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Geometry of pseudocharacters

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Abstract

If G is a group, a pseudocharacter $f\colon G \to \mathbb{R}$ is a function which is "almost" a homomorphism. If G admits a nontrivial pseudocharacter f, we define the space of ends of G relative to f and show that if the space of ends is complicated enough, then G contains a nonabelian free group. We also construct a quasi-action by G on a tree whose space of ends contains the space of ends of G relative to f. This construction gives rise to examples of "exotic" quasi-actions on trees.

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1 Introduction

Let G be a finitely presented group. Following the terminology in [12] a *quasi-character*[1] f on a group G is a real valued function on G which is a "coarse homomorphism" in the sense that the quantity $f(xy)-f(x)-f(y)$ is bounded. The quasicharacter f is a *pseudocharacter* if in addition f is a homomorphism on each cyclic subgroup of G. Any quasicharacter differs from some pseudocharacter by a bounded function on G. Brooks [6] gave the first examples where this pseudocharacter could not be chosen to be a homomorphism. Brooks' examples were on a free group, but his methods have since been generalized to give many examples of such "nontrivial" quasicharacters on groups [7, 9, 11, 3]. Interesting applications of such existence results can be found in [2] and [14]. For additional information on pseudocharacters we refer the reader to [15, 2, 12] and the references therein.

Our study of the geometry of pseudocharacters is partly inspired by the work of Calegari in [8], where it is shown that if a pseudocharacter on the fundamental group of a 3–manifold satisfies some simple geometric hypotheses, then the 3–manifold must satisfy a weak form of the Geometrization Conjecture. The current paper is partially an attempt to understand what happens when we are given a pseudocharacter which does *not* satisfy Calegari's hypotheses. Despite this motivation, we make no assumptions on the groups considered in this paper, except requiring that they be finitely presented.

Here is a brief outline of the paper. In Section 2 we define the set of ends of a group relative to a pseudocharacter and establish some basic properties. In Section 3 we make the distinction between uniform, unipotent and bushy pseudocharacters. As a kind of warm-up for the next section we show that a group admitting a bushy pseudocharacter contains a nonabelian free subgroup. In Section 4, we prove the main theorem:

Theorem 4.20 *If $f\colon G \to \mathbb{R}$ is a pseudocharacter which is not uniform, then G admits a cobounded quasi-action on a bushy tree.*

We obtain this quasi-action via an isometric action on a space quasi-isometric

[1] Several competing terminologies exist in the literature. What we call *quasi-characters* are often called *quasi-morphisms* [15, 2] or *quasi-homomorphisms* [3, 14]; what we call pseudocharacters are then called *homogeneous* quasi-morphisms or quasi-homomorphisms. The term "pseudocharacter" seems to originate in the papers of Faĭziev and Shtern (eg [10, 20]). The interest in functions (into normed groups) which are "almost homomorphisms" goes back at least to Ulam [22].

to a tree. A (possibly new) characterization of such spaces is given in Theorem 4.6. Finally we use a result of Bestvina and Fujiwara to obtain:

Theorem 4.29 *If G admits a single bushy pseudocharacter, then $H_b^2(G;\mathbb{R})$ and the space of pseudocharacters on G both have dimension equal to $|\mathbb{R}|$.*

Section 5 contains some examples involving negatively curved 3–manifolds. Specifically we show that all but finitely many Dehn surgeries on the figure eight knot have fundamental groups admitting bushy pseudocharacters. This gives the following:

Corollary 5.7 *There are infinitely many closed 3–manifold groups which quasi-act coboundedly on bushy trees but which admit no nontrivial isometric action on any $\mathbb{R}$–tree.*

This corollary can be thought of as an "irrigidity" result about quasi-actions on bushy trees, to be contrasted with the rigidity result of [18] about quasi-actions on *bounded valence* bushy trees.

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2 Definition of $E(f)$

2.1 $E(f,S)$ as a set

Definition 2.1 If G is a finitely presented group, then $f\colon G \to \mathbb{R}$ is a *pseudocharacter* if it has the following properties:

- $f(\alpha^n) = nf(\alpha)$ for all $\alpha \in G$, $n \in \mathbb{Z}$.
- $\delta f(\alpha,\beta) = f(\alpha) + f(\beta) - f(\alpha\beta)$ is bounded independent of α and β. We use $\|\delta f\|$ to denote the smallest nonnegative C so that $|\delta f(\alpha,\beta)| \leq C$ for all α, β in G.

We fix a group G and a pseudocharacter $f\colon G \to \mathbb{R}$. In order to better understand f, we will define a G–set $E(f)$, which may be thought of as the set of ends of G relative to f.

Let S be a finite generating set for G. For simplicity, we assume that there is a presentation $G = \langle S, R\rangle$ which is triangular, that is, every word in R has length three. It is not hard to show that any finitely presented group admits a finite triangular presentation. We will first define a set $E(f,S)$ and then show it is independent of S.

Definition 2.2 If S is a generating set for G, let $\epsilon_{f,S} = \sup_{s\in S}\{|f(s)|\} + \|\delta f\|$.

If $\Gamma(G,S)$ is the Cayley graph associated to the generating set S, we can extend f affinely over the edges of $\Gamma(G,S)$. Notice that $\epsilon_{f,S}$ gives an upper bound on the absolute value of the difference between $f(x)$ and $f(y)$ in terms of the distance in $\Gamma(G,S)$ between x and y. Namely, $|f(x) - f(y)| \leq \epsilon_{f,S} d_\Gamma(x,y)$, if d_Γ is the distance in the Cayley graph.

Definition 2.3 If $\phi\colon \mathbb{R}_+ \to \Gamma(G,S)$ is an infinite ray, we define the *sign* of ϕ to be

$$\sigma_f(\phi) = \begin{cases} +1 & \text{if } \lim_{t\to\infty} f\circ\phi(t) = \infty \\ -1 & \text{if } \lim_{t\to\infty} f\circ\phi(t) = -\infty \\ 0 & \text{otherwise.} \end{cases}$$

If f is understood, we simply write $\sigma(\phi)$. If w is some infinite word in the generators S, there is a path $\phi_w\colon \mathbb{R}_+ \to \Gamma(G,S)$ beginning at 1 and realizing the word. We define $\sigma(w) = \sigma(\phi_w)$. If g is a group element, we let $\sigma(g)$ be the sign of $f(g)$. Notice that if we pick a word w representing g then $\sigma(www\ldots) = \sigma(w^\infty) = \sigma(g)$.

We give two equivalent definitions of $E(f,S)$.

Definition 2.4 (Version 1)

$$E(f,S) = \big\{\phi\colon \mathbb{R}_+ \to \Gamma(G,S) \text{ continuous} \mid \sigma(\phi) \in \{+1,-1\}\big\}/\sim$$

We will say $\phi_1 \sim_C \phi_2$ if $\sigma(\phi_1) = \sigma(\phi_2)$ and for all D with $\sigma(\phi_1)D > C$ there is a path $\delta\colon [0,1] \to \Gamma(G,S)$ such that:

- $\delta(0) \in \phi_1(\mathbb{R}_+)$.
- $\delta(1) \in \phi_2(\mathbb{R}_+)$.
- $|f\circ\delta(t) - D| \leq C$ for all $t \in [0,1]$.

The path δ will be referred to as a *connecting path*. We say $\phi_1 \sim \phi_2$ if $\phi_1 \sim_C \phi_2$ for some C. This is an equivalence relation.

Definition 2.5 (Version 2)

$$E(f,S) = \{w = w_1 w_2 w_3 \dots \mid w_i \in S \cup S^{-1}\ \forall i \in \mathbb{Z} \text{ and } \sigma(w) \in \{+1,-1\}\}/\sim$$

We say $w = w_1 w_2 \dots \sim_C v = v_1 v_2 \dots$ if $\sigma(w) = \sigma(v)$ and for all D with $\sigma(w)D > C$ there is a word $d = d_1 \dots d_n$ in the letters $S \cup S^{-1}$ such that:

- $w_p d = v_p$ in G for some prefix w_p of w and some prefix v_p of v.
- $|f(w_p d_p) - D| \leq C$ for all prefixes d_p of d.

The word d will be referred to as a *connecting word*. We say $w \sim v$ if $w \sim_C v$ for some C. Again, this is an equivalence relation.

Lemma 2.6 *There is a canonical bijection between $E(f,S)$ (version 1) and $E(f,S)$ (version 2).*

Proof Let E_1 be the set described in Definition 2.4, and let E_2 be the set described in Definition 2.5. If $[\phi] \in E_1$, note that altering ϕ on any compact subset of $\mathbb{R}_+$ does not change its equivalence class. Thus we may assume $\phi(0)$ is the identity element of G. The equivalence class of ϕ is also left undisturbed by proper homotopies and arbitrary reparameterizations, and so we may assume that ϕ is a unit speed path with no backtracking. Such a representative ϕ determines an infinite word in the generators $w = w_1 w_2 w_3 \dots$ with each $w_i = \phi(i-1)^{-1}\phi(i) \in S \cup S^{-1}$. The reader may check that this recipe for building a word from a path gives a well-defined map from E_1 to E_2, and that this map is a bijection. ◻

Roughly speaking, if one thinks of the Cayley graph as divided up into thick slabs of group elements all sent to roughly the same real values, and two paths pass to infinity through the same slabs, we identify the paths.

Definition 2.7 If $[\phi] \in E(f,S)$ and $\sigma(\phi) = 1$ we say that $[\phi]$ is *positive*. If $\sigma(\phi) = -1$ we say that $[\phi]$ is *negative*. We denote by $E(f,S)^+$ the set of positive elements and by $E(f,S)^-$ the set of negative elements.

Remark 2.8 Notice that neither Definition 2.4 nor Definition 2.5 depend on anything but S and f up to multiplication by a nonzero real number. Consequently, $E(f,S)^+$ and $E(f,S)^-$ only depend on S and f up to multiplication by a positive real number. Thus in what follows we do not hesitate to scale f by a positive real number whenever convenient.

Example 2.9 If G is a free abelian group generated by S and f is any nontrivial homomorphism, then $E(f,S)$ contains precisely two elements, one positive and one negative. Thus $|E(f,S)^+| = |E(f,S)^-| = 1$.

Lemma 2.10 *The action of G on the Cayley graph induces an action on $E(f,S)$.*

Proof This is clearest looking at Definition 2.4. One must only check that $g\phi \sim g\phi'$ if $\phi \sim \phi'$, where ϕ and ϕ' are infinite rays in the Cayley graph with $\sigma(\phi)$ and $\sigma(\phi')$ nonzero. Suppose that $\phi \sim \phi'$. Then $\phi \sim_C \phi'$ for some $C > 0$. Let $C' = C + |f(g)| + \|\delta f\|$, and suppose that $\sigma(\phi)D > C'$. Then in particular $\sigma(\phi)D > C$, and so there is a connecting path δ with $\delta(0) \in \phi(\mathbb{R}_+)$, $\delta(1) \in \phi'(\mathbb{R}_+)$, and satisfying $|f \circ \delta(t) - D| \leq C$ for all $t \in [0,1]$. Let δ' be the same path, translated by g. Then $\delta'(0) \in g\phi(\mathbb{R}_+)$ and $\delta'(1) \in g\phi'(\mathbb{R}_+)$. Since $|f \circ \delta'(t) - (f \circ \delta(t) + f(g))| \leq \|\delta f\|$, we have $|f \circ \delta'(t) - D| \leq |f \circ \delta(t) + f(g) - D| + \|\delta f\| \leq |f \circ \delta(t) - D| + |f(g)| + \|\delta f\| \leq C + |f(g)| + \|\delta f\| = C'$. Thus if $\phi \sim_C \phi'$, then $g\phi \sim_{C'} g\phi'$, and so $g\phi \sim g\phi'$. □

2.2 Topology of $E(f,S)$

We next describe the topology on $E(f,S)$, by describing a basis of open sets in terms of Definition 2.4 above.

Definition 2.11 Let I be some interval in $\mathbb{R}$ of diameter bigger than $\epsilon_{f,S}$. Let B be some component of $f^{-1}(I) \subset \Gamma(G,S)$. Let C be some connected component of $\Gamma(G,S) \setminus B$. We define

$$U_{B,C} = \{[\phi] \in E(f,S) \mid \text{image}(\phi) \subset C\}$$

We make $E(f,S)$ a topological space with the collection of all such $U_{B,C}$ as a basis.

Note that if C and C' are distinct components of the complement of B, then $U_{B,C} \cap U_{B,C'}$ is empty.

It turns out that $E(f,S)$ is Hausdorff and totally disconnected. In fact, we will show the following:

Proposition 2.12 *There is a simplicial tree T and a map $i\colon E(f,S) \to \partial T$ which is a homeomorphism onto its image. (By ∂T we mean the Gromov boundary of T.)*

As noted in Remark 2.8, $E(f,S)$ is unchanged if we scale f by a nonzero real number. By scaling appropriately, we can ensure that $\epsilon_{f,S} < \frac{1}{4}$, and that $f^{-1}\big(\mathbb{Z}+\frac{1}{2}\big)$ contains no element of G. These assumptions will be made for the rest of the section.

Since we chose a triangular presentation of G, we may equivariantly add 2–simplices to $\Gamma(G,S)$ to obtain a simply connected 2–complex $\widetilde{K}$ corresponding to the presentation we started with. G acts on $\widetilde{K}$ with quotient K, where K is a one-vertex 2–complex with one edge for each element of S, and one triangular cell attached for each relation in our presentation. The function f may be extended affinely over the 2–simplices of $\widetilde{K}$ to give a function $f\colon \widetilde{K} \to \mathbb{R}$. Since $\tau = f^{-1}\big(\mathbb{Z}+\frac{1}{2}\big)$ misses the 0–skeleton of $\widetilde{K}$, and since we have scaled f so that $|f(v) - f(w)| \leq \epsilon_{f,S} < \frac{1}{4}$ whenever v and w are endpoints of the same edge, $f^{-1}\big(\mathbb{Z}+\frac{1}{2}\big)$ intersects each 2–cell either not at all or in a single normal arc. Thus τ is a union of possibly infinite tracks in $\widetilde{K}$. Each such track τ separates $\widetilde{K}$ into two components, and has a product neighborhood $\eta(\tau) = \tau \times \big(-\frac{1}{2},\frac{1}{2}\big)$ in the complement of the 0–skeleton of $\widetilde{K}$ (see Figure 1). As there may be vertices arbitrarily close to τ, the topological product

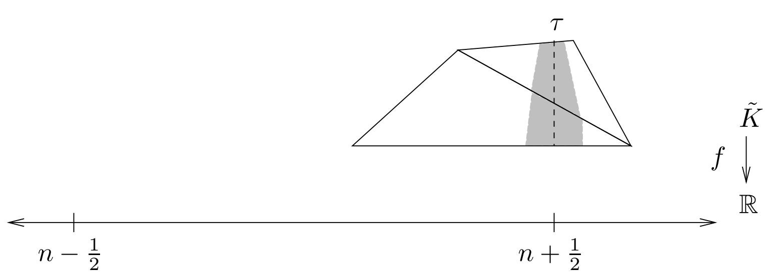

Figure 1: Each track τ has a product neighborhood.

neighborhood $\eta(\tau)$ must be allowed to vary in width from 2–cell to 2–cell and is not necessarily a component of $f^{-1}(I)$ for any interval I.

We obtain a quotient space T of $\widetilde{K}$ by smashing each component of $\eta(\tau)$ to an interval and each component of the complement of $\eta(\tau)$ to a point. Let $\pi\colon \widetilde{K} \to T$ be the quotient map. Clearly T is a simplicial graph. Since the preimage of each point under π is connected and $\widetilde{K}$ is simply connected, T must be simply connected. In particular, T is a tree. We refer to the preimage of an open edge of T as an *edge space* of $\widetilde{K}$ and to the preimage of a vertex as a *vertex space* of $\widetilde{K}$. We define a map $\overline{f}\colon T \to \mathbb{R}$ as follows. If v is a vertex of T, then the associated vertex space lies in $f^{-1}\big(n-\frac{1}{2}, n+\frac{1}{2}\big)$ for some n; we let $\overline{f}(v) = n$ and define $\overline{f}$ on edges by extending affinely. Note that

$|f(x) - \overline{f} \circ \pi(x)|$ is bounded independently of $x \in \widetilde{K}$.

Remark 2.13 Because $f^{-1}(\mathbb{Z} + \frac{1}{2})$ intersects any edge of $\widetilde{K}^1 = \Gamma(G, S)$ in at most a single point and any 2–cell in at most a single normal arc, each vertex space must contain vertices of $\widetilde{K}$, and any two vertices contained in the same vertex space are actually connected by a path in the intersection of $\widetilde{K}^1$ and the vertex space. Since therefore the components of $\Gamma(G,S) \setminus f^{-1}(\mathbb{Z} + \frac{1}{2})$ are in one to one correspondence with the vertex spaces, the particular pattern of 2–simplices added to form $\widetilde{K}$ is unimportant to the structure of T.

We now define a map from $E(f, S)$ to ∂T. Let $v_0 \in T$ be the vertex whose associated vertex space contains the identity element of G. Since T is a tree, ∂T can be identified with the set of geodesic rays in T starting at v_0. Given some element $[\phi] \in E(f, S)$, we will associate such a geodesic ray. First notice that we may assume that the image of ϕ contains $1 \in G$. Now consider the image of $\pi \circ \phi$ in T. By our choice of ϕ, this image contains v_0.

Lemma 2.14 *Suppose that $[\phi] \in E(f, S)$ and that the image of ϕ contains 1. Then the image of $\pi \circ \phi$ contains a unique geodesic ray starting at v_0.*

Proof Since $\lim_{t\to\infty} \overline{f} \circ \pi \circ \phi(t) = \lim_{t\to\infty} f \circ \phi(t) = \pm\infty$, and $|\overline{f}(v) - \overline{f}(w)| \leq d(v, w)$ for v, w in T, $\pi \circ \phi$ must eventually leave any finite diameter part of T. Let $B_R = \{x \in T \mid d(x, \pi \circ \phi(0)) < R\}$ for $R > 0$. For any $R \geq 0$ there is some t so that $\pi \circ \phi([t, \infty))$ lies in a single component C of $T \setminus B_R$. Let x_R be the point in C closest to $\pi \circ \phi(0)$. Then $\gamma\colon \mathbb{R}_+ \to T$ given by $\gamma(t) = x_t$ is the unique geodesic ray starting at $\pi \circ \phi(0) = v_0$ which is contained in the image of $\pi \circ \phi$. □

We define a map $i\colon E(f, S) \to \partial T$ by sending $[\phi]$ to this ray.

Lemma 2.15 *The map i is well-defined.*

Proof Suppose that $[\phi] = [\phi']$, but images of the paths $\pi \circ \phi$ and $\pi \circ \phi'$ contain distinct infinite rays r and r'. For clarity, we assume that $[\phi]$ is positive. The proof for $[\phi]$ negative is much the same.

We may modify ϕ and ϕ' so that the image of $\pi \circ \phi$ is r and the image of $\pi \circ \phi'$ is r'. Furthermore, we may adjust ϕ and ϕ' so that r intersects r' in a single point, v. Let $N_1 = \overline{f}(v)$. Since $\phi \sim \phi'$ there is some $C > 0$ so that $\phi \sim_C \phi'$. Let $N_2 > C$.

Let $N = |N_1| + N_2 + 1$. Since $N > C$, there is some $t \geq 0$, some $t' \geq 0$, and some path $\delta\colon [0,1] \to \Gamma(G,S)$ so that $\delta(0) = \phi(t)$, $\delta(1) = \phi'(t')$, and $|f \circ \delta(x) - N| \leq C$ for all $x \in [0,1]$. But this path δ must necessarily pass through $\pi^{-1}(v)$, and so there is some $x \in [0,1]$ so that $f \circ \delta(x) < N_1 + \frac{1}{2}$. But this implies that $|f \circ \delta(x) - N| > N_2 > C$, a contradiction. □

Lemma 2.16 *The map i is injective.*

Proof Suppose $i([\phi]) = i([\phi'])$. Let γ be a unit speed geodesic ray in T contained in $\pi \circ \phi(\mathbb{R}_+) \cap \pi \circ \phi'(\mathbb{R}_+)$. We have $\lim_{t\to\infty} \overline{f}\gamma(t) = \lim_{t\to\infty} f \circ \phi(t) = \lim_{t\to\infty} f \circ \phi'(t)$, so $\sigma(\phi) = \sigma(\phi')$. As in the last lemma, we assume for clarity that ϕ and ϕ' are positive.

We may assume that $\gamma(0)$ is a vertex of T, and so $\gamma(k)$ is a vertex of T for k any nonnegative integer. Let $N = \overline{f}(\gamma(0))$. By truncating γ if necessary, we may assume that N is positive and is the smallest value taken by $\overline{f} \circ \gamma$. For every integer n between N and ∞, there is a vertex v_n in the image of γ with $\overline{f}(v_n) = n$. Both ϕ and ϕ' must pass through the vertex space $V_n \subset \widetilde{K}$ associated to v_n.

For each $n \in \mathbb{Z}$, $n \geq N$, pick points x_n and x'_n on the intersections of the paths ϕ and ϕ' with V_n. Note that by Remark 2.13, x_n and x'_n are connected by a path δ_n in the intersection of $\widetilde{K}^1$ with V_n.

We claim that $\phi \sim_C \phi'$ for $C = N + 2$. Suppose $D > C$, and let $[D]$ be the integer part of D. Since $[D] > N$, there are points $x_{[D]}$ and $x'_{[D]}$ on the images of ϕ and ϕ' respectively. These points are connected by a path $\delta_{[D]}$ which lies entirely in the intersection of $\widetilde{K}^1$ with $V_{[D]}$. Since the image of $\delta_{[D]}$ lies entirely inside $V_{[D]}$, $|f \circ \delta(t) - D| < \frac{3}{2} < C$ for all t. Thus $\phi \sim_C \phi'$. In particular, $\phi \sim \phi'$. □

The following two lemmas complete the proof of Proposition 2.12.

Lemma 2.17 *The map i is continuous.*

Proof Recall that the topology on ∂T can be described by a basis of open sets, as follows: Let e be any open edge of T, and let T' be one of the two components of $T \setminus e$. There is a natural inclusion of $\partial T'$ into ∂T; the image of $\partial T'$ is a basic open set. The topology on ∂T is generated by such sets.

Let $U = \partial T'$ be an element of the basis described above. Let v be the unique vertex in $(T \setminus T') \cap \overline{e}$, where $\overline{e}$ is the closed edge whose interior is e. If

$N(v) = \{x \in T \mid d(x,v) < \frac{1}{2}\}$, then $B = \pi^{-1}(N(v))$ is a path component of the preimage $f^{-1}(\overline{f}(v) - \frac{1}{2}, \overline{f}(v) + \frac{1}{2})$, as it contains the entire vertex space corresponding to v and half of each edge space adjacent to v. Furthermore, $\pi^{-1}(T')$ lies entirely in the complement of B. Let C be the component of the complement of B containing $\pi^{-1}(T')$, and let $U_{B,C}$ be the basic open set in $E(f,S)$ defined by C (as in Definition 2.11). Then clearly $i^{-1}(U) = U_{B,C}$. Thus i is continuous. □

Lemma 2.18 *The map i is an open map onto its image, topologized as a subset of ∂T.*

Proof If $U_{B,C}$ be given as in Definition 2.11, then $\pi(B)$ is some connected subset of T. Thus $\pi(B)$ is contained in some minimal (possibly infinite) subtree Z. Because f and $\overline{f} \circ \pi$ are boundedly different from one another, and every point in Z is distance at most 1 from $\pi(B)$, any geodesic in T representing an element of $i(E(f))$ must eventually leave Z forever. If e is any edge with precisely one endpoint in Z, then let T_e be the component of $T \setminus \mathrm{Int}(e)$ which does not contain Z.

Claim 2.19 *$\partial T_e \cap i(E(f,S)) \subset i(U_{B,C})$ or $\partial T_e \cap i(U_{B,C}) = \emptyset$.*

Proof To prove the claim, suppose there is some point x in $\partial T_e \cap i(U_{B,C})$ and suppose that $y \in \partial T_e \cap i(E(f,S))$. Both $i^{-1}(x)$ and $i^{-1}(y)$ are represented by paths ϕ_x and ϕ_y whose images lie entirely in $\pi^{-1}(T_e)$. But since $B \subset \pi^{-1}(Z)$, the space $\pi^{-1}(T_e)$ is contained entirely in a single complementary component of B. Since $x \in i(U_{B,C})$, this complementary component is C, and so $[\phi_y] \in U_{B,C}$. Therefore $y \in i(U_{B,C})$. □

By the claim, $i(U_{B,C})$ can be expressed as a union of basic open sets in $i(E(f,S))$. The lemma follows. □

Corollary 2.20 *$E(f,S)$ is metrizable. In particular, $E(f,S)$ is Hausdorff.*

2.3 Invariance under change of generators

We have been using f to refer both to the pseudocharacter and to its extension to $\Gamma(G,S)$. For this subsection we need to deal with distinct generating sets, so we will temporarily refer to the extension of f to a particular Cayley graph $\Gamma(G,S)$ as f_S.

Let T be another finite triangular generating set for G. We choose equivariant maps $\tau\colon \Gamma(G,S) \to \Gamma(G,T)$ and $\upsilon\colon \Gamma(G,T) \to \Gamma(G,S)$ which are the identity on G and send each edge to a constant speed path. Let N be the maximum length of the image of a single edge under τ or υ. It is not hard to establish that both τ and υ are continuous $(N, 2N+2)$ quasi-isometries which are quasi-inverses of one another. In fact, we have

$$d(x,y)/N - (N + 1/N) \leq d(\tau x, \tau y) \leq N d(x,y)$$

and similar inequalities for υ.

We define a map $\overline{\tau}\colon E(f,S) \to E(f,T)$ by $\overline{\tau}([\phi]) = [\tau \circ \phi]$, and define $\overline{\upsilon}$ similarly.

Lemma 2.21 *The maps $\overline{\tau}$ and $\overline{\upsilon}$ are well-defined.*

Proof Suppose $[\phi] = [\phi'] \in E(f,S)$. Then $\phi \sim_C \phi'$ for some $C > 0$. If δ is a connecting path between ϕ and ϕ', then $\tau \circ \delta$ gives a connecting path between $\tau \circ \phi$ and $\tau \circ \phi'$. The only trouble is that f_T may vary on $\tau \circ \delta$ more than f_S varies on δ. On the other hand, $\tau \circ \delta$ never gets further than $\frac{N}{2}$ (in $\Gamma(G,T)$ from the group elements contained in the image of δ, and so if $\phi \sim_C \phi'$, then $\tau \circ \phi \sim_{C'} \tau \circ \phi'$ for $C' = C + \frac{N}{2}\epsilon_{f,T}$. Thus $[\tau \circ \phi] = [\tau \circ \phi']$. The proof for $\overline{\upsilon}$ is identical. □

Lemma 2.22 *The maps $\overline{\tau}$ and $\overline{\upsilon}$ are bijections.*

Proof $\overline{\tau}$ and $\overline{\upsilon}$ are inverses of one another. □

Lemma 2.23 *The maps $\overline{\tau}$ and $\overline{\upsilon}$ are open.*

Proof Let $U_{B,C} \subset E(f,S)$ be a basic open set. There is some interval $[a,b] \subset \mathbb{R}$ so that B is a connected component of $f_S^{-1}[a,b] \subset \Gamma(G,S)$ and C is a component of $\Gamma(G,S) \setminus B$. We wish to show that $\overline{\tau}(U_{B,C})$ is open in $E(f,T)$.

Since τ is continuous, $\tau(B)$ is connected. As no edge of B has image of length more than N, $f_T \circ \tau(B) \subset (a - N\epsilon_{f,T}, b + N\epsilon_{f,T})$. Thus $\tau(B) \subset B'$ a connected component of $f_T^{-1}[a - (N+N^2+2)\epsilon_{f,T}, b + (N+N^2+2)\epsilon_{f,T}]$. We have chosen the constants here so that the distance between any point in the complement of B' and any point in $\tau(B)$ is at least N^2+2.

We claim that $\overline{\tau}(U_{B,C})$ is a union of open sets of the form $U_{B',C'}$ where C' is a component of the complement of B'. The claim follows if each such $U_{B',C'}$ is either contained in or disjoint from $\overline{\tau}(U_{B,C})$.

Suppose $[\phi_1]$ and $[\phi_2]$ are in $U_{B',C'}$. Since $\overline{\tau}$ is onto and removing an initial segment does not change the equivalence class of a path, we may suppose $\phi_i = \tau \circ \psi_i$, where each ψ_i has image entirely in the complement of B and each ϕ_i has image entirely in C'. Thus there is a path δ in C' connecting $\phi_1(0)$ to $\phi_2(0)$. The path $\sigma \circ \delta$ therefore runs from $\psi_1(0)$ to $\psi_2(0)$.

If $\psi_1(0)$ and $\psi_2(0)$ were in different components of $\Gamma(G,S) \setminus B$, then $\sigma \circ \delta$ would pass through B. But then $\sigma \circ \delta$ must pass through a vertex v of B (since B is not contained in an edge). Because

$$d(x,y)/N - (N + 1/N) \leq d(\sigma x, \sigma y)$$

the distance between the path δ and $\tau(v)$ is less than $N^2 + 1$. But this contradicts the assertion that δ lies entirely outside of B'.

Since ψ_1 and ψ_2 are infinite paths in the same component of the complement of B, either both $[\psi_1]$ and $[\psi_2]$ are in $U_{B,C}$, or neither is. Likewise, either both $[\phi_1]$ and $[\phi_2]$ are in $\overline{\tau}(U_{B,C})$ or neither is, establishing the claim.

Again, the proof for $\overline{\upsilon}$ is identical. □

Corollary 2.24 *$\overline{\tau}$ is a homeomorphism.*

3 The action of G on $E(f)$

3.1 Dynamics

Let S be a generating set for G. The group G acts on $E(f) = E(f,S)$ via the action on the Cayley graph (Lemma 2.10).

Lemma 3.1 *G acts on $E(f)$ by homeomorphisms.*

Proof Let $U_{B,C}$ be a basic open set, so that B is a connected component of $f^{-1}[a,b] \subset \Gamma(G,S)$ and C is a connected component of the complement of B. Let $g \in G$. Note that $g(B) \subset f^{-1}[a + f(g) - \|\delta f\|, b + f(g) + \|\delta f\|]$. Let B' be the connected component of $f^{-1}[a + f(g) - \|\delta f\|, b + f(g) + \|\delta f\|]$ containing $g(B)$. If C' is a complementary component of B' we wish to claim that either $U_{B',C'} \subset g(U_{B,C})$ or $U_{B',C'}$ is disjoint from $g(U_{B,C})$. It will follow that g acts on $E(f)$ by an open map. Since g^{-1} must do likewise, it follows that G acts by homeomorphisms.

To establish the claim, suppose that $[\phi_1]$ and $[\phi_2]$ are in $U_{B',C'}$. We may assume that the images of ϕ_1 and ϕ_2 lie entirely in C'. It follows that $g^{-1}\phi_1$ and $g^{-1}\phi_2$ have image entirely in the same complementary component of B, and the claim is established. □

Now that we have established that G acts by homeomorphisms of $E(f)$ we look more closely at the dynamics of this action.

Definition 3.2 If a and r are fixed points of a group element g, we say that a is *attracting* and r is *repelling* if for any neighborhood U of a and any neighborhood V of r, we have $g^n(E(f) \setminus V) \subset U$ for all n sufficiently large.

Lemma 3.3 *If $g \in G$ and $f(g) \neq 0$, then g has exactly two fixed points in $E(f)$, one attracting and one repelling.*

Proof It is convenient to use Definition 2.5 here. Let w be a word in the letters $S \cup S^{-1}$ representing g. Since $f(g) \neq 0$, the words $[w^\infty] = [www\ldots]$ and $[\overline{w}^\infty] = [\overline{www}\ldots]$ are elements of $E(f)$. Both are clearly fixed by g.

Let U be an open set containing $[w^\infty]$, and let V be an open set containing $[\overline{w}^\infty]$. Without loss of generality, both U and V are basic open sets $U = U_{B,C}$ and $V = V_{D,E}$.

Let γ be the bi-infinite line made by taking the path from 1 to g described by w and translating it by powers of g. The group $\langle g\rangle \subset G$ acts on γ as $\mathbb{Z}$ acts on $\mathbb{R}$. Figure 2 shows approximately how all this might look in G.

Since f restricted to γ is a continuous quasi-isometry, $D \cap \gamma$ is a compact set. Thus there is some N_1 so that $g^n(D\cap\gamma)$ is in C for all $n > N_1$. The barrier spaces B and C are components of the preimages of closed intervals under f. So in particular there are intervals $[a,b]$ and $[c,d]$ so that $f(B) = [a,b]$ and $f(D) = [c,d]$. It is easy to check that $f(g^nD) \subset [c+nf(g)-\|\delta f\|, d+nf(g)+\|\delta f\|]$. Thus there is some N_2 so that $g^nD\cap B$ is empty for any $n > N_2$. Let $N = \max\{N_1, N_2\}$. It is clear that $g^n(D) \subset C$ for any $n > N$.

Let $e \in E(f) \setminus V$, and suppose that $n > N$. We claim that $g^n e \in U_{B,C}$. For let $\phi\colon G \to \Gamma(G,S)$ be a path so that $[\phi] = e$. We may assume that ϕ maps entirely into the complement of D, and thus that $g^n\phi$ maps entirely into the complement of g^nD. But if $g^n[\phi] \notin U_{B,C}$, then eventually the image of $g^n[\phi]$ leaves the half-space C. Since g^nD is contained entirely in C this means that $g^n\phi$ maps entirely into the component of the complement of g^nD which also contains B. In other words, $g^n\phi$ maps into the same component of the complement of g^nD as $\overline{w}^\infty$ does, namely g^nE. But this implies that $e \in U_{D,E}$, a contradiction to our original choice of D and E. □

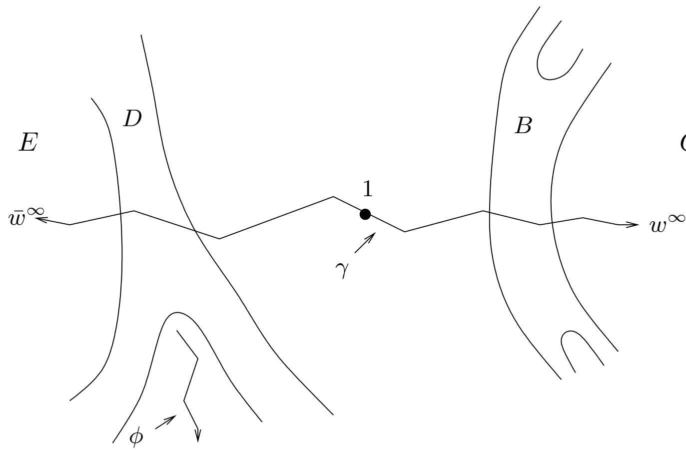

Figure 2: Possible arrangement of the barrier spaces

3.2 The bushy case

Definition 3.4 Let $E(f)^+ \subset E(f)$ be the set of positive elements of $E(f)$, and let $E(f)^-$ be the set of negative elements.

Remark 3.5 So long as there exists some g with $f(g) \neq 0$, then $E(f)^+$ and $E(f)^-$ are nonempty. Indeed, if w is any word representing g, then the infinite word $w^\infty = www\dots$ determines an element of $E(f)^\pm$, depending on whether $f(g)$ is positive or negative. Similarly the infinite word $\overline{w}^\infty = \overline{www}\dots$ determines an element of $E(f)^\mp$. Neither of these elements actually depends on w; we abuse notation slightly by writing them as $[g^\infty]$ and $[g^{-\infty}]$ respectively.

Definition 3.6 Let f be a pseudocharacter. If $|E(f)| = 2$ we say f is *uniform*. If $|E(f)^+| = 1$ or $|E(f)^-| = 1$ but f is not uniform, we say f is *unipotent*. Otherwise we say that f is *bushy*.

In [8], Calegari shows that if f is uniform and G is the fundamental group of a closed irreducible 3–manifold, then G satisfies the Weak Geometrization Conjecture. Thus one of our goals is to give information about what happens if f is not uniform.

Remark 3.7 The terminology here is slightly different from [8]. What is here called *uniform* is called *weakly uniform* in [8]. For f to be uniform in the sense of [8], its coarse level sets must be coarsely simply connected.

Following [8], we define an *unambiguously positive* element of G to be an element g with $f(g) > \|\delta f\|$. Note that if g is unambiguously positive and h is any element of G, then $f(hg) > f(h)$. If S is any triangular set of generators, we may alter S so that it contains an unambiguously positive element and is still triangular. By Corollary 2.24 this has no effect on the G–set $E(f)$. It is convenient in what follows to assume that S contains an unambiguously positive element. In this case we say that the generating set S is *unambiguous*.

Lemma 3.8 *If $E(f)$ is bushy, then there are group elements g_1, g_2, and g_3 so that $[g_1^\infty] \neq [g_2^\infty]$ and $[g_1^{-\infty}] \neq [g_3^{-\infty}]$ and $f(g_i) > 0$ for $i \in \{1,2,3\}$.*

Proof Let g_1 be an unambiguously positive element of S, where S is some fixed unambiguous triangular generating set. Then $[g_1^\infty] \in E(f)^+$ (see Remark 3.5 above). By assumption there is some $\phi\colon \mathbb{R}_+ \to \Gamma(G,S)$ with $[\phi] \in E(f)^+$ but $[\phi] \neq [g_1^\infty]$. We may assume that $\phi(0) = 1$. There is some $M > 0$ so that $f \circ \phi(t) > -M$ for all t.

Let B_R be the component of $f^{-1}[-R,+R]$ containing 1. If R is sufficiently large, then B_R always separates $[g_1^\infty]$ from $[\phi]$ in the sense that the two paths g_1^∞ and ϕ are eventually in different components of the complement of B_R.

We choose R large enough so that B_R separates $[g_1^\infty]$ from $[\phi]$ and so that R is much larger than M or $\|\delta f\|$. We may also choose R so that ϕ crosses the frontier of B_R in an edge of $\Gamma(G,S)$. Then the first group element h which ϕ passes through after leaving B_R has $f(h) > R$. Let $g_2 = hg_1^N$ where $N > \frac{99R}{f(g)-\|\delta f\|}$, so that $f(g_2) > 100R$. See Figure 3. We claim that $[g_2^\infty]$ is separated from $[g_1^\infty]$ by B_R. We can represent g_2 by a word $w = w_p g_1^N$ where w_p is just the word traversed by the initial part of ϕ. Note that $f \circ \phi$ never decreases by more than $2R$ on this initial segment. Let $\psi\colon \mathbb{R}_+ \to \Gamma(G,S)$ be the path representing $[g_2^\infty]$ which traverses the infinite word w^∞ at unit speed starting at 1. If ψ were to cross back over B_R after getting to g_2, we would have to have $f \circ \psi(t) < R$ for some $t > \text{length}(w)$. But since $f \circ \psi$ can decrease by no more than $2R$, this is impossible.

Thus $[g_2^\infty] \neq [g_1^\infty]$ and both $f(g_1)$ and $f(g_2)$ are positive. The proof of the existence of g_3 is almost identical. □

Theorem 3.9 *Let G be a finitely presented group. If there is a bushy pseudocharacter on G, then G contains a non-abelian free subgroup.*

Proof Let $f\colon G \to \mathbb{R}$ be a bushy pseudocharacter. To prove the theorem, it suffices to find g and g' in $G \backslash f^{-1}(0)$ with disjoint fixed point sets in $E(f)$. If we

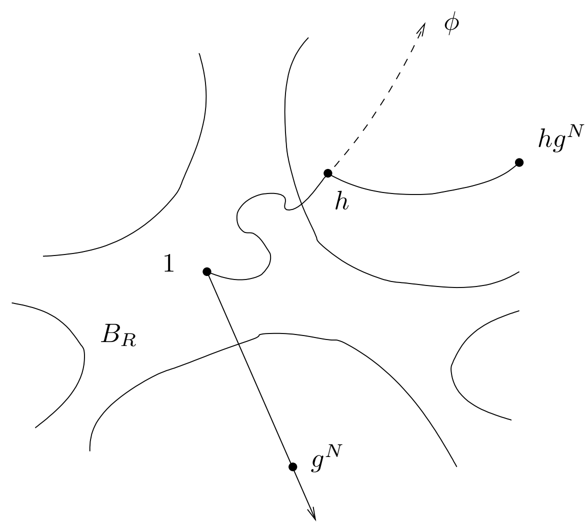

Figure 3: ϕ may wiggle around a bit inside B_R but the word representing $g_2 = hg^N$ is still "coarsely monotone" with respect to f.

can find such elements, then the Ping-Pong Lemma (see for instance [5, p467]) and the dynamics described in Lemma 3.3 ensure that high enough powers of these elements generate a free group.

Let g_1, g_2, and g_3 be as in the proof of Lemma 3.8. By taking powers we may assume that $f(g_i) > \|\delta f\|$ for $i \in \{1,2,3\}$. If no two of these have disjoint fixed point sets, then we must have $[g_3^\infty] = [g_1^\infty] = A$ and $[g_2^{-\infty}] = [g_1^{-\infty}] = B$. But then $g_2 g_3(A) \neq A$ and $g_2 g_3(B) \neq B$, and so we may set $g = g_1$ and $g' = g_2 g_3$. These clearly have disjoint fixed point sets. Furthermore $f(g) = f(g_1) > 0$ and $f(g') \geq f(g_2) + f(g_3) - \|\delta f\| > 0$. □

4 Quasi-actions on trees

In this section, we show that G acts on a Gromov hyperbolic graph quasi-isometric to a simplicial tree Γ, and that $E(f)$ embeds in the ends of Γ. If f is not uniform, this implies that G quasi-acts on the bushy tree Γ in the sense given in [18]:

Definition 4.1 A (K,C)*–quasi-isometry* is a (not necessarily continuous) function $q\colon X \to Y$ between metric spaces so that the following are true:

(1) For all $x_1,\ x_2 \in X$

$$d(x_1, x_2)/K - C \leq d(q(x_1), q(x_2)) \leq Kd(x_1, x_2) + C.$$

(2) The map q is *coarsely onto*, that is, every $y \in Y$ is distance at most C from some point in $q(X)$.

Definition 4.2 A (K, C)–*quasi-action* of a group G on a metric space X is a map $A\colon G \times X \to X$, denoted $A(g, x) \mapsto gx$, so that the following hold:

(1) For each g, $A(g, -)\colon G \to G$ is a (K, C) quasi-isometry.

(2) For each $x \in X$ and $g,\ h \in G$, we have $d(A(g, A(h, x)), A(gh, x)) \leq C$. (In other words, $d(g(hx), (gh)x) \leq C$.)

We call a quasi-action *cobounded* if for every $x \in X$, the map $A(-, x)\colon G \to X$ is coarsely onto.

Definition 4.3 Two quasi-actions $A_1\colon G \times X \to X$ and $A_2\colon G \times Y \to Y$ are called *quasi-conjugate* if there is a quasi-isometry $f\colon X \to Y$ so that for some $C \geq 0$ we have $d(f(A_1(g, x)), A_2(g, f(x)) \leq C$ for all $x \in X$. The map f is called a *quasi-conjugacy*.

In contrast to the quasi-actions discussed in [18], the quasi-actions on trees arising from pseudocharacters are not in general quasi-conjugate to actions on trees. We discuss "exotic" quasi-actions on trees further in Section 5.

4.1 Spaces Quasi-isometric to Trees

It is helpful to develop a characterization of geodesic metric spaces quasi-isometric to simplicial trees. We will call a geodesic space a *quasi-tree* if it is quasi-isometric to some simplicial tree. One reason to be interested in quasi-trees is the following observation, which was previously known to Kevin Whyte and probably to Gromov and others:

Proposition 4.4 *Any quasi-action on a geodesic metric space X is quasi-conjugate to an action on some connected graph quasi-isometric to X. Conversely, any isometric action on a geodesic metric space quasi-isometric to X is quasi-conjugate to some quasi-action on X.*

Sketch proof Suppose we have a (K,C)–quasi-action of a group G on the space X. Let Y be a graph with vertex set equal to $G \times X$, and connect (g,x) to (g',x') with an edge (of length one) whenever there is some $h \in G$ so that $d((hg)x,(hg')x') < 2C$. Define an action of G on the vertices of Y by $g(h,x) = (gh,x)$. Note that two vertices connected by an edge will always be mapped to two vertices connected by an edge, so this action extends to an isometric action on Y. Let $f\colon X \to Y$ be the function $f(x) = (1,x)$. It is not too hard to show that f quasi-conjugates the original quasi-action on X to the action on Y.

Conversely, suppose that X and Y are quasi-isometric spaces, and suppose $q\colon Y \to X$ and $p\colon X \to Y$ are (K,C)–quasi-isometries which are C–quasi-inverses of one another (that is, $d(y,p(q(y)))$ and $d(x,q(p(x)))$ are bounded above by C for all $y \in Y$ and $x \in X$). Given an isometric action of G on Y, it is straightforward to check that $A\colon G \times X \to X$ given by $A(g,x) = q(g(p(x)))$ is a $(K^2, KC + C)$–quasi-action. □

In particular, any quasi-action on a simplicial tree is quasi-conjugate to an isometric action on a quasi-tree and any isometric action on a quasi-tree is quasi-conjugate to a quasi-action on a simplicial tree.

The following lemma is well known (see, for example [5, p401]):

Lemma 4.5 *For all $K \geq 1$, $C \geq 0$, and $\delta \geq 0$ there is an $R(\delta,K,C)$ so that:*

If X is a δ–hyperbolic metric space (e.g. a quasi-tree), γ is a (K,C)–quasi-geodesic segment in X, and γ' is a geodesic segment with the same endpoints, then the images of γ' and γ are Hausdorff distance less than R from one another.

Theorem 4.6 *Let Y be a geodesic metric space. The following are equivalent:*

(1) *Y is quasi-isometric to some simplicial tree Γ.*

(2) (Bottleneck Property) *There is some $\Delta > 0$ so that for all x, y in Y there is a midpoint $m = m(x,y)$ with $d(x,m) = d(y,m) = \frac{1}{2}d(x,y)$ and the property that any path from x to y must pass within less than Δ of the point m.*

Proof
(1) ⇒ (2) Let $q\colon Y \to \Gamma$ be a (K,C)–quasi-isometry, where Γ is a simplicial tree. Note that since Γ is 0–hyperbolic, and Y is quasi-isometric to Γ, Y is

δ–hyperbolic for some δ. Let x and y be two points of Y, joined by some geodesic segment γ. Let m be the midpoint of γ, and suppose that α is some other path from x to y.

The image of a path under a (K,C)–quasi-isometry is a C–quasi-path. In other words, though the path need not be continuous, it can make "jumps" of length at most C. Therefore any point on the unique geodesic σ from $q(x)$ to $q(y)$ in Γ is no more than $\frac{C}{2}$ from the image of $q \circ \alpha$. Furthermore $q \circ \gamma$ is a (K,C)–quasi-geodesic. Thus by Lemma 4.5, the distance from $q(m)$ to σ is less than $R = R(\delta, K, C)$.

Let p be the point on σ closest to $q(m)$. There is some point $z \in Y$ on α so that $d(q(z), p) \leq \frac{C}{2}$. Since $d(p, q(m)) < R$ we have

$$d(q(z), q(m)) < C/2 + R$$

which implies

$$d(z, m) < K(C/2 + R) + C.$$

In other words, the path α must pass within $K\left(\frac{C}{2} + R\right) + C$ of the point m, so we may set $\Delta = K\left(\frac{C}{2} + R\right) + C$.

(2) $\Rightarrow$ (1) Given a geodesic metric space with the Bottleneck Property, we inductively construct a simplicial tree and a quasi-isometry from it to Y. At each stage of the construction we have a map $\beta_k \colon \Gamma_k \to Y$ where Γ_k is a tree of diameter $2k$. We let V_k be the image of the vertices of Γ_k under β_k, and let N_k be a large neighborhood of V_k. We refer to the set of path components of the complement of N_k as $\mathcal{C}_{k+1}$. Each element of $\mathcal{C}_{k+1}$ gives rise to a vertex in $\Gamma_{k+1} \setminus \Gamma_k$.

Step 0 Let $R = 20\Delta$. Pick some base point $* \in Y$. We set $V_0 = \{*\}$, and Γ_0 equal to a single point p_*. For each i there will be a natural identification of V_i with the vertices of Γ_i. We define $\beta_0 \colon \Gamma_0 \to Y$ so that the image of β_0 is $*$. We define $\mathcal{C}_0 = \{Y\}$.

Step k Let $k \geq 1$, and define $d_i \colon Y \to \mathbb{R}$ by $d_k(x) = d(x, V_{k-1})$. Let $N_{k-1} = \{x \in Y \mid d_i(x) < R\}$ and let $\mathcal{C}_k$ be the set of path components of $d_i^{-1}([R, \infty))$. If $C \in \mathcal{C}_k$, let $\mathrm{Front}(C) = \{x \in C \mid d_i(x) = R\}$. Because d_i is continuous and Y is path-connected, $\mathrm{Front}(C)$ is nonempty. We pick some $v_C \in \mathrm{Front}(C)$ for each C in $\mathcal{C}_k$, and let $U_k = \{v_C \mid C \in \mathcal{C}_k\}$.

The new set V_k is equal to $V_{k-1} \cup U_k$. To construct Γ_k, we add one vertex p_v to Γ_{k-1} for each $v \in U_k$. The point v is contained in exactly one element of $\mathcal{C}_{k-1}$, and this element contains exactly one element w of V_{k-1}. We connect the new vertex p_v to the old vertex p_w by a single edge. The map β_k is defined to be equal to β_{k-1} on Γ_{k-1}, and is extended to map a new edge between p_w and p_v to a geodesic segment in Y joining w to v. This completes Step k. See Figure 4 for an example of what this might look like.

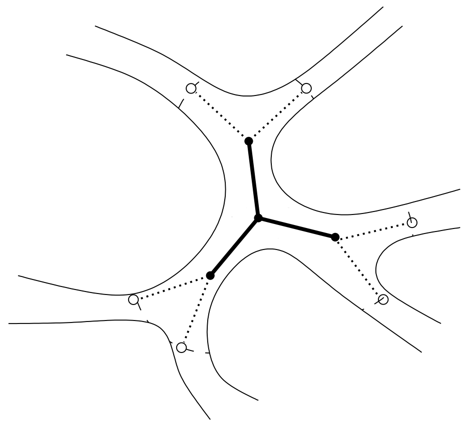

Figure 4: A stage in the construction of Γ. The image of Γ_{k-1} is a thick tripod. The elements of U_k are open circles, and the new edges of Γ_k are mapped to the dotted segments.

After the induction is completed, we have a map $\beta\colon \bigcup_k \Gamma_k = \Gamma \to Y$ which is defined to be equal to β_k on each Γ_k. The image of the 0–skeleton of Γ in Y is $V = \bigcup_k V_k$. We will show that $\beta\colon \Gamma \to Y$ is a quasi-isometry by showing that β restricted to the 0–skeleton of Γ is a quasi-isometry. We will use the following lemma:

Lemma 4.7 *Let $v \in U_i = V_i \setminus V_{i-1}$, and suppose that p_v is connected to $p_w \in \Gamma_{i-1}$ by an edge. Then the following assertions hold:*

(1) $R \leq d(v,w) \leq R + 6\Delta$

(2) *If $v \in C \in \mathcal{C}_i$ and $p \in \mathrm{Front}(C)$, then $d(v,p) \leq 6\Delta$.*

Proof We prove both assertions simultaneously by induction.

Step 1 If $i = 1$, then $V_{i-1} = V_0$ is a single point, so assertion (1) holds with $d(v,w) = R$.

We prove assertion (2) by way of contradiction. Namely, suppose that $d(v,p) > 6\Delta$. There is a point m equidistant from v and p so that any path from v to p passes within less than Δ of m. As there is a path β in C connecting v to

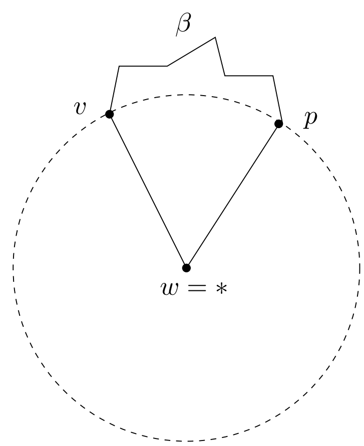

Figure 5: Assertion (2) Step 1. Where is the midpoint between p and v?

p, we must have $d(m,C) = \inf_{c \in C}\{d(m,c)\} < \Delta$. The situation is shown in Figure 5.

On the other hand, there is another path from v to p consisting of a geodesic segment $[v,w]$ from v to w and another geodesic segment $[w,p]$ from w back to p. By the Bottleneck property, m must lie inside a Δ–neighborhood of this path. In other words, there is some point z on the path $[v,w] \cup [w,p]$ so that $d(z,m) < \Delta$. Since $d(v,p) > 6\Delta$, it follows that $d(v,m)$ and $d(m,p)$ are both strictly greater than 3Δ, and thus $d(z,C) = d(z,\{v,p\}) > 2\Delta$ by the triangle inequality. Since $d(m,z) < \Delta$, we have $d(m,C) > 2\Delta - \Delta = \Delta$, a contradiction.

Step i We again prove assertion (1) first, now assuming that $v \in U_i$ is in the same $D \in \mathcal{C}_{i-1}$ as $w \in V_{i-1}$. Because $d(v, V_{i-1}) = R$, for any $\epsilon > 0$ there is some $w' \in V_{i-1}$ with $R \le d(v,w') \le R + \epsilon$. If for all $\epsilon > 0$ we can choose $w' = w$, then $d(v,w) = R$ and we are done, so assume that $w' \ne w$. Then w' is not contained in D, so a geodesic path from v to w' must pass through some point $d \in \mathrm{Front}(D)$ (see Figure 6). By the induction hypothesis (2), $d(d,w) \le 6\Delta$, so $d(v,w) \le R + \epsilon + 6\Delta$. Letting ϵ tend to zero, we obtain assertion (1).

To prove assertion (2) we again argue by way of contradiction. Let $p \in \mathrm{Front}(C)$ be such that $d(p,v) > 6\Delta$. We see as before that the midpoint m provided by the Bottleneck Property must satisfy $d(m,C) < \Delta$.

To obtain the contradictory inequality in this case requires some extra maneuvers. Let $\epsilon > 0$. Then we may find w_1 and w_2 in V_{i-1} so that $d(v,w_1) \le R + \epsilon$

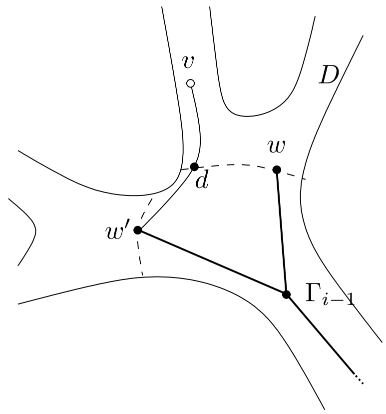

Figure 6: Assertion (1) Step i

and $d(p, w_2) \leq R + \epsilon$. These points w_1 and w_2 are connected by a path σ in $\beta(\Gamma_{i-1})$ which is the image of a geodesic path in the tree Γ_{i-1}. Together with geodesics $[v, w_1]$ and $[w_2, p]$, this path σ gives a path between v and p (see Figure 7). The midpoint m must lie in a Δ–neighborhood of this path. Let z

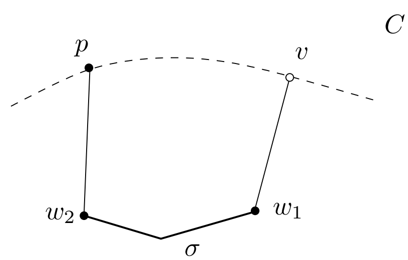

Figure 7: Assertion (2) Step i

be a point on the path $[v, w_1] \cup \sigma \cup [w_2, p]$ which is less than Δ from m. We claim first that z cannot lie on σ. Certainly z cannot be an element of V_{i-1}, as $d(V_{i-1}, C) = R = 20\Delta$. Suppose then that z is in the interior of an edge of σ. Using the triangle inequality and the induction hypothesis (each such edge must have length between R and $R + 6\Delta$), we see that $d(z, C) \geq 7\Delta$, so we would have $d(m, C) \geq 6\Delta$, a contradiction.

The only remaining possibility is that z lies on one of the geodesic segments $[v, w_1]$ or $[w_2, p]$. We may suppose that z lies on $[v, w_1]$. Assume that q is an arbitrary point in C. We will argue from the triangles shown in Figure 8. By

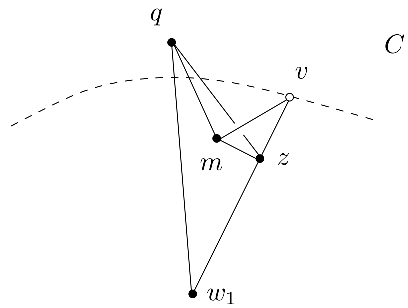

Figure 8: Assertion (2) Step i (continued)

assumption, $d(z,m) < \Delta$. Since $d(v,m) > 3\Delta$, $\triangle vmz$ gives $d(z,v) > 2\Delta$. We also have $d(v,w_1) \leq R+\epsilon$, and so since z lies on a geodesic from w_1 to v we get that $d(z,w_1) < R+\epsilon-2\Delta$. Since $q \in C$, we have $d(w_1,q) \geq R$, and so $\triangle qzw_1$ gives $d(z,q) > 2\Delta - \epsilon$. Finally $\triangle mqz$ gives $d(m,q) > \Delta - \epsilon$. Letting ϵ tend to zero gives $d(m,C) \geq \Delta$. This contradiction establishes Assertion (2). □

That β is coarsely onto follows easily from Lemma 4.7. Indeed, suppose that $x \in Y$ is not contained in an R–neighborhood of $V \subset \beta(\Gamma)$. Because Y is geodesic, we may find an x which is distance exactly R from V. There is then some i and some $v \in V_i$ so that $d(x,v) < R+\Delta$. Of course, x must lie in some component of the complement of $N_i = \{x \mid d(x,V_i) < R\}$, in other words there is some $C \in \mathcal{C}_{i+1}$ with $x \in C$. Let w_C be the element of U_{i+1} corresponding to this component. A geodesic path from x to v must pass through $\mathrm{Front}(C)$ at some point p. Since $d(x,v) < R+\Delta$ and $d(v,C) \geq R$, we have $d(p,x) < \Delta$. But by Lemma 4.7, $d(p,w_C) < 6\Delta$, so $d(x,w_C) < 7\Delta < R$, contradicting our choice of x.

The images of the edges of Γ allow us to get an upper bound on $d(\beta(x),\beta(y))$ where x and y are vertices of Γ. By Lemma 4.7 the image of each such edge is a geodesic of length less than or equal to $R+6\Delta$. Thus

$$d(\beta(x),\beta(y)) \leq (R+6\Delta)d(x,y) = 26\Delta d(x,y).$$

Let x and y be vertices of Γ. These are joined in Γ by a unique geodesic σ. For any $p \in \Gamma$ we can define $D(p)$ to be the minimum i so that $p \in \Gamma_i$. If p on σ minimizes D and is not an endpoint of σ, we refer to p as the *turnaround vertex*. Note that σ contains at most one turnaround vertex, as Γ is simply connected.

Lemma 4.8 *If z is a vertex on σ but is not a turnaround vertex, then any geodesic from $\beta(x)$ to $\beta(y)$ passes within 6Δ of $\beta(z)$.*

Proof Without loss of generality, we may assume $D(x) > D(z)$ and D is non-increasing from x to z. Otherwise we may switch x and y to ensure this is the case. Let $C \in \mathcal{C}_{D(z)}$ be the component of $Y \setminus N_{D(z)-1}$ containing z. We claim that $\beta(x) \in C$ but $\beta(y) \notin C$.

We see that $\beta(x) \in C$ by induction on $d(x,z)$. Let $z = z_0, z_1, \ldots, z_N = x$ be the sequence of vertices on σ joining z to x. Let $C = C_0, C_1, \ldots, C_N$ be the corresponding complementary components $C_i \in \mathcal{C}_{D(z)+i}$ from the construction of Γ. For $i \geq 1$ we have $C_i \subset C_{i-1}$ by construction, so $\beta(x) \in C$.

If $\beta(y)$ were in C, then there would be a path from y back to z on which D was non-increasing, and thus z would be a turnaround vertex.

Since $\beta(x) \in C$ but $\beta(y) \notin C$, any geodesic from $\beta(x)$ to $\beta(y)$ must pass through $\mathrm{Front}(C)$, and so by Lemma 4.7, the geodesic must pass within 6Δ of z. □

By Lemma 4.8 a geodesic from $\beta(x)$ to $\beta(y)$ must pass within 6Δ of each $\beta(z)$ where z is a vertex of Γ between x and y which is not the turnaround vertex. Images under β of successive vertices are at least R apart, so by picking points on the geodesic with 6Δ of the images of the vertices (except for the turnaround vertex) we see that

$$d(\beta(x), \beta(y)) \geq (R - 12\Delta)(d(x,y) - 2) = 8\Delta d(x,y) - 16\Delta.$$

Combining this result with the previously obtained upper bound we get

$$8\Delta d(x,y) - 16\Delta \leq d(\beta(x), \beta(y)) \leq 26\Delta d(x,y).$$

In particular, β is a quasi-isometric embedding which is R–almost onto, so it is a quasi-isometry. □

4.2 Pseudocharacters and Quasi-actions

Just as a homomorphism $\chi\colon G \to \mathbb{R}$ gives rise to a G–action on $\mathbb{R}$ via $g(x) = \chi(g) + x$ for $g \in G$ and $x \in \mathbb{R}$, a pseudocharacter $f\colon G \to \mathbb{R}$ gives rise to a $(1, \|\delta f\|)$–quasi-action of G on $\mathbb{R}$ via $g(x) = f(g) + x$. Roughly, in this section we attempt to "lift" this quasi-action to a quasi-action on T, the tree defined in Section 2.2. It is not immediately clear whether the quasi-action should lift, because the image of a vertex space (in $\widetilde{K}$) after action by a group element might intersect infinitely many vertex spaces. The point of the construction in this section is that we may "collapse" enough of T to get a quasi-action, and still be left with a complicated enough tree so that $E(f)$ embeds in its Gromov boundary.

Recall the definition of the tree T. We pick an (unambiguous) triangular generating set S. We then scale f so that $f(G)$ misses $\mathbb{Z}+\frac{1}{2}$ and so that f changes by at most $\frac{1}{4}$ over each edge. We then build a tree with vertex set in one-to-one correspondence with the components of $\widetilde{K}\setminus f^{-1}\bigl(\mathbb{Z}+\frac{1}{2}\bigr)$. The edges correspond to components of $f^{-1}\bigl(\mathbb{Z}+\frac{1}{2}\bigr)$, each of which is some possibly infinite track which separates $\widetilde{K}$ into two components.

Definition 4.9 We now define a graph X, which we will later show is a quasi-tree. Let V be the set of components of $f^{-1}\bigl(\mathbb{Z}+\frac{1}{2}\bigr)$. Then V is in one-to-one correspondence with the set of edges of T. Let X be the simplicial graph with vertex set equal to $G\times V$ and the following edge condition: Two distinct vertices (g,τ) and (g',τ') are to be connected by an edge if there is some h so that $hg(\tau)$ and $hg'(\tau')$ are contained in the same connected component of $f^{-1}\bigl[n-\frac{3}{2},n+\frac{1}{2}\bigr]$ for some $n\in\mathbb{Z}$. We endow the zero-skeleton X^0 with a G–action by setting $g(g_0,\tau_0)=(gg_0,\tau_0)$. Since this action respects the edge condition on pairs of vertices, it extends to an action on X.

Remark 4.10 The relationship between X and T is actually somewhat unclear. For every $x\in T$ we choose $e(x)$ to be an arbitrary edge adjacent to x. If $\varpi\colon T\to X$ is given by $\varpi(x)=(1,e(x))$, then ϖ is coarsely surjective and coarsely Lipschitz (in fact, $d(\varpi(x),\varpi(y))\leq d(x,y)+1$). If ϖ were a quasi-isometry, then Theorem 4.20 would follow immediately: G would quasi-act on the tree T. Usually, though, ϖ is not a quasi-isometry; the preimages of bounded sets do not even need to be bounded.

Lemma 4.11 *X is connected.*

Proof For every edge e of T, there is a vertex $(1,\tau_e)$ where τ_e is the component of $f^{-1}\bigl(\mathbb{Z}+\frac{1}{2}\bigr)$ corresponding to e. If e_1 and e_2 are adjacent edges of T, then $(1,\tau_{e_1})$ and $(1,\tau_{e_2})$ are certainly connected by an edge in X. Thus the vertices $\{1\}\times V$ are all in the same connected component of X.

Let (g,τ) be some vertex of X. As G acts by isomorphisms of the complex $\widetilde{K}$, $g\tau$ is, like τ, some track in $\widetilde{K}$. For any point $x\in\widetilde{K}$, $|f(gx)-f(g)-f(x)|\leq\|\delta f\|$, since f on $\widetilde{K}$ is obtained from f on G by affinely extending over each cell. In particular, since $f(\tau)$ is a point, $f(g\tau)$ has diameter less than or equal to $2\|\delta f\|$ in $\mathbb{R}$. Since by assumption $\|\delta f\|$ is much less than one, $f(g\tau)$ is contained in the interval $\left[n-\frac{3}{2}, n+\frac{1}{2}\right]$ for some $n \in \mathbb{Z}$. Since $g\tau$ is connected, it is therefore contained in a connected component of $f^{-1}\left[n-\frac{3}{2}, n+\frac{1}{2}\right]$. The boundary of this set contains at least one component τ' of $f^{-1}(\mathbb{Z})$, and so (g,τ) is connected to $(1,\tau')$ by an edge of X. Thus all the vertices of X are contained in the same connected component, and X is connected. □

G clearly acts simplicially on X. If we regard X as a path metric space with each edge having length 1, then G acts isometrically on X.

Proposition 4.12 *G acts coboundedly on X.*

Proof Let (g_0,τ_0) be a vertex of X. We will show that every other vertex of X is distance at most 1 from the orbit of (g_0,τ_0). Let (g_1,τ_1) be another vertex of X. For $i \in \{0,1\}$, let e_i be an edge which intersects $g_i\tau_i$, and let $h_i \in G$ be an endpoint of e_i. Then $h_i^{-1}g_i\tau_i$ is a track which passes through an edge adjacent to 1. Thus $\sup|f(h_i^{-1}g_i\tau_i)| \leq \epsilon_f + \|\delta f\| < 2\epsilon_f < \frac{1}{2}$. Since both tracks pass through edges adjacent to 1, a single component of $f^{-1}\left[-\frac{3}{2},\frac{1}{2}\right]$ contains both $h_0^{-1}g_0\tau_0$ and $h_1^{-1}g_1\tau_1$. According to Definition 4.9, this means that $d((h_0^{-1}g_0,\tau_0),(h_1^{-1}g_1,\tau_1)) \leq 1$. Since G acts on X by isometries, $d(h_1h_0^{-1}(g_0,\tau_0),(g_1,\tau_1)) \leq 1$ and the proposition is proved. □

Lemma 4.13 *Suppose (g_1,τ_1) and $(g_2,\tau_2) \in X$ and suppose that $g_1\tau_1 \cap g_2\tau_2$ is nonempty. Then $d((g_1,\tau_1),(g_2,\tau_2)) \leq 1$.*

Proof Suppose that (g_1,τ_1) and (g_2,τ_2) are distinct, and let h be a vertex of some 2–cell in $\widetilde{K}$ through which $g_1\tau_1$ and $g_2\tau_2$ both pass. Both $h^{-1}g_1\tau_1$ and $h^{-1}g_2\tau_2$ pass through a 2–cell adjacent to 1, and so $\sup|f(h^{-1}g_1\tau_1 \cup h^{-1}g_2\tau_2)| < \epsilon_f < \frac{1}{4}$. Thus $h^{-1}g_1\tau_1 \cup h^{-1}g_2\tau_2$ is contained in a single component of $f^{-1}\left[-\frac{3}{2},\frac{1}{2}\right]$, and $d((g_1,\tau_1),(g_2,\tau_2)) = d(h^{-1}(g_1,\tau_1),h^{-1}(g_2,\tau_2)) = 1$. □

Lemma 4.14 *Let (g_a,τ_a), (g_b,τ_b), $(g_c,\tau_c) \in X$ be such that $g_b\tau_b$ separates $g_a\tau_a$ from $g_c\tau_c$ in $\widetilde{K}$. Then any path from (g_a,τ_a) to (g_c,τ_c) passes within 2 of (g_b,τ_b).*

Proof Let $(g_a,\tau_a) = (g_0,\tau_0),(g_1,\tau_1),\ldots,(g_n,\tau_n) = (g_c,\tau_c)$ be the vertices of a path in X connecting (g_a,τ_a) to (g_c,τ_c). If $g_k\tau_k$ intersects $g_b\tau_b$ for any k then we have $d((g_k,\tau_k),(g_b,\tau_b)) \leq 1$ by Lemma 4.13. Thus we may assume that $g_k\tau_k$ is disjoint from $g_b\tau_b$ for all k. Since $g_b\tau_b$ separates $g_0\tau_0$ from $g_n\tau_n$ there is some k for which $g_k\tau_k$ and $g_{k+1}\tau_{k+1}$ are separated by $g_b\tau_b$. Since (g_k,τ_k) is connected to (g_{k+1},τ_{k+1}) by an edge of X, there is some $h\in G$, some $n\in\mathbb{Z}$, and some connected component B of $f^{-1}\left[n-\frac{3}{2},n+\frac{1}{2}\right]$ so that $hg_k\tau_k\cup hg_{k+1}\tau_{k+1}\subset B$. Since B is path-connected, there is some path $\gamma\colon[0,1]\to B$ with $\gamma(0)\in hg_k\tau_k$ and $\gamma(1)\in hg_{k+1}\tau_k$. Of course this path must cross $hg_b\tau_b$, so $hg_b\tau_b\cap B$ is nonempty. If $hg_b\tau_b$ were contained in B, we would have $hg_b\tau_b\cup hg_k\tau_k\subset B$, and so (g_b,τ_b) and (g_k,τ_k) would be connected by an edge, implying $d((g_b,\tau_b),(g_k,\tau_k))=1$. If on the other hand $hg_b\tau_b$ is not completely contained in B, then it must intersect some boundary component τ of B. Since $\tau=1\cdot\tau$ and $hg_b\tau_b$ intersect, we have $d((1,\tau),(hg_b,\tau_b))\le 1$ which implies $d((h^{-1},\tau),(g_b,\tau_b))\le 1$. Since $\tau\cup hg_k\tau_k\subset B$, we also have $d((h^{-1},\tau),(g_k,\tau_k))=1$. By the triangle inequality, $d((g_k,\tau_k),(g_b,\tau_b))\le 2$, establishing the lemma. □

Theorem 4.15 *The space X satisfies the Bottleneck Property of Theorem 4.6 for $\Delta=10$.*

Proof Let x, $y\in X$, and let m be the midpoint of some geodesic segment γ joining x to y. We may assume that $d(x,y)>20$, otherwise the Bottleneck Property is satisfied trivially for $\Delta=10$.

The proof of Lemma 4.11 shows that any vertex of X is distance at most 1 from some vertex of the form $(1,\tau)$. As any point of X is distance at most $\frac{1}{2}$ from some vertex, there exist $(1,\tau)$ and $(1,\tau')$ so that $d(x,(1,\tau))\le\frac{3}{2}$ and $d(y,(1,\tau'))\le\frac{3}{2}$.

Let e, e' be the edges in T corresponding to τ and τ', and let m_e, $m_{e'}$ be the midpoints of these edges. These points are connected by a unique geodesic passing through some sequence of edges $e=e_0,e_1,\dots,e_n=e'$ in T. If τ_i is the track in $\widetilde{K}$ associated to e_i, note that $d((1,\tau_i),(1,\tau_j))\le|i-j|$. In particular, $d((1,\tau_i),(1,\tau_{i+1}))=1$ for all i. Thus the sequence of edges in T defines a path in X leading from $(1,\tau)=(1,\tau_0)$ to $(1,\tau')=(1,\tau_n)$. We extend this path with geodesic segments of length less than or equal to $\frac{3}{2}$ to obtain a path from x to y, most of whose vertices lie in $1\times V$. This path clearly contains some point z so that $\min\{d(x,z),d(y,z)\}\ge\frac{d(x,y)}{2}>10$. Thus there is some vertex $(1,\tau_k)$ such that

$$\min\{d(x,(1,\tau_k)),d(y,(1,\tau_k))\}\ge\frac{d(x,y)}{2}-\frac{1}{2}>\frac{19}{2}.$$

Since $(1,\tau_k)$ is far from x and y, k is not equal to 0 or n, and so τ_k separates τ from τ' in $\widetilde{K}$. Thus by Lemma 4.14 any path from $(1,\tau)$ to $(1,\tau')$ must pass within 2 of $(1,\tau_k)$.

Let σ be any path from x to y. This path can be extended by adding segments of length at most $\frac{3}{2}$ at each end to give a path $\overline{\sigma}$ from $(1,\tau)$ to $(1,\tau')$. By the previous paragraph, $\overline{\sigma}$ must pass within 2 of $(1,\tau_k)$. Since the appended segments are very far (at least 8) from $(1,\tau_k)$, this means that σ must pass within 2 of $(1,\tau_k)$.

By the same argument, the geodesic γ passes within 2 of $(1,\tau_k)$. Let z be a point on γ which is within 2 of $(1,\tau_k)$. By the triangle inequality, $\min\{d(x,z),d(y,z)\} \geq \frac{d(x,y)}{2} - \frac{5}{2}$. Thus $d(z,m) \leq \frac{5}{2}$, where m is the midpoint of γ. Thus $d(\sigma,m) \leq \frac{11}{2} < 10$, establishing the theorem. □

Corollary 4.16 *The space X is quasi-isometric to a simplicial tree Γ. Since G acts coboundedly on X, G quasi-acts coboundedly on Γ.*

For the next lemma, we need the language of Gromov products in metric spaces. Recall that if (M,m) is a pointed metric space, and x, $y \in M$, then the Gromov product of x with y is defined to be $(x.y) = \frac{1}{2}(d(m,x)+d(m,y)-d(x,y))$. If M is Gromov hyperbolic, we say that a sequence $\{x_i\}$ of points in M *converges at infinity* if $\lim_{i,j\to\infty}(x_i.x_j) = \infty$. One can then define ∂M, the Gromov boundary of M, to be the set of sequences converging at infinity modulo the equivalence relation: $\{x_i\} \sim \{y_i\}$ if $\lim_{i,j\to\infty}(x_i.y_j) = \infty$. For more detail, see [5, Chapter III.H] and [13].

Lemma 4.17 *There is an injective map from $E(f)$ to ∂X.*

Proof Fix some base point $(1,\nu) \in X$. All Gromov products in X will be taken with respect to this base point. Let $[\phi] \in E(f)$. The path ϕ passes through some sequence of tracks in $f^{-1}\big(\mathbb{Z}+\frac{1}{2}\big)$. We choose a subsequence $\{\tau_i\}$ of these tracks so that for each i the track τ_i separates ν from τ_{i+1}. Recall that these τ_i may be identified with edges of the tree T from Section 2.2. Choosing them the way we have ensures that they all lie on a geodesic ray in T. It should be clear that this geodesic ray limits on $i([\phi])$ where i is the map from Proposition 2.12.

Because $[\phi] \in E(f)$, we necessarily have $\lim_{i\to\infty} f(\tau_i) = \pm\infty$. We claim that the sequence $\{x_i\} = \{(1,\tau_i)\}$ converges at infinity. To show this we must show that $(x_i.x_j) \to \infty$. Let $0 < i \leq j$. Then

$$(x_i.x_j) = \frac{1}{2}\big[d(x_i,(1,\nu)) + d(x_j,(1,\nu)) - d(x_i,x_j)\big].$$

If $i = j$ then clearly $(x_i.x_j) = d(x_i, (1, \nu))$. Otherwise, τ_i is between ν and τ_j, so any path (in particular a geodesic) from $(1, \nu)$ to x_j must pass within distance 2 of x_i, by Lemma 4.14. Thus we have

$$d(x_j, (1, \nu)) \geq d(x_i, x_j) + d(x_i, (1, \nu)) - 4$$

which implies

$$(x_i.x_j) \geq \frac{1}{2}\big(2d(x_i, (1, \nu)) - 4\big) = d(x_i, (1, \nu)) - 2.$$

Since $d(x_i, (1, \nu)) \geq \frac{1}{2}|f(\tau_i) - f(\nu)| \to \infty$ as $i \to \infty$, so does $(x_i.x_j)$ and so $\{x_i\}$ converges to some point in ∂X. An almost identical calculation shows that this point does not depend on the sequence of tracks we chose. We define a map $\bar{i}\colon E(f) \to \partial X$ by defining $\bar{i}([\phi])$ to be the point in ∂X to which this sequence converges.

We next claim that $\bar{i}$ is injective. Suppose that $[\phi]$ and $[\phi']$ are distinct elements of $E(f)$. By Proposition 2.12 we may identify $[\phi]$ and $[\phi']$ with elements of ∂T. There is a unique bi-infinite sequence of edges of T (ie a geodesic) joining $[\phi]$ to $[\phi']$, so that one of every triple of edges separates the remaining two from one another. We will abuse notation slightly by identifying an edge of T with the track associated to it. Let ω be the edge adjacent to the geodesic between $[\phi]$ and $[\phi']$ which is closest to ν. We may choose the sequences

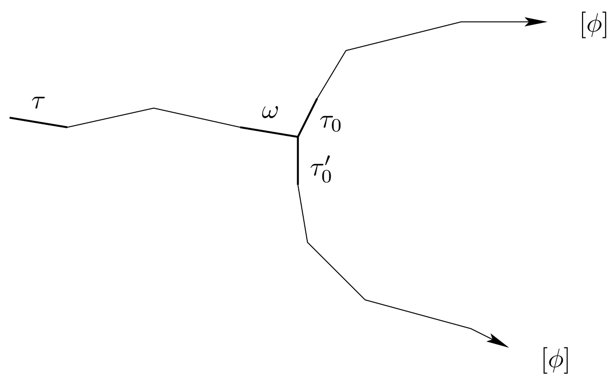

Figure 9: Arrangement of edges in T

$x_i = (1, \tau_i)$ and $x_i' = (1, \tau_i')$ representing $\bar{i}([\phi])$ and $\bar{i}([\phi'])$ so that τ_0 and τ_0' are as shown in Figure 9. Let $i,\ j \geq 1$. Then τ_i is separated by τ_0 from ν and τ_j' and τ_j' is separated by τ_0' from τ_i and ν. Transporting this arrangement into X we get Figure 10. In this figure, all the edges of the inner triangle have length 1. Lemma 4.14 ensures that all the edges leading from the inner

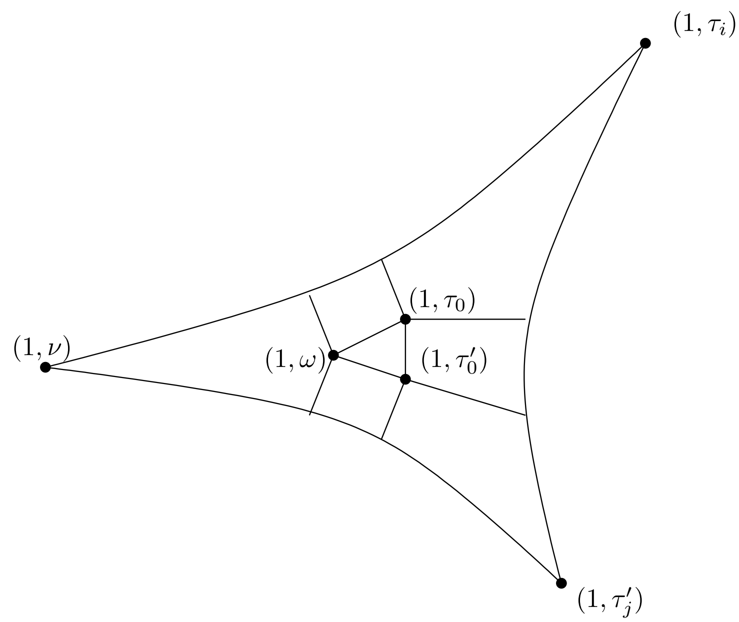

Figure 10: Computing the Gromov product in X

to the outer triangle have length at most 2. A computation then shows that $(x_i.x_j') \leq d((1,\nu),(1,\omega)) + 11$, and thus $i([\phi]) \neq i([\phi'])$. □

Definition 4.18 A tree or quasi-tree X is said to be *bushy*, with *bushiness constant* B, if removing any metric ball of radius B from X leaves a space with at least three unbounded path components.

Remark 4.19 If two quasi-trees are quasi-isometric and one is bushy, then clearly the other must also be bushy, though possibly with a different bushiness constant.

Theorem 4.20 *If $f\colon G \to \mathbb{R}$ is a pseudocharacter which is not uniform, then G admits a cobounded quasi-action on a bushy tree.*

Proof By Corollary 4.16, the space X defined in 4.9 is quasi-isometric to a tree Γ. If f is uniform, $E(f)$ contains at least three points. By Lemma 4.17 this implies that $\partial\Gamma \cong \partial X$ contains at least three points. Let γ_1, γ_2, and γ_3 be geodesic rays in Γ starting at some fixed point $p \in \Gamma$ and tending to these three points in $\partial\Gamma$. As these points are distinct, there is some $R > 0$ so that

$\gamma_1[R,\infty)\cap\gamma_2[R,\infty)\cap\gamma_3[R,\infty)$ is empty. Thus removing an R–ball centered at p from Γ leaves a space with at least three unbounded path components. Since G quasi-acts coboundedly on Γ (Corollary 4.16), there is some constant B so that removing *any* B–ball leaves a space with at least three unbounded path components. ◻

Remark 4.21 Recall that the pseudocharacter f gives rise to a $(1,\|\delta f\|)$–quasi-action on $\mathbb{R}$. The function $\varpi\colon X\to\mathbb{R}$ given by $\varpi((g,\tau))=f(g)+f(\tau)$ is coarsely equivariant in an obvious sense. Say that a G–quasi-tree (or tree with a G–quasi-action) Λ is *maximal with respect to* f if Λ admits a coarsely equivariant map to $\mathbb{R}$ and if whenever $q\colon \Lambda'\to\Lambda$ is a coarsely equivariant coarsely surjective map from another G–quasi-tree or tree with a G–quasi-action Λ', then q must be a quasi-conjugacy. It might be interesting to investigate the following questions: Do maximal quasi-trees exist? Under what conditions is our quasi-tree X maximal?

4.3 Space of Pseudocharacters

In this subsection we show that if there is a bushy pseudocharacter on G, then the space of pseudocharacters on G is actually infinite-dimensional. We first recall the terminology and main result of [3]. We consider the action of G on the quasi-tree X defined in the last subsection. Since X is quasi-isometric to a 0–hyperbolic space (a tree) it is δ–hyperbolic for some δ. The following definitions make sense whenever G is a group acting on a δ–hyperbolic graph X (see [3] for more details – Definitions 4.22–4.24 are quoted nearly verbatim from there).

Definition 4.22 Call an isometry g of X *hyperbolic* if it admits a (K,L)–quasi-axis for some K, and L. That is, there is a bi-infinite (K,L)–quasi-geodesic which is mapped to itself by a nontrivial translation. This quasi-geodesic is said to be given the *g–orientation* if it is oriented so that g acts as a positive translation. Note that any two (K,L)–quasi-axes of g are within some universal $B=B(\delta,K,L)$ of one another (by an elementary extension of Lemma 4.5), and any sufficiently long (K,L)–quasi-geodesic arc in a B–neighborhood of a quasi-axis for g inherits a natural g–orientation.

Definition 4.23 If g_1 and g_2 are hyperbolic elements of G, write $g_1\sim g_2$ if for an arbitrarily long segment J in a (K,L)–quasi-axis for g_1 there is a $g\in G$ such that $g(J)$ is within $B(\delta,K,L)$ of a (K,L)–quasi-axis of g_2 and

$g\colon J \to g(J)$ is orientation-preserving with respect to the g_2–orientation on $g(J)$.

Definition 4.24 Two hyperbolic isometries g_1 and g_2 are said to be *independent* if their quasi-axes do not contain rays which are a finite Hausdorff distance apart. Equivalently the fixed point sets of g_1 and g_2 in ∂X are disjoint. An action is *nonelementary* if there are group elements which act as independent hyperbolic isometries.

Definition 4.25 A *Bestvina-Fujiwara* action is a nonelementary action of a group G on a hyperbolic graph X so that there exist independent g_1, $g_2 \in G$ so that $g_1 \not\sim g_2$.

Theorem 4.26 [3] *If G admits a Bestvina-Fujiwara action, then $H^2_b(G;\mathbb{R})$ and the space of pseudocharacters on G both have dimension equal to $|\mathbb{R}|$.*

Proposition 4.27 *If $f\colon G \to \mathbb{R}$ is a bushy pseudocharacter, then the action on X described in Definition 4.9 is a Bestvina-Fujiwara action.*

Proof First note that if $g \in G$ and $f(g) > 0$, then g acts as a hyperbolic isometry of X. Indeed, let $x_0 = (g_0, \tau_0)$ be a vertex of X, and let $x_n = g^n x_0 = (g^n g_0, \tau_0)$. We choose a constant speed geodesic path from x_0 to x_1 and translate it to get a map $\gamma\colon \mathbb{R} \to X$ so that $\gamma(n) = x_n$ for all $n \in \mathbb{Z}$. For s, $t \in \mathbb{R}$ we clearly have $d(\gamma(s), \gamma(t)) \leq D|s-t|$, where D is the distance between x_0 and x_1.

Let $\overline{f}(g,\tau) = f(g) + f(\tau)$. Suppose x, x' are vertices of X which are connected by an edge. It is straightforward to see that $|\overline{f}(x) - \overline{f}(x')| \leq 2 + 4\|\delta f\|$. This gives an upper bound for the gradient of $\overline{f}$ on X. Since $|\overline{f}(x_m) - \overline{f}(x_n)| \geq |m-n|f(g) - \|\delta f\|$, we get $d(\gamma(n), \gamma(m)) \geq (|m-n|f(g) - \|\delta f\|)/(2+4\|\delta f\|)$. For arbitrary s, $t \in \mathbb{R}$ we have

$$d(\gamma(s), \gamma(t)) > \frac{f(g)}{2+4\|\delta f\|}|s-t| - \left(\frac{\|\delta f\|}{2+4\|\delta f\|} + 2K\right).$$

Thus γ is a quasi-axis for g.

Claim 4.28 *If $g_1 \sim g_2$, and $\sigma(g_1) \neq 0$, then $\sigma(g_2) = \sigma(g_1)$, where $\sigma(g)$ is the sign of $f(g)$ as in Definition 2.3.*

Proof Assume for simplicity that $\sigma(g_1) = 1$. It is sufficient to show that $f(g_2^N) > 0$ for some $N \geq 1$. For $i \in \{1, 2\}$, let γ_i be a (K, L)–quasi-axis for g_i, parameterized so that for some point x_i $\gamma(n) = g_i^n(x_i)$.

For $N > 0$ there is an $h = h_N$ in G so that $h(\gamma_2[0, N])$ is in a $B(\delta, K, L)$–neighborhood of γ_1, where δ is the thinness constant for X. Furthermore, if N is large enough, and $\gamma_1(p)$ and $\gamma_1(q)$ are the closest elements of $\gamma_1(\mathbb{Z})$ to $h\gamma_2(0)$ and $h\gamma_2(N)$ respectively, then $q > p$. We assume that N is at least this large. Indeed, by choosing N large enough, we can ensure that $q - p$ is as large as we like.

Since $|\overline{f}(g_1^n(x_1)) - (nf(g_1) + \overline{f}(x_1))|$ is bounded, we can therefore ensure that $\overline{f}(\gamma(q)) - \overline{f}(\gamma(p))$ is arbitrarily large. As the endpoints of $h\gamma_2[0, N]$ are at most $B(\delta, K, L) + d(x_1, g_1 x_1)$ from the points $\gamma_1(p)$ and $\gamma_2(q)$, and the gradient of $\overline{f}$ is bounded, it follows that we can choose N to make $\overline{f}(h\gamma_2(N)) - \overline{f}(h\gamma_2(0))$ very large. Thus we can make $\overline{f}(\gamma_2(N)) - \overline{f}(\gamma_1(0))$, and finally $f(g_2^N)$, as large as we like. In particular, we may find N so that $f(g_2^N)$ is positive. □

In Theorem 3.9, we showed that there are elements g and g', with $\sigma(g) = \sigma(g') = 1$, so that g and g' have disjoint fixed point sets in $E(f)$. Thus g^{-1} and g' act independently and hyperbolically on X. Furthermore, $\sigma(g') \neq \sigma(g^{-1})$, and so by the claim, $g' \not\sim g^{-1}$. □

The following theorem is an immediate corollary of Proposition 4.27 and Theorem 4.26:

Theorem 4.29 *If G admits a single bushy pseudocharacter, then $H_b^2(G; \mathbb{R})$ and the space of pseudocharacters on G both have dimension equal to $|\mathbb{R}|$.*

5 Examples

In [18], it is shown that any quasi-action on a bounded valence bushy tree is quasi-conjugate to an action on a (possibly different) bounded valence bushy tree. One might conjecture that some kind of analogous statement holds in the unbounded valence case. An immediate obstacle to such a conjecture is that any isometric action on an $\mathbb{R}$–tree gives rise to a quasi-action on a simplicial tree. One way to see this is that $\mathbb{R}$–trees clearly satisfy the bottleneck property of Theorem 4.6, and thus are quasi-isometric to simplicial trees. One might still ask the following question:

Question 5.1 *Is every quasi-action on a bushy tree by a finitely presented group quasi-conjugate to an action on an $\mathbb{R}$–tree?*

Kevin Whyte pointed out the following simple example after seeing an earlier version of this paper:

Example 5.2 Let F be the graph with vertex set equal to $\overline{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ and so that two vertices $\frac{p}{q}$ and $\frac{r}{s}$ are connected with an edge whenever $ps - qr = \pm 1$ (Formally we think of ∞ as $\frac{1}{0}$.). This is usually called the Farey graph. The group $PSL(2,\mathbb{Z})$ acts on the vertices by Möbius transformations, and preserves the edge condition. It is not hard to see that F satisfies the bottleneck condition of Theorem 4.6, and is thus a quasi-tree. In fact, F is quasi-isometric to an infinite valence tree. The action of $PSL(2,\mathbb{Z})$ on F thus induces a quasi-action on an infinite valence tree, via Proposition 4.4.

Proposition 5.3 *The action of $PSL(2,\mathbb{Z})$ on the Farey graph is not quasi-conjugate to an isometric action on any $\mathbb{R}$–tree.*

Sketch proof Suppose $PSL(2,\mathbb{Z})$ acts isometrically on an $\mathbb{R}$–tree Λ, and that this action is quasi-conjugate to the action on the Farey graph F. Note that $PSL(2,\mathbb{Z})$ is generated by the elements $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} / \{\pm I\}$ and $B = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} / \{\pm I\}$. Both A and B must fix points in Λ, as they fix points in F. (A fixes $\frac{1}{0}$ and B fixes $\frac{0}{1}$.) The element $(AB)^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} / \{\pm I\}$ has finite order, and so it must also it must also fix a point in Λ. It then follows (eg from [19, Corollary 1 on p64]) that A and B must have a common fixed point. Since A and B generate, this implies that $PSL(2,\mathbb{Z})$ fixes a point in Λ. Thus every orbit in Λ has finite diameter. This would imply that every orbit in F has finite diameter, which is easily seen to be false. □

Thus the answer to Question 5.1 is no. Of course $PSL(2,\mathbb{Z})$ admits a cocompact action on the (infinite diameter) Bass-Serre tree coming from the splitting $PSL(2,\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$. One might still ask the following question:

Question 5.4 *Does every group with a cobounded quasi-action on a bushy tree also act nontrivially and isometrically on some tree?*

We will show that the answer to this question is also no.

Suppose that M is a closed Riemannian manifold with all sectional curvatures ≤ -1. Here is one way to generate examples of pseudocharacters on $\pi_1(M)$. Let ω be any 1–form on M. If $g \in \pi_1(M)$ we let γ_g be the unique closed geodesic in its free homotopy class. Let $f_\omega\colon \pi_1(M) \to \mathbb{R}$ be given by

$$f_\omega(g) = \int_{\gamma_g} \omega. \tag{1}$$

We claim that f_ω is a pseudocharacter on G. It is clear from the definition that f_ω is conjugacy invariant and a homomorphism on each cyclic subgroup. To see it is a coarse homomorphism on G, we compute for $g,\ h \in \pi_1(M)$,

$$\begin{aligned}\delta f_\omega(g,h) &= f_\omega(gh) - f_\omega(g) - f_\omega(h)\\ &= \int_{\gamma_{gh}\cup -\gamma_g \cup -\gamma_h} \omega = \int_F d\omega\end{aligned}$$

where F is a (not necessarily embedded) pair of pants in M like the one shown in Figure 11. The quantity $\int_F d\omega$ only depends on F up to homotopy in M so

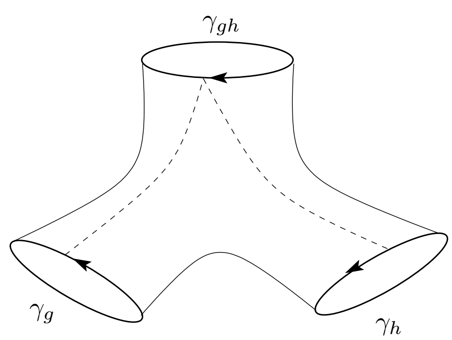

Figure 11: Pants

we may assume that F is triangulated as in Figure 12 and that each triangle of F has been straightened. This is to say that each edge of the triangulation has been made geodesic, and each face is a union of geodesics issuing from one vertex and terminating at the opposite edge. One may show using the Gauss-Bonnet theorem that the area of any straight triangle in M is $\leq \pi$. Thus the area of F is $\leq 5\pi$ after straightening. Since M is compact, $d\omega$ is bounded, and so $\int_F d\omega$ is bounded above by some multiple of the area of F. Thus $\|\delta f_\omega\| < \infty$.

We would like to use the preceding construction to obtain a bushy pseudocharacter on a group with no nontrivial action on any tree. Thurston showed in [21]

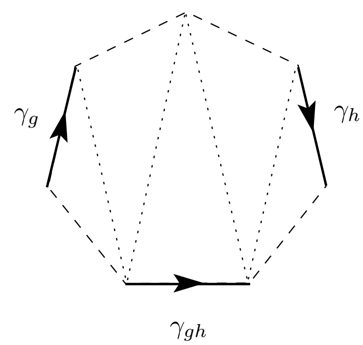

Figure 12: Pants cut open and triangulated with straight triangles

how to obtain (via Dehn filling) many negatively curved three-manifolds whose fundamental groups cannot act nontrivially on any tree.

Proposition 5.5 *All but finitely many fillings of the figure eight knot complement have fundamental groups which admit bushy pseudocharacters.*

Proof Let M be the complement of the figure eight knot in S^3. In [21], Thurston showed that M admits a complete hyperbolic metric of finite volume. In [1], Bart shows that M contains a closed, immersed, totally geodesic surface Σ which remains π_1–injective after all but at most thirteen fillings. The main tool used in Bart's proof is the Gromov-Thurston 2π Theorem, in which an explicit negatively curved metric is constructed on the filled manifold [4].

Let $M(\gamma)$ be one of the fillings which can be given a negatively curved metric, and let $G = \pi_1 M(\gamma)$. We assume that $M(\gamma)$ is endowed with the Riemannian metric given by the 2π theorem. Outside of a neighborhood of the core curve of the filling solid torus, this metric is isometric to the hyperbolic metric on M with a neighborhood of the cusp removed. Thus the surface Σ remains totally geodesic in $M(\gamma)$, and does not intersect the filling solid torus. Furthermore, the core curve c of the filling solid torus is a closed geodesic.

Let ω be a one-form on $M(\gamma)$, supported inside the filling solid torus, so that $\int_c \omega > 0$, and define $f_\omega\colon \pi_1(M(\gamma)) \to \mathbb{R}$ as in equation 1. We will show that f_ω is a bushy pseudocharacter.

We fix a generating set S for G and consider the Cayley graph $\Gamma = \Gamma(G, S)$. Since $M(\gamma)$ is compact, Γ is quasi-isometric to the universal cover $\widetilde{M(\gamma)}$. Since $M(\gamma)$ has a negatively curved Riemannian metric, Γ must be negatively curved in the sense of Gromov; further, the Gromov boundary $\partial\Gamma$ may be identified with the 2–sphere which is the visual boundary of $\widetilde{M}(\gamma)$ [5].

The inclusion of Σ into $M(\gamma)$ induces an inclusion of the surface group $F = \pi_1(\Sigma)$ into G; we note that some neighborhood $N(F)$ of F separates Γ into two unbounded complementary components. As Σ is totally geodesic in $M(\gamma)$, the Gromov boundary $\partial F \cong S^1$ embeds in ∂G and cuts ∂G into two open disks D_1 and D_2.

If g is any element conjugate into F, then its geodesic representative actually lies in Σ, and so $f_\omega(g) = 0$.

Let $g_c \in G$ be some element whose geodesic representative is c. Then $[g_c^\infty] \in E(f)^+$, and $\{g_c^n\}_{n\in\mathbb{N}}$ limits to some point $e_c \in \partial G$. The orbit of this point under the action of G on ∂G is dense [13], so we may find h_1 and h_2 in G so that $h_i e_c \in D_i$ for $i \in \{1,2\}$.

For each $i \in \{1,2\}$, let $\phi_i\colon \mathbb{R}_+ \to \Gamma$ be a constant speed path with $\phi_i(0) = h_i$ and $\phi_i(n) = h_i g_c^n$. Then $\lim_{t\to\infty} \phi_i(t) = h_i e_c$, and $[\phi_i] \in E(f)^+$ for each i.

We claim that $[\phi_1] \neq [\phi_2]$ in $E(f)$. Indeed, if $[\phi_1]$ and $[\phi_2]$ are equal, then (applying Definition 2.4) there are connecting paths $\delta\colon [0,1] \to \Gamma$ between ϕ_1 and ϕ_2 so that $\sup(f\circ\delta([0,1]))$ is large, but the diameter of $f\circ\delta([0,1])$ is small. Because the paths ϕ_1 and ϕ_2 go into separate components of the complement of $N(F)$, these connecting paths must pass through $N(F)$ (Figure 13). Since

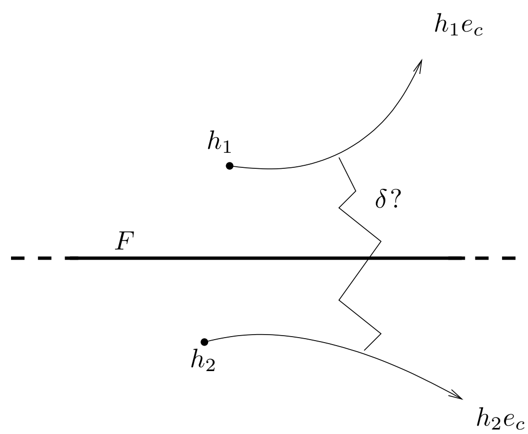

Figure 13: ϕ_1 and ϕ_2 cannot represent the same element of $E(f)$.

f_ω is zero on F, it is bounded on $N(F)$, and these connecting paths cannot exist.

A similar argument shows that we can find distinct elements of $E(f_\omega)^-$. □

Remark 5.6 The proof of the preceding proposition can be applied to fillings of any hyperbolic 3–manifold with an immersed closed incompressible surface, so long as the surface is quasi-Fuchsian.

It is proved in [17] that if a 3–manifold group acts on an $\mathbb{R}$–tree non-trivially, then it must split as either an amalgamated free product or as an HNN extension. By a standard argument, this implies that the 3–manifold is either reducible or Haken. Since all but finitely many fillings of the figure eight knot complement are non-Haken, we have the following corollary:

Corollary 5.7 *There are infinitely many closed* 3*–manifold groups which quasi-act coboundedly on bushy trees but which admit no nontrivial isometric action on any* $\mathbb{R}$*–tree.*

Remark 5.8 In a future article [16] we will look more closely at rigidity and irrigidity of group quasi-actions on trees, and in particular give examples of groups (for example $SL(n,\mathbb{Z})$ for $n > 2$) which do not quasi-act coboundedly on any (infinite diameter) tree.

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