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ON PROJECTIVE LIMITS OF REAL C^*- AND JORDAN OPERATOR ALGEBRAS[1]

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In the present paper a real and Jordan analogues of complex locally C^*-algebras are introduced. Their definitions and basic properties are discussed.

1. Introduction

Projective limits of Banach algebras have been studied sporadically by many authors since 1952, when they were first introduced by Arens [2] and Michael [11]. Projective limits of complex C^*-algebras were first mentioned by Arens [2]. They have since been studied under various names by Wenjen [20], Sya Do-Shin [18], Brooks [4], Inoue [10], Schmüdgen [17], Fritzsche [6–7], Fragoulopoulou [5], Phillips [15], etc. We will follow Inoue [10] in the usage of the name «*locally C^*-algebras*» for these objects.

At the same time, in parallel with the theory of complex C^*-algebras, a theory of their real and Jordan analogues, namely real C^*-algebras and JB-algebras, has been actively developed by various authors (for references, see for example [3, 8, 12]).

In the view of aforementioned, it is therefore interesting to extend existing theory to the case of real and Jordan analogues of complex locally C^*-algebras. The present paper (first in a sequence under preparation) is devoted to definitions and basic properties of such analogues, which we call *real locally C^**- and *locally JB-algebras*.

2. Preliminaries

In this section we give some preliminaries on complex locally C^*-algebras.

Definition 1. Let $\mathfrak{A}$ be a locally convex algebra over $\mathbb{C}$. $\mathfrak{A}$ is a *locally m-convex* iff there exists a basis of neighborhoods of zero entirely composed of convex idempotent sets U_i ($U_i^2 \subset U_i$).

In every locally convex topological space the topology can be defined by a basis of continuous seminorms (see [16]). If the case the algebra over $\mathbb{C}$ is a locally m-convex one, the basis can be chosen in such a way that each seminorm is a submultiplicative one (see [11]). In every locally m-convex algebra over $\mathbb{C}$, the multiplication law is jointly continuous, and if

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the algebra has a unit, the inversion is continuous on the subalgebra of invertible elements (see [2]).

Definition 2. An involution on an algebra $\mathfrak{A}$ over $\mathbb{C}$ is defined as a conjugate anti-isomorphism of period two, or:

$$* : \mathfrak{A} \to \mathfrak{A}, \quad x \mapsto x^*,$$

that satisfies the following conditions:

$$(x+y)^* = x^* + y^*,$$

$$(\lambda x)^* = \overline{\lambda} x^*,$$

$$(xy)^* = y^* x^*,$$

$$(x^*)^* = x,$$

for all $x, y \in \mathfrak{A}$, and each $\lambda \in \mathbb{C}$. An algebra over $\mathbb{C}$ in which there is an involution defined is called a *complex involutive algebra* or $*$-algebra over $\mathbb{C}$.

Definition 3. An element x in a topological algebra $\mathfrak{A}$ over $\mathbb{C}$ is called *bounded*, if there exists $\lambda \in \mathbb{C}$, $\lambda \neq 0$, such that the sets $\{(\lambda x)^n : n \in \mathbb{N}\}$, is bounded.

Definition 4. A locally convex $*$-algebra over $\mathbb{C}$ with unit is called *regular* if for every $x \in \mathfrak{A}$, $(\mathbf{1} + x^*x)$ is invertible. If in addition the element $(\mathbf{1} + x^*x)^{-1}$ is bounded, then the algebra is called *symmetric*.

Definition 5. A symmetric element, i. e. $x = x^*$, of a complex topological $*$-algebra with unit is called *Hermitian*, iff its spectrum is contained in $\mathbb{R}$. If every symmetric element is Hermitian, then involution is called Hermitian.

If the algebra is convex $*$-algebra with unit and continuous inversion, and in addition it is regular, then it is symmetric as well (see [1]).

Definition 6. A submultiplicative seminorm p defined on a $*$-algebra $\mathfrak{A}$ over $\mathbb{C}$ is called *regular*, if it satisfies the following condition:

$$p(x^*x) = p(x)^2,$$

for every $x \in \mathfrak{A}$.

Let Λ be a set of indices, directed by a relation (reflexive, transitive, antisymmetric) $\preceq$. Let $\{\mathfrak{A}_\alpha, \alpha \in \Lambda\}$ be a family of C^*-algebras, and g_α^β be, for $\alpha \preceq \beta$, a continuous linear mappings $g_\alpha^\beta : \mathfrak{A}_\beta \to \mathfrak{A}_\alpha$, so that $g_\alpha^\alpha(x_\alpha) = x_\alpha$, for all $\alpha \in \Lambda$, and $g_\alpha^\beta \circ g_\beta^\gamma = g_\alpha^\gamma$, whenever $\alpha \preceq \beta \preceq \gamma$.

Let Γ be the collections $\{g_\alpha^\beta\}$ of all such transformations. Let $\mathfrak{A}$ be a $*$-subalgebra of the direct product algebra

$$\prod_{\alpha \in \Lambda} \mathfrak{A}_\alpha,$$

so that for its elements $x_\alpha = g_\alpha^\beta(x_\beta)$, for all $\alpha \preceq \beta$, where $x_\alpha \in \mathfrak{A}_\alpha$, and $x_\beta \in \mathfrak{A}_\beta$.

Definition 7. The $*$-algebra $\mathfrak{A}$ above is called a *Hausdorff projective limit* of the family $\{\mathfrak{A}_\alpha, \alpha \in \Lambda\}$, relatively to the collection $\Gamma = \{g_\alpha^\beta : \alpha, \beta \in \Lambda, \alpha \preceq \beta\}$, and is denoted by $\varprojlim g_\alpha^\beta \mathfrak{A}_\beta$.

It is well known (see, for example [19]) that for each $\beta \in \Lambda$ there is a natural projection $\pi_\beta : \mathfrak{A} \longrightarrow \mathfrak{A}_\beta$, defined by $\pi_\beta(\{x_\alpha\}) = x_\beta$, and each projection π_α for all $\alpha \in \Lambda$ is continuous.

Definition 8. A topological $*$-algebra $\mathfrak{A}$ over $\mathbb{C}$ is called a *locally C^*-algebra* if there exists a projective family of C^*-algebras $\{\mathfrak{A}_\alpha; g_\alpha^\beta; \alpha, \beta \in \Lambda\}$, so that $\mathfrak{A} = \varprojlim g_\alpha^\beta \mathfrak{A}_\beta$.

Theorem 1. *A topological $*$-algebra $\mathfrak{A}$ over $\mathbb{C}$ is a locally C^*-algebra iff $\mathfrak{A}$ is a complete topological $*$-algebra in which there exists a basis of continuous seminorms composed entirely by regular seminorms.*

3. Topological $*$-algebras which are Hausdorff Projective Limits of real C^*-algebras

In this section we introduce a class of real $*$-algebras that are real analogues of the complex locally C^*-algebras.

DEFINITION 9. A real $*$-algebra R is called a *real locally C^*-algebra*, if there exists a projective family

$$\{\mathfrak{R}_\alpha; g_\alpha^\beta; \alpha, \beta \in \Lambda\}$$

of real C^*-algebras, with Λ be a set of indices, directed by a relation (reflexive, transitive, antisymmetric) «$\preceq$», so that

$$\mathfrak{R} = \varprojlim g_\alpha^\beta \mathfrak{R}_\beta.$$

EXAMPLE 1. Every real C^*-algebra is a real locally C^*-algebra.

EXAMPLE 2. A closed $*$-subalgebra of a real locally C^*-algebra is a real locally C^*-algebra.

EXAMPLE 3. The product $\prod\limits_{\alpha \in \mathbb{I}} \mathfrak{R}_\alpha$ of real C^*-algebras $\mathfrak{R}_\alpha$, with the product topology, is a real locally C^*-algebra.

EXAMPLE 4. Let X be a compactly generated Hausdorff space (this means that a subset $Y \subset X$ is closed iff $Y \cap K$ is closed for every compact subset $K \subset X$, see [21]). Then the algebra $C(X)$ of all continuous, not necessarily bounded real-valued functions on X, with the topology of uniform convergence on compact subsets, is a real locally C^*-algebra. It is known that all metrizable spaces and all locally compact Hausdorff spaces are compactly generated (see [21]).

Theorem 2. *A topological $*$-algebra $\mathfrak{R}$ is a real locally C^*-algebra iff it is complete $*$-algebra in which there exists a basis of continuous seminorms composed entirely by regular seminorms.*

In definition of real C^*-algebras (see [3, 12]), it is required that its complexification be a complex C^*-algebra. For the real locally C^*-algebras, however, the analogous property will hold automatically, thus, the following theorem is valid:

Theorem 3. *Let $\mathfrak{R}$ be a real locally C^*-algebra, then $\mathfrak{A} = \mathfrak{R} + i\mathfrak{R}$, is a complex locally C^*-algebra.*

Let $S(\mathfrak{R})$ be the set of all continuous regular seminorms on a real locally C^*-algebra $\mathfrak{R}$.

DEFINITION 10. Let $\mathfrak{R}$ be a real locally C^*-algebra. Then an element $a \in \mathfrak{R}$ is called *bounded*, if

$$\|a\|_\infty = \{\sup \rho(a) : \rho \in S(\mathfrak{R})\} < \infty.$$

The set of all bounded elements of $\mathfrak{R}$ is denoted by $b(\mathfrak{R})$.

The following theorem gives a description of the set of bounded elements in a real locally C^*-algebra.

Theorem 4. *Let $\mathfrak{R}$ be a real locally C^*-algebra. Then the set $b(\mathfrak{R})$ of bounded elements of $\mathfrak{R}$ be a real C^*-algebra in the norm $\|\cdot\|_\infty$.*

Proposition 1. *Let $\mathfrak{R}$ be a Real locally C^*-algebra, and $x \in \mathfrak{R}$ be normal. Then x is bounded iff $sp(x)$ is bounded.*

Corollary 1. *The set $b(\mathfrak{R})$ is dense in $\mathfrak{R}$.*

4. Topological Jordan Algebras which are Hausdorff Projective Limits of *JB*-algebras

In this section we introduce a class of Jordan algebras that are Jordan analogues of complex locally C^*-algebras.

DEFINITION 11. A Jordan algebra $\mathcal{A}$ is called a *locally JB-algebra*, if there exists a projective family $\{\mathcal{A}_\alpha; g_\alpha^\beta; \alpha, \beta \in \Lambda\}$ of *JB*-algebras $\mathcal{A}_\alpha$, with Λ be a set of indices, directed by a relation (reflexive, transitive, antisymmetric) $\preceq$, so that $\mathcal{A} = \varprojlim g_\alpha^\beta \mathcal{A}_\beta$.

EXAMPLE 5. Every *JB*-algebra is a locally *JB*-algebra.

EXAMPLE 6. A closed Jordan subalgebra of a locally *JB*-algebra is a locally *JB*-algebra.

EXAMPLE 7. The self-adjoint part of any complex locally C^*-algebra is a locally *JB*-algebra.

EXAMPLE 8. The self-adjoint part of any real locally C^*-algebra is a locally *JB*-algebra.

EXAMPLE 9. The product $\prod_{\alpha \in \mathbb{I}} \mathcal{A}_\alpha$ of *JB*-algebras $\mathcal{A}_\alpha$, with the product topology, is a locally *JB*-algebra.

EXAMPLE 10. Let X be a compactly generated Hausdorff space (this means that a subset $Y \subset X$ is closed iff $Y \cap K$ is closed for every compact subset $K \subset X$, see [21]). Then the algebra $C(X)$ of all continuous, not necessarily bounded real-valued functions on X with the topology of uniform convergence on compact subsets, is a locally *JB*-algebra.

Theorem 5. *A Jordan topological algebra over* $\mathbb{R}$ *is a locally JB-algebra iff it is complete Jordan topological algebra in which there exists a basis of continuous seminorms composed entirely by regular seminorms.*

DEFINITION 12. Let $\mathcal{A}$ be a locally *JB*-algebra. Then an element $a \in \mathcal{A}$ is called *bounded*, if

$$\|a\|_\infty = \{\sup \rho(a) : \rho \in S(\mathcal{A})\} < \infty.$$

The set of all bounded elements of $\mathcal{A}$ is denoted by $b(\mathcal{A})$.

The following theorem gives a description of the set of bounded elements in a locally *JB*-algebra.

Theorem 6. *Let* $\mathcal{A}$ *be a locally JB-algebra. Then the set* $b(\mathcal{A})$ *of bounded elements of* $\mathcal{A}$ *is a JB-algebra in the norm* $\|.\|_\infty$.

Proposition 2. *Let* $\mathcal{A}$ *be a locally JB-algebra, and* $x \in \mathcal{A}$ *be normal. Then* x *is bounded iff* $sp(x)$ *is bounded.*

Corollary 2. *The set* $b(\mathcal{A})$ *is dense in* $\mathcal{A}$.

5. Connection between real, complex $*$- and Jordan algebras which are Hausdorff Projective Limits

It is well known that the real C^*-algebras and the *JB*-algebras are related to the so-called enveloping C^*-algebra through the actions of an $*$-antiautomorphism of period 2 on it (see, for example [8]). Analogous results below extend the known results to the case of real locally C^*-algebras and the locally *JB*-algebras respectively.

Theorem 7. *Let* $\mathfrak{R}$ *be a real locally* C^**-algebra. Then there exists a complex locally* C^**-algebra* $\mathfrak{A}$*, so that* $\mathfrak{R}$ *is topologically and algebraically isomorphic to the set*

$$\{x \in \mathfrak{A} : \alpha(x^*) = x\},$$

where α *is an* $*$*-antiautomorphism of period* 2 *of* $\mathfrak{A}$.

Theorem 8. *Let $\mathcal{A}$ be a locally JB-algebra. Then there exists a Jordan ideal $\mathcal{A}_{ex} \subset \mathcal{A}$, and a complex locally C^*-algebra $\mathfrak{A}$, so that the factor-algebra $\mathcal{A}/\mathcal{A}_{ex}$ is topologically and algebraically isomorphic to the set*

$$\{x \in \mathfrak{A} : \alpha(x) = x\},$$

where α is an $*$-antiautomorphism of period 2 of $\mathfrak{A}$.

6. Abelian real locally C^*- and locally JB-algebras

In [10] Inoue described an abelian locally C^*-algebra by showing that it is isomorphic to a certain function algebra. Analogous results are true for real and Jordan analogues of abelian locally C^*-algebras.

Let $\mathfrak{R}$ be an abelian real locally C^*-algebra, $\mathfrak{R}^*$ be the dual space of $\mathfrak{R}$, and

$$F(\mathfrak{R}) = \{f \in \mathfrak{R}^* : f(xy) = f(x)f(y) \text{ for all } x, y \in \mathfrak{R}\}.$$

Theorem 9. *An abelian real locally C^*-algebra $\mathfrak{R}$ is isomorphic to the real locally C^*-algebra $C_0(F(\mathfrak{R}))$.*

Let $\mathcal{A}$ be an abelian locally JB-algebra, $\mathcal{A}^*$ be the dual space of $\mathcal{A}$, and

$$F(\mathcal{A}) = \{f \in \mathcal{A}^* : f(xy) = f(x)f(y) \text{ for all } x, y \in \mathcal{A}.\}$$

Theorem 10. *An abelian locally JB-algebra $\mathcal{A}$ is isomorphic to the locally JB-algebra $C_0(F(\mathcal{A}))$.*

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