# Part II

ISSN 1464-8997 (on line) 1464-8989 (printed) 

Geometry & Topology Monographs
Volume 3: Invitation to higher local fields
Part II, section 1, pages 199–213

# 1. Higher dimensional local fields and $L$-functions

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## 1.0. Introduction

**1.0.1.** Recall [P1], [FP] that if $X$ is a scheme of dimension $n$ and

$$X_0 \subset X_1 \subset \dots X_{n-1} \subset X_n = X$$

is a flag of irreducible subschemes ($\dim(X_i) = i$), then one can define a ring

$$K_{X_0,\dots,X_{n-1}}$$

associated to the flag. In the case where everything is regularly embedded, the ring is an $n$-dimensional local field. Then one can form an adelic object

$$\mathbb{A}_X = \prod{}' K_{X_0,\dots,X_{n-1}}$$

where the product is taken over all the flags with respect to certain restrictions on components of adeles [P1], [Be], [Hu], [FP].

**Example.** Let $X$ be an algebraic projective irreducible surface over a field $k$ and let $P$ be a closed point of $X$, $C \subset X$ be an irreducible curve such that $P \in C$.

If $X$ and $C$ are smooth at $P$, then we let $t \in \mathcal{O}_{X,P}$ be a local equation of $C$ at $P$ and $u \in \mathcal{O}_{X,P}$ be such that $u|_C \in \mathcal{O}_{C,P}$ is a local parameter at $P$. Denote by $\mathcal{C}$ the ideal defining the curve $C$ near $P$. Now we can introduce a two-dimensional local field $K_{P,C}$ attached to the pair $P, C$ by the following procedure including completions and localizations:

$$\begin{array}{rcl}
\widehat{\mathcal{O}}_{X,P} & = & k(P)[[u,t]] \supset \mathcal{C} = (t) \\
| & & \\
(\widehat{\mathcal{O}}_{X,P})_{\mathcal{C}} & = & \text{discrete valuation ring with residue field } k(P)((u)) \\
| & & \\
\widehat{\mathcal{O}}_{P,C} := \widehat{(\widehat{\mathcal{O}}_{X,P})_{\mathcal{C}}} & = & k(P)((u))[[t]] \\
| & & \\
K_{P,C} := \operatorname{Frac}(\widehat{\mathcal{O}}_{P,C}) & = & k(P)((u))((t))
\end{array}$$

Note that the left hand side construction is meaningful *without* any smoothness condition.

Let $K_P$ be the minimal subring of $K_{P,C}$ which contains $k(X)$ and $\widehat{\mathcal{O}}_{X,P}$. The ring $K_P$ is not a field in general. Then $K \subset K_P \subset K_{P,C}$ and there is another intermediate subring $K_C = \mathrm{Frac}\,(\mathcal{O}_C) \subset K_{P,C}$. Note that in dimension 2 there is a duality between points $P$ and curves $C$ (generalizing the classical duality between points and lines in projective geometry). We can compare the structure of adelic components in dimension one and two:

$$\begin{tikzcd}[arrows=dash] K_P \arrow[dd] \\ \\ K \end{tikzcd} \qquad\qquad \begin{tikzcd}[arrows=dash] & K_{P,C} \arrow[dl] \arrow[dr] & \\ K_P \arrow[dr] & & K_C \arrow[dl] \\ & K & \end{tikzcd}$$

**1.0.2.** In the one-dimensional case for every character $\chi\colon \mathrm{Gal}(K^{\mathrm{ab}}/K) \to \mathbb{C}^*$ we have the composite

$$\chi'\colon \mathbb{A}^* = {\prod}' K_x^* \xrightarrow{\text{reciprocity map}} \mathrm{Gal}(K^{\mathrm{ab}}/K) \xrightarrow{\chi} \mathbb{C}^*.$$

J. Tate [T] and independently K. Iwasawa introduced an analytically defined $L$-function

$$L(s,\chi,f) = \int_{\mathbb{A}^*} f(a)\chi'(a)|a|^s d^*a,$$

where $d^*$ is a Haar measure on $\mathbb{A}^*$ and the function $f$ belongs to the Bruhat–Schwartz space of functions on $\mathbb{A}$ (for the definition of this space see for instance [W1, Ch. VII]). For a special choice of $f$ and $\chi = 1$ we get the $\zeta$-function of the scheme $X$

$$\zeta_X(s) = \prod_{x\in X} (1 - N(x)^{-s})^{-1},$$

if $\dim(X) = 1$ (adding the archimedean multipliers if necessary). Here $x$ runs through the closed points of the scheme $X$ and $N(x) = |k(x)|$. The product converges for $\mathrm{Re}(s) > \dim X$. For $L(s,\chi,f)$ they proved the analytical continuation to the whole $s$-plane and the functional equation

$$L(s,\chi,f) = L(1-s,\chi^{-1},\widehat{f}),$$

using Fourier transformation ($f \mapsto \widehat{f}$) on the space $\mathbb{A}_X$ (cf. [T], [W1], [W2]).

**1.0.3.** Schemes can be classified according to their dimension

| $\dim(X)$ | geometric case | arithmetic case |
|---|---|---|
| ... | ... | ... |
| 2 | algebraic surface $/\mathbb{F}_q$ | arithmetic surface |
| 1 | algebraic curve $/\mathbb{F}_q$ | arithmetic curve |
| 0 | $\mathrm{Spec}(\mathbb{F}_q)$ | $\mathrm{Spec}(\mathbb{F}_1)$ |

where $\mathbb{F}_1$ is the "field of one element".

The analytical method works for the row of the diagram corresponding to dimension one. The problem to prove analytical continuation and functional equation for the $\zeta$-function of arbitrary scheme $X$ (Hasse–Weil conjecture) was formulated by A. Weil [W2] as a generalization of the previous Hasse conjecture for algebraic curves over fields of algebraic numbers, see [S1],[S2]. It was solved in the geometric situation by A. Grothendieck who employed cohomological methods [G]. Up to now there is no extension of this method to arithmetic schemes (see, however, [D]). On the other hand, a remarkable property of the Tate–Iwasawa method is that it can be simultaneously applied to the fields of algebraic numbers (arithmetic situation) and to the algebraic curves over a finite field (algebraic situation).

For a long time the author has been advocating (see, in particular, [P4], [FP]) the following:

**Problem.** *Extend Tate–Iwasawa's analytic method to higher dimensions.*

The higher adeles were introduced exactly for this purpose. In dimension one the adelic groups $\mathbb{A}_X$ and $\mathbb{A}_X^*$ are locally compact groups and thus we can apply the classical harmonic analysis. The starting point for that is the measure theory on locally compact local fields such as $K_P$ for the schemes $X$ of dimension 1. So we have the following:

**Problem.** *Develop a measure theory and harmonic analysis on $n$-dimensional local fields.*

Note that $n$-dimensional local fields are not locally compact topological spaces for $n > 1$ and by Weil's theorem the existence of the Haar measure on a topological group implies its locally compactness [W3, Appendix 1].

In this work several first steps in answering these problems are described.

## 1.1. Riemann–Hecke method

When one tries to write the $\zeta$-function of a scheme $X$ as a product over local fields attached to the flags of subvarieties one meets the following obstacle. For dimension greater than one the local fields are parametrized by flags and not by the closed points itself as in the Euler product. This problem is primary to any problems with the measure and integration. I think we have to return to the case of dimension one and reformulate the Tate–Iwasawa method. Actually, it means that we have to return to the Riemann–Hecke approach [He] known long before the work of Tate and Iwasawa. Of course, it was the starting point for their approach.

The main point is a reduction of the integration over ideles to integration over a single (or finitely many) local field.

Let $C$ be a smooth irreducible complete curve defined over a field $k = \mathbb{F}_q$.

Put $K = k(C)$. For a closed point $x \in C$ denote by $K_x$ the fraction field of the completion $\widehat{\mathcal{O}}_x$ of the local ring $\mathcal{O}_x$.

Let $P$ be a fixed smooth $k$-rational point of $C$. Put $U = C \setminus P$, $A = \Gamma(U, \mathcal{O}_C)$. Note that $A$ is a discrete subgroup of $K_P$.

A classical method to calculate $\zeta$-function is to write it as a Dirichlet series instead of the Euler product:

$$\zeta_C(s) = \sum_{I \in \mathrm{Div}\,(\mathcal{O}_C)} |I|_C^s$$

where $\mathrm{Div}\,(\mathcal{O}_C)$ is the semigroup of effective divisors, $I = \sum_{x \in X} n_x x$, $n_x \in \mathbb{Z}$ and $n_x = 0$ for almost all $x \in C$,

$$|I|_C = \prod_{x \in X} q^{-\sum n_x |k(x):k|}.$$

Rewrite $\zeta_C(s)$ as

$$\zeta_U(s)\zeta_P(s) = \Bigg(\sum_{I \subset U} |I|_U^s\Bigg)\Bigg(\sum_{\mathrm{supp}(I) = P} |I|_P^s\Bigg).$$

Denote $A' = A \setminus \{0\}$. For the sake of simplicity assume that $\mathrm{Pic}(U) = (0)$ and introduce $A''$ such that $A'' \cap k^* = (1)$ and $A' = A''k^*$. Then for every $I \subset U$ there is a unique $b \in A''$ such that $I = (b)$. We also write $|b|_* = |(b)|_*$ for $* = P, U$. Then from the product formula $|b|_C = 1$ we get $|b|_U = |b|_P^{-1}$. Hence

$$\zeta_C(s) = \Bigg(\sum_{b \in A''} |b|_U^s\Bigg)\Bigg(\sum_{m \geqslant 0} q^{-ms}\Bigg) = \Bigg(\sum_{b \in A''} |b|_P^{-s}\Bigg) \int_{a \in K_P^*} |a|_P^s f_+(a) d^* a$$

where in the last equality we have used local Tate's calculation, $f_+ = i^*\delta_{\widehat{\mathcal{O}}_P}$, $i: K_P^* \to K_P$, $\delta_{\widehat{\mathcal{O}}_P}$ is the characteristic function of the subgroup $\widehat{\mathcal{O}}_P$, $d^*(\widehat{\mathcal{O}}_P^*) = 1$. Therefore

$$\begin{aligned}\zeta_C(s) &= \sum_{b\in A''}\int_{a\in K_P^*} |ab^{-1}|_P^s f_+(a) d^*a \\ &= \sum_{b\in A''}\int_{c=ab^{-1}} |c|_P^s f_+(bc) d^*c = \int_{K_P^*} |c|_P^s F(c) d^*c,\end{aligned}$$

where $F(c) = \sum_{b\in A'} f_+(bc)$.

Thus, the calculation of $\zeta_C(s)$ is reduced to integration over the single local field $K_P$. Then we can proceed further using the Poisson summation formula applied to the function $F$.

This computation can be rewritten in a more functorial way as follows

$$\zeta_C(s) = \langle |\ |^s, f_0\rangle_G \cdot \langle |\ |^s, f_1\rangle_G = \langle |\ |^s, i^*(F)\rangle_{G\times G} = \langle |\ |^s, j_*\circ i^*(F)\rangle_G,$$

where $G = K_P^*$, $\langle f, f'\rangle_G = \int_G ff'dg$ and we introduced the functions $f_0 = \delta_{A''} =$ sum of Dirac's $\delta_a$ over all $a \in A''$ and $f_1 = \delta_{\mathcal{O}_P}$ on $K_P$ and the function $F = f_0 \otimes f_1$ on $K_P \times K_P$. We also have the norm map $|\ |: G \to \mathbb{C}^*$, the convolution map $j: G\times G \to G$, $j(x,y) = x^{-1}y$ and the inclusion $i: G\times G \to K_P \times K_P$.

For the appropriate classes of functions $f_0$ and $f_1$ there are $\zeta$-functions with a functional equation of the following kind

$$\zeta(s, f_0, f_1) = \zeta(1-s, \widehat{f}_0, \widehat{f}_1),$$

where $\widehat{f}$ is a Fourier transformation of $f$. We will study the corresponding spaces of functions and operations like $j_*$ or $i^*$ in subsection 1.3.

**Remark 1.** We assumed that $\mathrm{Pic}(U)$ is trivial. To handle the general case one has to consider the curve $C$ with several points removed. Finiteness of the $\mathrm{Pic}^0(C)$ implies that we can get an open subset $U$ with this property.

## 1.2. Restricted adeles for dimension 2

**1.2.1.** Let us discuss the situation for dimension one once more. We consider the case of the algebraic curve $C$ as above.

One-dimensional adelic complex

$$K \oplus \prod_{x\in C} \widehat{\mathcal{O}}_x \to {\prod}'_{x\in C} K_x$$

can be included into the following commutative diagram

$$\begin{array}{ccc} K \oplus \prod_{x\in C} \widehat{\mathcal{O}}_x & \longrightarrow & \prod'_{x\in C} K_x \\ \big\downarrow & & \big\downarrow \\ K \oplus \widehat{\mathcal{O}}_P & \longrightarrow & \prod'_{x\neq P} K_x/\widehat{\mathcal{O}}_x \oplus K_P \end{array}$$

where the vertical map induces an isomorphism of cohomologies of the horizontal complexes. Next, we have a commutative diagram

$$\begin{array}{ccc} K \oplus \widehat{\mathcal{O}}_P & \longrightarrow & \prod'_{x\neq P} K_x/\widehat{\mathcal{O}}_x \oplus K_P \\ \big\downarrow & & \big\downarrow \\ K/A & \longrightarrow & \prod'_{x\neq P} K_x/\widehat{\mathcal{O}}_x \end{array}$$

where the bottom horizontal arrow is an isomorphism (the surjectivity follows from the strong approximation theorem). This shows that the complex $A \oplus \widehat{\mathcal{O}}_P \to K_P$ is quasi-isomorphic to the full adelic complex. The construction can be extended to an arbitrary locally free sheaf $\mathcal{F}$ on $C$ and we obtain that the complex

$$W \oplus \widehat{\mathcal{F}}_P \to \widehat{\mathcal{F}}_P \otimes_{\widehat{\mathcal{O}}_P} K_P,$$

where $W = \Gamma(\mathcal{F}, C \setminus P) \subset K$, computes the cohomology of the sheaf $\mathcal{F}$.

This fact is essential for the analytical approach to the $\zeta$-function of the curve $C$. To understand how to generalize it to higher dimensions we have to recall another applications of this diagram, in particular, the so called Krichever correspondence from the theory of integrable systems.

Let $z$ be a local parameter at $P$, so $\widehat{\mathcal{O}}_P = k[[z]]$. The Krichever correspondence assigns points of infinite dimensional Grassmanians to $(C, P, z)$ and a torsion free coherent sheaf of $\mathcal{O}_C$-modules on $C$. In particular, there is an injective map from classes of triples $(C, P, z)$ to $A \subset k((z))$. In [P5] it was generalized to the case of algebraic surfaces using the higher adelic language.

**1.2.2.** Let $X$ be a projective irreducible algebraic surface over a field $k$, $C \subset X$ be an irreducible projective curve, and $P \in C$ be a smooth point on both $C$ and $X$.

In dimension two we start with the adelic complex

$$\mathbb{A}_0 \oplus \mathbb{A}_1 \oplus \mathbb{A}_2 \to \mathbb{A}_{01} \oplus \mathbb{A}_{02} \oplus \mathbb{A}_{12} \to A_{012},$$

where

$$A_0 = K = k(X), \qquad A_1 = \prod_{C\subset X} \widehat{\mathcal{O}}_C, \qquad A_2 = \prod_{x\in X} \widehat{\mathcal{O}}_x,$$

$$A_{01} = \prod\nolimits'_{C\subset X} K_C,\ A_{02} = \prod\nolimits'_{x\in X} K_x,\ A_{12} = \prod\nolimits'_{x\in C} \widehat{\mathcal{O}}_{x,C},\ A_{012} = \mathbb{A}_X = \prod\nolimits' K_{x,C}.$$

In fact one can pass to another complex whose cohomologies are the same as of the adelic complex and which is a generalization of the construction for dimension one. We have to make the following assumptions: $P \in C$ is a smooth point on both $C$ and $X$, and the surface $X \setminus C$ is affine. The desired complex is

$$A \oplus A_C \oplus \widehat{\mathcal{O}}_P \to B_C \oplus B_P \oplus \widehat{\mathcal{O}}_{P,C} \to K_{P,C}$$

where the rings $B_x$, $B_C$, $A_C$ and $A$ have the following meaning. Let $x \in C$. Let

$B_x = \bigcap\limits_{D \neq C} (K_x \cap \widehat{\mathcal{O}}_{x,D})$ where the intersection is taken inside $K_x$;

$B_C = K_C \cap ( \bigcap\limits_{x \neq P} B_x)$ where the intersection is taken inside $K_{x,C}$;

$A_C = B_C \cap \widehat{\mathcal{O}}_C$, $A = K \cap (\bigcap_{x \in X \setminus C} \widehat{\mathcal{O}}_x)$.

This can be easily extended to the case of an arbitrary torsion free coherent sheaf $\mathcal{F}$ on $X$.

**1.2.3.** Returning back to the question about the $\zeta$-function of the surface $X$ over $k = \mathbb{F}_q$ we suggest to write it as the product of three Dirichlet series

$$\zeta_X(s) = \zeta_{X\setminus C}(s)\zeta_{C\setminus P}(s)\zeta_P(s) = \Big(\sum_{I \subset X \setminus C} |I|_X^s\Big)\Big(\sum_{I \subset C\setminus P} |I|_X^s\Big)\Big(\sum_{I \subset \mathrm{Spec}(\widehat{\mathcal{O}}_{P,C})} |I|_X^s\Big).$$

Again we can assume that the surface $U = X \setminus C$ has the most trivial possible structure. Namely, $\mathrm{Pic}(U) = (0)$ and $\mathrm{Ch}(U) = (0)$. Then every rank 2 vector bundle on $U$ is trivial. In the general case one can remove finitely many curves $C$ from $X$ to pass to the surface $U$ satisfying these properties (the same idea was used in the construction of the higher Bruhat–Tits buildings attached to an algebraic surface [P3, sect. 3]).

Therefore any zero-ideal $I$ with support in $X \setminus C$, $C \setminus P$ or $P$ can be defined by functions from the rings $A$, $A_C$ and $\mathcal{O}_P$, respectively. The fundamental difference between the case of dimension one and the case of surfaces is that zero-cycles $I$ and ideals of finite colength on $X$ are not in one-to-one correspondence.

**Remark 2.** In [P2], [FP] we show that the functional equation for the $L$-function on an algebraic surface over a finite field can be rewritten using the $K_2$-adeles. Then it has the same shape as the functional equation for algebraic curves written in terms of $\mathbb{A}^*$-adeles (as in [W1]).

## 1.3. Types for dimension 1

We again discuss the case of dimension one. If $D$ is a divisor on the curve $C$ then the Riemann–Roch theorem says

$$l(D) - l(K_C - D) = \deg(D) + \chi(\mathcal{O}_C),$$

where as usual $l(D) = \dim\ \Gamma(C, \mathcal{O}_X(D))$ and $K_C$ is the canonical divisor. If $\mathbb{A} = \mathbb{A}_C$ and $\mathbb{A}_1 = \mathbb{A}(D)$ then

$$H^1(C, \mathcal{O}_X(D)) = \mathbb{A}/(\mathbb{A}(D) + K), \qquad H^0(C, \mathcal{O}_X(D)) = \mathbb{A}(D) \cap K$$

where $K = \mathbb{F}_q(C)$. We have the following topological properties of the groups:

| | |
|---|---|
| $\mathbb{A}$ | locally compact group, |
| $\mathbb{A}(D)$ | compact group, |
| $K$ | discrete group, |
| $\mathbb{A}(D) \cap K$ | finite group, |
| $\mathbb{A}(D) + K$ | group of finite index of $\mathbb{A}$. |

The group $\mathbb{A}$ is dual to itself. Fix a rational differential form $\omega \in \Omega^1_K$, $\omega \neq 0$ and an additive character $\psi$ of $\mathbb{F}_q$. The following bilinear form

$$\langle (f_x), (g_x) \rangle = \sum_x \operatorname{res}_x(f_x g_x \omega), \ (f_x), (g_x) \in \mathbb{A}$$

is non-degenerate and defines an auto-duality of $\mathbb{A}$.

If we fix a Haar measure $dx$ on $\mathbb{A}$ then we also have the Fourier transform

$$f(x) \mapsto \widehat{f}(x) = \int_{\mathbb{A}} \psi(\langle x, y \rangle) f(y) dy$$

for functions on $\mathbb{A}$ and for distributions $F$ defined by the Parseval equality

$$(\widehat{F}, \widehat{\phi}) = (F, \phi).$$

One can attach some functions and/or distributions to the subgroups introduced above

$$\begin{aligned}
\delta_D &= \text{the characteristic function of } \mathbb{A}(D) \\
\delta_{H^1} &= \text{the characteristic function of } \mathbb{A}(D) + K \\
\delta_K &= \sum_{\gamma \in K} \delta_\gamma \quad \text{where } \delta_\gamma \text{ is the delta-function at the point } \gamma \\
\delta_{H^0} &= \sum_{\gamma \in \mathbb{A}(D) \cap K} \delta_\gamma .
\end{aligned}$$

There are two fundamental rules for the Fourier transform of these functions

$$\widehat{\delta}_D = \operatorname{vol}(\mathbb{A}(D)) \delta_{\mathbb{A}(D)^\perp},$$

where

$$\mathbb{A}(D)^{\perp} = \mathbb{A}((\omega) - D),$$

and

$$\widehat{\delta}_{\Gamma} = \mathrm{vol}(\mathbb{A}/\Gamma)^{-1}\delta_{\Gamma^{\perp}}$$

for any discrete co-compact group $\Gamma$. In particular, we can apply that to $\Gamma = K = \Gamma^{\perp}$. We have

$$(\delta_K, \delta_D) = \#(K \cap \mathbb{A}(D)) = q^{l(D)},$$
$$(\widehat{\delta}_K, \widehat{\delta}_D) = \mathrm{vol}(\mathbb{A}(D))\mathrm{vol}(\mathbb{A}/K)^{-1}(\delta_K, \delta_{K_C - D}) = q^{\deg D} q^{\chi(\mathcal{O}_C)} q^{l(K_C - D)}$$

and the Parseval equality gives us the Riemann–Roch theorem.

The functions in these computations can be classified according to their types. There are four types of functions which were introduced by F. Bruhat in 1961 [Br].

Let $V$ be a finite dimensional vector space over the adelic ring $\mathbb{A}$ (or over an one-dimensional local field $K$ with finite residue field $\mathbb{F}_q$ ). We put

$$\mathcal{D} = \{\text{locally constant functions with compact support}\},$$
$$\mathcal{E} = \{\text{locally constant functions}\},$$
$$\mathcal{D}' = \{\text{dual to } \mathcal{D} \ = \text{ all distributions}\},$$
$$\mathcal{E}' = \{\text{dual to } \mathcal{E} \ = \text{ distributions with compact support}\}.$$

Every $V$ has a filtration $P \supset Q \supset R$ by compact open subgroups such that all quotients $P/Q$ are finite dimensional vector spaces over $\mathbb{F}_q$.

If $V, V'$ are the vector spaces over $\mathbb{F}_q$ of finite dimension then for every homomorphism $i\colon V \to V'$ there are two maps

$$\mathcal{F}(V) \xrightarrow{i_*} \mathcal{F}(V'), \qquad \mathcal{F}(V') \xrightarrow{i^*} \mathcal{F}(V),$$

of the spaces $\mathcal{F}(V)$ of all functions on $V$ (or $V'$ ) with values in $\mathbb{C}$. Here $i^*$ is the standard inverse image and $i_*$ is defined by

$$i_* f(v') = \begin{cases} 0, & \text{if } v' \notin \mathrm{im}(i) \\ \sum_{v \mapsto v'} f(v), & \text{otherwise.} \end{cases}$$

The maps $i_*$ and $i^*$ are dual to each other.

We apply these constructions to give a more functorial definition of the Bruhat spaces. For any triple $P$, $Q$, $R$ as above we have an epimorphism $i\colon P/R \to P/Q$ with the corresponding map for functions $\mathcal{F}(P/Q) \xrightarrow{i^*} \mathcal{F}(P/R)$ and a monomorphism $j\colon Q/R \to P/R$ with the map for functions $\mathcal{F}(Q/R) \xrightarrow{j_*} \mathcal{F}(P/R)$.

Now the Bruhat spaces can be defined as follows

$$\begin{aligned}
\mathcal{D} &= \varinjlim_{j_*} \varinjlim_{i^*} \mathcal{F}(P/Q),\\
\mathcal{E} &= \varprojlim_{j^*} \varinjlim_{i^*} \mathcal{F}(P/Q),\\
\mathcal{D}' &= \varprojlim_{j^*} \varprojlim_{i_*} \mathcal{F}(P/Q),\\
\mathcal{E}' &= \varinjlim_{j_*} \varprojlim_{i_*} \mathcal{F}(P/Q).
\end{aligned}$$

The spaces don't depend on the choice of the chain of subspaces $P, Q, R$. Clearly we have

$$\begin{aligned}
\delta_D &\in \mathcal{D}(\mathbb{A}),\\
\delta_K &\in \mathcal{D}'(\mathbb{A}),\\
\delta_{H^0} &\in \mathcal{E}'(\mathbb{A}),\\
\delta_{H^1} &\in \mathcal{E}(\mathbb{A}).
\end{aligned}$$

We have the same relations for the functions $\delta_{\mathcal{O}_P}$ and $\delta_{A''}$ on the group $K_P$ considered in section 1.

The Fourier transform preserves the spaces $\mathcal{D}$ and $\mathcal{D}'$ but interchanges the spaces $\mathcal{E}$ and $\mathcal{E}'$. Recalling the origin of the subgroups from the adelic complex we can say that, in dimension one the types of the functions have the following structure

$$\begin{tikzcd}
& \mathcal{E} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathcal{D} \arrow[dr, dash] & & \mathcal{D}' \arrow[dl, dash] \\
& \mathcal{E}' &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathbf{01} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbf{1} \arrow[dr, dash] & & \mathbf{0} \arrow[dl, dash] \\
& \emptyset &
\end{tikzcd}$$

corresponding to the full simplicial division of an edge. The Fourier transform is a reflection of the diagram with respect to the middle horizontal axis.

The main properties of the Fourier transform we need in the proof of the Riemann-Roch theorem (and of the functional equation of the $\zeta$-function) can be summarized as the commutativity of the following cube diagram

$$\begin{tikzcd}
& \mathcal{D}\otimes\mathcal{D}' \arrow[rr, "j_*"] \arrow[dl, leftrightarrow] \arrow[dd, "i^*"] & & \mathcal{E} \arrow[dl, leftrightarrow] \arrow[dd, "\alpha^*"] \\
\mathcal{D}\otimes\mathcal{D}' \arrow[rr, "i^*"] \arrow[dd, "j_*"] & & \mathcal{E}' \arrow[dd, "\beta_*"] & \\
& \mathcal{E}' \arrow[rr, "\beta_*"] \arrow[dl, leftrightarrow] & & \mathcal{F}(\mathbb{F}_1) \arrow[dl, dash] \\
\mathcal{E} \arrow[rr, "\alpha^*"] & & \mathcal{F}(\mathbb{F}_1) &
\end{tikzcd}$$

coming from the exact sequence

$$\mathbb{A} \xrightarrow{i} \mathbb{A} \oplus \mathbb{A} \xrightarrow{j} \mathbb{A},$$

with $i(a) = (a, a)$, $j(a, b) = a - b$, and the maps

$$\mathbb{F}_1 \xrightarrow{\alpha} \mathbb{A} \xrightarrow{\beta} \mathbb{F}_1$$

with $\alpha(0) = 0$, $\beta(a) = 0$. Here $\mathbb{F}_1$ is the field of one element, $\mathcal{F}(\mathbb{F}_1) = \mathbb{C}$ and the arrows with heads on both ends are the Fourier transforms.

In particular, the commutativity of the diagram implies the Parseval equality used above:

$$\begin{aligned}
&\langle \widehat{F}, \widehat{G} \rangle = \beta_* \circ i^*(\widehat{F} \otimes \widehat{G}) \\
&= \beta_* \circ i^*(\widehat{F \otimes G}) = \beta_* j_* \widehat{(F \otimes G)} \\
&= \alpha^* \circ j_*(F \otimes G) = \beta_* \circ i^*(F \otimes G) \\
&= \langle F, G \rangle.
\end{aligned}$$

**Remark 3.** These constructions can be extended to the function spaces on the groups $G(\mathbb{A})$ or $G(K)$ for a local field $K$ and a group scheme $G$.

## 1.4. Types for dimension 2

In order to understand the types of functions in the case of dimension 2 we have to look at the adelic complex of an algebraic surface. We will use physical notations and denote a space by the discrete index which corresponds to it. Thus the adelic complex can be written as

$$\emptyset \to \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2} \to \mathbf{01} \oplus \mathbf{02} \oplus \mathbf{12} \to \mathbf{012},$$

where $\emptyset$ stands for the augmentation map corresponding to the inclusion of $H^0$. Just as in the case of dimension one we have a duality of $\mathbb{A} = \mathbb{A}_{012} = \mathbf{012}$ with itself defined by a bilinear form

$$\langle (f_{x,C}), (g_{x,C}) \rangle = \sum_{x,C} \mathrm{res}_{x,C}(f_{x,C} g_{x,C}\, \omega), \qquad (f_{x,C}), (g_{x,C}) \in \mathbb{A}$$

which is also non-degenerate and defines the autoduality of $\mathbb{A}$.

It can be shown that

$$\mathbb{A}_0 = \mathbb{A}_{01} \cap \mathbb{A}_{02}, \quad \mathbb{A}_{01}^{\perp} = \mathbb{A}_{01}, \quad \mathbb{A}_{02}^{\perp} = \mathbb{A}_{02}, \quad \mathbb{A}_0^{\perp} = \mathbb{A}_{01} \oplus \mathbb{A}_{02},$$

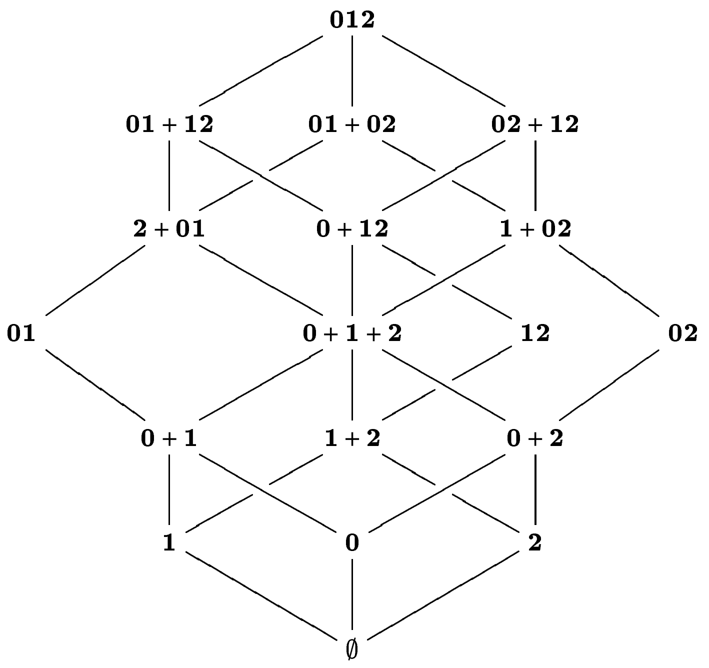
and so on. The proofs depend on the following residue relations for a rational differential form $\omega \in \Omega^2_{k(X)}$

$$\text{for all } x \in X \qquad \sum_{C \ni x} \operatorname{res}_{x,C}(\omega) = 0,$$
$$\text{for all } C \subset X \qquad \sum_{x \in C} \operatorname{res}_{x,C}(\omega) = 0.$$

We see that the subgroups appearing in the adelic complex are not closed under the duality. It means that the set of types in dimension two will be greater then the set of types coming from the components of the adelic complex. Namely, we have:

**Theorem 1** ([P4]). *Fix a divisor $D$ on an algebraic surface $X$ and let $\mathbb{A}_{12} = \mathbb{A}(D)$. Consider the lattice $\mathfrak{L}$ of the commensurability classes of subspaces in $\mathbb{A}_X$ generated by subspaces $\mathbb{A}_{01}, \mathbb{A}_{02}, \mathbb{A}_{12}$.*

*The lattice $\mathfrak{L}$ is isomorphic to a free distributive lattice in three generators and has the structure shown in the diagram.*




**Remark 4.** Two subspaces $V, V'$ are called commensurable if $(V + V')/V \cap V'$ is of finite dimension. In the one-dimensional case *all* the subspaces of the adelic complex are commensurable (even the subspaces corresponding to different divisors). In this case we get a free distributive lattice in two generators (for the theory of lattices see [Bi]).

Just as in the case of curves we can attach to every node some space of functions (or distributions) on $\mathbb{A}$. We describe here a particular case of the construction, namely, the space $\mathcal{F}_{02}$ corresponding to the node $\mathbf{02}$. Also we will consider not the full adelic group but a single two-dimensional local field $K = \mathbb{F}_q((u))((t))$.

In order to define the space we use the filtration in $K$ by the powers $\mathcal{M}^n$ of the maximal ideal $\mathcal{M} = \mathbb{F}_q((u))[[t]]t$ of $K$ as a discrete valuation (of rank 1) field. Then we try to use the same procedure as for the local field of dimension 1 (see above).

If $P \supset Q \supset R$ are the elements of the filtration then we need to define the maps

$$\mathcal{D}(P/R) \xrightarrow{i_*} \mathcal{D}(P/Q), \qquad \mathcal{D}(P/R) \xrightarrow{j^*} \mathcal{D}(Q/R)$$

corresponding to an epimorphism $i\colon P/R \to P/Q$ and a monomorphism $j\colon Q/R \to P/R$. The map $j^*$ is a restriction of the locally constant functions with compact support and it is well defined. To define the direct image $i_*$ one needs to integrate along the fibers of the projection $i$. To do that we have to choose a Haar measure on the fibers for *all* $P$, $Q$, $R$ in a consistent way. In other words, we need a system of Haar measures on all quotients $P/Q$ and by transitivity of the Haar measures in exact sequences it is enough to do that on all quotients $\mathcal{M}^n/\mathcal{M}^{n+1}$.

Since $\mathcal{O}_K/\mathcal{M} = \mathbb{F}_q((u)) = K_1$ we can first choose a Haar measure on the residue field $K_1$. It will depend on the choice of a fractional ideal $\mathcal{M}^i_{K_1}$ normalizing the Haar measure. Next, we have to extend the measure on all $\mathcal{M}^n/\mathcal{M}^{n+1}$. Again, it is enough to choose a second local parameter $t$ which gives an isomorphism

$$t^n\colon \mathcal{O}_K/\mathcal{M} \to \mathcal{M}^n/\mathcal{M}^{n+1}.$$

Having made these choices we can put as above

$$\mathcal{F}_{02} = \varprojlim{}_{j^*} \varprojlim{}_{i_*} \mathcal{D}(P/Q)$$

where the space $\mathcal{D}$ was introduced in the previous section.

We see that contrary to the one-dimensional case the space $\mathcal{F}_{02}$ is not intrinsically defined. But the choice of all additional data can be easily controlled.

**Theorem 2** ([P4])**.** *The set of the spaces $\mathcal{F}_{02}$ is canonically a principal homogeneous space over the valuation group $\Gamma_K$ of the field $K$.*

Recall that $\Gamma_K$ is non-canonically isomorphic to the lexicographically ordered group $\mathbb{Z} \oplus \mathbb{Z}$.

One can extend this procedure to other nodes of the diagram of types. In particular, for $\mathbf{012}$ we get the space which does not depend on the choice of the Haar measures.

The standard subgroup of the type $\mathbf{02}$ is $B_P = \mathbb{F}_p[[u]]((t))$ and it is clear that

$$\delta_{B_P} \in \mathcal{F}_{02}.$$

The functions $\delta_{B_C}$ and $\delta_{\widehat{\mathcal{O}}_{P,C}}$ have the types $\mathbf{01}$, $\mathbf{12}$ respectively.

**Remark 5.** Note that the whole structure of all subspaces in $\mathbb{A}$ or $K$ corresponding to different divisors or coherent sheaves is more complicated. The spaces $\mathbb{A}(D)$ of type $\mathbf{12}$ are no more commensurable. To describe the whole lattice one has to introduce several equivalence relations (commensurability up to compact subspace, a locally compact subspace and so on).

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ISSN 1464-8997 (on line) 1464-8989 (printed) 

Geometry & Topology Monographs
Volume 3: Invitation to higher local fields
Part II, section 2, pages 215–221

# 2. Adelic constructions for direct images of differentials and symbols

*Denis Osipov*

## 2.0. Introduction

Let $X$ be a smooth algebraic surface over a perfect field $k$.

Consider pairs $x \in C$, $x$ is a closed point of $X$, $C$ is either an irreducible curve on $X$ which is smooth at $x$, or an irreducible analytic branch near $x$ of an irreducible curve on $X$. As in the previous section 1 for every such pair $x \in C$ we get a two-dimensional local field $K_{x,C}$.

If $X$ is a projective surface, then from the adelic description of Serre duality on $X$ there is a local decomposition for the trace map $H^2(X, \Omega^2_X) \to k$ by using a two-dimensional residue map $\mathrm{res}_{K_{x,C}/k(x)}\colon \Omega^2_{K_{x,C}/k(x)} \to k(x)$ (see [P1]).

From the adelic interpretation of the divisors intersection index on $X$ there is a similar local decomposition for the global degree map from the group $CH^2(X)$ of algebraic cycles of codimension 2 on $X$ modulo the rational equivalence to $\mathbb{Z}$ by means of explicit maps from $K_2(K_{x,C})$ to $\mathbb{Z}$ (see [P3]).

Now we pass to the relative situation. Further assume that $X$ is any smooth surface, but there are a smooth curve $S$ over $k$ and a smooth projective morphism $f\colon X \to S$ with connected fibres. Using two-dimensional local fields and explicit maps we describe in this section a local decomposition for the maps

$$f_*\colon H^n(X, \Omega^2_X) \to H^{n-1}(S, \Omega^1_S), \quad f_*\colon H^n(X, \mathcal{K}_2(X)) \to H^{n-1}(S, \mathcal{K}_1(S))$$

where $\mathcal{K}$ is the Zariski sheaf associated to the presheaf $U \to K(U)$. The last two groups have the following geometric interpretation:

$$H^n(X, \mathcal{K}_2(X)) = CH^2(X, 2-n), \quad H^{n-1}(S, \mathcal{K}_1(S)) = CH^1(S, 2-n)$$

where $CH^2(X, 2-n)$ and $CH^1(S, 1-n)$ are higher Chow groups on $X$ and $S$ (see [B]). Note also that $CH^2(X, 0) = CH^2(X)$, $CH^1(S, 0) = CH^1(S) = \mathrm{Pic}(S)$, $CH^1(S, 1) = H^0(S, \mathcal{O}^*_S)$.

*Published 10 December 2000:* 

Let $s = f(x) \in S$. There is a canonical embedding $f^*\colon K_s \to K_{x,C}$ where $K_s$ is the quotient of the completion of the local ring of $S$ at $s$.

Consider two cases:

(1) $C \neq f^{-1}(s)$. Then $K_{x,C}$ is non-canonically isomorphic to $k(C)_x((t_C))$ where $k(C)_x$ is the completion of $k(C)$ at $x$ and $t_C$ is a local equation of $C$ near $x$.

(2) $C = f^{-1}(s)$. Then $K_{x,C}$ is non-canonically isomorphic to $k(x)((u))((t_s))$ where $\{u = 0\}$ is a transversal curve at $x$ to $f^{-1}(s)$ and $t_s \in K_s$ is a local parameter at $s$, i.e. $k(s)((t_s)) = K_s$.

## 2.1. Local constructions for differentials

**Definition.** For $K = k((u))((t))$ let $U = u^i k[[u,t]]dk[[u,t]] + t^j k((u))[[t]]dk((u))[[t]]$ be a basis of neighbourhoods of zero in $\Omega^1_{k((u))[[t]]/k}$ (compare with 1.4.1 of Part I). Let $\widetilde{\Omega}^1_K = \Omega^1_{K/k}/(K \cdot \cap U)$ and $\widetilde{\Omega}^n_K = \wedge^n \widetilde{\Omega}^1_K$. Similarly define $\widetilde{\Omega}^n_{K_s}$.

Note that $\widetilde{\Omega}^2_{K_{x,C}}$ is a one-dimensional space over $K_{x,C}$; and $\widetilde{\Omega}^n_{K_{x,C}}$ does not depend on the choice of a system of local parameters of $\widehat{\mathcal{O}}_x$, where $\widehat{\mathcal{O}}_x$ is the completion of the local ring of $X$ at $x$.

**Definition.** For $K = k((u))((t))$ and $\omega = \sum_i \omega_i(u) \wedge t^i dt = \sum_i u^i du \wedge \omega'_i(t) \in \widetilde{\Omega}^2_K$ put

$$\operatorname{res}_t(\omega) = \omega_{-1}(u) \in \widetilde{\Omega}^1_{k((u))},$$
$$\operatorname{res}_u(\omega) = \omega'_{-1}(t) \in \widetilde{\Omega}^1_{k((t))}.$$

Define a relative residue map

$$f_*^{x,C}\colon \widetilde{\Omega}^2_{K_{x,C}} \to \widetilde{\Omega}^1_{K_s}$$

as

$$f_*^{x,C}(\omega) = \begin{cases} \operatorname{Tr}_{k(C)_x/K_s} \operatorname{res}_{t_C}(\omega) & \text{if } C \neq f^{-1}(s) \\ \operatorname{Tr}_{k(x)((t_s))/K_s} \operatorname{res}_u(\omega) & \text{if } C = f^{-1}(s). \end{cases}$$

The relative residue map doesn't depend on the choice of local parameters.

**Theorem** (reciprocity laws for relative residues)**.** *Fix $x \in X$. Let $\omega \in \widetilde{\Omega}^2_{K_x}$ where $K_x$ is the minimal subring of $K_{x,C}$ which contains $k(X)$ and $\widehat{\mathcal{O}}_x$. Then*

$$\sum_{C \ni x} f_*^{x,C}(\omega) = 0.$$

*Fix $s \in S$. Let $\omega \in \widetilde{\Omega}^2_{K_F}$ where $K_F$ is the completion of $k(X)$ with respect to the discrete valuation associated with the curve $F = f^{-1}(s)$. Then*

$$\sum_{x\in F} f_*^{x,F}(\omega) = 0.$$

See [O].

## 2.2. The Gysin map for differentials

**Definition.** In the notations of subsection 1.2.1 in the previous section put

$$\Omega^1_{\mathbb{A}_S} = \{(f_s dt_s) \in \prod_{s\in S} \widetilde{\Omega}^1_{K_s}, \quad v_s(f_s) \geqslant 0 \text{ for almost all } s \in S\}$$

where $t_s$ is a local parameter at $s$, $v_s$ is the discrete valuation associated to $t_s$ and $K_s$ is the quotient of the completion of the local ring of $S$ at $s$. For a divisor $I$ on $S$ define

$$\Omega_{\mathbb{A}_S}(I) = \{(f_s) \in \Omega^1_{\mathbb{A}_S} : v_s(f_s) \geqslant -v_s(I) \quad \text{for all } s \in S\}.$$

Recall that the $n$-th cohomology group of the following complex

$$\begin{array}{ccc} \Omega^1_{k(S)/k} \oplus \Omega^1_{\mathbb{A}_S}(0) & \longrightarrow & \Omega^1_{\mathbb{A}_S} \\ (f_0, f_1) & \longmapsto & f_0 + f_1. \end{array}$$

is canonically isomorphic to $H^n(S, \Omega^1_S)$ (see [S, Ch.II]).

The sheaf $\Omega^2_X$ is invertible on $X$. Therefore, Parshin's theorem (see [P1]) shows that similarly to the previous definition and definition in 1.2.2 of the previous section for the complex $\Omega^2(\mathcal{A}_X)$

$$\begin{array}{ccccc} \Omega^2_{A_0} \oplus \Omega^2_{A_1} \oplus \Omega^2_{A_2} & \longrightarrow & \Omega^2_{A_{01}} \oplus \Omega^2_{A_{02}} \oplus \Omega^2_{A_{12}} & \longrightarrow & \Omega^2_{A_{012}} \\ (f_0, f_1, f_2) & \longmapsto & (f_0 + f_1, f_2 - f_0, -f_1 - f_2) & & \\ & & (g_1, g_2, g_3) & \longmapsto & g_1 + g_2 + g_3 \end{array}$$

where

$$\Omega^2_{A_i} \subset \Omega^2_{A_{ij}} \subset \Omega^2_{A_{012}} = \Omega^2_{\mathbb{A}_X} = {\prod_{x\in C}}' \widetilde{\Omega}^2_{K_{x,C}} \subset \prod_{x\in C} \widetilde{\Omega}^2_{K_{x,C}}$$

there is a canonical isomorphism

$$H^n(\Omega^2(\mathcal{A}_X)) \simeq H^n(X, \Omega^2_X).$$

Using the reciprocity laws above one can deduce:

**Theorem.** *The map* $f_* = \sum_{C\ni x, f(x)=s} f_*^{x,C}$ *from* $\Omega^2_{\mathbb{A}_X}$ *to* $\Omega^1_{\mathbb{A}_S}$ *is well defined. It maps the complex* $\Omega^2(\mathcal{A}_X)$ *to the complex*

$$0 \longrightarrow \Omega^1_{k(S)/k} \oplus \Omega^1_{\mathbb{A}_S}(0) \longrightarrow \Omega^1_{\mathbb{A}_S}.$$

*It induces the map* $f_*\colon H^n(X,\Omega^2_X) \to H^{n-1}(S,\Omega^1_S)$ *of 2.0.*

See [O].

## 2.3. Local constructions for symbols

Assume that $k$ is of characteristic 0.

**Theorem.** *There is an explicitly defined symbolic map*

$$f_*(\ ,\ )_{x,C}\colon K^*_{x,C} \times K^*_{x,C} \to K^*_s$$

(*see remark below* ) *which is uniquely determined by the following properties*

$$N_{k(x)/k(s)}\ t_{K_{x,C}}(\alpha,\beta,f^*\gamma) = t_{K_s}(f_*(\alpha,\beta)_{x,C},\gamma) \quad \textit{for all } \alpha,\beta \in K^*_{x,C},\ \gamma \in K^*_s$$

*where* $t_{K_{x,C}}$ *is the tame symbol of the two-dimensional local field* $K_{x,C}$ *and* $t_{K_s}$ *is the tame symbol of the one-dimensional local field* $K_s$ (*see 6.4.2 of Part I*);

$$\mathrm{Tr}_{\,k(x)/k(s)}(\alpha,\beta,f^*(\gamma)]_{K_{x,C}} = (f_*(\alpha,\beta)_{x,C},\gamma]_{K_s} \quad \textit{for all } \alpha,\beta \in K^*_{x,C},\ \gamma \in K_s$$

*where* $(\alpha,\beta,\gamma]_{K_{x,C}} = \mathrm{res}_{K_{x,C}/k(x)}(\gamma d\alpha/\alpha \wedge d\beta/\beta)$ *and* $(\alpha,\beta]_{K_s} = \mathrm{res}_{K_s/k(s)}(\alpha d\beta/\beta)$.

The map $f_*(\ ,\ )_{x,C}$ induces the map

$$f_*(\ ,\ )_{x,C}\colon K_2(K_{x,C}) \to K_1(K_s).$$

**Corollary** (reciprocity laws). *Fix a point* $s \in S$. *Let* $F = f^{-1}(s)$.

*Let* $\alpha,\beta \in K^*_F$. *Then*

$$\prod_{x\in F} f_*(\alpha,\beta)_{x,F} = 1.$$

*Fix a point* $x \in F$. *Let* $\alpha,\beta \in K^*_x$. *Then*

$$\prod_{C\ni x} f_*(\alpha,\beta)_{x,C} = 1.$$

**Remark.** If $C \neq f^{-1}(s)$ then $f_*(\ ,\ )_{x,C} = N_{k(C)_x/K_s}\, t_{K_{x,C}}$ where $t_{K_{x,C}}$ is the tame symbol with respect to the discrete valuation of rank 1 on $K_{x,C}$.

If $C = f^{-1}(s)$ then $f_*(\ ,\ )_{x,C} = N_{k(x)((t_s))/K_s}(\ ,\ )_f$ where $(\ ,\ )_f^{-1}$ coincides with Kato's residue homomorphism [K, §1]. An explicit formula for $(\ ,\ )_f$ is constructed in [O, Th.2].

## 2.4. The Gysin map for Chow groups

Assume that $k$ is of arbitrary characteristic.

**Definition.** Let $K_2'(\mathbb{A}_X)$ be the subset of all $(f_{x,C}) \in K_2(K_{x,C})$, $x \in C$ such that

(a) $f_{x,C} \in K_2(\mathcal{O}_{x,C})$ for almost all irreducible curves $C$ where $\mathcal{O}_{x,C}$ is the ring of integers of $K_{x,C}$ with respect to the discrete valuation of rank 1 on it;

(b) for all irreducible curves $C \subset X$, all integers $r \geqslant 1$ and almost all points $x \in C$

$$f_{x,C} \in K_2(\mathcal{O}_{x,C}, \mathcal{M}_C^r) + K_2(\widehat{\mathcal{O}}_x[t_C^{-1}]) \subset K_2(K_{x,C})$$

where $\mathcal{M}_C$ is the maximal ideal of $\mathcal{O}_{x,C}$ and $K_2(A, J) = \ker(K_2(A) \to K_2(A/J))$.

This definition is similar to the definition of [P2].

**Definition.** Using the diagonal map of $K_2(K_C)$ to $\prod_{x\in C} K_2(K_{x\in C})$ and of $K_2(K_x)$ to $\prod_{C\ni x} K_2(K_{x\in C})$ put

$$K_2'(A_{01}) = K_2'(\mathbb{A}_X) \cap \text{image of} \prod_{C\subset X} K_2(K_C),$$
$$K_2'(A_{02}) = K_2'(\mathbb{A}_X) \cap \text{image of} \prod_{x\in X} K_2(K_x),$$
$$K_2'(A_{12}) = K_2'(\mathbb{A}_X) \cap \text{image of} \prod_{x\in C} K_2(\mathcal{O}_{x,C}),$$
$$K_2'(A_0) = K_2(k(X)),$$
$$K_2'(A_1) = K_2'(\mathbb{A}_X) \cap \text{image of} \prod_{C\subset X} K_2(\mathcal{O}_C),$$
$$K_2'(A_2) = K_2'(\mathbb{A}_X) \cap \text{image of} \prod_{x\in X} K_2(\widehat{\mathcal{O}}_x)$$

where $\mathcal{O}_C$ is the ring of integers of $K_C$.

Define the complex $K_2(\mathcal{A}_X)$:

$$K_2'(A_0) \oplus K_2'(A_1) \oplus K_2'(A_2) \to K_2'(A_{01}) \oplus K_2'(A_{02}) \oplus K_2'(A_{12}) \to K_2'(A_{012})$$
$$(f_0, f_1, f_2) \mapsto (f_0 + f_1, f_2 - f_0, -f_1 - f_2)$$
$$(g_1, g_2, g_3) \quad \mapsto \quad g_1 + g_2 + g_3$$

where $K_2'(A_{012}) = K_2'(\mathbb{A}_X)$.

Using the Gersten resolution from $K$-theory (see [Q, §7]) one can deduce:

**Theorem.** *There is a canonical isomorphism*

$$H^n(K_2(\mathcal{A}_X)) \simeq H^n(X, \mathcal{K}_2(X)).$$

Similarly one defines $K_1'(\mathbb{A}_S)$. From $H^1(S, \mathcal{K}_1(S)) = H^1(S, \mathcal{O}_S^*) = \mathrm{Pic}(S)$ (or from the approximation theorem) it is easy to see that the $n$-th cohomology group of the following complex

$$\begin{array}{ccc} K_1(k(S)) \oplus \sum_{s\in S} K_1(\widehat{\mathcal{O}}_s) & \longrightarrow & K_1'(\mathbb{A}_S) \\ (f_0, f_1) & \longmapsto & f_0 + f_1. \end{array}$$

is canonically isomorphic to $H^n(S, \mathcal{K}_1(S))$ (here $\widehat{\mathcal{O}}_s$ is the completion of the local ring of $C$ at $s$).

Assume that $k$ is of characteristic 0.

Using the reciprocity law above and the previous theorem one can deduce:

**Theorem.** *The map* $f_* = \sum_{C\ni x, f(x)=s} f_*(\ ,\ )_{x,C}$ *from* $K_2'(\mathbb{A}_X)$ *to* $K_1'(\mathbb{A}_S)$ *is well defined. It maps the complex* $K_2(\mathcal{A}_X)$ *to the complex*

$$0 \longrightarrow K_1(k(S)) \oplus \sum_{s\in S} K_1(\widehat{\mathcal{O}}_s) \longrightarrow K_1'(\mathbb{A}_S).$$

*It induces the map* $f_*\colon H^n(X, \mathcal{K}_2(X)) \to H^{n-1}(S, \mathcal{K}_1(S))$ *of 2.0.*

*If* $n = 2$, *then the last map is the direct image morphism* (*Gysin map*) *from* $CH^2(X)$ *to* $CH^1(S)$.

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*ISSN 1464-8997 (on line) 1464-8989 (printed)* 

Geometry & Topology Monographs
Volume 3: Invitation to higher local fields
Part II, section 3, pages 223–237

# 3. The Bruhat–Tits buildings over higher dimensional local fields

*A. N. Parshin*

## 3.0. Introduction

A generalization of the Bruhat–Tits buildings for the groups $PGL(V)$ over $n$-dimensional local fields was introduced in [P1]. The main object of the classical Bruhat–Tits theory is a simplicial complex attached to any reductive algebraic group $G$ defined over a field $K$. There are two parallel theories in the case where $K$ has no additional structure or $K$ is a local (or more generally, complete discrete valuation) field. They are known as the *spherical* and *euclidean* buildings correspondingly (see subsection 3.2 for a brief introduction, [BT1], [BT2] for original papers and [R], [T1] for the surveys).

In the generalized theory of buildings they correspond to local fields of dimension zero and of dimension one. The construction of the Bruhat–Tits building for the group $PGL(2)$ over two-dimensional local field was described in detail in [P2]. Later V. Ginzburg and M. Kapranov extended the theory to arbitrary reductive groups over a two-dimensional local fields [GK]. Their definition coincides with ours for $PGL(2)$ and is different for higher ranks. But it seems that they are closely related (in the case of the groups of type $A_l$). It remains to develop the theory for arbitrary reductive groups over local fields of dimension greater than two.

In this work we describe the structure of the higher building for the group $PGL(3)$ over a two-dimensional local field. We refer to [P1], [P2] for the motivation of these constructions.

This work contains four subsections. In 3.1 we collect facts about the Weyl group. Then in 3.2 we briefly describe the building for $PGL(2)$ over a local field of dimension not greater than two; for details see [P1], [P2]. In 3.3 we study the building for $PGL(3)$ over a local field $F$ of dimension one and in 3.4 we describe the building over a two-dimensional local field.

We use the notations of section 1 of Part I.

If $K$ is an $n$-dimensional local field, let $\Gamma_K$ be the valuation group of the discrete valuation of rank $n$ on $K^*$; the choice of a system of local parameters $t_1, \dots, t_n$ of $K$ induces an isomorphism of $\Gamma_K$ and the lexicographically ordered group $\mathbb{Z}^{\oplus n}$.

Let $K$ ($K = K_2,\ K_1,\ K_0 = k$) be a two-dimensional local field. Let $O = O_K$, $M = M_K$, $\mathcal{O} = \mathcal{O}_K$, $\mathcal{M} = \mathcal{M}_K$ (see subsection 1.1 of Part I). Then $O = \mathrm{pr}^{-1}(\mathcal{O}_{K_1})$, $M = \mathrm{pr}^{-1}(\mathcal{M}_{K_1})$ where $\mathrm{pr}\colon \mathcal{O}_K \to K_1$ is the residue map. Let $t_1, t_2$ be a system of local parameters of $K$.

If $K \supset \mathcal{O}$ is the fraction field of a ring $\mathcal{O}$ we call $\mathcal{O}$-submodules $J \subset K$ fractional $\mathcal{O}$-ideals (or simply fractional ideals).

The ring $O$ has the following properties:

(i) $O/M \simeq k, \quad K^* \simeq \langle t_1 \rangle \times \langle t_2 \rangle \times O^*, \quad O^* \simeq k^* \times (1 + M)$;

(ii) every finitely generated fractional $O$-ideal is principal and equal to

$$P(i,j) = (t_1^i t_2^j) \quad \text{for some } i, j \in \mathbb{Z}$$

(for the notation $P(i,j)$ see loc.cit.);

(iii) every infinitely generated fractional $O$-ideal is equal to

$$P(j) = \mathcal{M}_K^j = \langle t_1^i t_2^j : i \in \mathbb{Z} \rangle \quad \text{for some } j \in \mathbb{Z}$$

(see [FP], [P2] or section 1 of Part I). The set of these ideals is totally ordered with respect to the inclusion.

## 3.1. The Weyl group

Let $B$ be the image of

$$\begin{pmatrix} O & O & \dots & O \\ M & O & \dots & O \\ & & \dots & \\ M & M & \dots & O \end{pmatrix}$$

in $PGL(m, K)$. Let $N$ be the subgroup of monomial matrices.

**Definition 1.** Let $T = B \bigcap N$ be the image of

$$\begin{pmatrix} O^* & \dots & 0 \\ & \ddots & \\ 0 & \dots & O^* \end{pmatrix}$$

in $G$.

The group

$$W = W_{K/K_1/k} = N/T$$

is called the *Weyl group*.

There is a rich structure of subgroups in $G$ which have many common properties with the theory of BN-pairs. In particular, there are Bruhat, Cartan and Iwasawa decompositions (see [P2]).

The Weyl group $W$ contains the following elements of order two

$$s_i = \begin{pmatrix} 1 & \dots & 0 & & & 0 & \dots & 0 \\ & \ddots & & & & & & \\ 0 & \dots & 1 & & & & & 0 \\ 0 & \dots & & 0 & 1 & & \dots & 0 \\ 0 & & & 1 & 0 & & \dots & 0 \\ 0 & \dots & & & & 1 & \dots & 0 \\ & & & & & & \ddots & \\ 0 & \dots & 0 & & & 0 & \dots & 1 \end{pmatrix}, \quad i = 1, \dots, m-1;$$

$$w_1 = \begin{pmatrix} 0 & 0 & \dots & 0 & t_1 \\ 0 & 1 & \dots & 0 & 0 \\ & & \dots & & \\ & & \dots & & \\ & & \dots & 1 & 0 \\ t_1^{-1} & 0 & \dots & 0 & 0 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 0 & 0 & \dots & 0 & t_2 \\ 0 & 1 & \dots & 0 & 0 \\ & & \dots & & \\ & & \dots & & \\ & & \dots & 1 & 0 \\ t_2^{-1} & 0 & \dots & 0 & 0 \end{pmatrix}.$$

The group $W$ has the following properties:

(i) $W$ is generated by the set $S$ of its elements of order two,

(ii) there is an exact sequence

$$0 \to E \to W_{K/K_1/k} \to W_K \to 1,$$

where $E$ is the kernel of the addition map

$$\underbrace{\Gamma_K \oplus \cdots \oplus \Gamma_K}_{m \text{ times}} \to \Gamma_K$$

and $W_K$ is isomorphic to the symmetric group $S_m$;

(iii) the elements $s_i$, $i = 1, \dots, m-1$ define a splitting of the exact sequence and the subgroup $\langle s_1, \dots, s_{m-1} \rangle$ acts on $E$ by permutations.

In contrast with the situation in the theory of BN-pairs the pair $(W, S)$ is not a Coxeter group and furthermore there is no subset $S$ of involutions in $W$ such that $(W, S)$ is a Coxeter group (see [P2]).

## 3.2. Bruhat–Tits building for $PGL(2)$ over a local field of dimension $\leqslant 2$

In this subsection we briefly recall the main constructions. For more details see [BT1], [BT2], [P1], [P2].

**3.2.1.** Let $k$ be a field (which can be viewed as a 0-dimensional local field). Let $V$ be a vector space over $k$ of dimension two.

**Definition 2.** The *spherical building* of $PGL(2)$ over $k$ is a zero-dimensional complex

$$\Delta(k) = \Delta(PGL(V), k)$$

whose vertices are lines in $V$.

The group $PGL(2, k)$ acts on $\Delta(k)$ transitively. The Weyl group (in this case it is of order two) acts on $\Delta(k)$ and its orbits are *apartments* of the building.

**3.2.2.** Let $F$ be a complete discrete valuation field with residue field $k$. Let $V$ be a vector space over $F$ of dimension two. We say that $L \subset V$ is a lattice if $L$ is an $\mathcal{O}_F$-module. Two submodules $L$ and $L'$ belong to the same class $\langle L \rangle \sim \langle L' \rangle$ if and only if $L = aL'$, with $a \in F^*$.

**Definition 3.** The *euclidean building* of $PGL(2)$ over $F$ is a one-dimensional complex $\Delta(F/k)$ whose vertices are equivalence classes $\langle L \rangle$ of lattices. Two classes $\langle L \rangle$ and $\langle L' \rangle$ are connected by an edge if and only if for some choice of $L, L'$ there is an exact sequence

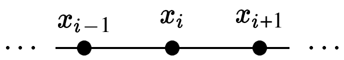
$$0 \to L' \to L \to k \to 0.$$

Denote by $\Delta_i(F/k)$ the set of $i$-dimensional simplices of the building $\Delta(F/k)$.

The following *link property* is important:

Let $P \in \Delta_0(F/k)$ be represented by a lattice $L$. Then the link of $P$ (= the set of edges of $\Delta(F/k)$ going from $P$) is in one-to-one correspondence with the set of lines in the vector space $V_P = L/\mathcal{M}_F L$ (which is $\Delta(PGL(V_P), k)$).

The orbits of the Weyl group $W$ (which is in this case an infinite group with two generators of order two) are infinite sets consisting of $x_i = \langle L_i \rangle$, $L_i = \mathcal{O}_F \oplus \mathcal{M}_F^i$.




An element $w$ of the Weyl group acts in the following way: if $w \in E = \mathbb{Z}$ then $w$ acts by translation of even length; if $w \notin E$ then $w$ acts as an involution with a unique fixed point $x_{i_0}$: $w(x_{i+i_0}) = x_{i_0-i}$.

To formalize the connection of $\Delta(F/k)$ with $\Delta(F)$ we define a *boundary* point of $\Delta(F/k)$ as a class of half-lines such that the intersection of every two half-lines from the class is a half-line in both of them. The set of the boundary points is called the *boundary* of $\Delta(F/k)$.

There is an isomorphism between $PGL(2,F)$-sets $\Delta(F)$ and the boundary of $\Delta(F/k)$: if a half-line is represented by $L_i = \mathcal{O}_F \oplus \mathcal{M}_F^i,\ i > 0$, then the corresponding vertex of $\Delta(F)$ is the line $F \oplus (0)$ in $V$.

It seems reasonable to slightly change the notations to make the latter isomorphisms more transparent.

**Definition 4** ([P1]). Put $\Delta_.[0](F/k) =$ the complex of classes of $\mathcal{O}_F$-submodules in $V$ isomorphic to $F \oplus \mathcal{O}_F$ (so $\Delta_.[0](F/k)$ is isomorphic to $\Delta(F)$) and put

$$\Delta_.[1](F/k) = \Delta(F/k).$$

Define the *building of* $PGL(2)$ *over* $F$ as the union

$$\Delta_.(F/k) = \Delta_.[1](F/k) \bigcup \Delta_.[0](F/k)$$

and call the subcomplex $\Delta_.[0](F/k)$ the *boundary* of the building. The discrete topology on the boundary can be extended to the whole building.

**3.2.3.** Let $K$ be a two-dimensional local field.

Let $V$ be a vector space over $K$ of dimension two. We say that $L \subset V$ is a lattice if $L$ is an $O$-module. Two submodules $L$ and $L'$ belong to the same class $\langle L\rangle \sim \langle L'\rangle$ if and only if $L = aL'$, with $a \in K^*$.

**Definition 5** ([P1]). Define the vertices of the building of $PGL(2)$ over $K$ as

$$\Delta_0[2](K/K_1/k) = \text{classes of } O\text{-submodules } L \subset V\colon\ L \simeq O \oplus O$$
$$\Delta_0[1](K/K_1/k) = \text{classes of } O\text{-submodules } L \subset V\colon\ L \simeq O \oplus \mathcal{O}$$
$$\Delta_0[0](K/K_1/k) = \text{classes of } O\text{-submodules } L \subset V\colon\ L \simeq O \oplus K.$$

Put

$$\Delta_0(K/K_1/k) = \Delta_0[2](K/K_1/k) \bigcup \Delta_0[1](K/K_1/k) \bigcup \Delta_0[0](K/K_1/k).$$

A set of $\{L_\alpha\},\ \alpha \in I$, of $O$-submodules in $V$ is called a *chain* if

(i) for every $\alpha \in I$ and for every $a \in K^*$ there exists an $\alpha' \in I$ such that $aL_\alpha = L_{\alpha'}$,

(ii) the set $\{L_\alpha, \alpha \in I\}$ is totally ordered by the inclusion.

A chain $\{L_\alpha, \alpha \in I\}$ is called a *maximal chain* if it cannot be included in a strictly larger set satisfying the same conditions (i) and (ii).

We say that $\langle L_0\rangle, \langle L_1\rangle, \ldots, \langle L_m\rangle$ belong to a *simplex* of dimension $m$ if and only if the $L_i,\ i = 0, 1, ..., m$ belong to a maximal chain of $\mathcal{O}_F$-submodules in $V$. The faces and the degeneracies can be defined in a standard way (as a deletion or repetition of a vertex). See [BT2].

Let $\{L_\alpha\}$ be a maximal chain of $O$-submodules in the space $V$. There are exactly three types of maximal chains ([P2]):

(i) if the chain contains a module $L$ isomorphic to $O \oplus O$ then all the modules of the chain are of that type and the chain is uniquely determined by its segment

$$\cdots \supset O \oplus O \supset M \oplus O \supset M \oplus M \supset \dots .$$

(ii) if the chain contains a module $L$ isomorphic to $O \oplus \mathcal{O}$ then the chain can be restored from the segment:

$$\cdots \supset O \oplus \mathcal{O} \supset O \oplus P(1,0) \supset O \oplus P(2,0) \supset \cdots \supset O \oplus \mathcal{M} \supset \dots$$

(recall that $P(1,0) = M$).

(iii) if the chain contains a module $L$ isomorphic to $\mathcal{O} \oplus \mathcal{O}$ then the chain can be restored from the segment:

$$\cdots \supset \mathcal{O} \oplus \mathcal{O} \supset P(1,0) \oplus \mathcal{O} \supset P(2,0) \oplus \mathcal{O} \supset \cdots \supset \mathcal{M} \oplus \mathcal{O} \supset \dots .$$

## 3.3. Bruhat–Tits building for $PGL(3)$ over a local field $F$ of dimension 1

Let $G = PGL(3)$.

Let $F$ be a one-dimensional local field, $F \supset \mathcal{O}_F \supset \mathcal{M}_F$, $\mathcal{O}_F/\mathcal{M}_F \simeq k$.

Let $V$ be a vector space over $F$ of dimension three. Define lattices in $V$ and their equivalence similarly to the definition of 3.2.2.

First we define the vertices of the building and then the simplices. The result will be a simplicial set $\Delta_.(G, F/k)$.

**Definition 6.** The *vertices* of the Bruhat–Tits building:

$$\begin{aligned}
\Delta_0[1](G, F/k) &= \{\text{classes of } \mathcal{O}_F\text{-submodules } L \subset V : L \simeq \mathcal{O}_F \oplus \mathcal{O}_F \oplus \mathcal{O}_F\},\\
\Delta_0[0](G, F/k) &= \{\text{classes of } \mathcal{O}_F\text{-submodules } L \subset V : L \simeq \mathcal{O}_F \oplus \mathcal{O}_F \oplus F\\
&\qquad \text{or } L \simeq \mathcal{O}_F \oplus F \oplus F\},\\
\Delta_0(G, F/k) &= \Delta_0[1](G, F/k) \cup \Delta_0[0](G, F/k).
\end{aligned}$$

We say that the points of $\Delta_0[1]$ are *inner* points, the points of $\Delta_0[0]$ are *boundary* points. Sometimes we delete $G$ and $F/k$ from the notation if this does not lead to confusion.

We have defined the vertices only. For the simplices of higher dimension we have the following:

**Definition 7.** Let $\{L_\alpha, \alpha \in I\}$ be a set of $\mathcal{O}_F$-submodules in $V$. We say that $\{L_\alpha, \alpha \in I\}$ is a *chain* if

(i) for every $\alpha \in I$ and for every $a \in K^*$ there exists an $\alpha' \in I$ such that $aL_\alpha = L_{\alpha'}$,
(ii) the set $\{L_\alpha, \alpha \in I\}$ is totally ordered by the inclusion.

A chain $\{L_\alpha, \alpha \in I\}$ is called a *maximal chain* if it cannot be included in a strictly larger set satisfying the same conditions (i) and (ii).

We say that $\langle L_0\rangle, \langle L_1\rangle, \dots, \langle L_m\rangle$ belong to a *simplex* of dimension $m$ if and only if the $L_i,\ i = 0, 1, ..., m$ belong to a maximal chain of $\mathcal{O}_F$-submodules in $V$. The faces and the degeneracies can be defined in a standard way (as a deletion or repetition of a vertex). See [BT2].

To describe the structure of the building we first need to determine all types of the maximal chains. Proceeding as in [P2] (for $PGL(2)$) we get the following result.

**Proposition 1.** *There are exactly three types of maximal chains of $\mathcal{O}_F$-submodules in the space $V$:*

(i) *the chain contains a module isomorphic to $\mathcal{O}_F \oplus \mathcal{O}_F \oplus \mathcal{O}_F$. Then all the modules from the chain are of that type and the chain has the following structure:*

$$\dots \supset \mathcal{M}_F^i L \supset \mathcal{M}_F^i L' \supset \mathcal{M}_F^i L'' \supset \mathcal{M}_F^{i+1} L \supset \mathcal{M}_F^{i+1} L' \supset \mathcal{M}_F^{i+1} L'' \supset \dots$$

*where $\langle L\rangle, \langle L'\rangle, \langle L''\rangle \in \Delta_0(G, F/k)[1]$ and $L \simeq \mathcal{O}_F \oplus \mathcal{O}_F \oplus \mathcal{O}_F$, $L' \simeq \mathcal{O}_F \oplus \mathcal{O}_F \oplus \mathcal{M}_F,\ L'' \simeq \mathcal{O}_F \oplus \mathcal{M}_F \oplus \mathcal{M}_F$.*

(ii) *the chain contains a module isomorphic to $\mathcal{O}_F \oplus \mathcal{O}_F \oplus F$. Then the chain has the following structure:*

$$\dots \supset \mathcal{M}_F^i L \supset \mathcal{M}_F^i L' \supset \mathcal{M}_F^{i+1} L \supset \dots$$

*where $\langle L\rangle, \langle L'\rangle \in \Delta_0(G, F/k)[0]$ and $L \simeq \mathcal{O}_F \oplus \mathcal{O}_F \oplus F,\ L' \simeq \mathcal{M}_F \oplus \mathcal{O}_F \oplus F$.*

(iii) *the chain contains a module isomorphic to $\mathcal{O}_F \oplus F \oplus F$. Then the chain has the following structure:*

$$\dots \supset \mathcal{M}_F^i L \supset \mathcal{M}_F^{i+1} L \supset \dots$$

*where $\langle L\rangle \in \Delta_0(G, F/k)[0]$.*

We see that the chains of the first type correspond to two-simplices, of the second type — to edges and the last type represent some vertices. It means that the simplicial set $\Delta_.$ is a disconnected union of its subsets $\Delta_.[m],\ m = 0, 1$. The dimension of the subset $\Delta_.[m]$ is equal to one for $m = 0$ and to two for $m = 1$.

Usually the buildings are defined as combinatorial complexes having a system of subcomplexes called apartments (see, for example, [R], [T1], [T2]). We show how to introduce them for the higher building.

**Definition 8.** Fix a basis $e_1, e_2, e_3 \in V$. The *apartment* defined by this basis is the following set

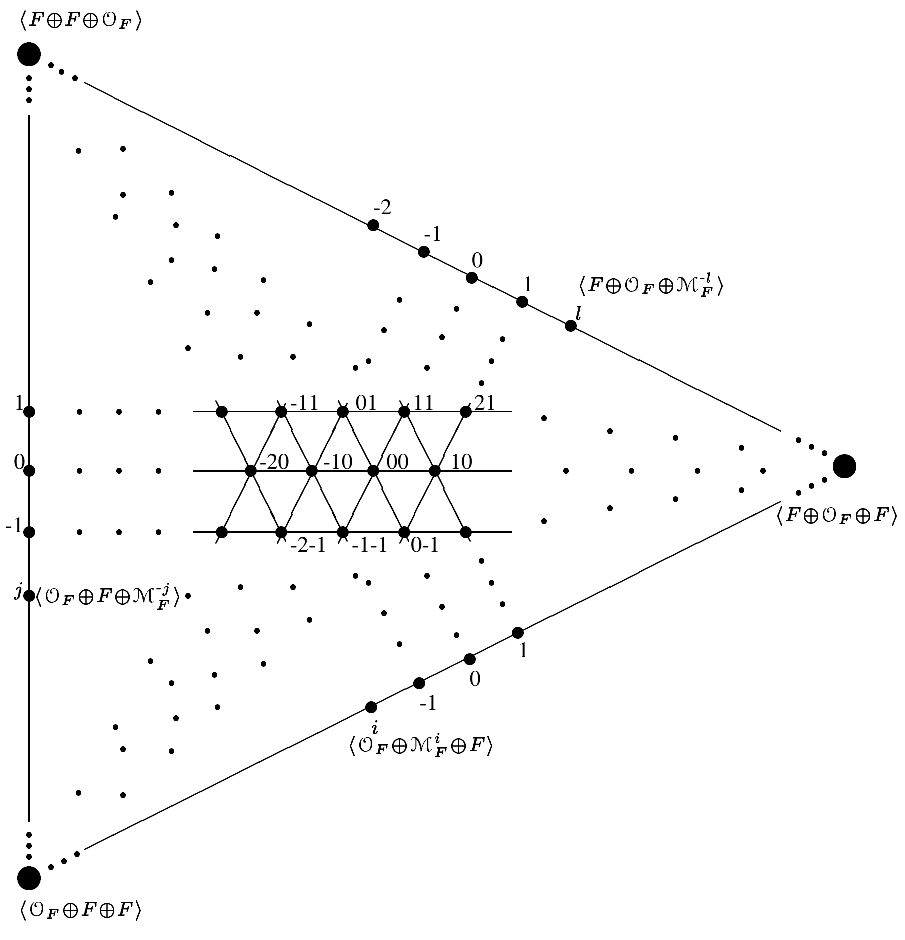
$$\Sigma_. = \Sigma_.[1] \cup \Sigma_.[0],$$

where

$$\Sigma_0[1] = \{\langle L\rangle : L = a_1e_1 \oplus a_2e_2 \oplus a_3e_3,\ \text{where } a_1, a_2, a_3 \text{ are } \mathcal{O}_F\text{-submodules in } F \text{ isomorphic to } \mathcal{O}_F\}$$

$$\Sigma_0[0] = \{\langle L\rangle : L = a_1e_1 \oplus a_2e_2 \oplus a_3e_3,\ \text{where } a_1, a_2, a_3 \text{ are } \mathcal{O}_F\text{-submodules in } F \text{ isomorphic either to } \mathcal{O}_F \text{ or to } F \text{ and at least one } a_i \text{ is isomorphic to } F\}.$$

$\Sigma_.[m]$ is the minimal subcomplex having $\Sigma_0[m]$ as vertices.




It can be shown that the building $\Delta_.(G, F/k)$ is glued from the apartments, namely

$$\Delta_.(G, F/k) = \bigsqcup_{\text{all bases of } V} \Sigma_. \,/\, \text{an equivalence relation}$$

(see [T2]).

We can make this description more transparent by drawing all that in the picture above where the dots of different kinds belong to the different parts of the building. In contrast with the case of the group $PGL(2)$ it is not easy to draw the whole building and we restrict ourselves to an apartment.

Here the inner vertices are represented by the lattices

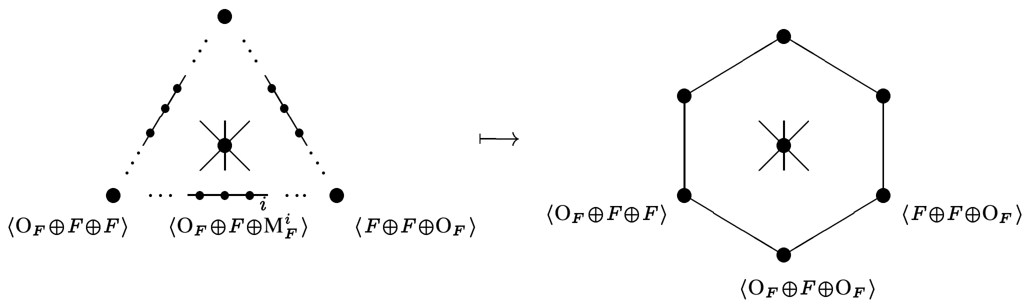
$$ij = \langle \mathcal{O}_F \oplus \mathcal{M}_F^i \oplus \mathcal{M}_F^j \rangle, \quad i, j \in \mathbb{Z}.$$

The definition of the boundary gives a topology on $\Delta_0(G, F/k)$ which is discrete on both subsets $\Delta_0[1]$ and $\Delta_0[0]$. The convergence of the inner points to the boundary points is given by the following rules:

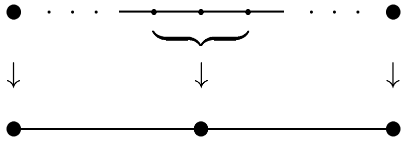
$$\langle \mathcal{O}_F \oplus \mathcal{M}_F^i \oplus \mathcal{M}_F^j \rangle \xrightarrow{j \to -\infty} \langle \mathcal{O}_F \oplus \mathcal{M}_F^i \oplus F \rangle,$$
$$\langle \mathcal{O}_F \oplus \mathcal{M}_F^i \oplus \mathcal{M}_F^j \rangle \xrightarrow{j \to \infty} \langle F \oplus F \oplus \mathcal{O}_F \rangle,$$

because $\langle \mathcal{O}_F \oplus \mathcal{M}_F^i \oplus \mathcal{M}_F^j \rangle = \langle \mathcal{M}_F^{-j} \oplus \mathcal{M}_F^{-j+i} \oplus \mathcal{O}_F \rangle$. The convergence in the other two directions can be defined along the same line (and it is shown on the picture). It is easy to extend it to the higher simplices.

Thus, there is the structure of a simplicial topological space on the apartment and then we define it on the whole building using the gluing procedure. This topology is stronger than the topology usually introduced to connect the inner part and the boundary together. The connection with standard "compactification" of the building is given by the following map:




This map is bijective on the inner simplices and on a part of the boundary can be described as



We note that the complex is not a CW-complex but only a closure finite complex. This "compactification" was used by G. Mustafin [M].

We have two kinds of connections with the buildings for other fields and groups. First, for the local field $F$ there are two local fields of dimension 0, namely $F$ and $k$. Then for every $P \in \Delta_0[1](PGL(V), F/k)$ the Link$(P)$ is equal to $\Delta_.(PGL(V_P), k)$ where $V_P = L/\mathcal{M}_F L$ if $P = \langle L \rangle$ and the Link$(P)$ is the boundary of the Star$(P)$. Since the apartments for the $PGL(3,k)$ are hexagons, we can also observe this property on the picture. The analogous relation with the building of $PGL(3,K)$ is more complicated. It is shown on the picture above.

The other relations work if we change the group $G$ but not the field. We see that three different lines go out from every inner point in the apartment. They represent the apartments of the group $PGL(2, F/k)$. They correspond to different embeddings of the $PGL(2)$ into $PGL(3)$.

Also we can describe the action of the Weyl group $W$ on an apartment. If we fix a basis, the extension

$$0 \to \Gamma_F \oplus \Gamma_F \to W \to S_3 \to 1$$

splits. The elements from $S_3 \subset W$ act either as rotations around the point 00 or as reflections. The elements of $\mathbb{Z} \oplus \mathbb{Z} \subset W$ can be represented as triples of integers (according to property (ii) in the previous subsection). Then they correspond to translations of the lattice of inner points along the three directions going from the point 00.

If we fix an embedding $PGL(2) \subset PGL(3)$ then the apartments and the Weyl groups are connected as follows:

$$\Sigma_.(PGL(2)) \subset \Sigma_.(PGL(3)),$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & W' & \longrightarrow & S_2 & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & W & \longrightarrow & S_3 & \longrightarrow & 1
\end{array}$$

where $W'$ is a Weyl group of the group $PGL(2)$ over the field $F/k$.

## 3.4. Bruhat–Tits building for $PGL(3)$ over a local field of dimension 2

Let $K$ be a two-dimensional local field. Denote by $V$ a vector space over $K$ of dimension three. Define lattices in V and their equivalence in a similar way to 3.2.3. We shall consider the following *types* of lattices:

$$\begin{array}{lll}
\Delta_0[2] & 222 & \langle O \oplus O \oplus O \rangle \\
\Delta_0[1] & 221 & \langle O \oplus O \oplus \mathcal{O} \rangle \\
 & 211 & \langle O \oplus \mathcal{O} \oplus \mathcal{O} \rangle \\
\Delta_0[0] & 220 & \langle O \oplus O \oplus K \rangle \\
 & 200 & \langle O \oplus K \oplus K \rangle
\end{array}$$

To define the buildings we repeat the procedure from the previous subsection.

**Definition 10.** The *vertices* of the Bruhat–Tits building are the elements of the following set:

$$\Delta_0(G, K/K_1/k) = \Delta_0[2] \cup \Delta_0[1] \cup \Delta_0[0].$$

To define the simplices of higher dimension we can repeat word by word Definitions 7 and 8 of the previous subsection replacing the ring $\mathcal{O}_F$ by the ring $O$ (note that we work only with the types of lattices listed above). We call the subset $\Delta[1]$ the *inner boundary* of the building and the subset $\Delta[0]$ the *external boundary*. The points in $\Delta[2]$ are the *inner* points.

To describe the structure of the building we first need to determine all types of the maximal chains. Proceeding as in [P2] for $PGL(2)$ we get the following result.

**Proposition 2.** *Let $\{L_\alpha\}$ be a maximal chain of $O$-submodules in the space $V$. There are exactly five types of maximal chains:*

(i) *If the chain contains a module $L$ isomorphic to $O \oplus O \oplus O$ then all the modules of the chain are of that type and the chain is uniquely determined by its segment*

$$\dots \supset O \oplus O \oplus O \supset M \oplus O \oplus O \supset M \oplus M \oplus O \supset M \oplus M \oplus M \supset \dots$$

(ii) *If the chain contains a module $L$ isomorphic to $O \oplus O \oplus \mathcal{O}$ then the chain can be restored from the segment:*

$$\dots \supset \text{“}\,\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}\,\text{”} \supset \dots \supset O \oplus O \oplus \mathcal{O} \supset M \oplus O \oplus \mathcal{O} \supset M \oplus M \oplus \mathcal{O} \supset \dots \supset \mathcal{M} \oplus \mathcal{M} \oplus \mathcal{O}$$
$$\text{quotient} \simeq K_1 \oplus K_1$$
$$= \mathcal{M} \oplus \mathcal{M} \oplus \mathcal{O} \supset \dots \supset \mathcal{M} \oplus \mathcal{M} \oplus O \supset \mathcal{M} \oplus \mathcal{M} \oplus M \supset \dots \supset \mathcal{M} \oplus \mathcal{M} \oplus \mathcal{M} \supset \dots$$
$$\text{quotient} \simeq K_1$$

*Here the modules isomorphic to $\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}$ do not belong to this chain and are inserted as in the proof of Proposition 1 of* [P2].

(iii) *All the modules $L_\alpha \simeq O \oplus \mathcal{O} \oplus \mathcal{O}$. Then the chain contains a piece*

$$\dots \supset \text{“}\,\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}\,\text{”} \supset \dots \supset O \oplus \mathcal{O} \oplus \mathcal{O} \supset M \oplus \mathcal{O} \oplus \mathcal{O} \supset \dots \supset \mathcal{M} \oplus \mathcal{O} \oplus \mathcal{O}$$
$$\text{quotient} \simeq K_1$$
$$= \mathcal{M} \oplus \mathcal{O} \oplus \mathcal{O} \supset \dots \supset \mathcal{M} \oplus O \oplus \mathcal{O} \supset \dots \supset \mathcal{M} \oplus \mathcal{M} \oplus \mathcal{O}$$
$$\text{quotient} \simeq K_1$$
$$= \mathcal{M} \oplus \mathcal{M} \oplus \mathcal{O} \supset \dots \supset \mathcal{M} \oplus \mathcal{M} \oplus O \supset \dots \supset \mathcal{M} \oplus \mathcal{M} \oplus \mathcal{M} \supset \dots$$
$$\text{quotient} \simeq K_1$$

*and can also be restored from it. Here the modules isomorphic to $\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}$ do not belong to this chain and are inserted as in the proof of Proposition 1 of* [P2].

(iv) *If there is an $L_\alpha \simeq O \oplus O \oplus K$ then one can restore the chain from*

$$\dots \supset O \oplus O \oplus K \supset M \oplus O \oplus K \supset M \oplus M \oplus K \supset \dots$$

(v) *If there is an* $L_\alpha \simeq O \oplus K \oplus K$ *then the chain can be written down as*

$$\dots \supset M^i \oplus K \oplus K \supset M^{i+1} \oplus K \oplus K \supset \dots$$

We see that the chains of the first three types correspond to two-simplices, of the fourth type — to edges of the external boundary and the last type represents a vertex of the external boundary. As above we can glue the building from apartments. To introduce them we can again repeat the corresponding definition for the building over a local field of dimension one (see Definition 4 of the previous subsection). Then the apartment $\Sigma_.$ is a union

$$\Sigma_. = \Sigma_.[2] \cup \Sigma_.[1] \cup \Sigma_.[0]$$

where the pieces $\Sigma_.[i]$ contain the lattices of the types from $\Delta_.[i]$.

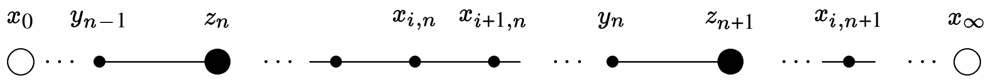
The combinatorial structure of the apartment can be seen from two pictures at the end of the subsection. There we removed the external boundary $\Sigma_.[0]$ which is simplicially isomorphic to the external boundary of an apartment of the building $\Delta_.(PGL(3), K/K_1/k)$. The dots in the first picture show a convergence of the vertices inside the apartment. As a result the building is a simplicial topological space.

We can also describe the relations of the building with buildings of the same group $G$ over the complete discrete valuation fields $K$ and $K_1$. In the first case there is a projection map

$$\pi: \Delta_.(G, K/K_1/k) \to \Delta_.(G, K/K_1).$$

Under this map the big triangles containing the simplices of type (i) are contracted into points, the triangles containing the simplices of type (ii) go to edges and the simplices of type (iii) are mapped isomorphically to simplices in the target space. The external boundary don't change.

The lines

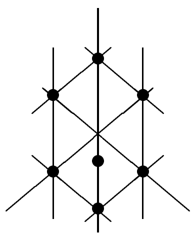



can easily be visualized inside the apartment. Only the big white dots corresponding to the external boundary are missing. We have three types of lines going from the inner points under the angle $2\pi/3$. They correspond to different embeddings of $PGL(2)$ into $PGL(3)$.

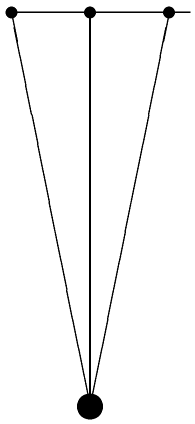
Using the lines we can understand the action of the Weyl group $W$ on an apartment. The subgroup $S_3$ acts in the same way as in 3.2. The free subgroup $E$ (see 3.1) has six types of translations along these three directions. Along each line we have two opportunities which were introduced for $PGL(2)$.

Namely, if $w \in \Gamma_K \simeq \mathbb{Z} \oplus \mathbb{Z} \subset W$ then $w = (0,1)$ acts as a shift of the whole structure to the right: $w(x_{i,n}) = x_{i,n+2}$, $w(y_n) = y_{n+2}$, $w(z_n) = z_{n+2}, w(x_0) = x_0$, $w(x_\infty) = x_\infty$.

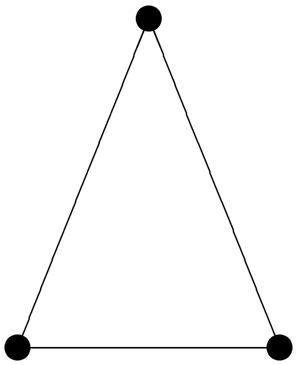
The element $w = (1,0)$ acts as a shift on the points $x_{i,n}$ but leaves fixed the points in the inner boundary $w(x_{i,n}) = x_{i+2,n}$, $w(y_n) = y_n$, $w(z_n) = z_n$, $w(x_0) = x_0$, $w(x_\infty) = x_\infty$, (see [P2, Theorem 5, v]).



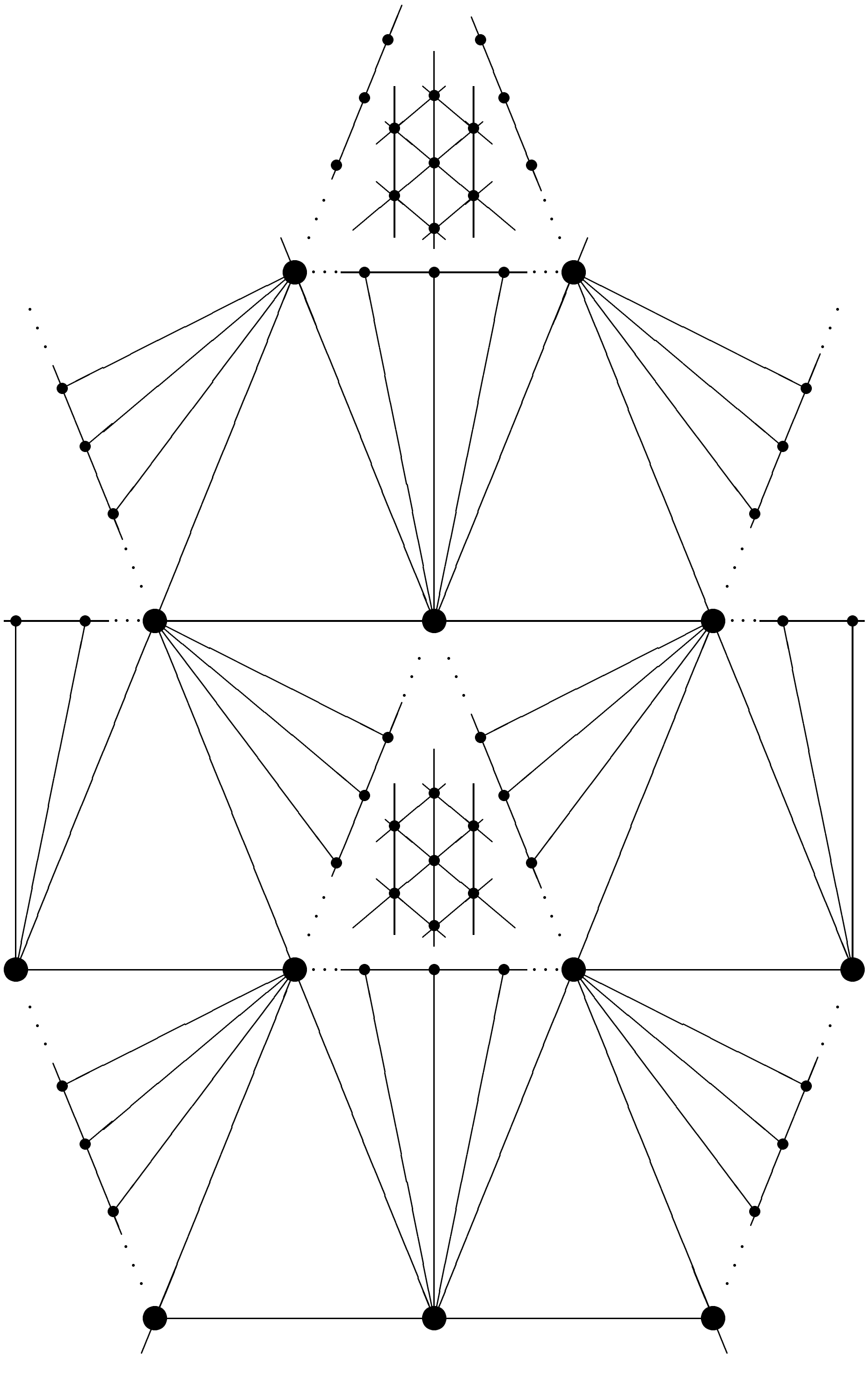
simplices of type (i)



simplices of type (ii)



simplices of type (iii)



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Geometry & Topology Monographs
Volume 3: Invitation to higher local fields
Part II, section 4, pages 239–253

# 4. Drinfeld modules and local fields of positive characteristic

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The relationship between local fields and Drinfeld modules is twofold. Drinfeld modules allow explicit construction of abelian and nonabelian extensions with prescribed properties of local and global fields of positive characteristic. On the other hand, $n$-dimensional local fields arise in the construction of (the compactification of) moduli schemes $X$ for Drinfeld modules, such schemes being provided with a natural stratification $X_0 \subset X_1 \subset \cdots X_i \cdots \subset X_n = X$ through smooth subvarieties $X_i$ of dimension $i$.

We will survey that correspondence, but refer to the literature for detailed proofs (provided these exist so far). An important remark is in order: The contents of this article take place in characteristic $p > 0$, and are in fact locked up in the characteristic $p$ world. No lift to characteristic zero nor even to schemes over $\mathbb{Z}/p^2$ is known!

## 4.1. Drinfeld modules

Let $L$ be a field of characteristic $p$ containing the field $\mathbb{F}_q$, and denote by $\tau = \tau_q$ raising to the $q$th power map $x \mapsto x^q$. If "$a$" denotes multiplication by $a \in L$, then $\tau a = a^q \tau$. The ring $\mathrm{End}(\mathbb{G}_{a/L})$ of endomorphisms of the additive group $\mathbb{G}_{a/L}$ equals $L\{\tau_p\} = \{\sum a_i \tau_p^i : a_i \in L\}$, the non-commutative polynomial ring in $\tau_p = (x \mapsto x^p)$ with the above commutation rule $\tau_p a = a^p \tau$. Similarly, the subring $\mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a/L})$ of $\mathbb{F}_q$-endomorphisms is $L\{\tau\}$ with $\tau = \tau_p^n$ if $q = p^n$. Note that $L\{\tau\}$ is an $\mathbb{F}_q$-algebra since $\mathbb{F}_q \hookrightarrow L\{\tau\}$ is central.

**Definition 1.** Let $\mathcal{C}$ be a smooth geometrically connected projective curve over $\mathbb{F}_q$. Fix a closed (but not necessarily $\mathbb{F}_q$-rational) point $\infty$ of $\mathcal{C}$. The ring $A = \Gamma(\mathcal{C} - \{\infty\}, \mathcal{O}_{\mathcal{C}})$ is called a *Drinfeld ring*. Note that $A^* = \mathbb{F}_q^*$.

*Published 10 December 2000:* 

**Example 1.** If $\mathcal{C}$ is the projective line $\mathbb{P}^1{}_{/\mathbb{F}_q}$ and $\infty$ is the usual point at infinity then $A = \mathbb{F}_q[T]$.

**Example 2.** Suppose that $p \neq 2$, that $\mathcal{C}$ is given by an affine equation $Y^2 = f(X)$ with a separable polynomial $f(X)$ of even positive degree with leading coefficient a non-square in $\mathbb{F}_q$, and that $\infty$ is the point above $X = \infty$. Then $A = \mathbb{F}_q[X,Y]$ is a Drinfeld ring with $\deg_{\mathbb{F}_q}(\infty) = 2$.

**Definition 2.** An $A$-*structure* on a field $L$ is a homomorphism of $\mathbb{F}_q$-algebras (in brief: an $\mathbb{F}_q$-ring homomorphism) $\gamma\colon A \to L$. Its $A$-*characteristic* $\mathrm{char}_A(L)$ is the maximal ideal $\ker(\gamma)$, if $\gamma$ fails to be injective, and $\infty$ otherwise. A *Drinfeld module structure* on such a field $L$ is given by an $\mathbb{F}_q$-ring homomorphism $\phi\colon A \to L\{\tau\}$ such that $\partial \circ \phi = \gamma$, where $\partial\colon L\{\tau\} \to L$ is the $L$-homomorphism sending $\tau$ to 0.

Denote $\phi(a)$ by $\phi_a \in \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a\,/L})$; $\phi_a$ induces on the additive group over $L$ (and on each $L$-algebra $M$) a *new* structure as an $A$-module:

$$a * x := \phi_a(x) \quad (a \in A, x \in M). \tag{4.1.1}$$

We briefly call $\phi$ a Drinfeld module over $L$, usually omitting reference to $A$.

**Definition 3.** Let $\phi$ and $\psi$ be Drinfeld modules over the $A$-field $L$. A *homomorphism* $u\colon \phi \to \psi$ is an element of $L\{\tau\}$ such that $u \circ \phi_a = \psi_a \circ u$ for all $a \in A$. Hence an *endomorphism* of $\phi$ is an element of the centralizer of $\phi(A)$ in $L\{\tau\}$, and $u$ is an *isomorphism* if $u \in L^* \hookrightarrow L\{\tau\}$ is subject to $u \circ \phi_a = \psi_a \circ u$.

Define $\deg\colon a \to \mathbb{Z} \cup \{-\infty\}$ and $\deg_\tau\colon L\{\tau\} \to \mathbb{Z} \cup \{-\infty\}$ by $\deg(a) = \log_q |A/a|$ ($a \neq 0$; we write $A/a$ for $A/aA$), $\deg(0) = -\infty$, and $\deg_\tau(f) =$ the well defined degree of $f$ as a "polynomial" in $\tau$. It is an easy exercise in Dedekind rings to prove the following

**Proposition 1.** *If $\phi$ is a Drinfeld module over $L$, there exists a non-negative integer $r$ such that* $\deg_\tau(\phi_a) = r\deg(a)$ *for all* $a \in A$; $r$ is called the *rank* $\mathrm{rk}(\phi)$ of $\phi$.

Obviously, $\mathrm{rk}(\phi) = 0$ means that $\phi = \gamma$, i.e., the $A$-module structure on $\mathbb{G}_{a\,/L}$ is the tautological one.

**Definition 4.** Denote by $\mathcal{M}^r(1)(L)$ the set of isomorphism classes of Drinfeld modules of rank $r$ over $L$.

**Example 3.** Let $A = \mathbb{F}_q[T]$ be as in Example 1 and let $K = \mathbb{F}_q(T)$ be its fraction field. Defining a Drinfeld module $\phi$ over $K$ or an extension field $L$ of $K$ is equivalent to specifying $\phi_T = T + g_1\tau + \cdots + g_r\tau^r \in L\{T\}$, where $g_r \neq 0$ and $r = \mathrm{rk}(\phi)$. In the special case where $\phi_T = T + \tau$, $\phi$ is called the *Carlitz module*. Two such Drinfeld modules $\phi$ and $\phi'$ are isomorphic over the algebraic closure $L^{\mathrm{alg}}$ of $L$ if and only if there is some $u \in L^{\mathrm{alg}\,*}$ such that $g_i' = u^{q^i-1}g_i$ for all $i \geqslant 1$. Hence $\mathcal{M}^r(1)(L^{\mathrm{alg}})$ can be described (for $r \geqslant 1$) as an open dense subvariety of a weighted projective space of dimension $r-1$ over $L^{\mathrm{alg}}$.

## 4.2. Division points

**Definition 5.** For $a \in A$ and a Drinfeld module $\phi$ over $L$, write ${}_a\phi$ for the subscheme of *$a$-division points* of $\mathbb{G}_{a\,/L}$ endowed with its structure of an $A$-module. Thus for any $L$-algebra $M$,

$$ {}_a\phi(M) = \{x \in M \colon \phi_a(x) = 0\}. $$

More generally, we put ${}_{\mathfrak{a}}\phi = \bigcap_{a \in \mathfrak{a}} \phi_a$ for an arbitrary (not necessarily principal) ideal $\mathfrak{a}$ of $A$. It is a finite flat group scheme of degree $\mathrm{rk}(\phi) \cdot \deg(\mathfrak{a})$, whose structure is described in the next result.

**Proposition 2** ([Dr], [DH, I, Thm. 3.3 and Remark 3.4])**.** *Let the Drinfeld module $\phi$ over $L$ have rank $r \geqslant 1$.*

(i) *If* $\mathrm{char}_A(L) = \infty$, *${}_{\mathfrak{a}}\phi$ is reduced for each ideal $\mathfrak{a}$ of $A$, and ${}_{\mathfrak{a}}\phi(L^{\mathrm{sep}}) = {}_{\mathfrak{a}}\phi(L^{\mathrm{alg}})$ is isomorphic with $(A/\mathfrak{a})^r$ as an $A$-module.*

(ii) *If $\mathfrak{p} = \mathrm{char}_A(L)$ is a maximal ideal, then there exists an integer $h$, the height* $\mathrm{ht}(\phi)$ *of $\phi$, satisfying $1 \leqslant h \leqslant r$, and such that ${}_{\mathfrak{a}}\phi(L^{\mathrm{alg}}) \simeq (A/\mathfrak{a})^{r-h}$ whenever $\mathfrak{a}$ is a power of $\mathfrak{p}$, and ${}_{\mathfrak{a}}\phi(L^{\mathrm{alg}}) \simeq (A/\mathfrak{a})^r$ if $(\mathfrak{a}, \mathfrak{p}) = 1$.*

The absolute Galois group $G_L$ of $L$ acts on ${}_{\mathfrak{a}}\phi(L^{\mathrm{sep}})$ through $A$-linear automorphisms. Therefore, any Drinfeld module gives rise to Galois representations on its division points. These representations tend to be "as large as possible".

The prototype of result is the following theorem, due to Carlitz and Hayes [H1].

**Theorem 1.** *Let $A$ be the polynomial ring $\mathbb{F}_q[T]$ with field of fractions $K$. Let $\rho\colon A \to K\{\tau\}$ be the Carlitz module, $\rho_T = T + \tau$. For any non-constant monic polynomial $a \in A$, let $K(a) := K({}_a\rho(K^{\mathrm{alg}}))$ be the field extension generated by the $a$-division points.*

(i) *$K(a)/K$ is abelian with group $(A/a)^*$. If $\sigma_b$ is the automorphism corresponding to the residue class of $b \bmod a$ and $x \in {}_a\rho(K^{\mathrm{alg}})$ then $\sigma_b(x) = \rho_b(x)$.*

(ii) *If $(a) = \mathfrak{p}^t$ is primary with some prime ideal $\mathfrak{p}$ then $K(a)/K$ is completely ramified at $\mathfrak{p}$ and unramified at the other finite primes.*

(iii) *If $(a) = \prod a_i$ ($1 \leqslant i \leqslant s$) with primary and mutually coprime $a_i$, the fields $K(a_i)$ are mutually linearly disjoint and $K = \otimes_{i \leqslant i \leqslant s} K(a_i)$.*

(iv) *Let $K_+(a)$ be the fixed field of $\mathbb{F}_q^* \hookrightarrow (A/a)^*$. Then $\infty$ is completely split in $K_+(a)/K$ and completely ramified in $K(a)/K_+(a)$.*

(v) *Let $\mathfrak{p}$ be a prime ideal generated by the monic polynomial $\pi \in A$ and coprime with $a$. Under the identification* $\mathrm{Gal}(K(a)/K) = (A/a)^*$, *the Frobenius element* $\mathrm{Frob}_{\mathfrak{p}}$ *equals the residue class of $\pi \bmod a$.*

Letting $a \to \infty$ with respect to divisibility, we obtain the field $K(\infty)$ generated over $K$ by all the division points of $\rho$, with group $\mathrm{Gal}(K(\infty)/K) = \varinjlim_a (A/a)^*$, which almost agrees with the group of finite idele classes of $K$. It turns out that $K(\infty)$ is the maximal abelian extension of $K$ that is tamely ramified at $\infty$, i.e., we get a constructive version of the class field theory of $K$. Hence the theorem may be seen both as a global variant of Lubin–Tate's theory and as an analogue in characteristic $p$ of the Kronecker–Weber theorem on cyclotomic extensions of $\mathbb{Q}$.

There are vast generalizations into two directions:

(a) abelian class field theory of arbitrary global function fields $K = \mathrm{Frac}(A)$, where $A$ is a Drinfeld ring.

(b) systems of nonabelian Galois representations derived from Drinfeld modules.

As to (a), the first problem is to find the proper analogue of the Carlitz module for an arbitrary Drinfeld ring $A$. As will result e.g. from Theorem 2 (see also (4.3.4)), the isomorphism classes of rank-one Drinfeld modules over the algebraic closure $K^{\mathrm{alg}}$ of $K$ correspond bijectively to the (finite!) class group $\mathrm{Pic}(A)$ of $A$. Moreover, these Drinfeld modules $\rho^{(\mathfrak{a})}$ ( $\mathfrak{a} \in \mathrm{Pic}(A)$ ) may be defined with coefficients in the ring $\mathcal{O}_{H_+}$ of $A$-integers of a certain abelian extension $H_+$ of $K$, and such that the leading coefficients of all $\rho_a^{(\mathfrak{a})}$ are units of $\mathcal{O}_{H_+}$. Using these data along with the identification of $H_+$ in the dictionary of class field theory yields a generalization of Theorem 1 to the case of arbitrary $A$. In particular, we again find an explicit construction of the class fields of $K$ (subject to a tameness condition at $\infty$ ). However, in view of class number problems, the theory (due to D. Hayes [H2], and superbly presented in [Go2, Ch.VII]) has more of the flavour of complex multiplication theory than of classical cyclotomic theory.

Generalization (b) is as follows. Suppose that $L$ is a finite extension of $K = \mathrm{Frac}(A)$, where $A$ is a general Drinfeld ring, and let the Drinfeld module $\phi$ over $L$ have rank $r$. For each power $\mathfrak{p}^t$ of a prime $\mathfrak{p}$ of $A$, $G_L = \mathrm{Gal}(L^{\mathrm{sep}}/L)$ acts on ${}_{\mathfrak{p}^t}\phi \simeq (A/\mathfrak{p}^t)^r$. We thus get an action of $G_L$ on the $\mathfrak{p}$-adic *Tate module* $T_{\mathfrak{p}}(\phi) \simeq (A_{\mathfrak{p}})^r$ of $\phi$ (see [DH, I sect. 4]. Here of course $A_{\mathfrak{p}} = \varprojlim A/\mathfrak{p}^t$ is the $\mathfrak{p}$-adic completion of $A$ with field of fractions $K_{\mathfrak{p}}$. Let on the other hand $\mathrm{End}(\phi)$ be the endomorphism ring of $\phi$, which also acts on $T_{\mathfrak{p}}(\phi)$. It is straightforward to show that (i) $\mathrm{End}(\phi)$ acts faithfully and (ii) the two actions commute. In other words, we get an inclusion

$$i\colon \mathrm{End}(\phi) \otimes_A A_{\mathfrak{p}} \hookrightarrow \mathrm{End}_{G_L}(T_{\mathfrak{p}}(\phi)) \tag{4.2.1}$$

of finitely generated free $A_{\mathfrak{p}}$-modules. The plain analogue of the classical Tate conjecture for abelian varieties, proved 1983 by Faltings, suggests that $i$ is in fact bijective. This has been shown by Taguchi [Tag] and Tamagawa. Taking $\mathrm{End}(T_{\mathfrak{p}}(\phi)) \simeq \mathrm{Mat}(r, A_{\mathfrak{p}})$ and the known structure of subalgebras of matrix algebras over a field into account, this means that the subalgebra

$$K_{\mathfrak{p}}[G_L] \hookrightarrow \mathrm{End}(T_{\mathfrak{p}}(\phi) \otimes_{A_{\mathfrak{p}}} K_{\mathfrak{p}}) \simeq \mathrm{Mat}(r, K_{\mathfrak{p}})$$

generated by the Galois operators is as large as possible. A much stronger statement is obtained by R. Pink [P1, Thm. 0.2], who shows that the image of $G_L$ in $\mathrm{Aut}(T_\mathfrak{p}(\phi))$ has finite index in the centralizer group of $\mathrm{End}(\phi) \otimes A_\mathfrak{p}$. Hence if e.g. $\phi$ has no "complex multiplications" over $L^{\mathrm{alg}}$ (i.e., $\mathrm{End}_{L^{\mathrm{alg}}}(\phi) = A$; this is the generic case for a Drinfeld module in characteristic $\infty$), then the image of $G_L$ has finite index in $\mathrm{Aut}(T_\mathfrak{p}(\phi)) \simeq GL(r, A_\mathfrak{p})$. This is quite satisfactory, on the one hand, since we may use the Drinfeld module $\phi$ to construct large nonabelian Galois extensions of $L$ with prescribed ramification properties. On the other hand, the important (and difficult) problem of estimating the index in question remains.

## 4.3. Weierstrass theory

Let $A$ be a Drinfeld ring with field of fractions $K$, whose completion at $\infty$ is denoted by $K_\infty$. We normalize the corresponding absolute value $|\ | = |\ |_\infty$ as $|a| = |A/a|$ for $0 \neq a \in A$ and let $C_\infty$ be the completed algebraic closure of $K_\infty$, i.e., the completion of the algebraic closure $K_\infty^{\mathrm{alg}}$ with respect to the unique extension of $|\ |$ to $K_\infty^{\mathrm{alg}}$. By Krasner's theorem, $C_\infty$ is again algebraically closed ([BGS, p. 146], where also other facts on function theory in $C_\infty$ may be found). It is customary to indicate the strong analogies between $A, K, K_\infty, C_\infty, \ldots$ and $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \ldots$, e.g. $A$ is a discrete and cocompact subring of $K_\infty$. But note that $C_\infty$ fails to be locally compact since $|C_\infty : K_\infty| = \infty$.

**Definition 6.** A *lattice of rank* $r$ (an $r$-lattice in brief) in $C_\infty$ is a finitely generated (hence projective) discrete $A$-submodule $\Lambda$ of $C_\infty$ of projective rank $r$, where the discreteness means that $\Lambda$ has finite intersection with each ball in $C_\infty$. The *lattice function* $e_\Lambda\colon C_\infty \to C_\infty$ of $\Lambda$ is defined as the product

$$e_\Lambda(z) = z \prod_{0 \neq \lambda \in \Lambda} (1 - z/\lambda). \tag{4.3.1}$$

It is entire (defined through an everywhere convergent power series), $\Lambda$-periodic and $\mathbb{F}_q$-linear. For a non-zero $a \in A$ consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda & \longrightarrow & C_\infty & \xrightarrow{e_\Lambda} & C_\infty & \longrightarrow & 0 \\ & & {\scriptstyle a}\big\downarrow & & {\scriptstyle a}\big\downarrow & & {\scriptstyle \phi_a^\Lambda}\big\downarrow & & \\ 0 & \longrightarrow & \Lambda & \longrightarrow & C_\infty & \xrightarrow{e_\Lambda} & C_\infty & \longrightarrow & 0 \end{array} \tag{4.3.2}$$

with exact lines, where the left and middle arrows are multiplications by $a$ and $\phi_a^\Lambda$ is defined through commutativity. It is easy to verify that

(i) $\phi_a^\Lambda \in C_\infty\{\tau\}$,

(ii) $\deg_\tau(\phi_a^\Lambda) = r \cdot \deg(a)$,

(iii) $a \mapsto \phi_a^\Lambda$ is a ring homomorphism $\phi^\Lambda \colon A \to C_\infty\{\tau\}$, in fact, a Drinfeld module of rank $r$. Moreover, all the Drinfeld modules over $C_\infty$ are so obtained.

**Theorem 2** (Drinfeld [Dr, Prop. 3.1])**.**

(i) *Each rank-* $r$ *Drinfeld module* $\phi$ *over* $C_\infty$ *comes via* $\Lambda \mapsto \phi^\Lambda$ *from some* $r$*-lattice* $\Lambda$ *in* $C_\infty$.

(ii) *Two Drinfeld modules* $\phi^\Lambda$, $\phi^{\Lambda'}$ *are isomorphic if and only if there exists* $0 \neq c \in C_\infty$ *such that* $\Lambda' = c \cdot \Lambda$.

We may thus describe $\mathcal{M}^r(1)(C_\infty)$ (see Definition 4) as the space of $r$-lattices modulo similarities, i.e., as some generalized upper half-plane modulo the action of an arithmetic group. Let us make this more precise.

**Definition 7.** For $r \geqslant 1$ let $\mathbb{P}^{r-1}(C_\infty)$ be the $C_\infty$-points of projective $r-1$-space and $\Omega^r := \mathbb{P}^{r-1}(C_\infty) - \bigcup H(C_\infty)$, where $H$ runs through the $K_\infty$-rational hyperplanes of $\mathbb{P}^{r-1}$. That is, $\underline{\omega} = (\omega_1 : \ldots : \omega_r)$ belongs to *Drinfeld's half-plane* $\Omega^r$ if and only if there is no non-trivial relation $\sum a_i\omega_i = 0$ with coefficients $a_i \in K_\infty$.

Both point sets $\mathbb{P}^{r-1}(C_\infty)$ and $\Omega^r$ carry structures of analytic spaces over $C_\infty$ (even over $K_\infty$), and so we can speak of holomorphic functions on $\Omega^r$. We will not give the details (see for example [GPRV, in particular lecture 6]); suffice it to say that locally uniform limits of rational functions (e.g. Eisenstein series, see below) will be holomorphic.

Suppose for the moment that the class number $h(A) = |\mathrm{Pic}(A)|$ of $A$ equals one, i.e., $A$ is a principal ideal domain. Then each $r$-lattice $\Lambda$ in $C_\infty$ is free, $\Lambda = \sum_{1\leqslant i\leqslant r} A\omega_i$, and the discreteness of $\Lambda$ is equivalent with $\underline{\omega} := (\omega_1 : \ldots : \omega_r)$ belonging to $\Omega^r \hookrightarrow \mathbb{P}^{r-1}(C_\infty)$. Further, two points $\underline{\omega}$ and $\underline{\omega}'$ describe similar lattices (and therefore isomorphic Drinfeld modules) if and only if they are conjugate under $\Gamma := GL(r, A)$, which acts on $\mathbb{P}^{r-1}(C_\infty)$ and its subspace $\Omega^r$. Therefore, we get a canonical bijection

$$\Gamma \setminus \Omega^r \xrightarrow{\sim} \mathcal{M}^r(1)(C_\infty) \tag{4.3.3}$$

from the quotient space $\Gamma \setminus \Omega^r$ to the set of isomorphism classes $\mathcal{M}^r(1)(C_\infty)$.

In the general case of arbitrary $h(A) \in \mathbb{N}$, we let $\Gamma_i := GL(Y_i) \hookrightarrow GL(r, k)$, where $Y_i \hookrightarrow K^r$ $(1 \leqslant i \leqslant h(A))$ runs through representatives of the $h(A)$ isomorphism classes of projective $A$-modules of rank $r$. In a similar fashion (see e.g. [G1, II sect.1], [G3]), we get a bijection

$$\dot{\bigcup}_{1\leqslant i\leqslant h(A)} \Gamma_i \setminus \Omega^r \xrightarrow{\sim} \mathcal{M}^r(1)(C_\infty), \tag{4.3.4}$$

which can be made independent of choices if we use the canonical adelic description of the $Y_i$.

**Example 4.** If $r = 2$ then $\Omega = \Omega^2 = \mathbb{P}^1(C_\infty) - -\mathbb{P}^1(K_\infty) = C_\infty - K_\infty$, which rather corresponds to $\mathbb{C} - \mathbb{R} = H^+ \bigcup H^-$ (upper and lower complex half-planes) than to $H^+$ alone. The group $\Gamma := GL(2, A)$ acts via $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(z) = \frac{az+b}{cz+d}$, and thus gives rise to *Drinfeld modular forms* on $\Omega$ (see [G1]). Suppose moreover that $A = \mathbb{F}_q[T]$ as in Examples 1 and 3. We define *ad hoc* a modular form of weight $k$ for $\Gamma$ as a holomorphic function $f : \Omega \to C_\infty$ that satisfies

(i) $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and

(ii) $f(z)$ is bounded on the subspace $\{z \in \Omega : \inf_{x \in K_\infty} |z - x| > 1\}$ of $\Omega$.

Further, we put $M_k$ for the $C_\infty$-vector space of modular forms of weight $k$. (In the special case under consideration, (ii) is equivalent to the usual "holomorphy at cusps" condition. For more general groups $\Gamma$, e.g. congruence subgroups of $GL(2, A)$, general Drinfeld rings $A$, and higher ranks $r \geqslant 2$, condition (ii) is considerably more costly to state, see [G1].) Let

$$E_k(z) := \sum_{(0,0) \neq (a,b) \in A \times A} \frac{1}{(az + b)^k} \tag{4.3.5}$$

be the *Eisenstein series* of weight $k$. Due to the non-archimedean situation, the sum converges for $k \geqslant 1$ and yields a modular form $0 \neq E_k \in M_k$ if $k \equiv 0\ (q - 1)$. Moreover, the various $M_k$ are linearly independent and

$$M(\Gamma) := \bigoplus_{k \geqslant 0} M_k = C_\infty[E_{q-1}, E_{q^2-1}] \tag{4.3.6}$$

is a polynomial ring in the two algebraically independent Eisenstein series of weights $q-1$ and $q^2-1$. There is an *a priori* different method of constructing modular forms via Drinfeld modules. With each $z \in \Omega$, associate the 2-lattice $\Lambda_z := Az + A \hookrightarrow C_\infty$ and the Drinfeld module $\phi^{(z)} = \phi^{(\Lambda_z)}$. Writing $\phi_T^{(z)} = T + g(z)\tau + \Delta(z)\tau^2$, the coefficients $g$ and $\Delta$ become functions in $z$, in fact, modular forms of respective weights $q - 1$ and $q^2 - 1$. We have ([Go1], [G1, II 2.10])

$$g = (T^g - T)E_{q-1}, : \Delta = (T^{q^2} - T)E_{q^2-1} + (T^{q^2} - T^q)E_{q-1}^{q+1}. \tag{4.3.7}$$

The crucial fact is that $\Delta(z) \neq 0$ for $z \in \Omega$, but $\Delta$ vanishes "at infinity". Letting $j(z) := g(z)^{q+1}/\Delta(z)$ (which is a function on $\Omega$ invariant under $\Gamma$), the considerations of Example 3 show that $j$ is a complete invariant for Drinfeld modules of rank two. Therefore, the composite map

$$j : \Gamma \setminus \Omega \xrightarrow{\sim} \mathcal{M}^2(1)(C_\infty) \xrightarrow{\sim} C_\infty \tag{4.3.8}$$

is bijective, in fact, biholomorphic.

## 4.4. Moduli schemes

We want to give a similar description of $\mathcal{M}^r(1)(C_\infty)$ for $r \geqslant 2$ and arbitrary $A$, that is, to convert (4.3.4) into an isomorphism of analytic spaces. One proceeds as follows (see [Dr], [DH], [G3]):

(a) Generalize the notion of "Drinfeld $A$-module over an $A$-field $L$" to "Drinfeld $A$-module over an $A$-scheme $S \to \operatorname{Spec} A$". This is quite straightforward. Intuitively, a Drinfeld module over $S$ is a continuously varying family of Drinfeld modules over the residue fields of $S$.

(b) Consider the functor on $A$-schemes:

$$\mathcal{M}^r : S \longmapsto \left\{ \begin{array}{l} \text{isomorphism classes of rank-}r \\ \text{Drinfeld modules over } S \end{array} \right\}.$$

The naive initial question is to represent this functor by an $S$-scheme $M^r(1)$. This is impossible in view of the existence of automorphisms of Drinfeld modules even over algebraically closed $A$-fields.

(c) As a remedy, introduce rigidifying level structures on Drinfeld modules. Fix some ideal $0 \neq \mathfrak{n}$ of $A$. An $\mathfrak{n}$-*level structure* on the Drinfeld module $\phi$ over the $A$-field $L$ whose $A$-characteristic doesn't divide $\mathfrak{n}$ is the choice of an isomorphism of $A$-modules

$$\alpha : (A/\mathfrak{n})^r \xrightarrow{\sim} {}_{\mathfrak{n}}\phi(L)$$

(compare Proposition 2). Appropriate modifications apply to the cases where $\operatorname{char}_A(L)$ divides $\mathfrak{n}$ and where the definition field $L$ is replaced by an $A$-scheme $S$. Let $\mathcal{M}^r(\mathfrak{n})$ be the functor

$$\mathcal{M}^r(\mathfrak{n}) : S \longmapsto \left\{ \begin{array}{l} \text{isomorphism classes of rank-}r \\ \text{Drinfeld modules over } S \text{ endowed} \\ \text{with an } \mathfrak{n}\text{-level structure} \end{array} \right\}.$$

**Theorem 3** (Drinfeld [Dr, Cor. to Prop. 5.4])**.** *Suppose that $\mathfrak{n}$ is divisible by at least two different prime ideals. Then $\mathcal{M}^r(\mathfrak{n})$ is representable by a smooth affine $A$-scheme $M^r(\mathfrak{n})$ of relative dimension $r-1$.*

In other words, the scheme $M^r(\mathfrak{n})$ carries a "tautological" Drinfeld module $\phi$ of rank $r$ endowed with a level-$\mathfrak{n}$ structure such that pull-back induces for each $A$-scheme $S$ a bijection

(4.4.1) $$M^r(\mathfrak{n})(S) = \{\text{morphisms } (S, M^r(\mathfrak{n}))\} \xrightarrow{\sim} \mathcal{M}^r(\mathfrak{n})(S), \quad f \longmapsto f^*(\phi).$$

$M^r(\mathfrak{n})$ is called the (fine) *moduli scheme* for the moduli problem $\mathcal{M}^r(\mathfrak{n})$. Now the finite group $G(\mathfrak{n}) := GL(r, A/\mathfrak{n})$ acts on $\mathcal{M}^r(\mathfrak{n})$ by permutations of the level structures. By functoriality, it also acts on $M^r(\mathfrak{n})$. We let $M^r(1)$ be the quotient of $M^r(\mathfrak{n})$ by $G(\mathfrak{n})$ (which does not depend on the choice of $\mathfrak{n}$). It has the property that at least its $L$-valued points for algebraically closed $A$-fields $L$ correspond bijectively and functorially to $\mathcal{M}^r(1)(L)$. It is therefore called a *coarse moduli scheme* for $\mathcal{M}^r(1)$. Combining the above with (4.3.4) yields a bijection

$$\dot{\bigcup}_{1\leqslant i\leqslant h(A)} \Gamma_i \setminus \Omega^r \xrightarrow{\sim} M^r(1)(C_\infty), \tag{4.4.2}$$

which even is an isomorphism of the underlying analytic spaces [Dr, Prop. 6.6]. The most simple special case is the one dealt with in Example 4, where $M^2(1) = \mathbb{A}^1{}_{/A}$, the affine line over $A$.

## 4.5. Compactification

It is a fundamental question to construct and study a "compactification" of the affine $A$-scheme $M^r(\mathfrak{n})$, relevant for example for the Langlands conjectures over $K$, the cohomology of arithmetic subgroups of $GL(r,A)$, or the $K$-theory of $A$ and $K$. This means that we are seeking a proper $A$-scheme $\overline{M}^r(\mathfrak{n})$ with an $A$-embedding $M^r(\mathfrak{n}) \hookrightarrow \overline{M}^r(\mathfrak{n})$ as an open dense subscheme, and which behaves functorially with respect to the forgetful morphisms $M^r(\mathfrak{n}) \to M^r(\mathfrak{m})$ if $\mathfrak{m}$ is a divisor of $\mathfrak{n}$. For many purposes it suffices to solve the apparently easier problem of constructing similar compactifications of the generic fiber $M^r(\mathfrak{n}) \times_A K$ or even of $M^r(\mathfrak{n}) \times_A C_\infty$. Note that varieties over $C_\infty$ may be studied by analytic means, using the GAGA principle.

There are presently three approaches towards the problem of compactification:
(a) a (sketchy) construction of the present author [G2] of a compactification $\overline{M}_\Gamma$ of $M_\Gamma$, the $C_\infty$-variety corresponding to an arithmetic subgroup $\Gamma$ of $GL(r,A)$ (see (4.3.4) and (4.4.2)). We will return to this below;
(b) an analytic compactification similar to (a), restricted to the case of a polynomial ring $A = \mathbb{F}_q[T]$, but with the advantage of presenting complete proofs, by M. M. Kapranov [K];
(c) R. Pink's idea of a modular compactification of $M^r(\mathfrak{n})$ over $A$ through a generalization of the underlying moduli problem [P2].

Approaches (a) and (b) agree essentially in their common domain, up to notation and some other choices. Let us briefly describe how one proceeds in (a). Since there is nothing to show for $r = 1$, we suppose that $r \geqslant 2$.

We let $A$ be any Drinfeld ring. If $\Gamma$ is a subgroup of $GL(r,K)$ commensurable with $GL(r,A)$ (we call such $\Gamma$ *arithmetic subgroups*), the point set $\Gamma \setminus \Omega$ is the set of $C_\infty$-points of an affine variety $M_\Gamma$ over $C_\infty$, as results from the discussion of subsection 4.4. If $\Gamma$ is the congruence subgroup $\Gamma(\mathfrak{n}) = \{\gamma \in GL(r,A) \colon \gamma \equiv 1 \bmod \mathfrak{n}\}$, then $M_\Gamma$ is one of the irreducible components of $M^r(\mathfrak{n}) \times_A C_\infty$.

**Definition 8.** For $\underline{\omega} = (\omega_1, \colon \dots \colon \omega_r) \in \mathbb{P}^{r-1}(C_\infty)$ put

$$r(\underline{\omega}) := \dim_K(K\omega_1 + \dots + K\omega_r) \quad \text{and} \quad r_\infty(\underline{\omega}) := \dim_{K_\infty}(K_\infty\omega_1 + \dots + K_\infty\omega_r).$$

Then $1 \leqslant r_\infty(\underline{\omega}) \leqslant r(\underline{\omega}) \leqslant r$ and $\Omega^r = \{\underline{\omega} \mid r_\infty(\underline{\omega}) = r\}$. More generally, for $1 \leqslant i \leqslant r$ let

$$\Omega^{r,i} := \{\underline{\omega}: r_\infty(\underline{\omega}) = r(\underline{\omega}) = i\}.$$

Then $\Omega^{r,i} = \dot{\bigcup} \Omega_V$, where $V$ runs through the $K$-subspaces of dimension $i$ of $K^r$ and $\Omega_V$ is constructed from $V$ in a similar way as is $\Omega^r = \Omega_{K^r}$ from $C_\infty^r = (K^r) \otimes C_\infty$. That is, $\Omega_V = \{\underline{\omega} \in \mathbb{P}(V \otimes C_\infty) \hookrightarrow \mathbb{P}^{r-1}(C_\infty): r_\infty(\underline{\omega}) = r(\underline{\omega}) = i\}$, which has a natural structure as analytic space of dimension $\dim(V) - 1$ isomorphic with $\Omega^{\dim(V)}$. Finally, we let $\overline{\Omega}^r := \{\underline{\omega}: r_\infty(\underline{\omega}) = r(\underline{\omega})\} = \dot{\bigcup}_{1 \leqslant i \leqslant r} \Omega^{r,i}$.

$\overline{\Omega}^r$ along with its stratification through the $\Omega^{r,i}$ is stable under $GL(r, K)$, so this also holds for the arithmetic group $\Gamma$ in question. The quotient $\Gamma \setminus \overline{\Omega}^r$ turns out to be the $C_\infty$-points of the wanted compactification $\overline{M}_\Gamma$.

**Definition 9.** Let $P_i \hookrightarrow G := GL(r)$ be the maximal parabolic subgroup of matrices with first $i$ columns being zero. Let $H_i$ be the obvious factor group isomorphic $GL(r-i)$. Then $P_i(K)$ acts via $H_i(K)$ on $K^{r-i}$ and thus on $\Omega^{r-i}$. From

$$G(K)/P_i(K) \xrightarrow{\sim} \{\text{subspaces } V \text{ of dimension } r-i \text{ of } K^r\}$$

we get bijections

$$\begin{aligned} &G(K) \times_{P_i(K)} \Omega^{r-i} \xrightarrow{\sim} \Omega^{r,r-i}, \\ &(g, \omega_{i+1} : \ldots : \omega_r) \longmapsto (0 : \cdots : 0 : \omega_{i+1} : \ldots : \omega_r) g^{-1} \end{aligned} \tag{4.5.1}$$

and

$$\Gamma \setminus \Omega^{r,r-i} \xrightarrow{\sim} \dot{\bigcup}_{g \in \Gamma \setminus G(K)/P_i(K)} \Gamma(i,g) \setminus \Omega^{r-i}, \tag{4.5.2}$$

where $\Gamma(i,g) := P_i \cap g^{-1}\Gamma_g$, and the double quotient $\Gamma \setminus G(K)/P_i(K)$ is finite by elementary lattice theory. Note that the image of $\Gamma(i,g)$ in $H_i(K)$ (the group that effectively acts on $\Omega^{r-i}$) is again an arithmetic subgroup of $H_i(K) = GL(r-i, K)$, and so the right hand side of (4.5.2) is the disjoint union of analytic spaces of the same type $\Gamma' \setminus \Omega^{r'}$.

**Example 5.** Let $\Gamma = \Gamma(1) = GL(r, A)$ and $i = 1$. Then $\Gamma \setminus G(K)/P_1(K)$ equals the set of isomorphism classes of projective $A$-modules of rank $r-1$, which in turn (through the determinant map) is in one-to-one correspondence with the class group $\mathrm{Pic}(A)$.

Let $F_V$ be the image of $\Omega_V$ in $\Gamma \setminus \overline{\Omega}^r$. The different analytic spaces $F_V$, corresponding to locally closed subvarieties of $\overline{M}_\Gamma$, are glued together in such a way that $F_U$ lies in the Zariski closure $\overline{F}_V$ of $F_V$ if and only if $U$ is $\Gamma$-conjugate to a $K$-subspace of $V$. Taking into account that $F_V \simeq \Gamma' \setminus \Omega^{\dim(V)} = M_{\Gamma'}(C_\infty)$ for some arithmetic subgroup $\Gamma'$ of $GL(\dim(V), K)$, $\overline{F}_V$ corresponds to the compactification $\overline{M}_{\Gamma'}$ of $M_{\Gamma'}$.

The details of the gluing procedure are quite technical and complicated and cannot be presented here (see [G2] and [K] for some special cases). Suffice it to say that for each boundary component $F_V$ of codimension one, a vertical coordinate $t_V$ may be specified such that $F_V$ is locally given by $t_V = 0$. The result (we refrain from stating a "theorem" since proofs of the assertions below in full strength and generality are published neither in [G2] nor in [K]) will be a normal projective $C_\infty$-variety $\overline{M}_\Gamma$ provided with an open dense embedding $i\colon M_\Gamma \hookrightarrow \overline{M}_\Gamma$ with the following properties:

- $\overline{M}_\Gamma(C_\infty) = \Gamma \setminus \bar{\Omega}^r$, and the inclusion $\Gamma \setminus \Omega^r \hookrightarrow \Gamma \setminus \bar{\Omega}^r$ corresponds to $i$;
- $\overline{M}_\Gamma$ is defined over the same finite abelian extension of $K$ as is $M_\Gamma$;
- for $\Gamma' \hookrightarrow \Gamma$, the natural map $M_{\Gamma'} \to M_\Gamma$ extends to $\overline{M}_{\Gamma'} \to \overline{M}_\Gamma$;
- the $F_V$ correspond to locally closed subvarieties, and $\overline{F}_V = \cup F_U$, where $U$ runs through the $K$-subspaces of $V$ contained up to the action of $\Gamma$ in $V$;
- $\overline{M}_\Gamma$ is "virtually non-singular", i.e., $\Gamma$ contains a subgroup $\Gamma'$ of finite index such that $\overline{M}_{\Gamma'}$ is non-singular; in that case, the boundary components of codimension one present normal crossings.

Now suppose that $\overline{M}_\Gamma$ is non-singular and that $x \in \overline{M}_\Gamma(C_\infty) = \bigcup_{1\leqslant i\leqslant r} \Omega^{r,i}$ belongs to $\Omega^{r,1}$. Then we can find a sequence $\{x\} = X_0 \subset \cdots X_i \cdots \subset X_{r-1} = \overline{M}_\Gamma$ of smooth subvarieties $X_i = \overline{F}_{V_i}$ of dimension $i$. Any holomorphic function around $x$ (or more generally, any modular form for $\Gamma$) may thus be expanded as a series in $t_V$ with coefficients in the function field of $\overline{F}_{V_{r-1}}$, etc. Hence $\overline{M}_\Gamma$ (or rather its completion at the $X_i$) may be described through $(r-1)$-dimensional local fields with residue field $C_\infty$. The expansion of some standard modular forms can be explicitly calculated, see [G1, VI] for the case of $r = 2$. In the last section we shall present at least the vanishing orders of some of these forms.

**Example 6.** Let $A$ be the polynomial ring $\mathbb{F}_q[T]$ and $\Gamma = GL(r, A)$. As results from Example 3, (4.3.3) and (4.4.2),

$$M_\Gamma(C_\infty) = M^r(1)(C_\infty) = \{(g_1, \dots, g_r) \in C_\infty^r : g_r \neq 0\}/C_\infty^*,$$

where $C_\infty^*$ acts diagonally through $c(g_1, \dots, g_r) = (\dots, c^{q^i-1} g_i, \dots)$, which is the open subspace of weighted projective space $\mathbb{P}^{r-1}(q-1, \dots, q^r-1)$ with non-vanishing last coordinate. The construction yields

$$\overline{M}_\Gamma(C_\infty) = \mathbb{P}^{r-1}(q-1, \dots, q^r-1)(C_\infty) = \dot{\bigcup}_{1\leqslant i\leqslant r} M^i(1)(C_\infty).$$

Its singularities are rather mild and may be removed upon replacing $\Gamma$ by a congruence subgroup.

## 4.6. Vanishing orders of modular forms

In this final section we state some results about the vanishing orders of certain modular forms along the boundary divisors of $\overline{M}_\Gamma$, in the case where $\Gamma$ is either $\Gamma(1) = GL(r, A)$ or a full congruence subgroup $\Gamma(\mathfrak{n})$ of $\Gamma(1)$. These are relevant for the determination of $K$- and Chow groups, and for standard conjectures about the arithmetic interpretation of partial zeta values.

In what follows, we suppose that $r \geqslant 2$, and put $z_i := \frac{\omega_i}{\omega_r}$ $(1 \leqslant i \leqslant r)$ for the coordinates $(\omega_1 : \ldots : \omega_r)$ of $\underline{\omega} \in \Omega^r$. Quite generally, $\underline{a} = (a_1, \ldots, a_r)$ denotes a vector with $r$ components.

**Definition 10.** The *Eisenstein series* $E_k$ of weight $k$ on $\Omega^r$ is defined as

$$E_k(\underline{\omega}) := \sum_{0 \neq \underline{a} \in A^r} \frac{1}{(a_1 z_1 + \cdots + a_r z_r)^k}.$$

Similarly, we define for $\underline{u} \in \mathfrak{n}^{-1} \times \cdots \times \mathfrak{n}^{-1} \subset K^r$

$$E_{k,\underline{u}}(\underline{\omega}) = \sum_{\substack{0 \neq \underline{a} \in K^r \\ \underline{a} \equiv \underline{u} \bmod A^r}} \frac{1}{(a_1 z_1 + \cdots + a_r z_r)^k}.$$

These are modular forms for $\Gamma(1)$ and $\Gamma(\mathfrak{n})$, respectively, that is, they are holomorphic, satisfy the obvious transformation values under $\Gamma(1)$ (resp. $\Gamma(\mathfrak{n})$), and extend to sections of a line bundle on $\overline{M}_\Gamma$. As in Example 4, there is a second type of modular forms coming directly from Drinfeld modules.

**Definition 11.** For $\underline{\omega} \in \Omega^r$ write $\Lambda_{\underline{\omega}} = Az_1 + \cdots + Az_r$ and $e_{\underline{\omega}}$, $\phi^{\underline{\omega}}$ for the lattice function and Drinfeld module associated with $\Lambda_{\underline{\omega}}$, respectively. If $a \in A$ has degree $d = \deg(a)$,

$$\phi_a^{\underline{\omega}} = a + \sum_{1 \leqslant i \leqslant r \cdot d} \ell_i(a, \underline{\omega}) \tau^i.$$

The $\ell_i(a, \underline{\omega})$ are modular forms of weight $q^i - 1$ for $\Gamma$. This holds in particular for

$$\Delta_a(\underline{\omega}) := \ell_{rd}(a, \underline{\omega}),$$

which has weight $q^{rd} - 1$ and vanishes nowhere on $\Omega^r$. The functions $g$ and $\Delta$ in Example 4 merely constitute a very special instance of this construction. We further let, for $\underline{u} \in (\mathfrak{n}^{-1})^r$,

$$e_{\underline{u}}(\underline{\omega}) := e_{\underline{\omega}}(u_1 z_1 + \cdots + u_r z_r),$$

the $\mathfrak{n}$-division point of type $\underline{u}$ of $\phi^{\underline{\omega}}$. If $\underline{u} \notin A^r$, $e_{\underline{u}}(\underline{\omega})$ vanishes nowhere on $\Omega^r$, and it can be shown that in this case,

$$e_{\underline{u}}^{-1} = E_{1,\underline{u}}. \qquad (4.6.1)$$

We are interested in the behavior around the boundary of $\overline{M}_\Gamma$ of these forms. Let us first describe the set $\{\overline{F}_V\}$ of boundary divisors, i.e., of irreducible components, all of codimension one, of $\overline{M}_\Gamma - M_\Gamma$. For $\Gamma = \Gamma(1) = GL(r, A)$, there is a natural bijection

$$\{\overline{F}_V\} \xrightarrow{\sim} \mathrm{Pic}(A) \tag{4.6.2}$$

described in detail in [G1, VI 5.1]. It is induced from $V \mapsto$ inverse of $\Lambda^{r-1}(V \cap A^r)$. (Recall that $V$ is a $K$-subspace of dimension $r-1$ of $K^r$, thus $V \cap A^r$ a projective module of rank $r-1$, whose $(r-1)$-th exterior power $\Lambda^{r-1}(V \cap A^r)$ determines an element of $\mathrm{Pic}(A)$.) We denote the component corresponding to the class $(\mathfrak{a})$ of an ideal $\mathfrak{a}$ by $\overline{F}_{(\mathfrak{a})}$. Similarly, the boundary divisors of $\overline{M}_\Gamma$ for $\Gamma = \Gamma(\mathfrak{n})$ could be described via generalized class groups. We simply use (4.5.1) and (4.5.2), which now give

$$\{\overline{F}_V\} \xrightarrow{\sim} \Gamma(\mathfrak{n}) \setminus GL(r, K)/P_1(K). \tag{4.6.3}$$

We denote the class of $\nu \in GL(r, K)$ by $[\nu]$. For the description of the behavior of our modular forms along the $\overline{F}_V$, we need the partial zeta functions of $A$ and $K$. For more about these, see [W] and [G1, III].

**Definition 12.** We let

$$\zeta_K(s) = \sum |\mathfrak{a}|^{-s} = \frac{P(q^{-s})}{(1-q^{-s})(1-q^{1-s})}$$

be the zeta function of $K$ with numerator polynomial $P(X) \in \mathbb{Z}[X]$. Here the sum is taken over the positive divisors $\mathfrak{a}$ of $K$ (i.e., of the curve $\mathcal{C}$ with function field $K$). Extending the sum only over divisors with support in $\mathrm{Spec}(A)$, we get

$$\zeta_A(s) = \sum_{0 \neq \mathfrak{a} \subset A \text{ ideal}} |\mathfrak{a}|^{-s} = \zeta_K(s)(1 - q^{-d_\infty s}),$$

where $d_\infty = \deg_{\mathbb{F}_q}(\infty)$. For a class $\mathfrak{c} \in \mathrm{Pic}(A)$ we put

$$\zeta_{\mathfrak{c}}(s) = \sum_{\mathfrak{a} \in \mathfrak{c}} |\mathfrak{a}|^{-s}.$$

If finally $\mathfrak{n} \subset K$ is a fractional $A$-ideal and $t \in K$, we define

$$\zeta_{t \bmod \mathfrak{n}}(s) = \sum_{\substack{a \in K \\ a \equiv t \bmod \mathfrak{n}}} |a|^{-s}.$$

Among the obvious distribution relations [G1, III sect. 1] between these, we only mention

$$\zeta_{(\mathfrak{n}^{-1})}(s) = \frac{|\mathfrak{n}|^s}{q-1} \zeta_{0 \bmod \mathfrak{n}}(s). \tag{4.6.4}$$

We are now in a position to state the following theorems, which may be proved following the method of [G1, VI].

**Theorem 4.** *Let* $a \in A$ *be non-constant and* $\mathfrak{c}$ *a class in* $\mathrm{Pic}(A)$. *The modular form* $\Delta_a$ *for* $GL(r, A)$ *has vanishing order*

$$\mathrm{ord}_{\mathfrak{c}}(\Delta_a) = -(|a|^r - 1)\zeta_{\mathfrak{c}}(1 - r)$$

*at the boundary component* $\overline{F}_{\mathfrak{c}}$ *corresponding to* $\mathfrak{c}$.

**Theorem 5.** *Fix an ideal* $\mathfrak{n}$ *of* $A$ *and* $\underline{u} \in K^r - A^r$ *such that* $\underline{u} \cdot \mathfrak{n} \subset A^r$, *and let* $e_{\underline{u}}^{-1} = E_{1,\underline{u}}$ *be the modular form for* $\Gamma(\mathfrak{n})$ *determined by these data. The vanishing order* $\mathrm{ord}_{[\nu]}$ *of* $E_{1,\underline{u}}(\underline{\omega})$ *at the component corresponding to* $\nu \in GL(r, K)$ (*see* (4.6.2)) *is given as follows*: *let* $\pi_1 : K^r \to K$ *be the projection to the first coordinate and let* $\mathfrak{a}$ *be the fractional ideal* $\pi_1(A^r \cdot \nu)$. *Write further* $\underline{u} \cdot \nu = (v_1, \dots, v_r)$. *Then*

$$\mathrm{ord}_{[\nu]} E_{1,\underline{u}}(\omega) = \frac{|\mathfrak{n}|^{r-1}}{|\mathfrak{a}|^{r-1}}(\zeta_{v_1 \bmod \mathfrak{a}}(1 - r) - \zeta_{0 \bmod \mathfrak{a}}(1 - r)).$$

Note that the two theorems do not depend on the full strength of properties of $\overline{M}_\Gamma$ as stated without proofs in the last section, but only on the *normality* of $\overline{M}_\Gamma$, which is *proved* in [K] for $A = \mathbb{F}_q[T]$, and whose generalization to arbitrary Drinfeld rings is straightforward (even though technical).

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*ISSN 1464-8997 (on line) 1464-8989 (printed)* **255**

Geometry & Topology Monographs
Volume 3: Invitation to higher local fields
Part II, section 5, pages 255–262

# 5. Harmonic analysis on algebraic groups over two-dimensional local fields of equal characteristic

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In this section we review the main parts of a recent work [4] on harmonic analysis on algebraic groups over two-dimensional local fields.

## 5.1. Groups and buildings

Let $K$ ( $K = K_2$ whose residue field is $K_1$ whose residue field is $K_0$, see the notation in section 1 of Part I) be a two-dimensional local field of equal characteristic. Thus $K_2$ is isomorphic to the Laurent series field $K_1((t_2))$ over $K_1$. It is convenient to think of elements of $K_2$ as (formal) loops over $K_1$. Even in the case where $\mathrm{char}(K_1) = 0$, it is still convenient to think of elements of $K_1$ as (generalized) loops over $K_0$ so that $K_2$ consists of double loops.

Denote the residue map $\mathcal{O}_{K_2} \to K_1$ by $p_2$ and the residue map $\mathcal{O}_{K_1} \to K_0$ by $p_1$. Then the ring of integers $O_K$ of $K$ as of a two-dimensional local field (see subsection 1.1 of Part I) coincides with $p_2^{-1}(\mathcal{O}_{K_1})$.

Let $G$ be a split simple simply connected algebraic group over $\mathbb{Z}$ (e.g. $G = SL_2$ ). Let $T \subset B \subset G$ be a fixed maximal torus and Borel subgroup of $G$; put $N = [B, B]$, and let $W$ be the Weyl group of $G$. All of them are viewed as group schemes.

Let $L = \mathrm{Hom}(\mathbb{G}_m, T)$ be the coweight lattice of $G$; the Weyl group acts on $L$.

Recall that $I(K_1) = p_1^{-1}(B(\mathbb{F}_q))$ is called an Iwahori subgroup of $G(K_1)$ and $T(\mathcal{O}_{K_1})N(K_1)$ can be seen as the "connected component of unity" in $B(K_1)$. The latter name is explained naturally if we think of elements of $B(K_1)$ as being loops with values in $B$.

*Published 10 December 2000:* 

**Definition.** Put

$$D_0 = p_2^{-1} p_1^{-1}(B(\mathbb{F}_q)) \subset G(O_K),$$
$$D_1 = p_2^{-1}(T(\mathcal{O}_{K_1})N(K_1)) \subset G(O_K),$$
$$D_2 = T(\mathcal{O}_{K_2})N(K_2) \subset G(K).$$

Then $D_2$ can be seen as the "connected component of unity" of $B(K)$ when $K$ is viewed as a two-dimensional local field, $D_1$ is a (similarly understood) connected component of an Iwahori subgroup of $G(K_2)$, and $D_0$ is called a double Iwahori subgroup of $G(K)$.

A choice of a system of local parameters $t_1, t_2$ of $K$ determines the identification of the group $K^*/O_K^*$ with $\mathbb{Z} \oplus \mathbb{Z}$ and identification $L \oplus L$ with $L \otimes (K^*/O_K^*)$.

We have an embedding of $L \otimes (K^*/O_K^*)$ into $T(K)$ which takes $a \otimes (t_1^j t_2^j)$, $i, j \in \mathbb{Z}$, to the value on $t_1^i t_2^j$ of the 1-parameter subgroup in $T$ corresponding to $a$.

Define the action of $W$ on $L \otimes (K^*/O_K^*)$ as the product of the standard action on $L$ and the trivial action on $K^*/O_K^*$. The semidirect product

$$\widehat{\widehat{W}} = (L \otimes K^*/O_K^*) \rtimes W$$

is called the *double affine Weyl group* of $G$.

A (set-theoretical) lifting of $W$ into $G(O_K)$ determines a lifting of $\widehat{\widehat{W}}$ into $G(K)$.

**Proposition.** *For every $i, j \in \{0, 1, 2\}$ there is a disjoint decomposition*

$$G(K) = \dot{\bigcup}_{w \in \widehat{\widehat{W}}} D_i w D_j.$$

*The identification $D_i \backslash G(K)/D_j$ with $\widehat{\widehat{W}}$ doesn't depend on the choice of liftings.*

*Proof.* Iterated application of the Bruhat, Bruhat–Tits and Iwasawa decompositions to the local fields $K_2$, $K_1$.

For the Iwahori subgroup $I(K_2) = p_2^{-1}(B(K_1))$ of $G(K_2)$ the homogeneous space $G(K)/I(K_2)$ is the "affine flag variety" of $G$, see [5]. It has a canonical structure of an ind-scheme, in fact, it is an inductive limit of projective algebraic varieties over $K_1$ (the closures of the affine Schubert cells).

Let $B(G, K_2/K_1)$ be the *Bruhat–Tits building* associated to $G$ and the field $K_2$. Then the space $G(K)/I(K_2)$ is a $G(K)$-orbit on the set of flags of type (vertex, maximal cell) in the building. For every vertex $v$ of $B(G, K_2/K_1)$ its locally finite Bruhat–Tits building $\beta_v$ isomorphic to $B(G, K_1/K_0)$ can be viewed as a "microbuilding" of the *double Bruhat–Tits building* $B(G, K_2/K_1/K_0)$ of $K$ as a two-dimensional local field constructed by Parshin ([7], see also section 3 of Part II). Then the set $G(K)/D_1$ is identified naturally with the set of all the horocycles $\{w \in \beta_v : d(z, w) = r\}$, $z \in \partial\beta_v$ of the microbuildings $\beta_v$ (where the "distance" $d(z, \ )$ is viewed as an element of a natural $L$-torsor). The fibres of the projection $G(K)/D_1 \to G(K)/I(K_2)$ are $L$-torsors.

## 5.2. The central extension and the affine Heisenberg–Weyl group

According to the work of Steinberg, Moore and Matsumoto [6] developed by Brylinski and Deligne [1] there is a central extension

$$1 \to K_1^* \to \Gamma \to G(K_2) \to 1$$

associated to the tame symbol $K_2^* \times K_2^* \to K_1^*$ for the couple $(K_2, K_1)$ (see subsection 6.4.2 of Part I for the general definition of the tame symbol).

**Proposition.** *This extension splits over every* $D_i$, $0 \leqslant i \leqslant 2$.

*Proof.* Use Matsumoto's explicit construction of the central extension.

Thus, there are identifications of every $D_i$ with a subgroup of $\Gamma$. Put

$$\Delta_i = \mathcal{O}_{K_1}^* D_i \subset \Gamma, \qquad \Xi = \Gamma/\Delta_1.$$

The minimal integer scalar product $\Psi$ on $L$ and the composite of the tame symbol $K_2^* \times K_2^* \to K_1^*$ and the discrete valuation $v_{K_1}: K^* \to \mathbb{Z}$ induces a $W$-invariant skew-symmetric pairing $L \otimes K^*/O_K^* \times L \otimes K^*/O_K^* \to \mathbb{Z}$. Let

$$1 \to \mathbb{Z} \to \mathcal{L} \to L \otimes K^*/O_K^* \to 1$$

be the central extension whose commutator pairing corresponds to the latter skew-symmetric pairing. The group $\mathcal{L}$ is called the *Heisenberg group*.

**Definition.** The semidirect product

$$\widetilde{W} = \mathcal{L} \rtimes W$$

is called the *double affine Heisenberg–Weyl* group of $G$.

**Theorem.** *The group* $\widetilde{W}$ *is isomorphic to* $L_{\mathrm{aff}} \rtimes \widehat{W}$ *where* $L_{\mathrm{aff}} = \mathbb{Z} \oplus L$, $\widehat{W} = L \rtimes W$ *and*

$$w \circ (a, l') = (a, w(l)), \quad l \circ (a, l') = (a + \Psi(l, l'), l'), \qquad w \in W, \quad l, l' \in L, \quad a \in \mathbb{Z}.$$

*For every* $i, j \in \{0, 1, 2\}$ *there is a disjoint union*

$$\Gamma = \dot{\bigcup}_{w \in \widetilde{W}} \Delta_i w \Delta_j$$

*and the identification* $\Delta_i \backslash \Gamma / \Delta_j$ *with* $\widetilde{W}$ *is canonical.*

## 5.3. Hecke algebras in the classical setting

Recall that for a locally compact group $\Gamma$ and its compact subgroup $\Delta$ the Hecke algebra $\mathcal{H}(\Gamma,\Delta)$ can be defined as the algebra of compactly supported double $\Delta$-invariant continuous functions of $\Gamma$ with the operation given by the convolution with respect to the Haar measure on $\Gamma$. For $C = \Delta\gamma\Delta \in \Delta\backslash\Gamma/\Delta$ the Hecke correspondence $\Sigma_C = \{(\alpha\Delta,\beta\Delta) : \alpha\beta^{-1} \in C\}$ is a $\Gamma$-orbit of $(\Gamma/\Delta)\times(\Gamma/\Delta)$.

For $x \in \Gamma/\Delta$ put $\Sigma_C(x) = \Sigma_C \cap (\Gamma/\Delta)\times\{x\}$. Denote the projections of $\Sigma_C$ to the first and second component by $\pi_1$ and $\pi_2$.

Let $\mathcal{F}(\Gamma/\Delta)$ be the space of continuous functions $\Gamma/\Delta \to \mathbb{C}$. The operator

$$\tau_C\colon \mathcal{F}(\Gamma/\Delta) \to \mathcal{F}(\Gamma/\Delta), \quad f \to \pi_{2*}\pi_1^*(f)$$

is called the *Hecke operator* associated to $C$. Explicitly,

$$(\tau_C f)(x) = \int_{y\in\Sigma_C(x)} f(y)d\mu_{C,x},$$

where $\mu_{C,x}$ is the $\mathrm{Stab}(x)$-invariant measure induced by the Haar measure. Elements of the Hecke algebra $\mathcal{H}(\Gamma,\Delta)$ can be viewed as "continuous" linear combinations of the operators $\tau_C$, i.e., integrals of the form $\int \phi(C)\tau_C dC$ where $dC$ is some measure on $\Delta\backslash\Gamma/\Delta$ and $\phi$ is a continuous function with compact support. If the group $\Delta$ is also open (as is usually the case in the $p$-adic situation), then $\Delta\backslash\Gamma/\Delta$ is discrete and $\mathcal{H}(\Gamma,\Delta)$ consists of finite linear combinations of the $\tau_C$.

## 5.4. The regularized Hecke algebra $\mathcal{H}(\Gamma,\Delta_1)$

Since the two-dimensional local field $K$ and the ring $O_K$ are not locally compact, the approach of the previous subsection would work only after a new appropriate integration theory is available.

The aim of this subsection is to make sense of the Hecke algebra $\mathcal{H}(\Gamma,\Delta_1)$.

Note that the fibres of the projection $\Xi = \Gamma/\Delta_1 \to G(K)/I(K_2)$ are $L_{\mathrm{aff}}$-torsors and $G(K)/I(K_2)$ is the inductive limit of compact (profinite) spaces, so $\Xi$ can be considered as an object of the category $\mathcal{F}_1$ defined in subsection 1.2 of the paper of Kato in this volume.

Using Theorem of 5.2 for $i = j = 1$ we introduce:

**Definition.** For $(w,l) \in \widetilde{W} = L_{\mathrm{aff}} \rtimes \widehat{W}$ denote by $\Sigma_{w,l}$ the Hecke correspondence (i.e., the $\Gamma$-orbit of $\Xi\times\Xi$) associated to $(w,l)$. For $\xi \in \Xi$ put

$$\Sigma_{w,l}(\xi) = \{\xi' : (\xi,\xi') \in \Sigma_{w,l}\}.$$

The stabilizer $\mathrm{Stab}(\xi) \leqslant \Gamma$ acts transitively on $\Sigma_{w,l}(\xi)$.

**Proposition.** *$\Sigma_{w,l}(\xi)$ is an affine space over $K_1$ of dimension equal to the length of $w \in \widehat{W}$. The space of compex valued Borel measures on $\Sigma_{w,l}(\xi)$ is 1-dimensional. A choice of a* $\mathrm{Stab}(\xi)$*-invariant measure $\mu_{w,l,\xi}$ on $\Sigma_{w,l}(\xi)$ determines a measure $\mu_{w,l,\xi'}$ on $\Sigma_{w,l}(\xi')$ for every $\xi'$.*

**Definition.** For a continuous function $f:\Xi \to \mathbb{C}$ put

$$(\tau_{w,l}f)(\xi) = \int_{\eta\in\Sigma_{w,l}(\xi)} f(\eta)d\mu_{w,l,\xi}.$$

Since the domain of the integration is not compact, the integral may diverge. As a first step, we define the space of functions on which the integral makes sense. Note that $\Xi$ can be regarded as an $L_{\mathrm{aff}}$-torsor over the ind-object $G(K)/I(K_2)$ in the category $\mathrm{pro}(C_0)$, i.e., a compatible system of $L_{\mathrm{aff}}$-torsors $\Xi_\nu$ over the affine Schubert varieties $Z_\nu$ forming an exhaustion of $G(K)/I(K_1)$. Each $\Xi_\nu$ is a locally compact space and $Z_\nu$ is a compact space. In particular, we can form the space $\mathcal{F}_0(\Xi_\nu)$ of locally constant complex valued functions on $\Xi_\nu$ whose support is compact (or, what is the same, proper with respect to the projection to $Z_\nu$). Let $\mathcal{F}(\Xi_\nu)$ be the space of all locally constant complex functions on $\Xi_\nu$. Then we define $\mathcal{F}_0(\Xi) = \text{"}\varprojlim\text{"}\,\mathcal{F}_0(\Xi_\nu)$ and $\mathcal{F}(\Xi) = \text{"}\varprojlim\text{"}\,\mathcal{F}(\Xi_\nu)$. They are pro-objects in the category of vector spaces. In fact, because of the action of $L_{\mathrm{aff}}$ and its group algebra $\mathbb{C}[L_{\mathrm{aff}}]$ on $\Xi$, the spaces $\mathcal{F}_0(\Xi), \mathcal{F}(\Xi)$ are naturally pro-objects in the category of $\mathbb{C}[L_{\mathrm{aff}}]$-modules.

**Proposition.** *If $f = (f_\nu) \in \mathcal{F}_0(X)$ then* $\mathrm{Supp}(f_\nu) \cap \Sigma_{w,l}(\xi)$ *is compact for every $w, l, \xi, \nu$ and the integral above converges. Thus, there is a well defined Hecke operator*

$$\tau_{w,l}: \mathcal{F}_0(\Xi) \to \mathcal{F}(\Xi)$$

*which is an element of* $\mathrm{Mor}(\mathrm{pro}(\mathrm{Mod}_{\mathbb{C}[L_{\mathrm{aff}}]}))$. *In particular, $\tau_{w,l}$ is the shift by $l$ and $\tau_{w,l+l'} = \tau_{w,l'}\tau_{e,l}$.*

Thus we get Hecke operators as operators from one (pro-)vector space to another, bigger one. This does not yet allow to compose the $\tau_{w,l}$. Our next step is to consider certain infinite linear combinations of the $\tau_{w,l}$.

Let $T^\vee_{\mathrm{aff}} = \mathrm{Spec}(\mathbb{C}[L_{\mathrm{aff}}])$ be the "dual affine torus" of $G$. A function with finite support on $L_{\mathrm{aff}}$ can be viewed as the collection of coefficients of a polynomial, i.e., of an element of $\mathbb{C}[L_{\mathrm{aff}}]$ as a regular function on $T^\vee_{\mathrm{aff}}$. Further, let $Q \subset L_{\mathrm{aff}} \otimes \mathbb{R}$ be a strictly convex cone with apex $0$. A function on $L_{\mathrm{aff}}$ with support in $Q$ can be viewed as the collection of coefficients of a formal power series, and such series form a ring containing $\mathbb{C}[L_{\mathrm{aff}}]$. On the level of functions the ring operation is the convolution. Let $\mathcal{F}_Q(L_{\mathrm{aff}})$ be the space of functions whose support is contained in some translation of $Q$. It is a ring with respect to convolution.

Let $\mathbb{C}(L_{\mathrm{aff}})$ be the field of rational functions on $T^\vee_{\mathrm{aff}}$. Denote by $F^{\mathrm{rat}}_Q(L_{\mathrm{aff}})$ the subspace in $F_Q(L_{\mathrm{aff}})$ consisting of functions whose corresponding formal power series are expansions of rational functions on $T^\vee_{\mathrm{aff}}$.

If $A$ is any $L_{\mathrm{aff}}$-torsor (over a point), then $\mathcal{F}_0(A)$ is an (invertible) module over $\mathcal{F}_0(L_{\mathrm{aff}}) = \mathbb{C}[L_{\mathrm{aff}}]$ and we can define the spaces $\mathcal{F}_Q(A)$ and $\mathcal{F}^{\mathrm{rat}}_Q(A)$ which will be modules over the corresponding rings for $L_{\mathrm{aff}}$. We also write $\mathcal{F}^{\mathrm{rat}}(A) = \mathcal{F}_0(A) \otimes_{\mathbb{C}[L_{\mathrm{aff}}]} \mathbb{C}(L_{\mathrm{aff}})$.

We then extend the above concepts "fiberwise" to torsors over compact spaces (objects of $\mathrm{pro}(C_0)$) and to torsors over objects of $\mathrm{ind}(\mathrm{pro}(C_0))$ such as $\Xi$.

Let $w \in \widehat{W}$. We denote by $Q(w)$ the image under $w$ of the cone of dominant affine coweights in $L_{\mathrm{aff}}$.

**Theorem.** *The action of the Hecke operator $\tau_{w,l}$ takes $\mathcal{F}_0(\Xi)$ into $\mathcal{F}^{\mathrm{rat}}_{Q(w)}(\Xi)$. These operators extend to operators*

$$\tau^{\mathrm{rat}}_{w,l} : \mathcal{F}^{\mathrm{rat}}(\Xi) \to \mathcal{F}^{\mathrm{rat}}(\Xi).$$

Note that the action of $\tau^{\mathrm{rat}}_{w,l}$ involves a kind of regularization procedure, which is hidden in the identification of the $\mathcal{F}^{\mathrm{rat}}_{Q(w)}(\Xi)$ for different $w$, with subspaces of the same space $\mathcal{F}^{\mathrm{rat}}(\Xi)$. In practical terms, this involves summation of a series to a rational function and re-expansion in a different domain.

Let $\mathcal{H}_{\mathrm{pre}}$ be the space of finite linear combinations $\sum_{w,l} a_{w,l}\tau_{w,l}$. This is not yet an algebra, but only a $\mathbb{C}[L_{\mathrm{aff}}]$-module. Note that elements of $\mathcal{H}_{\mathrm{pre}}$ can be written as finite linear combinations $\sum_{w\in\widehat{W}} f_w(t)\tau_w$ where $f_w(t) = \sum_l a_{w,l}t^l$, $t \in T^\vee_{\mathrm{aff}}$, is the polynomial in $\mathbb{C}[L_{\mathrm{aff}}]$ corresponding to the collection of the $a_{w,l}$. This makes the $\mathbb{C}[L_{\mathrm{aff}}]$-module structure clear. Consider the tensor product

$$\mathcal{H}_{\mathrm{rat}} = \mathcal{H}_{\mathrm{pre}} \otimes_{\mathbb{C}[L_{\mathrm{aff}}]} \mathbb{C}(L_{\mathrm{aff}}).$$

Elements of this space can be considered as finite linear combinations $\sum_{w\in\widehat{W}} f_w(t)\tau_w$ where $f_w(t)$ are now rational functions. By expanding rational functions in power series, we can consider the above elements as certain infinite linear combinations of the $\tau_{w,l}$.

**Theorem.** *The space $\mathcal{H}_{\mathrm{rat}}$ has a natural algebra structure and this algebra acts in the space $\mathcal{F}^{\mathrm{rat}}(\Xi)$, extending the action of the $\tau_{w,l}$ defined above.*

The operators associated to $\mathcal{H}_{\mathrm{rat}}$ can be viewed as certain integro-difference operators, because their action involves integration (as in the definition of the $\tau_{w,l}$) as well as inverses of linear combinations of shifts by elements of $L$ (these combinations act as difference operators).

**Definition.** The regularized Hecke algebra $\mathcal{H}(\Gamma, \Delta_1)$ is, by definition, the subalgebra in $\mathcal{H}_{\mathrm{rat}}$ consisting of elements whose action in $\mathcal{F}_{\mathrm{rat}}(\Xi)$ preserves the subspace $\mathcal{F}_0(\Xi)$.

## 5.5. The Hecke algebra and the Cherednik algebra

In [2] I. Cherednik introduced the so-called double affine Hecke algebra $\mathrm{Cher}_q$ associated to the root system of $G$. As shown by V. Ginzburg, E. Vasserot and the author [3], $\mathrm{Cher}_q$ can be thought as consisting of finite linear combinations $\sum_{w\in\widehat{W}_{\mathrm{ad}}} f_w(t)[w]$ where $W_{\mathrm{ad}}$ is the affine Weyl group of the adjoint quotient $G_{\mathrm{ad}}$ of $G$ (it contains $\widehat{W}$ ) and $f_w(t)$ are rational functions on $T^\vee_{\mathrm{aff}}$ satisfying certain residue conditions. We define the modified Cherednik algebra $\ddot{H}_q$ to be the subalgebra in $\mathrm{Cher}_q$ consisting of linear combinations as above, but going over $\widehat{W}\subset\widehat{W}_{\mathrm{ad}}$.

**Theorem.** *The regularized Hecke algebra $\mathcal{H}(\Gamma,\Delta_1)$ is isomorphic to the modified Cherednik algebra $\ddot{H}_q$. In particular, there is a natural action of $\ddot{H}_q$ on $\mathcal{F}_0(\Xi)$ by integro-difference operators.*

*Proof.* Use the principal series intertwiners and a version of Mellin transform. The information on the poles of the intertwiners matches exactly the residue conditions introduced in [3].

**Remark.** The only reason we needed to assume that the 2-dimensional local field $K$ has equal characteristic was because we used the fact that the quotient $G(K)/I(K_2)$ has a structure of an inductive limit of projective algebraic varieties over $K_1$. In fact, we really use only a weaker structure: that of an inductive limit of profinite topological spaces (which are, in this case, the sets of $K_1$-points of affine Schubert varieties over $K_1$ ). This structure is available for any 2-dimensional local field, although there seems to be no reference for it in the literature. Once this foundational matter is established, all the constructions will go through for any 2-dimensional local field.

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*ISSN 1464-8997 (on line) 1464-8989 (printed)* **263**

Geometry & Topology Monographs
Volume 3: Invitation to higher local fields
Part II, section 6, pages 263–272

# 6. $\Phi$-$\Gamma$-modules and Galois cohomology

*Laurent Herr*

## 6.0. Introduction

Let $G$ be a profinite group and $p$ a prime number.

**Definition.** A finitely generated $\mathbb{Z}_p$-module $V$ endowed with a continuous $G$-action is called a $\mathbb{Z}_p$-adic representation of $G$. Such representations form a category denoted by $\mathrm{Rep}_{\mathbb{Z}_p}(G)$; its subcategory $\mathrm{Rep}_{\mathbb{F}_p}(G)$ (respectively $\mathrm{Rep}_{p\text{-tor}}(G)$) of mod $p$ representations (respectively $p$-torsion representations) consists of the $V$ annihilated by $p$ (respectively a power of $p$).

**Problem.** To calculate in a simple explicit way the cohomology groups $H^i(G,V)$ of the representation $V$.

A method to solve it for $G = G_K$ ($K$ is a local field) is to use Fontaine's theory of $\Phi$-$\Gamma$-modules and pass to a simpler Galois representation, paying the price of enlarging $\mathbb{Z}_p$ to the ring of integers of a two-dimensional local field. In doing this we have to replace linear with semi-linear actions.

In this paper we give an overview of the applications of such techniques in different situations. We begin with a simple example.

## 6.1. The case of a field of positive characteristic

Let $E$ be a field of characteristic $p$, $G = G_E$ and $\sigma\colon E^{\mathrm{sep}} \to E^{\mathrm{sep}}$, $\sigma(x) = x^p$ the absolute Frobenius map.

**Definition.** For $V \in \mathrm{Rep}_{\mathbb{F}_p}(G_E)$ put $D(V) := (E^{\mathrm{sep}} \otimes_{\mathbb{F}_p} V)^{G_E}$; $\sigma$ acts on $D(V)$ by acting on $E^{\mathrm{sep}}$.

*Published 10 December 2000:* 

**Properties.**

(1) $\dim_E D(V) = \dim_{\mathbb{F}_p} V$;

(2) the "Frobenius" map $\varphi\colon D(V) \to D(V)$ induced by $\sigma \otimes \mathrm{id}_V$ satisfies:

a) $\varphi(\lambda x) = \sigma(\lambda)\varphi(x)$ for all $\lambda \in E$, $x \in D(V)$ (so $\varphi$ is $\sigma$-semilinear);

b) $\varphi(D(V))$ generates $D(V)$ as an $E$-vector space.

**Definition.** A finite dimensional vector space $M$ over $E$ is called an *étale* $\Phi$*-module* over $E$ if there is a $\sigma$-semilinear map $\varphi\colon M \to M$ such that $\varphi(M)$ generates $M$ as an $E$-vector space.

Étale $\Phi$-modules form an abelian category $\Phi M_E^{\text{ét}}$ (the morphisms are the linear maps commuting with the Frobenius $\varphi$).

**Theorem 1** (Fontaine, [F]). *The functor* $V \to D(V)$ *is an equivalence of the categories* $\mathrm{Rep}_{\mathbb{F}_p}(G_E)$ *and* $\Phi M_E^{\text{ét}}$.

We see immediately that $H^0(G_E, V) = V^{G_E} \simeq D(V)^{\varphi}$.

So in order to obtain an explicit description of the Galois cohomology of mod $p$ representations of $G_E$, we should try to derive in a simple manner the functor associating to an étale $\Phi$-module the group of points fixed under $\varphi$. This is indeed a much simpler problem because there is only one operator acting.

For $(M, \varphi) \in \Phi M_E^{\text{ét}}$ define the following complex of abelian groups:

$$C_1(M): \qquad 0 \to M \xrightarrow{\varphi - 1} M \to 0$$

($M$ stands at degree 0 and 1).

This is a functorial construction, so by taking the cohomology of the complex, we obtain a cohomological functor $(\mathcal{H}^i := H^i \circ C_1)_{i\in\mathbb{N}}$ from $\Phi M_E^{\text{ét}}$ to the category of abelian groups.

**Theorem 2.** *The cohomological functor* $(\mathcal{H}^i \circ D)_{i\in\mathbb{N}}$ *can be identified with the Galois cohomology functor* $(H^i(G_E, .))_{i\in\mathbb{N}}$ *for the category* $\mathrm{Rep}_{\mathbb{F}_p}(G_E)$. *So, if* $M = D(V)$ *then* $\mathcal{H}^i(M)$ *provides a simple explicit description of* $H^i(G_E, V)$.

*Proof of Theorem 2.* We need to check that the cohomological functor $(\mathcal{H}^i)_{i\in\mathbb{N}}$ is universal; therefore it suffices to verify that for every $i \geqslant 1$ the functor $\mathcal{H}^i$ is effaceable: this means that for every $(M, \varphi_M) \in \Phi M_E^{\text{ét}}$ and every $x \in \mathcal{H}^i(M)$ there exists an embedding $u$ of $(M, \varphi_M)$ in $(N, \varphi_N) \in \Phi M_E^{\text{ét}}$ such that $\mathcal{H}^i(u)(x)$ is zero in $\mathcal{H}^i(N)$. But this is easy: it is trivial for $i \geqslant 2$; for $i = 1$ choose an element $m$ belonging to the class $x \in M/(\varphi - 1)(M)$, put $N := M \oplus Et$ and extend $\varphi_M$ to the $\sigma$-semi-linear map $\varphi_N$ determined by $\varphi_N(t) := t + m$. $\square$

## 6.2. $\Phi$-$\Gamma$-modules and $\mathbb{Z}_p$-adic representations

**Definition.** Recall that a Cohen ring is an absolutely unramified complete discrete valuation ring of mixed characteristic $(0, p > 0)$, so its maximal ideal is generated by $p$.

We describe a general formalism, explained by Fontaine in [F], which lifts the equivalence of categories of Theorem 1 in characteristic 0 and relates the $\mathbb{Z}_p$-adic representations of $G$ to a category of modules over a Cohen ring, endowed with a "Frobenius" map and a group action.

Let $R$ be an algebraically closed complete valuation (of rank 1) field of characteristic $p$ and let $H$ be a normal closed subgroup of $G$. Suppose that $G$ acts continuously on $R$ by ring automorphisms. Then $F := R^H$ is a perfect closed subfield of $R$.

For every integer $n \geq 1$, the ring $W_n(R)$ of Witt vectors of length $n$ is endowed with the product of the topology on $R$ defined by the valuation and then $W(R)$ with the inverse limit topology. Then the componentwise action of the group $G$ is continuous and commutes with the natural Frobenius $\sigma$ on $W(R)$. We also have $W(R)^H = W(F)$.

Let $E$ be a closed subfield of $F$ such that $F$ is the completion of the $p$-radical closure of $E$ in $R$. Suppose there exists a Cohen subring $\mathcal{O}_{\mathcal{E}}$ of $W(R)$ with residue field $E$ and which is stable under the actions of $\sigma$ and of $G$. Denote by $\mathcal{O}_{\widehat{\mathcal{E}_{\mathrm{ur}}}}$ the completion of the integral closure of $\mathcal{O}_{\mathcal{E}}$ in $W(R)$: it is a Cohen ring which is stable by $\sigma$ and $G$, its residue field is the separable closure of $E$ in $R$ and $(\mathcal{O}_{\widehat{\mathcal{E}_{\mathrm{ur}}}})^H = \mathcal{O}_{\mathcal{E}}$.

The natural map from $H$ to $G_E$ is an isomorphism if and only if the action of $H$ on $R$ induces an isomorphism from $H$ to $G_F$. We suppose that this is the case.

**Definition.** Let $\Gamma$ be the quotient group $G/H$. An étale $\Phi$-$\Gamma$-module over $\mathcal{O}_{\mathcal{E}}$ is a finitely generated $\mathcal{O}_{\mathcal{E}}$-module endowed with a $\sigma$-semi-linear Frobenius map $\varphi\colon M \to M$ and a continuous $\Gamma$-semi-linear action of $\Gamma$ commuting with $\varphi$ such that the image of $\varphi$ generates the module $M$.

Étale $\Phi$-$\Gamma$-modules over $\mathcal{O}_{\mathcal{E}}$ form an abelian category $\Phi\Gamma M^{\text{ét}}_{\mathcal{O}_{\mathcal{E}}}$ (the morphisms are the linear maps commuting with $\varphi$). There is a tensor product of $\Phi$-$\Gamma$-modules, the natural one. For two objects $M$ and $N$ of $\Phi\Gamma M^{\text{ét}}_{\mathcal{O}_{\mathcal{E}}}$ the $\mathcal{O}_{\mathcal{E}}$-module $\mathrm{Hom}_{\mathcal{O}_{\mathcal{E}}}(M,N)$ can be endowed with an étale $\Phi$-$\Gamma$-module structure (see [F]).

For every $\mathbb{Z}_p$-adic representation $V$ of $G$, let $D_H(V)$ be the $\mathcal{O}_{\mathcal{E}}$-module $(\mathcal{O}_{\widehat{\mathcal{E}_{\mathrm{ur}}}} \otimes_{\mathbb{Z}_p} V)^H$. It is naturally an étale $\Phi$-$\Gamma$-module, with $\varphi$ induced by the map $\sigma \otimes \mathrm{id}_V$ and $\Gamma$ acting on both sides of the tensor product. From Theorem 2 one deduces the following fundamental result:

**Theorem 3** (Fontaine, [F]). *The functor $V \to D_H(V)$ is an equivalence of the categories $\mathrm{Rep}_{\mathbb{Z}_p}(G)$ and $\Phi\Gamma M^{\text{ét}}_{\mathcal{O}_{\mathcal{E}}}$.*

**Remark.** If $E$ is a field of positive characteristic, $\mathcal{O}_{\mathcal{E}}$ is a Cohen ring with residue field $E$ endowed with a Frobenius $\sigma$, then we can easily extend the results of the whole subsection 6.1 to $\mathbb{Z}_p$-adic representations of $G$ by using Theorem 3 for $G = G_E$ and $H = \{1\}$.

## 6.3. A brief survey of the theory of the field of norms

For the details we refer to [W], [FV] or [F].

Let $K$ be a complete discrete valuation field of characteristic $0$ with perfect residue field $k$ of characteristic $p$. Put $G = G_K = \mathrm{Gal}(K^{\mathrm{sep}}/K)$.

Let $\mathbb{C}$ be the completion of $K^{\mathrm{sep}}$, denote the extension of the discrete valuation $v_K$ of $K$ to $\mathbb{C}$ by $v_K$. Let $R^* = \varprojlim \mathbb{C}_n^*$ where $\mathbb{C}_n = \mathbb{C}$ and the morphism from $\mathbb{C}_{n+1}$ to $\mathbb{C}_n$ is raising to the $p$th power. Put $R := R^* \cup \{0\}$ and define $v_R((x_n)) = v_K(x_0)$. For $(x_n), (y_n) \in R$ define

$$(x_n) + (y_n) = (z_n) \qquad \text{where } z_n = \lim_m (x_{n+m} + y_{n+m})^{p^m} .$$

Then $R$ is an algebraically closed field of characteristic $p$ complete with respect to $v_R$ (cf. [W]). Its residue field is isomorphic to the algebraic closure of $k$ and there is a natural continuous action of $G$ on $R$. (Note that Fontaine denotes this field by $\mathrm{Fr}\, R$ in [F]).

Let $L$ be a Galois extension of $K$ in $K^{\mathrm{sep}}$. Recall that one can always define the ramification filtration on $\mathrm{Gal}(L/K)$ in the upper numbering. Roughly speaking, $L/K$ is an arithmetically profinite extension if one can define the lower ramification subgroups of $G$ so that the classical relations between the two filtrations for finite extensions are preserved. This is in particular possible if $\mathrm{Gal}(L/K)$ is a $p$-adic Lie group with finite residue field extension.

The field $R$ contains in a natural way the field of norms $N(L/K)$ of every arithmetically profinite extension $L$ of $K$ and the restriction of $v$ to $N(L/K)$ is a discrete valuation. The residue field of $N(L/K)$ is isomorphic to that of $L$ and $N(L/K)$ is stable under the action of $G$. The construction is functorial: if $L'$ is a finite extension of $L$ contained in $K^{\mathrm{sep}}$, then $L'/K$ is still arithmetically profinite and $N(L'/K)$ is a separable extension of $N(L/K)$. The direct limit of the fields $N(L'/K)$ where $L'$ goes through all the finite extensions of $L$ contained in $K^{\mathrm{sep}}$ is the separable closure $E^{\mathrm{sep}}$ of $E = N(L/K)$. It is stable under the action of $G$ and the subgroup $G_L$ identifies with $G_E$. The field $E^{\mathrm{sep}}$ is dense in $R$.

Fontaine described how to lift these constructions in characteristic $0$ when $L$ is the cyclotomic $\mathbb{Z}_p$-extension $K_\infty$ of $K$. Consider the ring of Witt vectors $W(R)$ endowed with the Frobenius map $\sigma$ and the natural componentwise action of $G$. Define the topology of $W(R)$ as the product of the topology defined by the valuation on $R$. Then one can construct a Cohen ring $\mathcal{O}_{\widehat{\mathcal{E}_{\mathrm{ur}}}}$ with residue field $E^{\mathrm{sep}}$ ( $E = N(L/K)$ ) such that:

(i) $\mathcal{O}_{\widehat{\mathcal{E}}_{\rm ur}}$ is stable by $\sigma$ and the action of $G$,

(ii) for every finite extension $L$ of $K_\infty$ the ring $(\mathcal{O}_{\widehat{\mathcal{E}}_{\rm ur}})^{G_L}$ is a Cohen ring with residue field $E$.

Denote by $\mathcal{O}_{\mathcal{E}(K)}$ the ring $(\mathcal{O}_{\widehat{\mathcal{E}}_{\rm ur}})^{G_{K_\infty}}$. It is stable by $\sigma$ and the quotient $\Gamma = G/G_{K_\infty}$ acts continuously on $\mathcal{O}_{\mathcal{E}(K)}$ with respect to the induced topology. Fix a topological generator $\gamma$ of $\Gamma$: it is a continuous ring automorphism commuting with $\sigma$. The fraction field of $\mathcal{O}_{\mathcal{E}(K)}$ is a two-dimensional standard local field (as defined in section 1 of Part I). If $\pi$ is a lifting of a prime element of $N(K_\infty/K)$ in $\mathcal{O}_{\mathcal{E}(K)}$ then the elements of $\mathcal{O}_{\mathcal{E}(K)}$ are the series $\sum_{i\in\mathbb{Z}} a_i\pi^i$, where the coefficients $a_i$ are in $W(k_{K_\infty})$ and converge $p$-adically to 0 when $i \to -\infty$.

## 6.4. Application of $\mathbb{Z}_p$-adic representations of $G$ to the Galois cohomology

If we put together Fontaine's construction and the general formalism of subsection 6.2 we obtain the following important result:

**Theorem 3'** (Fontaine, [F]). *The functor $V \to D(V) := (\mathcal{O}_{\widehat{\mathcal{E}}_{\rm ur}} \otimes_{\mathbb{Z}_p} V)^{G_{K_\infty}}$ defines an equivalence of the categories $\mathrm{Rep}_{\mathbb{Z}_p}(G)$ and $\Phi\Gamma M^{\text{ét}}_{\mathcal{O}_{\mathcal{E}(K)}}$.*

Since for every $\mathbb{Z}_p$-adic representation of $G$ we have $H^0(G,V) = V^G \simeq D(V)^\varphi$, we want now, as in paragraph 6.1, compute explicitly the cohomology of the representation using the $\Phi$-$\Gamma$-module associated to $V$.

For every étale $\Phi$-$\Gamma$-module $(M,\varphi)$ define the following complex of abelian groups:

$$C_2(M): \qquad 0 \to M \xrightarrow{\alpha} M \oplus M \xrightarrow{\beta} M \to 0$$

where $M$ stands at degree 0 and 2,

$$\alpha(x) = ((\varphi-1)x, (\gamma-1)x), \quad \beta((y,z)) = (\gamma-1)y - (\varphi-1)z.$$

By functoriality, we obtain a cohomological functor $(\mathcal{H}^i := H^i \circ C_2)_{i\in\mathbb{N}}$ from $\Phi\Gamma M^{\text{ét}}_{\mathcal{O}_{\mathcal{E}(K)}}$ to the category of abelian groups.

**Theorem 4** (Herr, [H]). *The cohomological functor $(\mathcal{H}^i \circ D)_{i\in\mathbb{N}}$ can be identified with the Galois cohomology functor $(H^i(G,.))_{i\in\mathbb{N}}$ for the category $\mathrm{Rep}_{p\text{-tor}}(G)$. So, if $M = D(V)$ then $\mathcal{H}^i(M)$ provides a simple explicit description of $H^i(G,V)$ in the $p$-torsion case.*

*Idea of the proof of Theorem 4.* We have to check that for every $i \geqslant 1$ the functor $\mathcal{H}^i$ is effaceable. For every $p$-torsion object $(M,\varphi_M) \in \Phi\Gamma M^{\text{ét}}_{\mathcal{O}_{\mathcal{E}(K)}}$ and every $x \in \mathcal{H}^i(M)$ we construct an explicit embedding $u$ of $(M,\varphi_M)$ in a certain $(N,\varphi_N) \in \Phi\Gamma M^{\text{ét}}_{\mathcal{O}_{\mathcal{E}(K)}}$

such that $\mathcal{H}^i(u)(x)$ is zero in $\mathcal{H}^i(N)$. For details see [H]. The key point is of topological nature: we prove, following an idea of Fontaine in [F], that there exists an open neighbourhood of 0 in $M$ on which $(\varphi - 1)$ is bijective and use then the continuity of the action of $\Gamma$. □

As an application of theorem 4 we can prove the following result (due to Tate):

**Theorem 5.** *Assume that $k_K$ is finite and let $V$ be in* $\mathrm{Rep}_{p\text{-tor}}(G)$. *Without using class field theory the previous theorem implies that $H^i(G,V)$ are finite, $H^i(G,V) = 0$ for $i \geqslant 3$ and*

$$\sum_{i=0}^{2} l(H^i(G,V)) = -|K:\mathbb{Q}_p|\; l(V),$$

*where $l(\,)$ denotes the length over $\mathbb{Z}_p$.*

See [H].

**Remark.** Because the finiteness results imply that the Mittag–Leffler conditions are satisfied, it is possible to generalize the explicit construction of the cohomology and to prove analogous results for $\mathbb{Z}_p$ (or $\mathbb{Q}_p$ )-adic representations by passing to the inverse limits.

## 6.5. A new approach to local class field theory

The results of the preceding paragraph allow us to prove without using class field theory the following:

**Theorem 6** (Tate's local duality)**.** *Let $V$ be in* $\mathrm{Rep}_{p\text{-tor}}(G)$ *and $n \in \mathbb{N}$ such that $p^n V = 0$. Put $V^*(1) :=$* $\mathrm{Hom}(V, \mu_{p^n})$. *Then there is a canonical isomorphism from $H^2(G, \mu_{p^n})$ to $\mathbb{Z}/p^n$ and the cup product*

$$H^i(G,V) \times H^{2-i}(G,V^*(1)) \xrightarrow{\cup} H^2(G,\mu_{p^n}) \simeq \mathbb{Z}/p^n$$

*is a perfect pairing.*

It is well known that a proof of the local duality theorem of Tate without using class field theory gives a construction of the reciprocity map. For every $n \geqslant 1$ we have by duality a functorial isomorphism between the finite groups $\mathrm{Hom}(G,\mathbb{Z}/p^n) = H^1(G,\mathbb{Z}/p^n)$ and $H^1(G,\mu_{p^n})$ which is isomorphic to $K^*/(K^*)^{p^n}$ by Kummer theory. Taking the inverse limits gives us the $p$-part of the reciprocity map, the most difficult part.

*Sketch of the proof of Theorem 6.* ([H2]).

a) Introduction of differentials:

Let us denote by $\Omega^1_c$ the $\mathcal{O}_{\mathcal{E}(K)}$-module of continuous differential forms of $\mathcal{O}_{\mathcal{E}}$ over $W(k_{K_\infty})$. If $\pi$ is a fixed lifting of a prime element of $E(K_\infty/K)$ in $\mathcal{O}_{\mathcal{E}(K)}$, then this module is free and generated by $d\pi$. Define the residue map from $\Omega^1_c$ to $W(k_{K_\infty})$ by res $\left(\sum_{i\in\mathbb{Z}} a_i\pi^i d\pi\right) := a_{-1}$; it is independent of the choice of $\pi$.

b) Calculation of some $\Phi$-$\Gamma$-modules:

The $\mathcal{O}_{\mathcal{E}(K)}$-module $\Omega^1_c$ is endowed with an étale $\Phi$-$\Gamma$-module structure by the following formulas: for every $\lambda \in \mathcal{O}_{\mathcal{E}(K)}$ we put:

$$p\varphi(\lambda d\pi) = \sigma(\lambda)d(\sigma(\pi)) \ \ , \ \ \gamma(\lambda d\pi) = \gamma(\lambda)d(\gamma(\pi)).$$

The fundamental fact is that there is a natural isomorphism of $\Phi$-$\Gamma$-modules over $\mathcal{O}_{\mathcal{E}(K)}$ between $D(\mu_{p^n})$ and the reduction $\Omega^1_{c,n}$ of $\Omega^1_c$ modulo $p^n$.

The étale $\Phi$-$\Gamma$-module associated to the representation $V^*(1)$ is $\widetilde{M} := \mathrm{Hom}(M, \Omega^1_{c,n})$, where $M = D(V)$. By composing the residue with the trace we obtain a surjective and continuous map $\mathrm{Tr}_n$ from $M$ to $\mathbb{Z}/p^n$. For every $f \in \widetilde{M}$, the map $\mathrm{Tr}_n \circ f$ is an element of the group $M^\vee$ of continuous group homomorphisms from $M$ to $\mathbb{Z}/p^n$. This gives in fact a group isomorphism from $\widetilde{M}$ to $M^\vee$ and we can therefore transfer the $\Phi$-$\Gamma$-module structure from $\widetilde{M}$ to $M^\vee$. But, since $k$ is finite, $M$ is locally compact and $M^\vee$ is in fact the Pontryagin dual of $M$.

c) Pontryagin duality implies local duality:

We simply dualize the complex $C_2(M)$ using Pontryagin duality (all arrows are strict morphisms in the category of topological groups) and obtain a complex:

$$C_2(M)^\vee : \qquad 0 \to M^\vee \xrightarrow{\beta^\vee} M^\vee \oplus M^\vee \xrightarrow{\alpha^\vee} M^\vee \to 0,$$

where the two $M^\vee$ are in degrees $0$ and $2$. Since we can construct an explicit quasi-isomorphism between $C_2(M^\vee)$ and $C_2(M)^\vee$, we easily obtain a duality between $\mathcal{H}^i(M)$ and $\mathcal{H}^{2-i}(M^\vee)$ for every $i \in \{0,1,2\}$.

d) The canonical isomorphism from $\mathcal{H}^2(\Omega^1_{c,n})$ to $\mathbb{Z}/p^n$:

The map $\mathrm{Tr}_n$ from $\Omega^1_{c,n}$ to $\mathbb{Z}/p^n$ factors through the group $\mathcal{H}^2(\Omega^1_{c,n})$ and this gives an isomorphism. But it is not canonical! In fact the construction of the complex $C_2(M)$ depends on the choice of $\gamma$. Fortunately, if we take another $\gamma$, we get a quasi-isomorphic complex and if we normalize the map $\mathrm{Tr}_n$ by multiplying it by the unit $-p^{v_p(\log\chi(\gamma))}/\log\chi(\gamma)$ of $\mathbb{Z}_p$, where $\log$ is the $p$-adic logarithm, $\chi$ the cyclotomic character and $v_p = v_{\mathbb{Q}_p}$, then everything is compatible with the change of $\gamma$.

e) The duality is given by the cup product:

We can construct explicit formulas for the cup product:

$$\mathcal{H}^i(M) \times \mathcal{H}^{2-i}(M^\vee) \xrightarrow{\cup} \mathcal{H}^2(\Omega^1_{c,n})$$

associated with the cohomological functor $(\mathcal{H}^i)_{i\in\mathbb{N}}$ and we compose them with the preceding normalized isomorphism from $\mathcal{H}^2(\Omega^1_{c,n})$ to $\mathbb{Z}/p^n$. Since everything is explicit, we can compare with the pairing obtained in c) and verify that it is the same up to a unit of $\mathbb{Z}_p$. □

**Remark.** Benois, using the previous theorem, deduced an explicit formula of Coleman's type for the Hilbert symbol and proved Perrin-Riou's formula for crystalline representations ([B]).

## 6.6. Explicit formulas for the generalized Hilbert symbol on formal groups

Let $K_0$ be the fraction field of the ring $W_0$ of Witt vectors with coefficients in a finite field of characteristic $p > 2$ and $\mathcal{F}$ a commutative formal group of finite height $h$ defined over $W_0$.

For every integer $n \geqslant 1$, denote by $\mathcal{F}[p^n]$ the $p^n$-torsion points in $\mathcal{F}(\mathcal{M}_C)$, where $\mathcal{M}_C$ is the maximal ideal of the completion $C$ of an algebraic closure of $K_0$. The group $\mathcal{F}[p^n]$ is isomorphic to $(\mathbb{Z}/p^n\mathbb{Z})^h$.

Let $K$ be a finite extension of $K_0$ contained in $K^{\text{sep}}$ and assume that the points of $\mathcal{F}[p^n]$ are defined over $K$. We then have a bilinear pairing:

$$( \, , \, ]_{\mathcal{F},n} \colon G_K^{\text{ab}} \times \mathcal{F}(\mathcal{M}_K) \to \mathcal{F}[p^n]$$

(see section 8 of Part I).

When the field $K$ contains a primitive $p^n$ th root of unity $\zeta_{p^n}$, Abrashkin gives an explicit description for this pairing generalizing the classical Brückner–Vostokov formula for the Hilbert symbol ([A]). In his paper he notices that the formula makes sense even if $K$ does not contain $\zeta_{p^n}$ and he asks whether it holds without this assumption. In a recent unpublished work, Benois proves that this is true.

Suppose for simplicity that $K$ contains only $\zeta_p$. Abrashkin considers in his paper the extension $\widetilde{K} := K(\pi^{p^{-r}}, r \geqslant 1)$, where $\pi$ is a fixed prime element of $K$. It is not a Galois extension of $K$ but is arithmetically profinite, so by [W] one can consider the field of norms for it. In order not to loose information given by the roots of unity of order a power of $p$, Benois uses the composite Galois extension $L := K_\infty \widetilde{K}/K$ which is arithmetically profinite. There are several problems with the field of norms $N(L/K)$, especially it is not clear that one can lift it in characteristic $0$ with its Galois action. So, Benois simply considers the completion $F$ of the $p$-radical closure of $E = N(L/K)$ and its separable closure $F^{\text{sep}}$ in $R$. If we apply what was explained in subsection 6.2 for $\Gamma = \text{Gal}(L/K)$, we get:

**Theorem 7.** *The functor* $V \to D(V) := (W(F^{\text{sep}}) \otimes_{\mathbb{Z}_p} V)^{G_L}$ *defines an equivalence of the categories* $\text{Rep}_{\mathbb{Z}_p}(G)$ *and* $\Phi\Gamma M^{\text{ét}}_{W(F)}$.

Choose a topological generator $\gamma'$ of $\mathrm{Gal}(L/K_\infty)$ and lift $\gamma$ to an element of $\mathrm{Gal}(L/\widetilde{K})$. Then $\Gamma$ is topologically generated by $\gamma$ and $\gamma'$, with the relation $\gamma\gamma' = (\gamma')^a\gamma$, where $a = \chi(\gamma)$ ($\chi$ is the cyclotomic character). For $(M,\varphi) \in \Phi\Gamma M^{\text{ét}}_{W(F)}$ the continuous action of $\mathrm{Gal}(L/K_\infty)$ on $M$ makes it a module over the Iwasawa algebra $\mathbb{Z}_p[[\gamma' - 1]]$. So we can define the following complex of abelian groups:

$$C_3(M): \qquad 0 \to M_0 \xrightarrow{\alpha\mapsto A_0\alpha} M_1 \xrightarrow{\alpha\mapsto A_1\alpha} M_2 \xrightarrow{\alpha\mapsto A_2\alpha} M_3 \to 0$$

where $M_0$ is in degree 0, $M_0 = M_3 = M$, $M_1 = M_2 = M^3$,

$$A_0 = \begin{pmatrix} \varphi - 1 \\ \gamma - 1 \\ \gamma' - 1 \end{pmatrix}, A_1 = \begin{pmatrix} \gamma - 1 & 1 - \varphi & 0 \\ \gamma' - 1 & 0 & 1 - \varphi \\ 0 & \gamma'^a - 1 & \delta - \gamma \end{pmatrix}, A_2 = ((\gamma')^a - 1 \;\; \delta - \gamma \;\; \varphi - 1)$$

and $\delta = ((\gamma')^a - 1)(\gamma' - 1)^{-1} \in \mathbb{Z}_p[[\gamma' - 1]]$.

As usual, by taking the cohomology of this complex, one defines a cohomological functor $(\mathcal{H}^i)_{i\in\mathbb{N}}$ from $\Phi\Gamma M^{\text{ét}}_{W(F)}$ in the category of abelian groups. Benois proves only that the cohomology of a $p$-torsion representation $V$ of $G$ injects in the groups $\mathcal{H}^i(D(V))$ which is enough to get the explicit formula. But in fact a stronger statement is true:

**Theorem 8.** *The cohomological functor* $(\mathcal{H}^i \circ D)_{i\in\mathbb{N}}$ *can be identified with the Galois cohomology functor* $(H^i(G,.))_{i\in\mathbb{N}}$ *for the category* $\mathrm{Rep}_{p\text{-tor}}(G)$.

*Idea of the proof.* Use the same method as in the proof of Theorem 4. It is only more technically complicated because of the structure of $\Gamma$. □

Finally, one can explicitly construct the cup products in terms of the groups $\mathcal{H}^i$ and, as in [B], Benois uses them to calculate the Hilbert symbol.

**Remark.** Analogous constructions (equivalence of category, explicit construction of the cohomology by a complex) seem to work for higher dimensional local fields. In particular, in the two-dimensional case, the formalism is similar to that of this paragraph; the group $\Gamma$ acting on the Φ- Γ-modules has the same structure as here and thus the complex is of the same form. This work is still in progress.

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ISSN 1464-8997 (on line) 1464-8989 (printed) 

Geometry & Topology Monographs
Volume 3: Invitation to higher local fields
Part II, section 7, pages 273–279

# 7. Recovering higher global and local fields from Galois groups — an algebraic approach

*Ido Efrat*

## 7.0. Introduction

We consider the following general problem: let $F$ be a known field with absolute Galois group $G_F$. Let $K$ be a field with $G_K \simeq G_F$. What can be deduced about the arithmetic structure of $K$?

As a prototype of this kind of questions we recall the celebrated Artin–Schreier theorem: $G_K \simeq G_{\mathbb{R}}$ if and only if $K$ is real closed. Likewise, the fields $K$ with $G_K \simeq G_E$ for some finite extension $E$ of $\mathbb{Q}_p$ are the $p$-adically closed fields (see [Ne], [P1], [E1], [K]). Here we discuss the following two cases:

1. *$K$ is a higher global field*
2. *$K$ is a higher local field*

## 7.1. Higher global fields

We call a field *finitely generated* (or a *higher global field*) if it is finitely generated over its prime subfield. The (proven) 0-dimensional case of Grothendieck's anabelian conjecture ([G1], [G2]) can be stated as follows:

*Let $K, F$ be finitely generated infinite fields. Any isomorphism $G_K \simeq G_F$ is induced in a functorial way by an (essentially unique) isomorphism of the algebraic closures of $K$ and $F$.*

This statement was proven:
- by Neukirch [Ne] for finite normal extensions of $\mathbb{Q}$;
- by Iwasawa (unpublished) and Uchida [U1–3] (following Ikeda [I]) for all global fields;
- by Pop [P2] and Spiess [S] for function fields in one variable over $\mathbb{Q}$;

- by Pop ([P3–5]) in general.

For recent results on the 1-*dimensional* anabelian conjecture – see the works of Mochizuki [M], Nakamura [N] and Tamagawa [T].

## 7.2. Earlier approaches

Roughly speaking, the above proofs in the 0-dimensional case are divided into a local part and a global part. To explain the local part, define the Kronecker dimension $\dim(K)$ of a field $K$ as $\mathrm{trdeg}(K/\mathbb{F}_p)$ if $\mathrm{char}(K) = p > 0$, and as $\mathrm{trdeg}(K/\mathbb{Q}) + 1$ if $\mathrm{char}(K) = 0$. Now let $v$ be a Krull valuation on $K$ (not necessarily discrete or of rank 1) with residue field $\overline{K}_v$. It is called 1-*defectless* if $\dim K = \dim \overline{K}_v + 1$. The main result of the local theory is the following *local correspondence*: given an isomorphism $\varphi: G_K \xrightarrow{\sim} G_F$, a closed subgroup $Z$ of $G_K$ is the decomposition group of some 1-defectless valuation $v$ on $K$ if and only if $\varphi(Z)$ is the decomposition group of some 1-defectless valuation $v'$ on $F$. The 'global theory' then combines the isomorphisms between the corresponding decomposition fields to construct the desired isomorphism of the algebraic closures (see [P5] for more details).

The essence of the local correspondence is clearly the detection of valuations on a field $K$ just from the knowledge of the group-theoretic structure of $G_K$. In the earlier approaches this was done by means of various *Hasse principles*; i.e., using the injectivity of the map

$$H(K) \to \prod_{v \in S} H(K_v^h)$$

for some cohomological functor $H$ and some set $S$ of non-trivial valuations on $K$, where $K_v^h$ is the henselization of $K$ with respect to $v$. Indeed, if this map is injective and $H(K) \neq 0$ then $H(K_v^h) \neq 0$ for at least one $v \in S$. In this way one finds "arithmetically interesting" valuations on $K$.

In the above-mentioned works the local correspondence was proved using known Hasse principles for:

(1) Brauer groups over global fields (Brauer, Hasse, Noether);
(2) Brauer groups over function fields in one variable over local fields (Witt, Tate, Lichtenbaum, Roquette, Sh. Saito, Pop);
(3) $H^3(G_K, \mathbb{Q}/\mathbb{Z}(2))$ over function fields in one variable over $\mathbb{Q}$ (Kato, Jannsen).

Furthermore, in his proof of the 0-dimensional anabelian conjecture in its general case, Pop uses a *model-theoretic* technique to transfer the Hasse principles in (2) to a more general context of conservative function fields in one variable over certain henselian valued fields. More specifically, by a deep result of Kiesler–Shelah, a property is elementary in a certain language (in the sense of the first-order predicate calculus) if and only if it is preserved by isomorphisms of models in the language, and both the property and its negation are preserved by nonprincipal ultrapowers. It turns out that in an appropriate setting, the Hasse principle for the Brauer groups satisfies these conditions, hence has an elementary nature. One can now apply model-completeness results on tame valued fields by F.-V. Kuhlmann [Ku].

This led one to the problem of finding an *algebraic* proof of the local correspondence, i.e., a proof which does not use non-standard arguments (see [S, p. 115]; other model-theoretic techniques which were earlier used in the global theory of [P2] were replaced by Spiess in [S] by algebraic ones).

We next explain how this can indeed be done (see [E3] for details and proofs).

## 7.3. Construction of valuations from $K$-theory

Our algebraic approach to the local correspondence is based on a $K$-theoretic (yet elementary) construction of valuations, which emerged in the early 1980's in the context of quadratic form theory (in works of Jacob [J], Ware [W], Arason–Elman–Jacob [AEJ], Hwang–Jacob [HJ]; see the survey [E2]). We also mention here the alternative approaches to such constructions by Bogomolov [B] and Koenigsmann [K]. The main result of (the first series of) these constructions is:

**Theorem 1.** *Let $p$ be a prime number and let $E$ be a field. Assume that* $\mathrm{char}\,(E) \neq p$ *and that $\langle -1, E^{*p}\rangle \leqslant T \leqslant E^*$ is an intermediate group such that:*

(a) *for all $x \in E^* \setminus T$ and $y \in T \setminus E^{*p}$ one has $\{x,y\} \neq 0$ in $K_2(E)$*

(b) *for all $x, y \in E^*$ which are $\mathbb{F}_p$-linearly independent mod $T$ one has $\{x,y\} \neq 0$ in $K_2(E)$.*

*Then there exists a valuation $v$ on $E$ with value group $\Gamma_v$ such that:*

(i) $\mathrm{char}\,(\overline{E}_v) \neq p$;

(ii) $\dim_{\mathbb{F}_p}(\Gamma_v/p) \geqslant \dim_{\mathbb{F}_p}(E^*/T) - 1$;

(iii) *either* $\dim_{\mathbb{F}_p}(\Gamma_v/p) = \dim_{\mathbb{F}_p}(E^*/T)$ *or* $\overline{E}_v \neq \overline{E}_v^p$.

In particular we have:

**Corollary.** *Let $p$ be a prime number and let $E$ be a field. Suppose that* $\mathrm{char}\,(E) \neq p$, $-1 \in E^{*p}$, *and that the natural symbolic map induces an isomorphism*

$$\wedge^2(E^*/E^{*p}) \xrightarrow{\sim} K_2(E)/p.$$

*Then there is a valuation $v$ on $E$ such that*

(i) $\mathrm{char}\,(\overline{E}_v) \neq p$;

(ii) $\dim_{\mathbb{F}_p}(\Gamma_v/p) \geqslant \dim_{\mathbb{F}_p}(E^*/E^{*p}) - 1$;

(iii) *either* $\dim_{\mathbb{F}_p}(\Gamma_v/p) = \dim_{\mathbb{F}_p}(E^*/E^{*p})$ *or* $\overline{E}_v \neq \overline{E}_v^p$.

We remark that the construction used in the proof of Theorem 1 is of a completely explicit and elementary nature. Namely, one chooses a certain intermediate group $T \leqslant H \leqslant E^*$ with $(H:T)|p$ and denotes

$$O^- = \{x \in E \setminus H \ : \ 1 - x \in T\}, \qquad O^+ = \{x \in H \ : \ xO^- \subset O^-\}.$$

It turns out that $O = O^- \cup O^+$ is a valuation ring on $E$, and the corresponding valuation $v$ is as desired.

The second main ingredient is the following henselianity criterion proven in [E1]:

**Proposition 1.** *Let $p$ be a prime number and let $(E,v)$ be a valued field with* $\mathrm{char}(\overline{E}_v) \neq p$*, such that the maximal pro-$p$ Galois group $G_{\overline{E}_v}(p)$ of $\overline{E}_v$ is infinite. Suppose that*

$$\sup_{E'} \mathrm{rk}(G_{E'}(p)) < \infty$$

*with $E'$ ranging over all finite separable extensions of $E$. Then $v$ is henselian.*

Here the *rank* $\mathrm{rk}(G)$ of a profinite group $G$ is its minimal number of (topological) generators.

After translating the Corollary to the Galois-theoretic language using Kummer theory and the Merkur'ev–Suslin theorem and using Proposition 1 we obtain:

**Proposition 2.** *Let $p$ be a prime number and let $E$ be a field such that* $\mathrm{char}(E) \neq p$. *Suppose that for every finite separable extension $E'$ of $E$ one has*
(1) $H^1(G_{E'}, \mathbb{Z}/p) \simeq (\mathbb{Z}/p)^{n+1}$;
(2) $H^2(G_{E'}, \mathbb{Z}/p) \simeq \wedge^2 H^1(G_{E'}, \mathbb{Z}/p)$ *via the cup product;*
(3) $\dim_{\mathbb{F}_p}(\Gamma_u/p) \leqslant n$ *for every valuation $u$ on $E'$.*

*Then there exists a henselian valuation $v$ on $E$ such that* $\mathrm{char}(\overline{E}_v) \neq p$ *and* $\dim_{\mathbb{F}_p}(\Gamma_v/p) = n$.

## 7.4. A Galois characterization of 1-defectless valuations

For a field $L$ and a prime number $p$, we recall that the virtual $p$-cohomological dimension $\mathrm{vcd}_p(G_L)$ is the usual $p$-cohomological dimension $\mathrm{cd}_p(G_L)$ if $\mathrm{char}(L) \neq 0$ and is $\mathrm{vcd}_p(G_{L(\sqrt{-1})})$ if $\mathrm{char}(L) = 0$.

**Definition.** Let $p$ be a prime number and let $L$ be a field with $n = \dim L < \infty$ and $\mathrm{char}(L) \neq p$. We say that $L$ is *$p$-divisorial* if there exist subfields $L \subset E \subset M \subset L^{\mathrm{sep}}$ such that
(a) $M/L$ is Galois;
(b) every $p$-Sylow subgroup of $G_M$ is isomorphic to $\mathbb{Z}_p$;
(c) the virtual $p$-cohomological dimension $\mathrm{vcd}_p(G_L)$ of $G_L$ is $n+1$;

(d) either $n = 1$ or $\mathrm{Gal}(M/L)$ has no non-trivial closed normal pro-soluble subgroups;
(e) for every finite separable extension $E'/E$ one has

$$H^1(G_{E'}, \mathbb{Z}/p) \simeq (\mathbb{Z}/p)^{n+1}, \quad H^2(G_{E'}, \mathbb{Z}/p) \simeq \wedge^2 H^1(G_{E'}, \mathbb{Z}/p)$$

via the cup product.

The main result is now:

**Theorem 2** ([E3]). *Let $p$ be a prime number and let $K$ be a finitely generated field of characteristic different from $p$. Let $L$ be an algebraic extension of $K$. Then the following conditions are equivalent:*
(i) *there exists a $1$-defectless valuation $v$ on $K$ such that $L = K_v^h$;*
(ii) *$L$ is a minimal $p$-divisorial separable algebraic extension of $K$.*

*Idea of proof.* Suppose first that $v$ is a $1$-defectless valuation on $K$. Take $L = K_v^h$ and let $M$ be a maximal unramified extension of $L$. Also let $w$ be a valuation on $K$ such that $\Gamma_w \simeq \mathbb{Z}^{\dim(K)}$, $\mathrm{char}(\overline{K}_w) \neq p$, and such that the corresponding valuation rings satisfy $\mathcal{O}_w \subset \mathcal{O}_v$. Let $K_w^h$ be a henselization of $(K, w)$ containing $L$ and take $E = K_w^h(\mu_p)$ ($E = K_w^h(\mu_4)$ if $p = 2$). One shows that $L$ is $p$-divisorial with respect to this tower of of extensions.

Conversely, suppose that $L$ is $p$-divisorial, and let $L \subset E \subset M \subset L^{\mathrm{sep}}$ be a tower of extensions as in the definition above. Proposition 2 gives rise to a henselian valuation $w$ on $E$ such that $\mathrm{char}(\overline{E}_w) \neq p$ and $\dim_{\mathbb{F}_p}(\Gamma_w/p) = \dim(K)$. Let $w_0$ be the unique valuation on $E$ of rank $1$ such that $\mathcal{O}_{w_0} \supset \mathcal{O}_w$, and let $u$ be its restriction to $L$. The unique extension $u_M$ of $w_0$ to $M$ is henselian. Since $M/L$ is normal, every extension of $u$ to $M$ is conjugate to $u_M$, hence is also henselian. By a classical result of F.-K. Schmidt, the non-separably closed field $M$ can be henselian with respect to at most one valuation of rank $1$. Conclude that $u$ is henselian as well. One then shows that it is $1$-defectless.

The equivalence of (i) and (ii) now follows from these two remarks, and a further application of F.-K. Schmidt's theorem.

The local correspondence now follows from the observation that condition (ii) of the Theorem is actually a condition on the closed subgroup $G_L$ of the profinite group $G_K$ (note that $\dim(L) = \mathrm{vcd}(G_K) - 1$).

## 7.5. Higher local fields

Here we report on a joint work with Fesenko [EF].

An analysis similar to the one sketched in the case of higher global fields yields:

**Theorem 3** ([EF]). *Let $F$ be an $n$-dimensional local field. Suppose that the canonical valuation on $F$ of rank $n$ has residue characteristic $p$. Let $K$ be a field such that $G_K \simeq G_F$. Then there is a henselian valuation $v$ on $K$ such that $\Gamma_v/l \simeq (\mathbb{Z}/l)^n$ for every prime number $l \neq p$ and such that $\mathrm{char}(\overline{K}_v) = p$ or $0$.*

**Theorem 4** ([EF]). *Let $q = p^r$ be a prime power and let $K$ be a field with $G_K \simeq G_{\mathbb{F}_q((t))}$. Then there is a henselian valuation $v$ on $K$ such that*
(1) *$\Gamma_v/l \simeq \mathbb{Z}/l$ for every prime number $l \neq p$;*
(2) *$\mathrm{char}(\overline{K}_v) = p$;*
(3) *the maximal prime-to-$p$ Galois group $G_{\overline{K}_v}(p')$ of $\overline{K}_v$ is isomorphic to $\prod_{l\neq p}\mathbb{Z}_l$;*
(4) *if $\mathrm{char}(K) = 0$ then $\Gamma_v = p\Gamma_v$ and $\overline{K}_v$ is perfect.*

Moreover, for every positive integer $d$ there exist valued fields $(K, v)$ as in Theorem 4 with characteristic $p$ and for which $\Gamma_v/p \simeq (\mathbb{Z}/p)^d$. Likewise there exist examples with $\Gamma_v \simeq \mathbb{Z}$, $G_{\overline{K}_v} \not\simeq \hat{\mathbb{Z}}$ and $\overline{K}_v$ imperfect, as well as examples with $\mathrm{char}(K) = 0$.

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Geometry & Topology Monographs
Volume 3: Invitation to higher local fields
Part II, section 8, pages 281–292

# 8. Higher local skew fields

*Alexander Zheglov*

$n$-dimensional local skew fields are a natural generalization of $n$-dimensional local fields. The latter have numerous applications to problems of algebraic geometry, both arithmetical and geometrical, as it is shown in this volume. From this viewpoint, it would be reasonable to restrict oneself to commutative fields only. Nevertheless, already in class field theory one meets non-commutative rings which are skew fields finite-dimensional over their center $K$. For example, $K$ is a (commutative) local field and the skew field represents elements of the Brauer group of the field $K$ (see also an example below). In [Pa] A.N. Parshin pointed out another class of non-commutative local fields arising in differential equations and showed that these skew fields possess many features of commutative fields. He defined a skew field of formal pseudo-differential operators in $n$ variables and studied some of their properties. He raised a problem of classifying non-commutative local skew fields.

In this section we treat the case of $n = 2$ and list a number of results, in particular a classification of certain types of 2-dimensional local skew fields.

## 8.1. Basic definitions

**Definition.** A skew field $K$ is called a *complete discrete valuation skew field* if $K$ is complete with respect to a discrete valuation (the residue skew field is not necessarily commutative). A field $K$ is called an *$n$-dimensional local skew field* if there are skew fields $K = K_n, K_{n-1}, \dots, K_0$ such that each $K_i$ for $i > 0$ is a complete discrete valuation skew field with residue skew field $K_{i-1}$.

**Examples.**

(1) Let $k$ be a field. Formal pseudo-differential operators over $k((X))$ form a 2-dimensional local skew field $K = k((X))((\partial_X^{-1}))$, $\partial_X X = X\partial_X + 1$. If $\mathrm{char}(k) = 0$ we get an example of a skew field which is an infinite dimensional vector space over its centre.

(2) Let $L$ be a local field of equal characteristic (of any dimension). Then an element of $\mathrm{Br}(L)$ is an example of a skew field which is finite dimensional over its centre.

From now on *let* $K$ be a two-dimensional local skew field. Let $t_2$ be a generator of $\mathcal{M}_{K_2}$ and $t_1'$ be a generator of $\mathcal{M}_{K_1}$. If $t_1 \in K$ is a lifting of $t_1'$ then $t_1, t_2$ is called a *system of local parameters* of $K$. We denote by $v_{K_2}$ and $v_{K_1}$ the (surjective) discrete valuations of $K_2$ and $K_1$ associated with $t_2$ and $t_1'$.

**Definition.** A two-dimensional local skew field $K$ is said to *split* if there is a section of the homomorphism $\mathcal{O}_{K_2} \to K_1$ where $\mathcal{O}_{K_2}$ is the ring of integers of $K_2$.

**Example (N. Dubrovin).** Let $\mathbb{Q}((u))\langle x, y\rangle$ be a free associative algebra over $\mathbb{Q}((u))$ with generators $x, y$. Let $I = \langle [x,[x,y]], [y,[x,y]]\rangle$. Then the quotient

$$A = \mathbb{Q}((u))\langle x, y\rangle / I$$

is a $\mathbb{Q}$-algebra which has no non-trivial zero divisors, and in which $z = [x,y] + I$ is a central element. Any element of $A$ can be uniquely represented in the form

$$f_0 + f_1 z + \ldots + f_m z^m$$

where $f_0, \ldots, f_m$ are polynomials in the variables $x, y$.

One can define a discrete valuation $w$ on $A$ such that $w(x) = w(y) = w(\mathbb{Q}((u))) = 0$, $w([x,y]) = 1$, $w(a) = k$ if $a = f_k z^k + \ldots + f_m z^m$, $f_k \neq 0$. The skew field $B$ of fractions of $A$ has a discrete valuation $v$ which is a unique extension of $w$. The completion of $B$ with respect to $v$ is a two-dimensional local skew field which does not split (for details see [Zh, Lemma 9]).

**Definition.** Assume that $K_1$ is a field. The homomorphism

$$\varphi_0\colon K^* \to \mathrm{Int}(K), \quad \varphi_0(x)(y) = x^{-1} y x$$

induces a homomorphism $\varphi\colon K_2^*/\mathcal{O}_{K_2}^* \to \mathrm{Aut}(K_1)$. The *canonical automorphism* of $K_1$ is $\alpha = \varphi(t_2)$ where $t_2$ is an arbitrary prime element of $K_2$.

**Definition.** Two two-dimensional local skew fields $K$ and $K'$ are *isomorphic* if there is an isomorphism $K \to K'$ which maps $\mathcal{O}_K$ onto $\mathcal{O}_{K'}$, $\mathcal{M}_K$ onto $\mathcal{M}_{K'}$ and $\mathcal{O}_{K_1}$ onto $\mathcal{O}_{K_1'}$, $\mathcal{M}_{K_1}$ onto $\mathcal{M}_{K_1'}$.

## 8.2. Canonical automorphisms of infinite order

**Theorem.**

(1) *Let $K$ be a two-dimensional local skew field. If $\alpha^n \neq \mathrm{id}$ for all $n \geqslant 1$ then* $\mathrm{char}\,(K_2) = \mathrm{char}\,(K_1)$*, $K$ splits and $K$ is isomorphic to a two-dimensional local skew field $K_1((t_2))$ where $t_2 a = \alpha(a) t_2$ for all $a \in K_1$.*

(2) *Let $K, K'$ be two-dimensional local skew fields and let $K_1, K_1'$ be fields. Let* $\alpha^n \neq \mathrm{id}$, $\alpha'^n \neq \mathrm{id}$ *for all $n \geqslant 1$. Then $K$ is isomorphic to $K'$ if and only if there is an isomorphism $f\colon K_1 \to K_1'$ such that $\alpha = f^{-1}\alpha' f$ where $\alpha, \alpha'$ are the canonical automorphisms of $K_1$ and $K_1'$.*

**Remarks.**

1. This theorem is true for any higher local skew field.
2. There are examples (similar to Dubrovin's example) of local skew fields which do not split and in which $\alpha^n = \mathrm{id}$ for some positive integer $n$.

*Proof.* (2) follows from (1). We sketch the proof of (1). For details see [Zh, Th.1].

If $\mathrm{char}\,(K) \neq \mathrm{char}\,(K_1)$ then $\mathrm{char}\,(K_1) = p > 0$. Hence $v(p) = r > 0$. Then for any element $t \in K$ with $v(t) = 0$ we have $ptp^{-1} \equiv \alpha^r(\bar t) \bmod \mathcal{M}_K$ where $\bar t$ is the image of $t$ in $K_1$. But on the other hand, $pt = tp$, a contradiction.

Let $F$ be the prime field in $K$. Since $\mathrm{char}\,(K) = \mathrm{char}\,(K_1)$ the field $F$ is a subring of $\mathcal{O} = \mathcal{O}_{K_2}$. One can easily show that there exists an element $c \in K_1$ such that $\alpha^n(c) \neq c$ for every $n \geqslant 1$ [Zh, Lemma 5].

Then any lifting $c'$ in $\mathcal{O}$ of $c$ is transcendental over $F$. Hence we can embed the field $F(c')$ in $\mathcal{O}$. Let $\bar L$ be a maximal field extension of $F(c')$ which can be embedded in $\mathcal{O}$. Denote by $L$ its image in $\mathcal{O}$. Take $\bar a \in K_1 \setminus \bar L$. We claim that there exists a lifting $a' \in \mathcal{O}$ of $\bar a$ such that $a'$ commutes with every element in $L$. To prove this fact we use the completeness of $\mathcal{O}$ in the following argument.

Take any lifting $a$ in $\mathcal{O}$ of $\bar a$. For every element $x \in L$ we have $axa^{-1} \equiv x \bmod \mathcal{M}_K$. If $t_2$ is a prime element of $K_2$ we can write

$$axa^{-1} = x + \delta_1(x) t_2$$

where $\delta_1(x) \in \mathcal{O}$. The map $\overline{\delta_1}\colon L \ni x \to \overline{\delta_1(x)} \in K_1$ is an $\alpha$-derivation, i.e.

$$\overline{\delta_1}(ef) = \overline{\delta_1}(e)\alpha(f) + e\overline{\delta_1}(f)$$

for all $e, f \in L$. Take an element $h$ such that $\alpha(h) \neq h$, then $\overline{\delta_1}(a) = g\alpha(a) - ag$ where $g = \overline{\delta_1}(h)/(\alpha(h) - h)$. Therefore there is $a_1 \in K_1$ such that

$$(1 + a_1 t_2) axa^{-1} (1 + a_1 t_2)^{-1} \equiv x \bmod \mathcal{M}_K^2.$$

By induction we can find an element $a' = \ldots \cdot (1 + a_1 t_2) a$ such that $a' x a'^{-1} = x$.

Now, if $\overline{a}$ is not algebraic over $\overline{L}$, then for its lifting $a' \in \mathcal{O}$ which commutes with $L$ we would deduce that $L(a')$ is a field extension of $F(c')$ which can be embedded in $\mathcal{O}$, which contradicts the maximality of $L$.

Hence $\overline{a}$ is algebraic and separable over $\overline{L}$. Using a generalization of Hensel's Lemma [Zh, Prop.4] we can find a lifting $a'$ of $\overline{a}$ such that $a'$ commutes with elements of $L$ and $a'$ is algebraic over $L$, which again leads to a contradiction.

Finally let $\overline{a}$ be purely inseparable over $\overline{L}$, $\overline{a}^{p^k} = \overline{x}$, $x \in L$. Let $a'$ be its lifting which commutes with every element of $L$. Then $a'^{p^k} - x$ commutes with every element of $L$. If $v_K(a'^{p^k} - x) = r \neq \infty$ then similarly to the beginning of this proof we deduce that the image of $(a'^{p^k} - x)c(a'^{p^k} - x)^{-1}$ in $K_1$ is equal to $\alpha^r(c)$ (which is distinct from $c$), a contradiction. Therefore, $a'^{p^k} = x$ and the field $L(a')$ is a field extension of $F(c')$ which can be embedded in $\mathcal{O}$, which contradicts the maximality of $L$.

Thus, $\overline{L} = K_1$.

To prove that $K$ is isomorphic to a skew field $K_1((t_2))$ where $t_2 a = \alpha(a) t_2$ one can apply similar arguments as in the proof of the existence of an element $a'$ such that $a'xa'^{-1} = x$ (see above). So, one can find a parameter $t_2$ with a given property. □

In some cases we have a complete classification of local skew fields.

**Proposition** ([Zh]). *Assume that $K_1$ is isomorphic to $k((t_1))$. Put*

$$\zeta = \alpha(t_1)t_1^{-1} \bmod \mathcal{M}_{K_1}.$$

*Put $i_\alpha = 1$ if $\zeta$ is not a root of unity in $k$ and $i_\alpha = v_{K_1}(\alpha^n(t_1) - t_1)$ if $\zeta$ is a primitive $n$th root. Assume that $k$ is of characteristic zero. Then there is an automorphism $f \in \mathrm{Aut}_k(K_1)$ such that $f^{-1}\alpha f = \beta$ where*

$$\beta(t_1) = \zeta t_1 + x t_1^{i_\alpha} + x^2 y t_1^{2i_\alpha - 1}$$

*for some $x \in k^*/k^{*(i_\alpha - 1)}$, $y \in k$.*

*Two automorphisms $\alpha$ and $\beta$ are conjugate if and only if*

$$(\zeta(\alpha), i_\alpha, x(\alpha), y(\alpha)) = (\zeta(\beta), i_\beta, x(\beta), y(\beta)).$$

*Proof.* First we prove that $\alpha = f\beta' f^{-1}$ where

$$\beta'(t_1) = \zeta t_1 + x t_1^{in+1} + y t_1^{2in+1}$$

for some natural $i$. Then we prove that $i_\alpha = i_{\beta'}$.

Consider a set $\{\alpha_i : i \in \mathbb{N}\}$ where $\alpha_i = f_i \alpha_{i-1} f_i^{-1}$, $f_i(t_1) = t_1 + x_i t_1^i$ for some $x_i \in k$, $\alpha_1 = \alpha$. Write

$$\alpha_i(t_1) = \zeta t_1 + a_{2,i} t_1^2 + a_{3,i} t_1^3 + \dots .$$

One can check that $a_{2,2} = x_2(\zeta^2 - \zeta) + a_{2,1}$ and hence there exists an element $x_2 \in k$ such that $a_{2,2} = 0$. Since $a_{j,i+1} = a_{j,i}$, we have $a_{2,j} = 0$ for all $j \geqslant 2$. Further, $a_{3,3} = x_3(\zeta^3 - \zeta) + a_{3,2}$ and hence there exists an element $x_3 \in k$ such that $a_{3,3} = 0$. Then $a_{3,j} = 0$ for all $j \geqslant 3$. Thus, any element $a_{k,k}$ can be made equal to zero if $n \nmid (k-1)$, and therefore $\alpha = f\tilde{\alpha}f^{-1}$ where

$$\tilde{\alpha}(t_1) = \zeta t_1 + \tilde{a}_{in+1}t_1^{in+1} + \tilde{a}_{in+n+1}t_1^{in+n+1} + \dots$$

for some $i$, $\tilde{a}_j \in k$. Notice that $\tilde{a}_{in+1}$ does not depend on $x_i$. Put $x = x(\alpha) = \tilde{a}_{in+1}$.

Now we replace $\alpha$ by $\tilde{\alpha}$. One can check that if $n|(k-1)$ then

$$a_{j,k} = a_{j,k-1} \qquad \text{for } 2 \leqslant j < k + in$$

and

$$a_{k+in,k} = x_k x(k - in - 1) + a_{k+in} + \text{ some polynomial which does not depend on } x_k.$$

From this fact it immediately follows that $a_{2in+1,in+1}$ does not depend on $x_i$ and for all $k \neq in+1$ $a_{k+in,k}$ can be made equal to zero. Then $y = y(\alpha) = a_{2in+1,in+1}$.

Now we prove that $i_\alpha = i_{\beta'}$. Using the formula

$$\beta'^n(t_1) = t_1 + nx(\alpha)\zeta^{-1}t_1^{in+1} + \dots$$

we get $i_{\beta'} = in+1$. Then one can check that $v_{K_1}(f^{-1}(\alpha^n - \mathrm{id})f) = v_{K_1}(\alpha^n - \mathrm{id}) = i_\alpha$. Since $\beta'^n - \mathrm{id} = f^{-1}(\alpha^n - \mathrm{id})f$, we get the identity $i_\alpha = i_{\beta'}$.

The rest of the proof is clear. For details see [Zh, Lemma 6 and Prop.5]. $\square$

## 8.3. Canonical automorphisms of finite order

### 8.3.1. Characteristic zero case.

Assume that

a two-dimensional local skew field $K$ splits,
$K_1$ is a field, $K_0 \subset Z(K)$,
$\mathrm{char}(K) = \mathrm{char}(K_0) = 0$,
$\alpha^n = \mathrm{id}$ for some $n \geqslant 1$,
for any convergent sequence $(a_j)$ in $K_1$ the sequence $(t_2 a_j t_2^{-1})$ converges in $K$.

**Lemma.** *$K$ is isomorphic to a two-dimensional local skew field $K_1((t_2))$ where*

$$t_2 a t_2^{-1} = \alpha(a) + \delta_i(a)t_2^i + \delta_{2i}(a)t_2^{2i} + \delta_{2i+n}(a)t_2^{2i+n} + \dots \qquad \textit{for all } a \in K_1$$

*where $n|i$ and $\delta_j : K_1 \to K_1$ are linear maps and*

$$\delta_i(ab) = \delta_i(a)\alpha(b) + \alpha(a)\delta_i(b) \qquad \textit{for every } a, b \in K_1.$$

*Moreover*

$$t_2^n a t_2^{-n} = a + \delta'_i(a)t_2^i + \delta'_{2i}(a)t_2^{2i} + \delta'_{2i+n}(a)t_2^{2i+n} + \dots$$

*where* $\delta'_j$ *are linear maps and* $\delta'_i$ *and* $\delta := \delta'_{2i} - ((i+1)/2)\delta'^2_i$ *are derivations.*

**Remark.** The following fact holds for the field $K$ of any characteristic: $K$ is isomorphic to a two-dimensional local skew field $K_1((t_2))$ where

$$t_2 a t_2^{-1} = \alpha(a) + \delta_i(a)t_2^i + \delta_{i+1}(a)t_2^{i+1} + \dots$$

where $\delta_j$ are linear maps which satisfy some identity. For explicit formulas see [Zh, Prop.2 and Cor.1].

*Proof.* It is clear that $K$ is isomorphic to a two-dimensional local skew field $K_1((t_2))$ where

$$t_2 a t_2^{-1} = \alpha(a) + \delta_1(a)t_2 + \delta_2(a)t_2^2 + \dots \qquad \text{for all } a$$

and $\delta_j$ are linear maps. Then $\delta_1$ is a $(\alpha^2, \alpha)$-derivation, that is $\delta_1(ab) = \delta_1(a)\alpha^2(b) + \alpha(a)\delta_1(b)$.

Indeed,

$$\begin{aligned} t_2 ab t_2^{-1} &= t_2 a t_2^{-1} t_2 b t_2^{-1} = (\alpha(a) + \delta_1(a)t_2 + \dots)(\alpha(b) + \delta_1(b)t_2 + \dots) \\ &= \alpha(a)\alpha(b) + (\delta_1(a)\alpha^2(b) + \alpha(a)\delta_1(b))t_2 + \dots = \alpha(ab) + \delta_1(ab)t_2 + \dots . \end{aligned}$$

From the proof of Theorem 8.2 it follows that $\delta_1$ is an inner derivation, i.e. $\delta_1(a) = g\alpha^2(a) - \alpha(a)g$ for some $g \in K_1$, and that there exists a $t_{2,2} = (1 + x_1 t_2)t_2$ such that

$$t_{2,2} a t_{2,2}^{-1} = \alpha(a) + \delta_{2,2}(a)t_{2,2}^2 + \dots .$$

One can easily check that $\delta_{2,2}$ is a $(\alpha^3, \alpha)$-derivation. Then it is an inner derivation and there exists $t_{2,3}$ such that

$$t_{2,3} a t_{2,3}^{-1} = \alpha(a) + \delta_{3,3}(a)t_{2,3}^3 + \dots .$$

By induction one deduces that if

$$t_{2,j} a t_{2,j}^{-1} = \alpha(a) + \delta_{n,j}(a)t_{2,j}^n + \dots + \delta_{kn,j}(a)t_{2,j}^{kn} + \delta_{j,j}(a)t_{2,j}^j + \dots$$

then $\delta_{j,j}$ is a $(\alpha^{j+1}, \alpha)$-derivation and there exists $t_{2,j+1}$ such that

$$t_{2,j+1} a t_{2,j+1}^{-1} = \alpha(a) + \delta_{n,j}(a)t_{2,j+1}^n + \dots + \delta_{kn,j}(a)t_{2,j+1}^{kn} + \delta_{j+1,j+1}(a)t_{2,j+1}^{j+1} + \dots .$$

The rest of the proof is clear. For details see [Zh, Prop.2, Cor.1, Lemmas 10, 3]. □

**Definition.** Let $i = v_{K_2}(\varphi(t_2^n)(t_1) - t_1) \in n\mathbb{N} \cup \infty$, ($\varphi$ is defined in subsection 8.1) and let $r \in \mathbb{Z}/i$ be $v_{K_1}(x) \bmod i$ where $x$ is the residue of $(\varphi(t_2^n)(t_1) - t_1)t_2^{-i}$. Put

$$a = \operatorname{res}_{t_1}\left(\frac{(\delta'_{2i} - \frac{i+1}{2}\delta'^2_i)(t_1)}{\delta'_i(t_1)^2} dt_1\right) \in K_0.$$

( $\delta'_i, \delta'_{2i}$ are the maps from the preceding lemma).

**Proposition.** *If $n = 1$ then $i, r$ don't depend on the choice of a system of local parameters; if $i = 1$ then $a$ does not depend on the choice of a system of local parameters; if $n \neq 1$ then $a$ depends only on the maps $\delta_{i+1}, \dots, \delta_{2i-1}$, $i, r$ depend only on the maps $\delta_j$, $j \notin n\mathbb{N}$, $j < i$.*

*Proof.* We comment on the statement first. The maps $\delta_j$ are uniquely defined by parameters $t_1, t_2$ and they depend on the choice of these parameters. So the claim that $i, r$ depend only on the maps $\delta_j$, $j \notin n\mathbb{N}$, $j < i$ means that $i, r$ don't depend on the choice of parameters $t_1, t_2$ which preserve the maps $\delta_j$, $j \notin n\mathbb{N}$, $j < i$.

Note that $r$ depends only on $i$. Hence it is sufficient to prove the proposition only for $i$ and $a$. Moreover it suffices to prove it for the case where $n \neq 1$, $i \neq 1$, because if $n = 1$ then the sets $\{\delta_j : j \notin n\mathbb{N}\}$ and $\{\delta_{i+1} : \dots, \delta_{2i-1}\}$ are empty.

It is clear that $i$ depends on $\delta_j$, $j \notin n\mathbb{N}$. Indeed, it is known that $\delta_1$ is an inner $(\alpha^2, \alpha)$-derivation (see the proof of the lemma). By [Zh, Lemma 3] we can change a parameter $t_2$ such that $\delta_1$ can be made equal $\delta_1(t_1) = t_1$. Then one can see that $i = 1$. From the other hand we can change a parameter $t_2$ such that $\delta_1$ can be made equal to 0. In this case $i > 1$. This means that $i$ depends on $\delta_1$. By [Zh, Cor.3] any map $\delta_j$ is uniquely determined by the maps $\delta_q$, $q < j$ and by an element $\delta_j(t_1)$. Then using similar arguments and induction one deduces that $i$ depends on other maps $\delta_j$, $j \notin n\mathbb{N}$, $j < i$.

Now we prove that $i$ does not depend on the choice of parameters $t_1, t_2$ which preserve the maps $\delta_j$, $j \notin n\mathbb{N}$, $j < i$.

Note that $i$ does not depend on the choice of $t_1$: indeed, if $t'_1 = t_1 + bz^j$, $b \in K_1$ then $z^n t'_1 z^{-n} = z^n t_1 z^{-n} + (z^n b z^{-n}) z^j = t'_1 + r$, where $r \in \mathcal{M}_K^i \setminus \mathcal{M}_K^{i+1}$. One can see that the same is true for $t'_1 = c_1 t_1 + c_2 t_2^2 + \dots$, $c_j \in K_0$.

Let $\delta_q$ be the first non-zero map for given $t_1, t_2$. If $q \neq i$ then by [Zh, Lemma 8, (ii)] there exists a parameter $t'_1$ such that $z t'_1 z^{-1} = {t'_1}^{\alpha} + \delta_{q+1}(t'_1) z^{q+1} + \dots$. Using this fact and Proposition 8.2 we can reduce the proof to the case where $q = i$, $\alpha(t_1) = \xi t_1$, $\alpha(\delta_i(t_1)) = \xi \delta_i(t_1)$ (this case is equivalent to the case of $n = 1$). Then we apply [Zh, Lemma 3] to show that

$$v_{K_2}((\phi(t'_2) - 1)(t_1)) = v_{K_2}((\phi(t_2) - 1)(t_1)),$$

for any parameters $t_2, t'_2$, i.e. $i$ does not depend on the choice of a parameter $t_2$. For details see [Zh, Prop.6].

To prove that $a$ depends only on $\delta_{i+1}, \dots, \delta_{2i-1}$ we use the fact that for any pair of parameters $t'_1, t'_2$ we can find parameters $t''_1 = t_1 + r$, where $r \in \mathcal{M}_K^i$, $t''_2$ such that corresponding maps $\delta_j$ are equal for all $j$. Then by [Zh, Lemma 8] $a$ does not depend on $t''_1$ and by [Zh, Lemma 3] $a$ depends on $t''_2 = t_2 + a_1 t_2^2 + \dots$, $a_j \in K_1$ if and only if $a_1 = \dots = a_{i-1}$. Using direct calculations one can check that $a$ doesn't depend on $t''_2 = a_0 t_2$, $a_0 \in K_1^*$.

To prove the fact it is sufficient to prove it for $t_1'' = t_1 + ct_1^h z^j$ for any $j < i,\ c \in K_0$. Using [Zh, Lemma 8] one can reduce the proof to the assertion that some identity holds. The identity is, in fact, some equation on residue elements. One can check it by direct calculations. For details see [Zh, Prop.7]. □

**Remark.** The numbers $i, r, a$ can be defined only for local skew fields which splits. One can check that the definition can not be extended to the skew field in Dubrovin's example.

**Theorem.**

(1) *$K$ is isomorphic to a two-dimensional local skew field $K_0((t_1))((t_2))$ such that*

$$t_2 t_1 t_2^{-1} = \xi t_1 + x t_2^i + y t_2^{2i}$$

*where $\xi$ is a primitive $n$ th root, $x = ct_1^r,\ c \in K_0^*/(K_0^*)^d$,*

$$y = (a + r(i+1)/2) t_1^{-1} x^2, \quad d = \gcd(r-1, i).$$

*If $n = 1,\ i = \infty$, then $K$ is a field.*

(2) *Let $K, K'$ be two-dimensional local skew fields of characteristic zero which splits; and let $K_1, K_1'$ be fields. Let $\alpha^n = \mathrm{id}$, $\alpha'^{n'} = \mathrm{id}$ for some $n, n' \geqslant 1$. Then $K$ is isomorphic to $K'$ if and only if $K_0$ is isomorphic to $K_0'$ and the ordered sets $(n, \xi, i, r, c, a)$ and $(n', \xi', i', r', c', a')$ coincide.*

*Proof.* (2) follows from the Proposition of 8.2 and (1). We sketch the proof of (1).

From Proposition 8.2 it follows that there exists $t_1$ such that $\alpha(t_1) = \xi t_1$; $\delta_i(t_1)$ can be represented as $ct_1^r a^i$. Hence there exists $t_2$ such that

$$t_2 t_1 t_2^{-1} = \xi t_1 + x t_2^i + \delta_{2i}(t_1) t_2^{2i} + \dots$$

Using [Zh, Lemma 8] we can find a parameter $t_1' = t_1 \bmod \mathcal{M}_K$ such that

$$t_2 t_1' t_2^{-1} = \xi t_1 + x t_2^i + y t_2^{2i} + \dots$$

The rest of the proof is similar to the proof of the lemma. Using [Zh, Lemma 3] one can find a parameter $t_2' = t_2 \bmod \mathcal{M}_K^2$ such that $\delta_j(t_1) = 0,\ j > 2i$. □

**Corollary.** *Every two-dimensional local skew field $K$ with the ordered set*

$$(n, \xi, i, r, c, a)$$

*is a finite-dimensional extension of a skew field with the ordered set $(1, 1, 1, 0, 1, a)$.*

**Remark.** There is a construction of a two-dimensional local skew field with a given set $(n, \xi, i, r, c, a)$.

**Examples.**

(1) The ring of formal pseudo-differential equations is the skew field with the set $(n = 1, \xi = 1, i = 1, r = 0, c = 1, a = 0)$.

(2) The elements of $\mathrm{Br}(L)$ where $L$ is a two-dimensional local field of equal characteristic are local skew fields. If, for example, $L$ is a $C_2$- field, they split and $i = \infty$. Hence any division algebra in $\mathrm{Br}(L)$ is cyclic.

**8.3.2. Characteristic $p$ case.**

**Theorem.** *Suppose that a two-dimensional local skew field $K$ splits, $K_1$ is a field, $K_0 \subset Z(K)$, $\mathrm{char}(K) = \mathrm{char}(K_0) = p > 2$ and $\alpha = \mathrm{id}$.*

*Then $K$ is a finite dimensional vector space over its center if and only if $K$ is isomorphic to a two-dimensional local skew field $K_0((t_1))((t_2))$ where*

$$t_2^{-1} t_1 t_2 = t_1 + x t_2^i$$

*with $x \in K_1^p$, $(i,p) = 1$.*

*Proof.* The "if" part is obvious. We sketch the proof of the "only if" part.

If $K$ is a finite dimensional vector space over its center then $K$ is a division algebra over a henselian field. In fact, the center of $K$ is a two-dimensional local field $k((u))((t))$. Then by [JW, Prop.1.7] $K_1/(Z(K))_1$ is a purely inseparable extension. Hence there exists $t_1$ such that $t_1^{p^k} \in Z(K)$ for some $k \in \mathbb{N}$ and $K \simeq K_0((t_1))((t_2))$ as a vector space with the relation

$$t_2 t_1 t_2^{-1} = t_1 + \delta_i(t_1) t_2^i + \dots$$

(see Remark 8.3.1). Then it is sufficient to show that $i$ is prime to $p$ and there exist parameters $t_1 \in K_1$, $t_2$ such that the maps $\delta_j$ satisfy the following property:

(*) If $j$ is not divisible by $i$ then $\delta_j = 0$. If $j$ is divisible by $i$ then $\delta_j = c_{j/i} \delta_i^{j/i}$ with some $c_{j/i} \in K_1$.

Indeed, if this property holds then by induction one deduces that $c_{j/i} \in K_0$, $c_{j/i} = ((i+1)\dots(i(j/i-1)+1))/(j/i)!$. Then one can find a parameter $t_2' = b t_2$, $b \in K_1$ such that $\delta_j'$ satisfies the same property and $\delta_i^2 = 0$. Then

$${t_2'}^{-1} t_1 t_2' = t_1 - \delta_i'(t_1) t_2^i.$$

First we prove that $(i,p) = 1$. To show it we prove that if $p|i$ then there exists a map $\delta_j$ such that $\delta_j(t_1^{p^k}) \neq 0$. To find this map one can use [Zh, Cor.1] to show that $\delta_{ip}(t_1^p) \neq 0$, $\delta_{ip^2}(t_1^{p^2}) \neq 0$, $\dots$, $\delta_{ip^k}(t_1^{p^k}) \neq 0$.

Then we prove that for some $t_2$ property (*) holds. To show it we prove that if property (*) does not hold then there exists a map $\delta_j$ such that $\delta_j(t_1^{p^k}) \neq 0$. To find this map we reduce the proof to the case of $i \equiv 1 \bmod p$. Then we apply the following idea.

Let $j \equiv 1 \bmod p$ be the minimal positive integer such that $\delta_j$ is not equal to zero on $K_1^{p^l}$. Then one can prove that the maps $\delta_m,\ kj \leqslant m < (k+1)j,\ k \in \{1, \dots, p-1\}$ satisfy the following property:

there exist elements $c_{m,k} \in K_1$ such that

$$(\delta_m - c_{m,1}\delta - \dots - c_{m,k}\delta^k)|_{K_1^{p^l}} = 0$$

where $\delta\colon K_1 \to K_1$ is a linear map, $\delta|_{K_1^{p^l}}$ is a derivation, $\delta(t_1^j) = 0$ for $j \notin p^l\mathbb{N}$, $\delta(t_1^{p^l}) = 1,\ c_{kj,k} = c(\delta_j(t_1^{p^l}))^k,\ c \in K_0$.

Now consider maps $\widetilde{\delta_q}$ which are defined by the following formula

$$t_2^{-1}at_2 = a + \widetilde{\delta_i}(a)t_2^i + \widetilde{\delta_{i+1}}(a)t_2^{i+1} + \dots, \quad a \in K_1.$$

Then $\widetilde{\delta_q} + \delta_q + \sum_{k=1}^{q-1}\delta_k\widetilde{\delta_{q-k}} = 0$ for any $q$. In fact, $\widetilde{\delta_q}$ satisfy some identity which is similar to the identity in [Zh, Cor.1]. Using that identity one can deduce that if

$j \equiv 1 \bmod p$ and there exists the minimal $m$ ($m \in \mathbb{Z}$) such that $\delta_{mp+2i}|_{K_1^{p^l}} \neq 0$ if $j \not| (mp+2i)$ and $\delta_{mp+2i}|_{K_1^{p^l}} \neq s\delta_j^{(2i+mp)/j}|_{K_1^{p^l}}$ for any $s \in K_1$ otherwise, and $\delta_q(t_1^{p^l}) = 0$ for $q < mp+2i,\ q \not\equiv 1 \bmod p$,

then

$(mp+2i)+(p-1)j$ is the minimal integer such that $\delta_{(mp+2i)+(p-1)j}|_{K_1^{p^{l+1}}} \neq 0$.

To complete the proof we use induction and [Zh, Lemma 3] to show that there exist parameters $t_1 \in K_1,\ t_2$ such that $\delta_q(t_1^{p^l}) = 0$ for $q \not\equiv 1,2 \bmod p$ and $\delta_j^2 = 0$ on $K_1^{p^l}$. □

**Corollary 1.** *If $K$ is a finite dimensional division algebra over its center then its index is equal to $p$.*

**Corollary 2.** *Suppose that a two-dimensional local skew field $K$ splits, $K_1$ is a field, $K_0 \subset Z(K)$, $\mathrm{char}(K) = \mathrm{char}(K_0) = p > 2$, $K$ is a finite dimensional division algebra over its center of index $p^k$.*

*Then either $K$ is a cyclic division algebra or has index $p$.*

*Proof.* By [JW, Prop. 1.7] $K_1/\overline{Z(K)}$ is the compositum of a purely inseparable extension and a cyclic Galois extension. Then the canonical automorphism $\alpha$ has order $p^l$ for some $l \in \mathbb{N}$. By [Zh, Lemma 10] (which is true also for $\mathrm{char}(K) = p > 0$), $K \simeq K_0((t_1))((t_2))$ with

$$t_2at_2^{-1} = \alpha(a) + \delta_i(a)t_2^i + \delta_{i+p^l}(a)t_2^{i+p^l} + \delta_{i+p^l}(a)t_2^{i+2p^l} + \dots$$

where $i \in p^l\mathbb{N}$, $a \in K_1$. Suppose that $\alpha \neq 1$ and $K_1$ is not a cyclic extension of $\overline{Z(K)}$. Then there exists a field $F \subset K_1$, $F \not\subset Z(K)$ such that $\alpha|_F = 1$. If $a \in F$ then for some $m$ the element $a^{p^m}$ belongs to a cyclic extension of the field $\overline{Z(K)}$, hence $\delta_j(a^{p^m}) = 0$ for all $j$. But we can apply the same arguments as in the proof of the preceding theorem to show that if $\delta_i \neq 0$ then there exists a map $\delta_j$ such that $\delta_j(a^{p^m}) \neq 0$, a contradiction. We only need to apply [Zh, Prop.2] instead of [Zh, Cor.1] and note that $\alpha\delta = x\delta\alpha$ where $\delta$ is a derivation on $K_1$, $x \in K_1$, $x \equiv 1 \bmod \mathcal{M}_{K_1}$, because $\alpha(t_1)/t_1 \equiv 1 \bmod \mathcal{M}_{K_1}$.

Hence $t_2 a t_2^{-1} = \alpha(a)$ and $K_1/\overline{Z(K)}$ is a cyclic extension and $K$ is a cyclic division algebra $(K_1(t_2^{p^k})/Z(K), \alpha, t_2^{p^k})$. □

**Corollary 3.** *Let $F = F_0((t_1))((t_2))$ be a two-dimensional local field, where $F_0$ is an algebraically closed field. Let $A$ be a division algebra over $F$.*

*Then $A \simeq B \otimes C$, where $B$ is a cyclic division algebra of index prime to $p$ and $C$ is either cyclic (as in Corollary 2) or $C$ is a local skew field from the theorem of index $p$.*

*Proof.* Note that $F$ is a $C_2$-field. Then $A_1$ is a field, $A_1/F_1$ is the compositum of a purely inseparable extension and a cyclic Galois extension, and $A_1 = F_0((u))$ for some $u \in A_1$. Hence $A$ splits. So, $A$ is a splitting two-dimensional local skew field.

It is easy to see that the index of $A$ is $|\overline{A} : \overline{F}| = p^q m$, $(m, p) = 1$. Consider subalgebras $B = C_A(F_1)$, $C = C_A(F_2)$ where $F_1 = F(u^{p^q})$, $F_2 = F(u^m)$. Then by [M, Th.1] $A \simeq B \otimes C$.

The rest of the proof is clear. □

Now one can easily deduce that

**Corollary 4.** *The following conjecture: the exponent of $A$ is equal to its index for any division algebra $A$ over a $C_2$-field $F$ (see for example* [PY, 3.4.5.]*)*
*has the positive answer for $F = F_0((t_1))((t_2))$.*

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*ISSN 1464-8997 (on line) 1464-8989 (printed)* 

Geometry & Topology Monographs
Volume 3: Invitation to higher local fields
Part II, section 9, pages 293–298

# 9. Local reciprocity cycles

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In this section we introduce a description of totally ramified Galois extensions of a local field with finite residue field (extensions have to satisfy certain arithmetical restrictions if they are infinite) in terms of subquotients of formal power series $\mathbb{F}_p^{\text{sep}}[[X]]^*$. This description can be viewed as a non-commutative local reciprocity map (which is not in general a homomorphism but a cocycle) which directly describes the Galois group in terms of certain objects related to the ground field. Abelian class field theory as well as metabelian theory of Koch and de Shalit [K], [KdS] (see subsection 9.4) are partial cases of this theory.

## 9.1. Group $U^{\diamond}_{N(\widehat{L/F})}$

Let $F$ be a local field with finite residue field. Denote by $\varphi \in G_F$ a lifting of the Frobenius automorphism of $F_{\text{ur}}/F$.

Let $F^{\varphi}$ be the fixed field of $\varphi$. The extension $F^{\varphi}/F$ is totally ramified.

**Lemma** ([KdS, Lemma 0.2])**.** *There is a unique norm compatible sequence of prime elements $\pi_E$ in finite subextensions $E/F$ of $F^{\varphi}/F$.*

*Proof.* Uniqueness follows from abelian local class field theory, existence follows from the compactness of the group of units. □

In what follows we fix $F^{\varphi}$ and consider Galois subextensions $L/F$ of $F^{\varphi}/F$. Assume that $L/F$ is *arithmetically profinite*, ie for every $x$ the ramification group $\text{Gal}(L/F)^x$ is open in $\text{Gal}(L/F)$ (see also subsection 6.3 of Part II). For instance, a totally ramified $p$-adic Lie extension is arithmetically profinite.

For an arithmetically profinite extension $L/F$ define its Hasse–Herbrand function $h_{L/F}\colon [0,\infty) \to [0,\infty)$ as $h_{L/F}(x) = \lim h_{M/F}(x)$ where $M/F$ runs over finite subextensions of $L/F$ (cf. [FV, Ch. III §5]).

*Published 10 December 2000:* 

If $L/F$ is infinite let $N(L/F)$ be the field of norms of $L/F$. It can be identified with $k_F((\Pi))$ where $\Pi$ corresponds to the norm compatible sequence $\pi_E$ (see subsection 6.3 of Part II, [W], [FV, Ch.III §5]).

Denote by $\varphi$ the automorphism of $N(L/F)_{\mathrm{ur}}$ and of its completion $\widehat{N(L/F)}$ corresponding to the Frobenius automorphism of $F_{\mathrm{ur}}/F$.

**Definition.** Denote by $U^{\diamond}_{\widehat{N(L/F)}}$ the subgroup of the group $U_{\widehat{N(L/F)}}$ of those elements whose $\widehat{F}$-component belongs to $U_F$. An element of $U^{\diamond}_{\widehat{N(L/F)}}$ such that its $\widehat{F}$-component is $\varepsilon \in U_F$ will be called a lifting of $\varepsilon$.

The group $U^{\diamond}_{\widehat{N(L/F)}}/U_{N(L/F)}$ is a direct product of a quotient group of the group of multiplicative representatives of the residue field $k_F$ of $F$, a cyclic group $\mathbb{Z}/p^a$ and a free topological $\mathbb{Z}_p$-module. The Galois group $\mathrm{Gal}(L/F)$ acts naturally on $U^{\diamond}_{\widehat{N(L/F)}}/U_{N(L/F)}$.

## 9.2. Reciprocity map $\mathcal{N}_{L/F}$

To motivate the next definition we interpret the map $\Upsilon_{L/F}$ (defined in 10.1 and 16.1) for a finite Galois totally ramified extension $L/F$ in the following way. Since in this case both $\pi_\Sigma$ and $\pi_L$ are prime elements of $L_{\mathrm{ur}}$, there is $\varepsilon \in U_{L_{\mathrm{ur}}}$ such that $\pi_\Sigma = \pi_L\varepsilon$. We can take $\tilde\sigma = \sigma\varphi$. Then $\pi_L^{\sigma-1} = \varepsilon^{1-\sigma\varphi}$. Let $\eta \in U_{\widehat{L}}$ be such that $\eta^{\varphi-1} = \varepsilon$. Since $(\eta^{\sigma\varphi-1}\varepsilon^{-1})^{\varphi-1} = (\eta^{(\sigma-1)\varphi})^{\varphi-1}$, we deduce that $\varepsilon = \eta^{\sigma\varphi-1}\eta^{(1-\sigma)\varphi}\rho$ with $\rho \in U_L$. Thus, for $\xi = \eta^{\sigma\varphi-1}$

$$\Upsilon_{L/F}(\sigma) \equiv N_{\Sigma/F}\pi_\Sigma \equiv N_{\widehat{L}/\widehat{F}}\xi \bmod N_{L/F}L^*, \qquad \xi^{1-\varphi} = \pi_L^{\sigma-1}.$$

**Definition.** For a $\sigma \in \mathrm{Gal}(L/F)$ let $U_\sigma \in U_{\widehat{N(L/F)}}$ be a solution of the equation

$$\boxed{U^{1-\varphi} = \Pi^{\sigma-1}}$$

(recall that $\mathrm{id} - \varphi\colon U_{\widehat{N(L/F)}} \to U_{\widehat{N(L/F)}}$ is surjective). Put

$$\mathcal{N}_{L/F}\colon \mathrm{Gal}(L/F) \to U^{\diamond}_{\widehat{N(L/F)}}/U_{N(L/F)}, \quad \mathcal{N}_{L/F}(\sigma) = U_\sigma \mod U_{N(L/F)}.$$

**Remark.** Compare the definition with Fontaine-Herr's complex defined in subsection 6.4 of Part II.

**Properties.**

(1) $\mathcal{N}_{L/F} \in Z^1(\mathrm{Gal}(L/F), U^{\diamond}_{\widehat{N(L/F)}}/U_{N(L/F)})$ is injective.

(2) For a finite extension $L/F$ the $\widehat{F}$-component of $\mathcal{N}_{L/F}(\sigma)$ is equal to the value $\Upsilon_{L/F}(\sigma)$ of the abelian reciprocity map $\Upsilon_{L/F}$ (see the beginning of 9.2).

(3) Let $M/F$ be a Galois subextension of $L/F$ and $E/F$ be a finite subextension of $L/F$. Then the following diagrams of *maps* are commutative:

$$\begin{array}{ccc} \mathrm{Gal}(L/E) & \xrightarrow{\mathcal{N}_{L/E}} & U^{\diamond}_{\widehat{N(L/E)}}/U_{N(L/E)} \\ \downarrow & & \downarrow \\ \mathrm{Gal}(L/F) & \xrightarrow{\mathcal{N}_{L/F}} & U^{\diamond}_{\widehat{N(L/F)}}/U_{N(L/F)} \end{array} \qquad \begin{array}{ccc} \mathrm{Gal}(L/F) & \xrightarrow{\mathcal{N}_{L/F}} & U^{\diamond}_{\widehat{N(L/F)}}/U_{N(L/F)} \\ \downarrow & & \downarrow \\ \mathrm{Gal}(M/F) & \xrightarrow{\mathcal{N}_{M/F}} & U^{\diamond}_{\widehat{N(M/F)}}/U_{N(M/F)}. \end{array}$$

(4) Let $U^{\diamond}_{n,\widehat{N(L/F)}}$ be the filtration induced from the filtration $U_{n,\widehat{N(L/F)}}$ on the field of norms. For an infinite arithmetically profinite extension $L/F$ with the Hasse–Herbrand function $h_{L/F}$ put $\mathrm{Gal}(L/F)_n = \mathrm{Gal}(L/F)^{h^{-1}_{L/F}(n)}$. Then $\mathcal{N}_{L/F}$ maps $\mathrm{Gal}(L/F)_n \setminus \mathrm{Gal}(L/F)_{n+1}$ into $U^{\diamond}_{n,\widehat{N(L/F)}}U_{N(L/F)} \setminus U^{\diamond}_{n+1,\widehat{N(L/F)}}U_{N(L/F)}$.

(6) The set $\mathrm{im}(\mathcal{N}_{L/F})$ is not closed in general with respect to multiplication in the group

$U_{\widehat{N(L/F)}}/U_{N(L/F)}$. Endow $\mathrm{im}(\mathcal{N}_{L/F})$ with a new group structure given by $x \star y = x\mathcal{N}^{-1}_{L/F}(x)(y)$. Then clearly $\mathrm{im}(\mathcal{N}_{L/F})$ is a group isomorphic to $\mathrm{Gal}(L/F)$.

**Problem.** What is $\mathrm{im}(\mathcal{N}_{L/F})$?

One method to solve the problem is described below.

## 9.3. Reciprocity map $\mathcal{H}_{L/F}$

**Definition.** Fix a tower of subfields $F = E_0 - E_1 - E_2 - \dots$, such that $L = \cup E_i$, $E_i/F$ is a Galois extension, and $E_i/E_{i-1}$ is cyclic of prime degree. We can assume that $|E_{i+1} : E_i| = p$ for all $i \geqslant i_0$ and $|E_{i_0} : E_0|$ is relatively prime to $p$.

Let $\sigma_i$ be a generator of $\mathrm{Gal}(E_i/E_{i-1})$. Denote

$$X_i = U_{\widehat{E}_i}^{\sigma_i - 1}.$$

The group $X_i$ is a $\mathbb{Z}_p$-submodule of $U_{1,\widehat{E}_i}$. It is the direct sum of a cyclic torsion group of order $p^{n_i}$, $n_i \geqslant 0$, generated by, say, $\alpha_i$ ($\alpha_i = 1$ if $n_i = 0$) and a free topological $\mathbb{Z}_p$-module $Y_i$.

We shall need a sufficiently “nice” injective map from characteristic zero or $p$ to characteristic $p$

$$f_i \colon U_{\widehat{E}_i}^{\sigma_i - 1} \to U_{\widehat{N(L/E_i)}} \to U_{N(L/F)}.$$

If $F$ is a local field of characteristic zero containing a non-trivial $p$th root $\zeta$ and $f_i$ is a homomorphism, then $\zeta$ is doomed to go to 1. Still, from certain injective maps (not homomorphisms) $f_i$ specifically defined below we can obtain a subgroup $\prod f_i(U_{\widehat{E_i}}^{\sigma_i-1})$ of $U^{\diamond}_{\widehat{N(L/F)}}$.

**Definition.** If $n_i=0$, set $A^{(i)}\in U_{\widehat{N(L/E_i)}}$ to be equal to 1.
If $n_i>0$, let $A^{(i)}\in U_{\widehat{N(L/E_i)}}$ be a lifting of $\alpha_i$ with the following restriction: $A^{(i)}_{\widehat{E_{i+1}}}$ is not a root of unity of order a power of $p$ (this condition can always be satisfied, since the kernel of the norm map is uncountable).

**Lemma** ([F])**.** *If* $A^{(i)}\neq 1$, *then* $\beta_{i+1}={A^{(i)}_{\widehat{E_{i+1}}}}^{p^{n_i}}$ *belongs to* $X_{i+1}$.

Note that every $\beta_{i+1}$ when it is defined doesn't belong to $X_{i+1}^p$. Indeed, otherwise we would have ${A^{(i)}_{\widehat{E_{i+1}}}}^{p^{n_i}}=\gamma^p$ for some $\gamma\in X_{i+1}$ and then ${A^{(i)}_{\widehat{E_{i+1}}}}^{p^{n_i-1}}=\gamma\zeta$ for a root $\zeta$ of order $p$ or 1. Taking the norm down to $\widehat{E_i}$ we get $\alpha_i^{p^{n_i-1}}=N_{\widehat{E_{i+1}}/\widehat{E_i}}\gamma=1$, which contradicts the definition of $\alpha_i$.

**Definition.** Let $\beta_{i,j}$, $j\geqslant 1$ be free topological generators of $Y_i$ which include $\beta_i$ whenever $\beta_i$ is defined. Let $B^{(i,j)}\in U_{\widehat{N(L/E_i)}}$ be a lifting of $\beta_{i,j}$ (i.e. ${B^{(i,j)}}_{\widehat{E_i}}=\beta_{i,j}$), such that if $\beta_{i,j}=\beta_i$, then $B^{(i,j)}_{\widehat{E_k}}=B^{(i)}_{\widehat{E_k}}={A^{(i-1)}_{\widehat{E_k}}}^{p^{n_i-1}}$ for $k\geqslant i$.

Define a map $X_i\to U_{\widehat{N(L/E_i)}}$ by sending a convergent product $\alpha_i^c\prod_j\beta_{i,j}^{c_j}$, where $0\leqslant c\leqslant n_i-1$, $c_j\in\mathbb{Z}_p$, to ${A^{(i)}}^c\prod_j {B^{(i,j)}}^{c_j}$ (the latter converges). Hence we get a map

$$f_i\colon U_{\widehat{E_i}}^{\sigma_i-1}\to U_{\widehat{N(L/E_i)}}\to U_{\widehat{N(L/F)}}$$

which depends on the choice of lifting. Note that $f_i(\alpha)_{\widehat{E_i}}=\alpha$.

Denote by $Z_i$ the image of $f_i$. Let

$$Z_{L/F}=Z_{L/F}(\{E_i,f_i\})=\left\{\prod_i z^{(i)}: z^{(i)}\in Z_i\right\},$$

$$Y_{L/F}=\{y\in U_{\widehat{N(L/F)}}: y^{1-\varphi}\in Z_{L/F}\}.$$

**Lemma.** *The product of* $z^{(i)}$ *in the definition of* $Z_{L/F}$ *converges.* $Z_{L/F}$ *is a subgroup of* $U^{\diamond}_{\widehat{N(L/F)}}$. *The subgroup* $Y_{L/F}$ *contains* $U_{N(L/F)}$.

**Theorem** ([F]). *For every* $(u_{\widehat{E_i}}) \in U^{\diamond}_{\widehat{N(L/F)}}$ *there is a unique automorphism* $\tau$ *in the group* $\mathrm{Gal}(L/F)$ *satisfying*

$$(u_{\widehat{E_i}})^{1-\varphi} \equiv \Pi^{\tau-1} \mod Z_{L/F}.$$

*If* $(u_{\widehat{E_i}}) \in Y_{L/F}$*, then* $\tau = 1$.

*Hint.* Step by step, passing from $\widehat{E_i}$ to $\widehat{E_{i+1}}$. □

**Remark.** This theorem can be viewed as a non-commutative generalization for finite $k$ of exact sequence $(*)$ of 16.2.

**Corollary.** *Thus, there is map*

$$\mathcal{H}_{L/F}: U^{\diamond}_{\widehat{N(L/F)}} \to \mathrm{Gal}(L/F), \quad \mathcal{H}_{L/F}((u_{\widehat{E_i}})) = \tau.$$

*The composite of* $\mathcal{N}_{L/F}$ *and* $\mathcal{H}_{L/F}$ *is the identity map of* $\mathrm{Gal}(L/F)$.

## 9.4. Main Theorem

**Theorem** ([F]). *Put*

$$\mathcal{H}_{L/F}: U^{\diamond}_{\widehat{N(L/F)}}/Y_{L/F} \to \mathrm{Gal}(L/F), \quad \mathcal{H}_{L/F}((u_{\widehat{E}})) = \tau$$

*where* $\tau$ *is the unique automorphism satisfying* $(u_{\widehat{E}})^{1-\varphi} \equiv \Pi^{\tau-1} \mod Z_{L/F}$. *The injective map* $\mathcal{H}_{L/F}$ *is a bijection. The bijection*

$$\mathcal{N}_{L/F}: \mathrm{Gal}(L/F) \to U^{\diamond}_{\widehat{N(L/F)}}/Y_{L/F}$$

*induced by* $\mathcal{N}_{L/F}$ *defined in 9.2 is a 1-cocycle.*

**Corollary.** Denote by $q$ the cardinality of the residue field of $F$. Koch and de Shalit [K], [KdS] constructed a sort of metabelian local class field theory which in particular describes totally ramified metabelian extensions of $F$ (the commutator group of the commutator group is trivial) in terms of the group

$$\mathfrak{n}(F) = \left\{(u \in U_F, \xi(X) \in \mathbb{F}_p^{\mathrm{sep}}[[X]]^*) : \xi(X)^{\varphi-1} = \{u\}(X)/X\right\}$$

with a certain group structure. Here $\{u\}(X)$ is the residue series in $\mathbb{F}_p^{\mathrm{sep}}[[X]]^*$ of the endomorphism $[u](X) \in O_F[[X]]$ of the formal Lubin–Tate group corresponding to $\pi_F$, $q$, $u$.

Let $M/F$ be the maximal totally ramified metabelian subextension of $F_\varphi$, then $M/F$ is arithmetically profinite. Let $R/F$ be the maximal abelian subextension of $M/F$. Every coset of $U^{\diamond}_{\widehat{N(M/F)}}$ modulo $Y_{M/F}$ has a unique representative in

$\mathrm{im}(\mathcal{N}_{M/F})$. Send a coset with a representative $(u_{\widehat{Q}}) \in U^{\diamond}_{\widehat{N(M/F)}}$ ($F \subset Q \subset M$, $|Q:F| < \infty$) satisfying $(u_{\widehat{Q}})^{1-\varphi} = (\pi_Q)^{\tau-1}$ with $\tau \in \mathrm{Gal}(M/F)$ to

$$\left(u_{\widehat{F}}^{-1}, (u_{\widehat{E}}) \in U^{\diamond}_{\widehat{N(R/F)}}\right) \quad (F \subset E \subset R, |E:F| < \infty).$$

It belongs to $\mathfrak{n}(F)$, so we get a map

$$g\colon U^{\diamond}_{\widehat{N(M/F)}}/Y_{M/F} \to \mathfrak{n}(F).$$

This map is a bijection [F] which makes Koch–de Shalit's theory a corollary of the main results of this section.

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*ISSN 1464-8997 (on line) 1464-8989 (printed)* 

Geometry & Topology Monographs
Volume 3: Invitation to higher local fields
Part II, section 10, pages 299–304

# 10. Galois modules and class field theory

*Boas Erez*

In this section we shall try to present the reader with a sample of several significant instances where, on the way to proving results in Galois module theory, one is lead to use class field theory. Conversely, some contributions of Galois module theory to class fields theory are hinted at. We shall also single out some problems that in our opinion deserve further attention.

## 10.1. Normal basis theorem

The Normal Basis Theorem is one of the basic results in the Galois theory of fields. In fact one can use it to obtain a proof of the fundamental theorem of the theory, which sets up a correspondence between subgroups of the Galois group and subfields. Let us recall its statement and give a version of its proof following E. Noether and M. Deuring (a very modern proof!).

**Theorem** (Noether, Deuring)**.** *Let $K$ be a finite extension of $\mathbb{Q}$. Let $L/K$ be a finite Galois extension with Galois group $G = \mathrm{Gal}(L/K)$. Then $L$ is isomorphic to $K[G]$ as a $K[G]$-module. That is: there is an $a \in L$ such that $\{\sigma(a)\}_{\sigma\in G}$ is a $K$-basis of $L$. Such an $a$ is called a normal basis generator of $L$ over $K$.*

*Proof.* Use the isomorphism

$$\varphi\colon L \otimes_K L \to L[G], \quad \varphi(x \otimes y) = \sum_{\sigma\in G} \sigma(x) y \sigma^{-1},$$

then apply the Krull–Schmidt theorem to deduce that this isomorphism descends to $K$. Note that an element $a$ in $L$ generates a normal basis of $L$ over $K$ if and only if $\varphi(a) \in L[G]^*$.

### 10.1.1. Normal integral bases and ramification.

Let us now move from dimension $0$ (fields) to dimension $1$, and consider rings of algebraic integers.

Let $p$ be a prime number congruent to 1 modulo an (odd) prime $l$. Let $L_1 = \mathbb{Q}(\mu_p)$, and let $K$ be the unique subfield of $L_1$ of degree $l$ over $\mathbb{Q}$. Then $G = \mathrm{Gal}(K/\mathbb{Q})$ is cyclic of order $l$ and $K$ is tamely ramified over $\mathbb{Q}$. One can construct a normal basis for the ring $\mathcal{O}_K$ of integers in $K$ over $\mathbb{Z}$: indeed if $\zeta$ denotes a primitive $p$-th root of unity, then $\zeta$ is a normal basis generator for $L_1/\mathbb{Q}$ and the trace of $\zeta$ to $K$ gives the desired normal integral basis generator. Let now $L_2 = \mathbb{Q}(\mu_{l^2})$. It is easy to see that there is no integral normal basis for $L_2$ over $\mathbb{Q}$. As noticed by Noether, this is related to the fact that $L_2$ is a wildly ramified extension of the rationals. However there is the following structure result, which gives a complete and explicit description of the Galois module structure of rings of algebraic integers in absolute abelian extensions.

**Theorem** (Leopoldt 1959)**.** *Let $K$ be an abelian extension of $\mathbb{Q}$. Let $G = \mathrm{Gal}(K/\mathbb{Q})$. Define*

$$\Lambda = \{\lambda \in \mathbb{Q}[G] : \lambda\mathcal{O}_K \subset \mathcal{O}_K\}$$

*where $\mathcal{O}_K$ is the ring of integers of $K$. Then $\mathcal{O}_K$ is isomorphic to $\Lambda$ as a $\Lambda$-module.*

Note that the statement is not true for an arbitrary global field, nor for general relative extensions of number fields. The way to prove this theorem is by first dealing with the case of cyclotomic fields, for which one constructs explicit normal basis generators in terms of roots of unity. In this step one uses the criterion involving the resolvent map $\varphi$ which we mentioned in the previous theorem. Then, for a general absolute abelian field $K$, one embeds $K$ into the cyclotomic field $\mathbb{Q}(f_K)$ with smallest possible conductor by using the Kronecker–Weber theorem, and one "traces the result down" to $K$. Here it is essential that the extension $\mathbb{Q}(f_K)/K$ is essentially tame. Explicit class field theory is an important ingredient of the proof of this theorem; and, of course, this approach has been generalized to other settings: abelian extensions of imaginary quadratic fields (complex multiplication), extensions of Lubin–Tate type, etc.

### 10.1.2. Factorizability.

While Leopoldt's result is very satisfactory, one would still like to know a way to express the relation there as a relation between the Galois structure of rings of integers in general Galois extensions and the most natural integral representation of the Galois group, namely that given by the group algebra. There is a very neat description of this which uses the notion of factorizability, introduced by A. Fröhlich and A. Nelson. This leads to an equivalence relation on modules which is weaker than local equivalence (genus), but which is non-trivial.

Let $G$ be a finite group, and let $S = \{H : H \leqslant G\}$. Let $T$ be an abelian group.

**Definition.** A map $f\colon S \to T$ is called *factorizable* if every relation of the form

$$\sum_{H \in S} a_H \operatorname{ind}_H^G 1 = 0$$

with integral coefficients $a_H$, implies the relation

$$\prod_{H \in S} f(H)^{a_H} = 1 .$$

**Example.** Let $G = \mathrm{Gal}(L/K)$, then the discriminant of $L/K$ defines a factorizable function (conductor-discriminant formula).

**Definition.** Let $i\colon M \to N$ be a morphism of $\mathcal{O}_K[G]$-lattices. The lattices $M$ and $N$ are said to be *factor-equivalent* if the map $H \to |L^H : i(M)^H|$ is factorizable.

**Theorem** (Fröhlich, de Smit)**.** *If $G = \mathrm{Gal}(L/K)$ and $K$ is a global field, then $\mathcal{O}_L$ is factor-equivalent to $\mathcal{O}_K[G]$.*

Again this result is based on the isomorphism induced by the resolvent map $\varphi$ and the fact that the discriminant defines a factorizable function.

### 10.1.3. Admissible structures.

Ideas related to factorizability have very recently been used to describe the Galois module structure of ideals in local field extensions. Here is a sample of the results.

**Theorem** (Vostokov, Bondarko)**.** *Let $K$ be a local field of mixed characteristic with finite residue field. Let $L$ be a finite Galois extension of $K$ with Galois group $G$.*

(1) *Let $I_1$ and $I_2$ be indecomposable $\mathcal{O}_K[G]$-submodules of $\mathcal{O}_L$. Then $I_1$ is isomorphic to $I_2$ as $\mathcal{O}_K[G]$-modules if and only if there is an $a$ in $K^*$ such that $I_1 = aI_2$.*
(2) *$\mathcal{O}_L$ contains decomposable ideals if and only if there is a subextension $E/L$ of $L/K$ such that $|L : E|\mathcal{O}_L$ contains the different $\mathcal{D}_{L/E}$.*
(3) *If $L$ is a totally ramified Galois $p$-extension of $K$ and $\mathcal{O}_L$ contains decomposable ideals, then $L/K$ is cyclic and $|L : K|\mathcal{O}_L$ contains the different $\mathcal{D}_{L/K}$.*

In fact what is remarkable with these results is that they do *not* involve class field theory.

## 10.2. Galois module theory in geometry

Let $X$ be a smooth projective curve over an algebraically closed field $k$. Let a finite group $G$ act on $X$. Put $Y = X/G$.

**Theorem** (Nakajima 1975)**.** *The covering $X/Y$ is tame if and only if for every line bundle $\mathcal{L}$ of sufficiently large degree which is stable under the $G$-action $H^0(X, \mathcal{L})$ is a projective $k[G]$-module.*

This is the precise analogue of Ullom's version of Noether's Criterion for the existence of a normal integral basis for ideals in a Galois extension of discrete valuation rings. In fact if $(X, G)$ is a *tame* action of a finite group $G$ on any reasonable proper scheme over a ring $A$ like $\mathbb{Z}$ or $\mathbb{F}_p$, then for any coherent $G$-sheaf $\mathcal{F}$ on $X$ one can define an equivariant Euler–Poincaré characteristic $\chi(\mathcal{F}, G)$ in the Grothendieck group $K_0(A[G])$ of finitely generated *projective* $A[G]$-modules. It is an outstanding problem to compute these equivariant Euler characteristics. One of the most important results in this area is the following. Interestingly it relies heavily on results from class field theory.

**Theorem** (Pappas 1998)**.** *Let $G$ be an abelian group and let $\mathcal{X}$ be an arithmetic surface over $\mathbb{Z}$ with a free $G$-action. Then $2\chi(\mathcal{O}_{\mathcal{X}}, G) = 0$ in $K_0(\mathbb{Z}[G])/\langle \mathbb{Z}[G] \rangle$.*

## 10.3. Galois modules and $L$-functions

Let a finite group $G$ act on a projective, regular scheme $X$ of dimension $n$ defined over the finite field $\mathbb{F}_q$ and let $Y = X/G$. Let $\zeta(X, t)$ be the zeta-function of $X$. Let $e_X$ be the $l$-adic Euler characteristic of $X$. Recall that

$$\zeta(X, t) = \pm (q^n t^2)^{-e_X/2} \zeta(X, q^{-n} t^{-1}), \quad e_X \cdot n = 2 \sum_{0 \leqslant i \leqslant n} (-1)^i (n - i) \chi(\Omega^i_{X/\mathbb{F}_q})$$

the latter being a consequence of the Hirzebruch–Riemann–Roch theorem and Serre duality. It is well known that the zeta-function of $X$ decomposes into product of $L$-functions, which also satisfy functional equations. One can describe the constants in these functional equations by "taking isotypic components" in the analogue of the above expression for $e_X \cdot n/2$ in terms of equivariant Euler-Poincaré characteristics. The results that have been obtained so far do *not* use class field theory in any important way. So we are lead to formulate the following problem:

**Problem.** Using Parshin's adelic approach (sections 1 and 2 of Part II) find another proof of these results.

Let us note that one of the main ingredients in the work on these matters is a formula on $\varepsilon$-factors of T. Saito, which generalizes one by S. Saito inspired by Parshin's results.

## 10.4. Galois structure of class formations

Let $K$ be a number field and let $L$ be a finite Galois extension of $K$, with Galois group $G = \mathrm{Gal}(L/K)$. Let $S$ be a finite set of primes including those which ramify in $L/K$ and the archimedean primes. Assume that $S$ is stable under the $G$-action. Put $\Delta S = \ker(\mathbb{Z}S \to \mathbb{Z})$. Let $U_S$ be the group of $S$-units of $L$. Recall that $U_S \otimes \mathbb{Q}$ is isomorphic to $\Delta S \otimes \mathbb{Q}$ as $\mathbb{Q}[G]$-modules. There is a well known exact sequence

$$0 \to U_S \to A \to B \to \Delta S \to 0$$

with finitely generated $A, B$ such that $A$ has finite projective dimension and $B$ is projective. The latter sequence is closely related to the fundamental class in global class field theory and the class $\Omega = (A) - (B)$ in the projective class group $\mathrm{Cl}\,(\mathbb{Z}[G]$ is clearly related to the Galois structure of $S$-units. There are local analogues of the above sequence, and there are analogous sequences relating (bits) of higher $K$-theory groups (the idea is to replace the pair $(U_S,\ \Delta S)$ by a pair $(K_i'(\mathcal{O}),\ K_{i-1}'(\mathcal{O}))$).

**Problem.** Using complexes of $G$-modules (as in section 11 of part I) can one generalize the local sequences to higher dimensional fields?

For more details see [E].

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