrum $H(A) : |\boldsymbol{F}| \to \boldsymbol{G}$ is defined and moreover

$$A = \bigcup_{F \in \boldsymbol{F}} \bigcap_{s \in F} R_s. \tag{2}$$

It is an essential point that I is a countable set and the family $\left\{\bigcap_F R_s\right\}$ is a fundamental system of nonempty compact subsets of A.

Let $\boldsymbol{G}$ be some category. We shall call a covariant functor $H_F : \Omega \to \boldsymbol{G}$ a *Hausdorff spectrum functor* if $\Omega = |\boldsymbol{F}|$ for some admissible class $\boldsymbol{F} \in \boldsymbol{D}$. If $\boldsymbol{F} = |\boldsymbol{F}|$ then H_F is a functor of the direct spectrum, while if $\boldsymbol{F} = \{|\boldsymbol{F}|\}$ (that is, $\boldsymbol{F}$ consists of a single element $|F| = |\boldsymbol{F}|$) then H_F is a functor of the inverse spectrum.

If $\boldsymbol{F}$ is an admissible class for Ω and the functor

$$h_{\mathscr{F}} : \begin{cases} |\mathscr{F}| \to \mathscr{G}, \\ s \mapsto X_s, \\ \left(s' \xrightarrow{\omega_{ss'}} s\right) \mapsto \left(X_s \to X_{s'}\right), \\ \left(F' \xrightarrow{\omega_{FF'}} F\right) \mapsto \left((X_s)_{s \in |F|} \to (X_{s'})_{s' \in |F'|}\right) \end{cases}$$

is injective on objects and morphisms (in the set-theoretic sense), then there exists a directed class

$$\left((X_s)_{s \in |F|}, q_{FF'}\right)_{F,F'} \in \boldsymbol{F}$$

of classes $(X_s,\, h_{s's})_{s,s' \in |F|}$ ($F \in |\boldsymbol{F}|$) which are directed in the dual category $\boldsymbol{G}^0$ and which satisfy the following conditions.

1°. The morphism $X_s \xrightarrow{h_{s's}} X_{s'}$ is chosen and fixed if and only if the morphism $s' \xrightarrow{\omega_{ss'}} s$ is chosen and then $h_{s's} : X_s \to X_{s'}$ is the only morphism.

2°. The diagram

$$\begin{tikzcd} X_s \arrow[r, "h_{s''s}"] \arrow[d, "h_{s's}"'] & X_{s''} \arrow[d, "h_{s's''}"] \\ X_{s'} \arrow[r, equal] & X_{s'} \end{tikzcd}$$

is commutative for all $s'' \xrightarrow{\omega_{s's''}} s' \xrightarrow{\omega_{ss'}} s$.

3°. If $(X_s)_{s \in |F|} \xrightarrow{q_{F'F}} (X_{s'})_{s' \in |F'|}$, then for each $X_{s'}$ ($s' \in |F'|$) there exists a unique morphism $h_{s's} : X_s \to X_s$ ($s \in |F|$). The collection of morphisms $h_{s's}$ ($s' \in |F'|$) defines the morphism $q_{F'F}$ so that we shall write $q_{F'F} = (h_{s's})_{F'F}$. Each set $F \in \boldsymbol{F}$ is a filter base of subsets $T \subset |F|$ and moreover for each $T \in F$ the class $(X_s, h_{s's})_T$ is directed in the category $\boldsymbol{G}^0$.

Definition 1. We shall call a class $(X_s, h_{s's})_{s,s' \in |F|}$ satisfying conditions 1°–3° a *Hausdorff spectrum over the category* $\boldsymbol{G}$ and we shall denote it by $\{X_s, \boldsymbol{F}, h_{s's}\}$.

The direct and inverse spectra of a family of objects are particular cases of Hausdorff spectra: it suffices to put $\boldsymbol{F} = |\boldsymbol{F}|$, $h_{s's} = q_{F'F}$ in the direct case and $\boldsymbol{F} = \{|\boldsymbol{F}|\}$, $h_{s's} : X_s \to X_{s'}$ ($s' \to s$), $q_{F'F} = i_{|F|} = i_{|F|}$ in the inverse case.

Under a suitable definition of spectral mapping (see the structure of the category $\boldsymbol{D}(\boldsymbol{F})$) the set of Hausdorff spectra over $\boldsymbol{G}$ forms a category which we denote by Spect $\boldsymbol{G}$. If $\boldsymbol{X} =$

$\{X_s, \boldsymbol{F}, h_{s's}\}$, $\boldsymbol{Y} = \{Y_p, \boldsymbol{F}^1, h_{p'p}\}$ are objects from Spect $\boldsymbol{G}$, then we shall say that two Hausdorff spectrum mappings $\omega_{YX} : \boldsymbol{X} \to \boldsymbol{Y}$ and $\omega'_{YX} : \boldsymbol{X} \to \boldsymbol{Y}$ are *equivalent* if for any $F \in \boldsymbol{F}$ there exists $F^* \in \boldsymbol{F}^1$ such that the diagram

$$\begin{tikzcd} X_s \arrow[r, "\omega_{ps}"] \arrow[d, "\omega'_{p's}"'] & Y_p \arrow[d, "h_{p^*p}"] \\ Y_{p'} \arrow[r, "h_{p^*p'}"] & Y_{p^*} \end{tikzcd}$$

is commutative for any $p^* \in |F^*|$.

Now let us consider a new category $\boldsymbol{H}(\boldsymbol{G})$ whose objects are the objects of the category Spect $\boldsymbol{G}$, but the set $\mathrm{Hom}_H(\boldsymbol{X}, \boldsymbol{Y})$ is formed by the equivalence classes of mappings $\omega_{YX} : \boldsymbol{X} \to \boldsymbol{Y}$. We shall denote such classes by $\|\omega_{YX}\|$.

For any objects $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \boldsymbol{H}$ the law of composition defines a bilinear mapping

$$\mathrm{Hom}_H(\boldsymbol{X}, \boldsymbol{Y}) \times \mathrm{Hom}_H(\boldsymbol{Y}, \boldsymbol{Z}) \to \mathrm{Hom}_H(\boldsymbol{X}, \boldsymbol{Z})$$

($\mathrm{Hom}_H(\boldsymbol{X}, \boldsymbol{Y})$ is an abelian group).

Definition 2. Let $\boldsymbol{X} = \{X_s, \boldsymbol{F}, h_{s's}\}$ be a Hausdorff spectrum over the category $\boldsymbol{G}$. We shall call an object Z of the category $\boldsymbol{G}$ a *categorical H-limit of the Hausdorff spectrum $\boldsymbol{X}$ over $\boldsymbol{G}$* if for any objects $A, B \in \boldsymbol{G}$ and spectral mappings $A \xrightarrow{a} \mathrm{X} \xrightarrow{b} B$ there exists a unique sequence in $\boldsymbol{G}$ $A \xrightarrow{\alpha} \mathrm{Z} \xrightarrow{\beta} B$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "\alpha"'] & X \arrow[d, "b"] \\ Z \arrow[r, "\beta"] & B \end{tikzcd}$$

is commutative in the category Spect $\boldsymbol{G}$.

The concepts of projective and inductive limits over the category $\boldsymbol{G}$ are special cases of categorical H-limits. For example, let $\boldsymbol{X}$ be the inverse spectrum of objects from $\boldsymbol{G}$. Then (Lim) holds and moreover any object X_s from $\boldsymbol{X}$ can be taken for $B \in \boldsymbol{G}$ with the identity morphism $b_s : X_s \to X_s$ forming the spectral mapping $b^s : \boldsymbol{X} \to X_s$ $(s \in |F|)$. Thus the following diagram is commutative

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "\alpha"'] & X \arrow[d, "b"] \\ Z \arrow[r, "\beta"] & X \end{tikzcd}$$

where $b = (b^s)$, $\beta = (\beta^s)$, $\beta^s : Z \to X_s$ $(s \in |F|)$, b is the identity morphism of the category Spect $\boldsymbol{G}$. Therefore the diagram

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "\alpha"'] & X \arrow[d, equal] \\ Z \arrow[r, "\beta"] & X \end{tikzcd}$$

is commutative for any object $A \in \boldsymbol{G}$.

The categorical H-limit of a Hausdorff spectrum (the functor Haus) exists in any semi-abelian category $\boldsymbol{G}$ with direct sums and products (for example, the category of vector spaces L, the category TLC of topological vector groups, the category TLC of locally convex spaces).

Let Ω be a countable set and $\boldsymbol{X} = \{X_s, \boldsymbol{F}, h_{s's}\}$ a regular Hausdorff spectrum in the category TLC; such a spectrum is said to be countable. A continuous linear image in the category TLC of an H-limit $X = \overleftarrow{\lim}_{\vec{\mathscr{F}}} h_{s's}X_s$ of Banach spaces X_s ($s \in |\boldsymbol{F}|$) is called an *H-space.* The class of H-spaces contains the Fréchet spaces and is stable with respect to the operations of passage to countable inductive and projective limits, closed subspaces and factor spaces. Moreover, a strengthened variant of the closed graph theorem holds for H-spaces. The class of H-spaces is the broadest of all the analogous classes known at this time, namely those of Rajkov, De Wilde, Hakamura, Zabrejko–Smirnov. A countable separated regular H-limit of a Hausdorff spectrum of H-spaces in the category TLC is an H-space [7].

Throughout this chapter Hausdorff spectra are assumed to be countable unless the contrary is explicitly stated.

2. Let Haus : $\boldsymbol{H}(TLC) \to L$ be the covariant additive Hausdorff limit functor from the semiabelian category $\boldsymbol{H}(TLC)$ to the abelian category L of vector spaces (over $\mathbb{R}$ or $\mathbb{C}$). We recall [11] that by an *injective resolvent $\boldsymbol{I}$ of an object $\boldsymbol{X} \in \boldsymbol{H}(TLC)$* we mean any sequence

$$0 \longrightarrow \mathrm{I}_0 \xrightarrow{i_0} \mathrm{I}_1 \xrightarrow{i_1} \ldots,$$

formed by injective objects and exact in its members $\boldsymbol{I}_k$, $k \geqslant 1$, with $\ker i_0 \cong \boldsymbol{X}$. Any two injective resolvents of the same object are homotopic to each other. Since there are many injective objects in the category $\boldsymbol{H}(TLC)$ [3], each object of this category has at least one injective resolvent. The right derivatives of the Hausdorff limit functor Haus are defined by the formula

$$\mathrm{Haus}^k(\boldsymbol{X}) = H^k(\mathrm{Haus}(\boldsymbol{I})) \quad (k = 0, 1, \ldots),$$

where $\boldsymbol{X} \in \boldsymbol{H}(TLC)$, $\boldsymbol{I}$ is any injective resolvent of $\boldsymbol{X}$, $\mathrm{Haus}(\boldsymbol{I})$ is the complex of morphisms of the category L obtained by application of the functor Haus to each morphism of the complex $\boldsymbol{I}$, and $H^k(\mathrm{Haus}(\boldsymbol{I}))$ ($k = 0, 1, \ldots$) are the homologies of the complex $\mathrm{Haus}(\boldsymbol{I})$. Each morphism $\boldsymbol{X} \to \boldsymbol{Y}$ of the category $\boldsymbol{H}(TLC)$ is covered by a morphism $\boldsymbol{I} \to \boldsymbol{Y}$ of the injective resolvents of the objects $\boldsymbol{X}$ and $\boldsymbol{Y}$ (see [11, Chapter V, § 1]). From this follows the existence of morphisms $\mathrm{Haus}^k(\boldsymbol{X}) \to \mathrm{Haus}^k(\boldsymbol{Y})$ so that the objects of $\mathrm{Haus}^k(\boldsymbol{X})$ do not depend on the choice of injective resolvent. On the other hand the functor Haus has injective type [3, p. 88], therefore the canonical isomorphism of functors holds:

$$\mathrm{Haus} \cong \mathrm{Haus}^0 .$$

Proposition 3. *For every free Hausdorff spectrum $\boldsymbol{E} \in \boldsymbol{H}(L)$*

$$\mathrm{Haus}^i(\boldsymbol{E}) = 0 \quad (i = 1, 2, \ldots).$$

We now compute the derived functors Haus^i ($i \geqslant 1$) in the following way (see [2, 10]). Let $\boldsymbol{X} = \{X_s, \boldsymbol{F}, h_{s's}\}$ be an arbitrary Hausdorff spectrum and $\boldsymbol{E}$ the free Hausdorff spectrum with generators X_s ($s \in |\boldsymbol{F}|$). Let us consider the sequence of Hausdorff spectrum mappings

$$0 \longrightarrow \boldsymbol{X} \xrightarrow{\omega_{EX}} \boldsymbol{E} \xrightarrow{\omega_{EE}} \boldsymbol{E} \longrightarrow 0, \qquad (D)$$

in which the components of the mapping ω_{EX} (i. e. the collection $(\omega_{TsT})_{T\in|\varphi(F)|}$, where $s_T \in T$ is the unique maximal element in T with respect to the direction relation) act according to the formula $\omega_{Ts_T} : x_{s_T} \mapsto (\hat{h}_{s's_T} x_{s_T})_{s' \in T}$, while the Hausdorff spectrum mapping $\omega_{EE} : \boldsymbol{E} \to \boldsymbol{E}$ is formed by means of the morphisms (T_n is a cofinal right-filtering sequence)

$$\omega_{T^* T_n} : (x_s)_{s \in T_n} \mapsto \big(x_{s^*} - \hat{h}_{s^* s_{T_n}} X_{s_{T_n}}\big)_{s^* \in T^*}$$

for any T^*, $T_n \in F$, $F \in \boldsymbol{F}$, $T_0 = \varnothing$, $T_{n-1} \subset T^* \subset T_n$, $s_{T_n} \not\subset T^*$ $(n = 1, 2, \ldots)$.

It is now clear that the sequence (D) is exact; following V. P. Palamodov [2] we shall call the sequence (D) the *canonical resolvent of the Hausdorff spectrum* $\boldsymbol{X}$.

Applying the functor Haus to the canonical resolvent (D) we obtain the sequence of locally convex spaces

$$0 \to \mathrm{Haus}(\mathscr{X}) \to \bigoplus_{\mathscr{F}} \prod_{F} X_s \to \bigoplus_{\mathscr{F}} \prod_{F} X_s\,,$$

where $\bigoplus_{\mathscr{F}} \prod_F X_s$ is the direct sum of the products of the X_s $(s \in |\boldsymbol{F}|)$ under the natural inductive limit topology; this sequence is acyclic and moreover exact from the left.

Proposition 4. *Let* Haus : $\boldsymbol{H}(TLC) \to L$ *and let*

$$0 \longrightarrow \boldsymbol{X} \xrightarrow{\omega_{YX}} \boldsymbol{Y} \xrightarrow{\omega_{ZY}} \boldsymbol{Z} \longrightarrow 0 \qquad (D')$$

be an exact sequence of Hausdorff spectra. Then the following exact connecting sequence is defined in the category L (δ^I $(i = 1, 2, \ldots)$ *are the connecting morphisms*)*:*

$$0 \longrightarrow \mathrm{Haus}(\boldsymbol{X}) \longrightarrow \mathrm{Haus}(\boldsymbol{Y}) \longrightarrow \mathrm{Haus}(\boldsymbol{Z}) \longrightarrow \mathrm{Haus}^1(\boldsymbol{X})$$
$$\longrightarrow \ldots \longrightarrow \mathrm{Haus}^{i-1}(\boldsymbol{Z}) \xrightarrow{\delta^{i-1}} \mathrm{Haus}^i(\boldsymbol{X}) \xrightarrow{\overline{\omega}^i_{YX}} \mathrm{Haus}^i(\boldsymbol{Y}) \xrightarrow{\overline{\omega}_{ZY}} \mathrm{Haus}^i(\boldsymbol{Z}) \xrightarrow{\delta^i} \ldots$$

3. In [1] and [2] V. P. Palamodov established the fundamental Theorems 11.1 and 11.2 giving necessary and sufficient conditions for the vanishing at zero $\mathrm{Pro}^1(\boldsymbol{X}) = 0$ for the functor Pro of the projective limit of a countable family of locally convex spaces. We aim to establish analogous conditions for the vanishing at zero $\mathrm{Haus}^1(\boldsymbol{X}) = 0$ for the Hausdorff limit functor and for the not necessarily countable case.

We recall that in questions concerning the stability of the class of H-spaces with respect to Hausdorff limits and also in the theorem about the representation of H-spaces by means of Banach spaces the assumption of regularity of the Hausdorff spectrum was an important condition. Here it will be necessary to impose the following condition. Let $\boldsymbol{X} = \{X_s, \boldsymbol{F}, h_{s's}\}_T$ be a Hausdorff spectrum of locally convex spaces and for each $T \in F$ let $V_F^T \subset \prod_F X_s$ be defined by

$$V_F^T = \Big\{x = (x_s) \in \prod_F X_s : x_{s'} = \hat{h}_{s's} x_s,\ s, s' \in T\Big\},$$

equipped with the projective topology with respect to the preimages $\pi_s^{-1} \tau_s$ $(s \in T)$, where $\pi_s : \prod_F X_s \to X_s$ is the canonical projection. The corresponding base of neighborhoods of zero for the projective topology generates the $TVG\,\big(\prod_F X_s, \sigma_{(T)}\big)$ $(T \in F)$.

Let us form the $TVG\,(\prod_F X_s, \sigma_{(F)})$ with base of neighborhoods of zero V_F^T $(T \in F)$. The Hausdorff spectrum $\boldsymbol{X}$ is said to be *regular* if $\big(\prod_F X_s, \sigma_{(F)}\big)$ satisfies the condition: convergence of a net $(a_\gamma)_{\gamma \in P}$ in the $TVG_s\big(\prod_F X_s, \sigma_{(T)}\big)$ $(T \in F)$ implies its convergence in the $TVG\,\big(\prod_F X_s, \sigma_{(F)}\big)$. If every X_s $(s \in |\boldsymbol{F}|)$ has the indiscrete topology, then it is not difficult to see that the first part of the condition for regularity is equivalent to completeness of $\big(\prod_F X_s, \sigma_{(F)}\big)$.

Theorem 1. *Let* $\boldsymbol{X}$ *be a regular Hausdorff spectrum of nonseparated parts over the category* TLC*. Then* $\mathrm{Haus}^1(\boldsymbol{X}) = 0$.

If $\boldsymbol{Y}$ is a regular Hausdorff spectrum over TLC and $\boldsymbol{X}$ is the Hausdorff spectrum of nonseparated parts, then it is easy to see that $\boldsymbol{X}$ is also a regular spectrum. In fact, bearing in mind the remark before the theorem, it is sufficient to establish the completeness of $\left(\prod_F X_s, \sigma_{(F)}\right)$; this TVG is embedded in the corresponding TVG $\left(\prod_F Y_s, \sigma^1_{(F)}\right)$. If $(a_\gamma)_{\gamma\in P}$ is fundamental under $\sigma_{(F)}$, then $a_\gamma \in a_{\gamma_0} + V_F^T$ $(\forall T \in F,\ \gamma \succ \gamma(T),\ \gamma_0 \succ \gamma(T))$ and because of the closedness of V_F^T in the latter TVG we obtain the inclusion $(a^* = \lim_P a_\gamma)$

$$a^* - a_{\gamma_0} \in V_F^T \quad (\forall T \in F,\ \gamma_0 \succ \gamma(T)),$$

which also implies the convergence of (a_γ) to a^* in $\left(\prod_F X_s, \sigma_{(F)}\right)$.

Thus, in the enunciation of Theorem 1 regularity of the Hausdorff spectrum $\boldsymbol{X}$ can be replaced by regularity of the Hausdorff spectrum $\boldsymbol{Y}$.

Theorem 2. *Let $\boldsymbol{Y}$ be a regular Hausdorff spectrum, $\boldsymbol{X}$ the Hausdorff spectrum of nonseparated parts of $\boldsymbol{Y}$ and $0 \to \boldsymbol{X} \to \boldsymbol{Y} \to \boldsymbol{Y}/\boldsymbol{X} \to 0$ an exact sequence of Hausdorff spectra. Then the sequence $0 \to \mathrm{Haus}(\boldsymbol{X}) \to \mathrm{Haus}(\boldsymbol{Y}) \to \mathrm{Haus}(\boldsymbol{Y}/\boldsymbol{X}) \to 0$ is exact in the category L.*

Let us continue our consideration of the question of exactness of the functor $\mathrm{Haus} : \boldsymbol{H}(TLC) \to L$ for an arbitrary exact sequence of Hausdorff spectra $0 \to \boldsymbol{X} \to \boldsymbol{Y} \to \boldsymbol{Z} \to 0$. From the proofs given above it is clear that a sufficient condition for the vanishing at zero $\mathrm{Haus}^1(\boldsymbol{X}) = 0$ is the completeness of the $TVG\left(\prod_F X_s, \sigma^*_{(F)}\right)$ for each $F \in \boldsymbol{F}$ (see Proposition 7.1 of [3]), where $\boldsymbol{I}^*_{(F)}$ is formed by the filtering V_F^T with respect to T. At the same time each space V_F^T is endowed with the linear topology defined by the inverse image $\sup_T \pi_s^{-1}\tau_s$ $(T \in F)$ forming at the same time the TVG $\left(\prod_F X_s, \sigma_{(F)}\right)$ so that the TVG $\left(\prod_F X_s, \sigma_{(F)}\right)$ is not in general metrizable. It turns out that completeness of the TVG $\left(\prod_F X_s, \sigma^*_{(F)}\right)$ is also a necessary condition for the vanishing at zero $\mathrm{Haus}^1(\boldsymbol{X}) = 0$.

Proposition 5. *Let $\boldsymbol{X} = \{X_s, \boldsymbol{F}, h_{s's}\}$ be a countable Hausdorff spectrum over the category L. Then in order that $\mathrm{Haus}^1(\boldsymbol{X}) = 0$ it is necessary and sufficient that the $TVG\left(\prod_F X_s, \sigma^*_{(F)}\right)$ is complete for each $F \in \boldsymbol{F}$.*

Theorem 3. *Let $\boldsymbol{X} = \{X_s, \boldsymbol{F}, h_{s's}\}$ be a countable Hausdorff spectrum over the category L. Then in order that $\mathrm{Haus}^1(\boldsymbol{X}) = 0$ it is necessary and sufficient that for each $F \in \boldsymbol{F}$ it is possible to define in $\prod_F X_s$ a quasinorm $\mu = \mu_F \geqslant 0$ such that*

(i) *the associated topological group $\left(\prod_F X_s, \tau^*_{(F)}\right)$ is complete, $\tau_F \geqslant \sigma^*_{(F)}$,*

(ii) *μ^*_F is continuous on $\left(\prod_F X_s, \sigma^*_{(F)}\right)$.*

$\lhd$ *Necessity.* This follows from the argument before the theorem, since on putting $\tau_F = \sigma^*_{(F)}$ and

$$\mu_F(x) = \sum_{k=1}^{\infty} 2^{-k} d_{T_k}(x),$$

where $d_{T_k}(x) = 0$ for $x \in V_F^{T_k}$ and $d_{T_k}(x) = 1$ for $x \in \prod_F X_s \backslash V_F^{T_k}$ $(k \in \mathbf{N})$, we obtain (i) and (ii).

Sufficiency. Let $Z_F = \bigcap_{k=1}^{\infty} V_F^{T_k}$ and let the factor space $\prod_F X_s/Z_F$ be endowed with the images of the topologies $\sigma^*_{(F)}$ and τ_F, so that, if

$$d_F(\xi) = \inf_{x\in\xi} \mu_F(x) \quad \text{and} \quad \tilde{d}_F(\xi) = \inf_{x\in\xi} \sum_{k=1}^{\infty} 2^{-k} d_{T_k}(x)\,,$$

the $MVG(\prod_F X_s/Z_F, d_F)$ is separated and complete and the $MVG(\prod_F X_s/Z_F, \tilde{d}_F)$ is separated. Thus on the $MVG(\prod_F X_s/Z_F, \tilde{d}_F)$ the functional d_F is countably semiadditive and

$$d_F^*(\xi) = \inf_{\xi_n \to \xi} \lim_{n\to\infty} d_F(\xi_n) = \inf_{x\in\xi} \mu_F^*(x)$$

is continuous on it. Hence by the Lemma on a countably semiadditive functional [8] we obtain $d_F = d_F^*$ and, consequently, the $MVG(\prod_F X_s/Z_F, \tilde{d}_F)$ is complete. But this means that the $TVG(\prod_F X_s, \sigma^*_{(F)})$ will be complete, which allows us to conclude on considering all $F \in \boldsymbol{F}$ that $\mathrm{Haus}^1(\boldsymbol{X}) = 0$. The Theorem is proved. $\triangleright$

In the case of a countable inverse spectrum, in particular, we obtain the first part of Theorem 11.1.1 of [1]; in the case of a direct spectrum $\boldsymbol{X}$ the topology τ_F is indiscrete for each singleton set $F \in \boldsymbol{F}$. Moreover, the famous lemma of V. P. Palamodov [1], which makes up the main part of the proof, is a special case of the lemma about a countably semiadditive functional [8].

In what follows φ_F^s denotes the filter topology on X_s $(s \in |F|)$, which is formed by the spaces $\{\hat{h}_{ss'} X_{s'}\}$ $(s' \in |F|)$. We note, however, that the product topology on $\prod_F X_s$ obtained from the topologies φ_F^s $(s \in |F|)$ does not in general coincide with the topology $\sigma^*_{(F)}$.

Sufficient conditions for the vanishing at zero $\mathrm{Haus}^1(\boldsymbol{X}) = 0$, which are more convenient for applications, are given in the following proposition.

Theorem 4. *Let $\boldsymbol{X} = \{X_s, \boldsymbol{F}, h_{s's}\}$ be a countable Hausdorff spectrum over the category L. In order that* $\mathrm{Haus}^1(\boldsymbol{X}) = 0$ *it is sufficient that for each $s \in |F|$ it is possible to define in X_s a family of quasinorms $\{\rho_{\beta_s}\}$ which determines a complete separated pseudotopological vector space (X_s, ρ_{β_s}), preserves the continuity of the morphisms $\hat{h}_{s's}$ and is such that for each $s \in |F|$, $F \in \boldsymbol{F}$ the following condition is satisfied:*

(A) *for some $\beta_s = \beta_s(F)$ the functional $\rho^*_{\beta_s}$ is continuous in the filter topology (X_s, φ_F^s).*

In particular, in the case of an inverse spectrum $\boldsymbol{X}$ we obtain Theorem 5.1 of [2] and moreover our assertion is even stronger in this case.

Theorem 5. *Let $\boldsymbol{X} = \{X_s, \boldsymbol{F}, h_{s's}\}$ be a countable Hausdorff spectrum of separated H-spaces over the category TLC. Then in order that* $\mathrm{Haus}^1(\boldsymbol{X}) = 0$ *it is necessary and sufficient that the spaces (X_s, φ_F^s) $(s \in |F|)$ are complete TVGs for each $F \in \boldsymbol{F}$.*

In the case of an inverse spectrum of Fréchet spaces Theorem 5 extends the criteria (F) and (R) of V. P. Palamodov's Corollary 11.4 in [1]. We note that in Theorem 5 it is separatedness of the pseudotopology which is actually required, therefore in general the H-space may be nonseparated.

Theorem 6. *Let $\boldsymbol{X} = \{X_s, \boldsymbol{F}, h_{s's}\}$ be a countable Hausdorff spectrum of H-spaces over the category TLC with separated associated pseudotopology $\{(\rho_s^{P_s})^*\}$ which preserves the continuity of the morphisms $h_{s's}$. Then in order that* $\mathrm{Haus}^1(\boldsymbol{X}) = 0$ *it is necessary and sufficient that for each $s \in |\boldsymbol{F}|$ there exists a quasinorm $\rho_s^{P_s}(F)$ $(s \in |F|)$ in X_s such that*

(A') *$(\rho_s^{P_s})^*$ is continuous in the filter topology φ_F^s and the system $\{\rho_s^{P_s}\}$ preserves the continuity of the morphisms $h_{s's}$.*

In particular the theorem by Retakh [9] follows from Theorem 6.

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ИСПОЛЬЗОВАНИЕ ГОМОЛОГИЧЕСКИХ МЕТОДОВ НА БАЗЕ ИТЕРИРОВАННЫХ СПЕКТРОВ В ФУНКЦИОНАЛЬНОМ АНАЛИЗЕ

Смирнов Е. И.

В статье водятся новые понятия функционального анализа: хаусдорфов спектр и хаусдорфов предел или H-предел хаусдорфова спектра в категории локально выпуклых пространств (или даже, в более общих полуабелевых категориях). Частными случаями регулярного хаусдорфова предела являются проективный и индуктивный пределы отделимых локально выпуклых пространств. Новый класс H-пространств содержит пространства Фреше и замкнут относительно операций взятия счетного индуктивного и проективного пределов, перехода к замкнутому подпространству и фактор-пространству. Более того, для H-пространств справедлив усиленный вариант теоремы о замкнутом графике. Доказаны теоремы об обращении в нуль первой производной функтора хаусдорфова предела средствами гомологической алгебры.

Ключевые слова: топология, спектр, замкнутый график, дифференциальные уравнения, гомологические методы, категория.