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Completions of $\mathbb{Z}/(p)$–Tate cohomology of periodic spectra

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Abstract

We construct splittings of some completions of the $\mathbb{Z}/(p)$–Tate cohomology of $E(n)$ and some related spectra. In particular, we split (a completion of) $tE(n)$ as a (completion of) a wedge of $E(n-1)$'s as a spectrum, where t is shorthand for the fixed points of the $\mathbb{Z}/(p)$–Tate cohomology spectrum (ie the Mahowald inverse limit $\lim_{\leftarrow k}(P_{-k} \wedge \Sigma E(n))$). We also give a multiplicative splitting of $tE(n)$ after a suitable base extension.

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1 Introduction

Preliminaries and notation

We fix a prime p, and an integer n. We use t to denote Mahowald's inverse limit construction $tE = \lim_{\leftarrow k}(P_{-k} \wedge \Sigma E)$, where P_{-k} stands for either $\mathbb{R}P^{\infty}_{-k}$ or its analogue for $B\mathbb{Z}/(p)$ when p is odd; see [20] or [26]. This is an abbreviation for the fixed points of the Greenlees–May Tate cohomology functor; we write tE for what would be denoted in [7] by $t_{\mathbf{Z}/(p)}(i_* E)^{\mathbf{Z}/(p)}$. In particular if E is a ring spectrum, then so is tE.

The starting point of this paper is a calculation on coefficients, which is by now well known (see Lemma 2.1). If E is a complex oriented ring spectrum in which the series $[p](x)$ is not a zero divisor in $E^*[\![x]\!] = E^* \mathbb{C}P^{\infty}$, then

$$\pi_{-*}(tE) = E^*((x))/([p](x)),$$

where $|x| = 2$ and $E^k = \pi_{-k}(E)$. Here we use the awkward grading to emphasize that the ring is related to $E^* \mathbb{C}P^{\infty}$ and $E^* B\mathbb{Z}/(p)$. Graded more conventionally, we have

$$\pi_*(tE) = E_*((x))/([p](x))$$

where $|x| = -2$.

Throughout this paper we will use $R((x))$ to denote the ring of formal Laurent series over R that are allowed to be infinite series in x, but only finite in x^{-1}. If E is a ring spectrum and x an indeterminate in degree $-d < 0$, we write $E((x))$ for the ring spectrum given by $\lim_{\leftarrow i} \bigvee_{j \geq i} \Sigma^{dj} E$. The multiplication is defined by the inverse limit of the obvious maps

$$\Big(\bigvee_{j \geq i_0} \Sigma^{dj} E\Big) \wedge \Big(\bigvee_{k \geq i_1} \Sigma^{dk} E\Big) \longrightarrow \bigvee_{l \geq i_0 + i_1} \Sigma^{ld} E. \tag{1.1}$$

This gives $\pi_*[E((x))] = E_*((x))$. (Note that the theoretical possibility of phantom maps in (1.1) means the multiplicative structure there may not be unique, but we are content to use whichever multiplicative structure might occur. Actually work of Hovey and Strickland shows that there are no phantom maps in this situation [14], so the multiplication is defined uniquely.) Note also that if E is connective,

$$\lim_{\leftarrow i} \bigvee_{j \geq i} \Sigma^{dj} E = \prod_{j \in \mathbf{Z}} \Sigma^{dj} E. \tag{1.2}$$

In the succeeding, if we refer to a multiplication on $\prod_{j\in\mathbb{Z}} \Sigma^{dj} E_*$ it will be the one coming from the equivalence (1.2), and the ring structure on $\prod_{j\in\mathbb{Z}} \Sigma^{dj} R_*$ for a connective graded ring R_* will be understood to be the one given by the additive isomorphism with $R_*((x))$.

If M_* is a flat module over E_*, we write $M_* \otimes E$ for the spectrum representing the homology theory

$$X \mapsto M_* \otimes_{E_*} E_*(X).$$

This is a module spectrum over E (again, it will not matter for us that the module structure is well-defined only up to phantom maps). We also use $[p](x)$ for the p–series of the formal group law over E_* given by the orientation. When necessary we will decorate $[p]$ with a subscript indicating the formal group law.

We will work with a number of different spectra E, all closely related. The cohomology theory closest to BP is $BP\langle n\rangle$, the version of BP with singularities satisfying $BP\langle n\rangle_* = \mathbb{Z}_{(p)}[v_1, \dots, v_n]$ where v_i is the ith Araki generator. The Johnson–Wilson theory $E(n)_* = v_n^{-1}BP\langle n\rangle_*$ has the obvious coefficient ring obtained from inverting $v_n \in BP\langle n\rangle_*$

We will also need to consider some simple variants of $E(n)$. We list the theories below and their coefficient rings. They are all flat over $E(n)$ and are thus determined by their coefficients:

$$\begin{aligned}
\hat{E}(n) &= L_{K(n)}E(n) \qquad \text{so} \\
\hat{E}(n)_* &= \mathbb{Z}_p[[v_1, \dots, v_{n-1}]][v_n, v_n^{-1}] = (E(n)_*)^\wedge_{I_{n-1}} \\
(E_n)_* &= \hat{E}(n)_*[u]/(u^{p^n-1} - v_n) \\
&\qquad \text{and in later sections} \\
E(n)[w]_* &= E(n)_*[w]/(w^{p^{n-1}-1} - v_{n-1}).
\end{aligned}$$

The "$\wedge$" is meant to suggest "complete." The reader should note that our choice of E_n so that $E_{n,*} = \mathbb{Z}_p[[u_1, \dots, u_{n-1}]][u, u^{-1}]$ conflicts with the common convention that E_n denote the same theory extended by the Witt vectors of $\mathbb{F}_{p^n}$. We make no use of that theory in this paper, so this should cause no confusion. The multiplicative structure on $\hat{E}(n)$ is given by the composite

$$\begin{aligned}
L_{K(n)}E(n) \wedge L_{K(n)}E(n) \to L_{K(n)}(L_{K(n)}E(n) \wedge L_{K(n)}E(n)) &= \\
L_{K(n)}(E(n) \wedge E(n)) \to L_{K(n)}E(n)&.
\end{aligned}$$

The other two spectra are finite extensions of $\hat{E}(n)$, so the multiplicative structure is determined in the obvious way from the coefficients.

History

Perhaps a few words of history are relevant here. Lin's theorem [18, 9] shows that for any finite CW–spectrum X, the map

$$X \to \varprojlim_{k} (P_{-k} \wedge \Sigma X) = tX$$

is p–completion. This is not true [3, 2] for $E = BP$ or various other BP–module spectra, though there is some predictable behavior, which is quite different from that of finite complexes. In particular, tX is quite large in these cases.

By a lovely but simple argument involving the thick subcategory theorem, Mahowald and Shick [22] showed (for $p = 2$) that if X is a type n finite complex, and v is a v_n–self map, then $t(v^{-1}X) \simeq *$. This is the starting point for a series of observations. For example, it turns out that $tK(n) \simeq *$; indeed [8] proves that $t_G K(n) \simeq *$ for any finite group G. Hopkins has shown that if X is of type n then $t(L_n X) \simeq *$ if X is type n (this is proved in [12]). Calculations of Hopkins and Mahowald (including a proof when $n = 1$) lead them to conjecture that if X is type $n-1$, then

$$t(L_n X) = L_{n-1}X^{\wedge}_{p} \vee \Sigma^{-1} L_{n-1}X^{\wedge}_{p}. \tag{1.3}$$

This is related to the chromatic splitting conjecture [11].

In light of (1.3), it seems worthwhile to investigate $tE(n)$. In [8] it is shown that $tE(n)$ is v_{n-1}–periodic in an appropriate sense. Using this, one sees in [12] that when X is finite, the *Bousfield class* of $t(L_n X)$ is compatible with the conjecture (1.3) (for $p = 2$, this is also proved by different techniques in [17]). The goal of the present paper is to give as precise a description as possible of $tE(n)$ in terms of familiar v_{n-1}–periodic spectra like $E(n-1)$. We hope that with the results of this paper in hand, it may be possible to make further headway on (1.3).

Results

The first step, in section 2, is a series of calculations of tE in terms of more familiar objects for various E. It turns out to be useful to complete with respect to $I_{n-1} = (p, \ldots, v_{n-2})$. Since $tE(n)$ is already $E(n-1)$ local, this amounts to localizing with respect to $K(n-1)$. This completion is in some sense not very dramatic; in particular if X is type $n-1$ as in (1.3), the completion leaves $t(L_n X)$ unchanged. The main result, for the case of $E(n)$, is:

Theorem A (Proposition 2.11) *There is an isomorphism of rings*

$$\pi_*\left(tE(n)\right)^\wedge_{I_{n-1}} \cong E(n-1)_*((x))^\wedge_{I_{n-1}}. \tag{1.4}$$

Section 3 gives splittings of the spectra $v_n^{-1}tBP\langle n\rangle$ and $tE(n)$ in the manner suggested by Theorem A. For $tE(n)$ the result is:

Theorem B (Theorem 3.10) *There is a map of spectra*

$$\lim_{\substack{\longleftarrow \\ i\in\mathbf{N}}} \bigvee_{j\le -i} \Sigma^{2j} E(n-1) \to tE(n)^\wedge_{I_{n-1}},$$

which after completion at I_{n-1} (or equivalently after localization with respect to $K(n-1)$) induces the isomorphism of Theorem A on homotopy groups.

We emphasize that the map in Theorem B is not multiplicative, even though the left hand side can be given a multiplication as in equation (1.1) and the right hand side is a ring spectrum by [7], and Theorem A gives an isomorphism of rings. Now the homotopy calculation in section 2 also shows that the obvious formal group law over $tE(n)$ has height $n-1$, so one might view the map

$$E(n) \to tE(n)$$

as a sort of Chern character, the classical Chern character being the case $n=1$. To make sense of this, one ought to use the calculations of section 2 to construct a map of ring spectra.

To do this, we give an isomorphism between the natural formal group law over $tE(n)$ and the Honda formal group law of height $n-1$. This can be done after a dramatic base extension, but in section 4 we show that it can then be done in a canonical way. Section 5 uses section 4 and the Lubin–Tate theory of lifts to construct an isomorphism of ring spectra between an extended version of $tE(n)$ and an extended version of $E(n-1)$.

Theorem C (Corollary 5.11) *These is a canonical equivalence of complex oriented ring spectra*

$$C_{\mathbf{F}_p((y))^{\mathrm{sep}}} \wedge_{C_{\mathbf{F}_p((y))}} TE \to C_{\mathbf{F}_p((y))^{\mathrm{sep}}} \wedge_{\mathbf{Z}_p} E_{n-1}.$$

The objects discussed in Theorem C are defined precisely in sections 4 and 5. Briefly, the ring $C_{\mathbf{F}_p((y))}$ is isomorphic to the p-adic completion of $\mathbb{Z}_p((y))$ (see Lemma 5.5). The ring $C_{\mathbf{F}_p((y))^{\mathrm{sep}}}$ is a complete discrete valuation ring with maximal ideal (p) and residue field $\mathbb{F}_p((y))^{\mathrm{sep}}$. TE is a finite extension of $tE(n)$

completed at I_{n-1} and defined in equation (4.3), and E_{n-1} is defined as usual to be a completed 2–periodic extension of $E(n-1)$ as usual so that (4.1) holds.

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2 Calculations on homotopy groups

The homotopy of tE for E complex oriented

We assume that E is complex oriented, and that $[p](x)$ is not a 0 divisor in $E^*[[x]]$. In [7, §16] it is shown that tE is an inverse limit of E smashed with Thom spectra over $B\mathbb{Z}/(p)$. In particular, [7, 16.3] shows that $\pi_* tE = E^*((x))/([p](x))$ as a module over $E^*[[x]]/([p](x)) = E^*(B\mathbb{Z}/(p)_+)$.

Now [7, 3.5] proves that the map

$$E^*(B\mathbb{Z}/(p)_+) \to \pi_* tE$$

is a map of rings. It follows that the element "x^{-1}" in $\pi_* tE = E^*((x))/([p](x))$ satisfies $x \cdot x^{-1} = 1$ in the ring $\pi_* tE$, so really is the inverse of x. From this one concludes that

Lemma 2.1 *There is an isomorphism of rings*

$$\pi_*(tE) \cong E^*((x))/([p](x)). \tag{2.2}$$

An isomorphism after completion

Recall $R((x))$ is $S^{-1}R[[x]]$ where S is the multiplicatively closed subset generated by x. If R is graded connected, and x has degree -2, then additively $R((x)) = \prod_{k\in\mathbf{Z}} \Sigma^{2k}R$. As remarked in the introduction, wherever this notation occurs, we will also consider $\prod_{k\in\mathbf{Z}} \Sigma^{2k}R$ to have the ring structure given by this isomorphism. Similarly, when E is a connective ring spectrum, we will consider the ring structure on $\prod_{k\in\mathbf{Z}} \Sigma^{2k}E = \varprojlim_i \bigvee_{k \le i} \Sigma^{2k}E$ as in (1.1).

The following observation motivates Conjecture 1.6 of [3] (which was corrected as Conjecture 1.2 of [2]).

Proposition 2.3

$$
\begin{aligned}
(tBP\langle n\rangle) &= BP\langle n\rangle((x))/([p](x)) \\
&\cong \prod_{k\in\mathbb{Z}} \Sigma^{2k} BP\langle n-1\rangle^{\wedge}_{p} \qquad (2.4)\\
&\cong BP\langle n-1\rangle^{\wedge}_{p}((x)).
\end{aligned}
$$

Proof The first line is equation (2.2). We wish to simplify the right hand side. For convenience, we introduce degree 0 elements

$$w_i = v_i x^{p^i-1}.$$

Then the ideal in the quotient above is generated by the relation

$$px +_F w_1x +_F w_2x +_F \cdots +_F w_nx = 0$$

where F is the formal group law over $BP\langle n\rangle$ induced by the orientation from BP. Expanding the formal sums and dividing by x, we can write the relation in the form

$$w_n = -(p + \text{ a formal series in } w_1, \dots, w_n),$$

and this equation implies

$$w_n \equiv -(p + w_1 + \dots + w_{n-1}) \quad \text{(mod decomposables)}. \qquad (2.5)$$

By iterating this relation some finite number of times (depending on the multi-index $(i_0, \dots, i_{n-1})$), we produce a polynomial W_n so that

$$w_n = W_n(w_1, \dots, w_{n-1}) \in \mathbb{Z}/(p^{i_0})[w_1, \dots, w_{n-1}]/(w_1^{i_1}, \dots, w_{n-1}^{i_{n-1}}).$$

We thus produce a power series $W_n \in \mathbb{Z}_p[[w_1, \dots, w_{n-1}]]$ so that

$$w_n = W_n(w_1, \dots, w_{n-1}) \in \mathbb{Z}_p[[w_1, \dots, w_{n-1}]] \subset \mathbb{Z}_p[v_1, \dots, v_{n-1}][\![x]\!].$$

Thus the map

$$(tBP\langle n\rangle) \to BP\langle n-1\rangle^{\wedge}_{p}((x))$$

which sends v_i to v_i $(i<n)$, x to x and v_n to $x^{-(p^n-1)}W_n(w_1, \dots, w_{n-1})$ is a well defined ring map. There is a map

$$BP\langle n-1\rangle((x)) \to (tBP\langle n\rangle)$$

defined in the obvious way, that becomes an inverse map upon extending it to $BP\langle n-1\rangle^{\wedge}_{p}((x))$. This extension exists, because the relation

$$[p](x) = 0$$

together with Araki's formula for the p–series [24, A2.2.4] allows us to write

$$px = [-1](v_1 x^p +_F \cdots +_F v_n x^{p^n}) \tag{2.6}$$

where F is the formal group law on $BP\langle n\rangle$ induced by the complex orientation from BP. Now dividing (2.6) by x lets us to write any p–adic integer as a power series in x. □

The conjecture of [3, 2] is that the isomorphism (2.4) is the effect on homotopy groups of an equivalence of spectra

$$tBP\langle n\rangle \simeq \prod_{k\in\mathbb{Z}} \Sigma^{2k} BP\langle n-1\rangle^{\wedge}_{p}.$$

This is proved for $n = 1$ in [3] and for $n = 2$ in [2] using the Adams spectral sequence. Of course in general an isomorphism in homotopy doesn't give an isomorphism of spectra, but it does if it is an isomorphism of MU–modules satisfying Landweber exactness. This suggests inverting v_n to produce an isomorphism of spectra. We shall show that there is an isomorphism of the expected form after an appropriate completion, taking as our starting point the spectra $v_n^{-1}(tBP\langle n\rangle)$ and $tE(n)$. The first point is that these spectra are not the same.

The map $BP\langle n\rangle \to E(n)$ gives a map $tBP\langle n\rangle \to tE(n)$, and since v_n is a unit on the right, this gives a map $v_n^{-1}tBP\langle n\rangle \to tE(n)$. On homotopy we get

$$\begin{aligned}\pi_*(v_n^{-1}(tBP\langle n\rangle)) &= v_n^{-1}[BP\langle n\rangle_*((x))/([p](x))]\\ &\hookrightarrow E(n)_*((x))/([p](x))\\ &= \pi_*(tE(n)) = \pi_*(t(v_n^{-1}BP\langle n\rangle)).\end{aligned}$$

To see that the inclusion is proper, set $r = |v_n|/|v_1| + 1$ and notice that the series

$$1 + v_n^{-1}v_1^{r}x^{p-1} + v_n^{-2}v_1^{2r}x^{2(p-1)} + \cdots$$

is an element of $E(n)_*((x))/([p](x))$ but not of $v_n^{-1}BP\langle n\rangle_*((x))/([p](x))$. This reflects the fact that t need not commute with direct limits (nor in fact does it generally commute with inverse limits).

We treat the case $v_n^{-1}(tBP\langle n\rangle)$ first.

We would like to extend the isomorphism of (2.4) to an isomorphism

$$v_n^{-1}tBP\langle n\rangle \to v_{n-1}^{-1}BP\langle n-1\rangle^{\wedge}_{p*}((x)). \tag{2.7}$$

For the map to exist we need the image of w_n under the isomorphism in (2.4) to be a unit after inverting v_{n-1} on the range of that map. This is false for $n > 1$. We can see this by checking that the image of w_n after inverting v_{n-1} is not a unit even modulo $(v_1, \ldots, v_{n-2})$. The range of (2.7) in dimension 0, modulo $(v_1, \ldots, v_{n-2})$, is $Z_p[[w_{n-1}]][w_{n-1}^{-1}]$. We have im $(xw_n) = -(px +_F w_{n-1}x)$, so

$$\text{im}\,(w_n) = -(p + w_{n-1}) \pmod{pw_{n-1}Z_p[[w_{n-1}]]}.$$

Since w_{n-1} is a unit in the range, the image of w_n is a unit if and only if the image of w_n divided by w_{n-1}, which we will abuse notation by writing w_n/w_{n-1}, is a unit. Continuing to work modulo $(v_1, \ldots, v_{n-2})$,

$$w_n/w_{n-1} = -(1 + \frac{p}{w_{n-1}}) \pmod{pZ_p[[w_{n-1}]]}.$$

Now if we examine the inclusion of rings

$$\colon Z_p[[w_{n-1}]][w_{n-1}^{-1}] \qquad (Z_p[[w_{n-1}]][w_{n-1}^{-1}])^\wedge_p,$$

we see that since $w_n/w_{n-1} = -1 + p\epsilon$, w_n/w_{n-1} is a unit in the completion, with inverse given by $-(1 + p\epsilon + p^2\epsilon^2 + \ldots)$. If we write this element as a power series in w_{n-1} plus a power series in w_{n-1}^{-1}, we see that $(w_n/w_{n-1})^{-1}$ is the sum of a power series in w_{n-1} with

$$\frac{a_1}{w_{n-1}} + \frac{a_2}{w_{n-1}^2} + \frac{a_3}{w_{n-1}^3} + \ldots$$

where $a_i = (-1)^i p^i \pmod{p^{i+1}}$. The power series in w_{n-1} is neccessarily in the image of , but $\frac{a_1}{w_{n-1}} + \frac{a_2}{w_{n-1}^2} + \frac{a_3}{w_{n-1}^3} + \ldots$ is clearly not. It follows that $(w_n/w_{n-1})^{-1}$ is not in the image of , and hence w_n/w_{n-1} is not a unit in $Z_p[[w_{n-1}]][w_{n-1}^{-1}]$.

Similarly, if the map existed, it could be an isomorphism only if v_{n-1} were a unit in the domain after inverting v_n. This also is not the case for $n > 1$, for similar reasons. When $n = 1$ both of these conditions are met, the map above exists, and is an isomorphism.

Proposition 2.8 *If the map from (2.4) is completed at the ideal*

$$I_{n-1} = (p, v_1, \ldots, v_{n-2})$$

then there is an isomorphism

$$(\;v_n^{-1}tBP\langle n\rangle)^\wedge_{I_{n-1}} \to (v_{n-1}^{-1}[BP\langle n-1\rangle^\wedge_p((x))])^\wedge_{I_{n-1}}. \tag{2.9}$$

Proof We follow the isomorphism of (2.4)

$$\begin{aligned}(tBP\langle n\rangle)_* &= BP\langle n\rangle_*((x))/([p](x)) \\ &\cong BP\langle n-1\rangle^\wedge_{p*}((x))\end{aligned}$$

by the inclusion

$$BP\langle n-1\rangle^\wedge_{p*}((x)) \to (v_{n-1}^{-1}[BP\langle n-1\rangle^\wedge_{p*}((x))])^\wedge_{I_{n-1}}$$

into the module obtained by inverting v_{n-1} and then completing. Now note that w_n has image as in (2.5). So dividing by $v_{n-1}x^{p^{n-1}} = w_{n-1}$ we see the image of w_n divided by w_{n-1} is -1 plus terms in the ideal I_{n-1} and terms in $xBP\langle n-1\rangle^\wedge_{p*}[[x]]$. This is a unit, so since we have inverted v_{n-1}, the image of w_n is a unit. This allows us to extend our map to the domain given by inverting v_n. Since the range is complete, we can also extend to the completion.

On the other hand, a similar argument allows us construct the inverse map from the inverse map of (2.4). □

Now we shall show that the isomorphism in (2.9) is induced by a map of spectra

$$(v_n^{-1}tBP\langle n\rangle)^\wedge_{I_{n-1}} \longrightarrow (v_{n-1}^{-1}\prod_{\mathbb{Z}}\Sigma^{2k}BP\langle n-1\rangle)^\wedge_{I_{n-1}}.$$

Both sides are ring spectra with obvious MU–module structures. They would be isomorphic as MU–algebras by the Landweber exact functor theorem if we could make them MU–algebras so that the coefficient isomorphism preserves the map from MU_*.

Let R be the ring spectrum on the right. In order to construct a map of ring spectra inducing the isomorphism in (2.9), it is necessary that the FGL induced by

$$v_n^{-1}tBP\langle n\rangle_* \to R_* \to R_*/(p,\ldots,v_{n-2})$$

be isomorphic to the "usual" FGL on $R_*/(I_{n-1})$ (induced by $BP\langle n-1\rangle_* \to R_*$). We cannot demonstrate such an isomorphism, but we can exhibit an isomorphism of spectra (that preserves neither the MU–module structure, nor the multiplicative structure). We do this in section 3.

The situation for $tE(n)$ is very similar. We can attempt to construct a map

$$tE(n)_* \to E(n-1)^\wedge_{p*}((x)) \qquad (2.10)$$

as in (2.4), but we immediately run into the problem that w_n, and hence v_n, does not go to a unit for $n > 1$. Also as before, v_{n-1} is not a unit on the left. The solution is the same; after completing both sides at I_{n-1} we can construct an isomorphism.

Proposition 2.11 *The map of (2.4) extends to an isomorphism*

$$E(n)_*((x))/[p](x)^{\wedge}_{I_{n-1}} \cong E(n-1)_*((x))^{\wedge}_{I_{n-1}} \tag{2.12}$$

where

$$(tv_n^{-1}BP\langle n\rangle)^{\wedge}_{I_{n-1}} = \pi_* tE(n)^{\wedge}_{I_{n-1}} = E(n)_*((x))/[p](x)^{\wedge}_{I_{n-1}}$$

and

$$E(n-1)_*((x))^{\wedge}_{I_{n-1}} = (v_{n-1}^{-1}BP\langle n-1\rangle)_*((x))^{\wedge}_{I_{n-1}}.$$

In the next section, we use this calculation to construct an isomorphism of spectra. We are able to do this without showing the corresponding formal group laws are isomorphic, so we do not get an isomorphism of MU–modules. In section 5, we extend scalars suitably to construct an isomorphism of formal group laws, yielding isomorphisms of MU–algebra spectra.

3 Structure of $E(n)$ as an $E(n-1)$–module spectrum

We begin with an algebraic observation.

Lemma 3.1 *Let $(A, \mathfrak{m})$ be a complete local ring, $k = A/\mathfrak{m}$. Let M be an A–module such that the map $\mathfrak{m} \otimes x \mapsto \mathfrak{m}x$ induces isomorphisms*

$$(\mathfrak{m}^r/\mathfrak{m}^{r+1}) \otimes_k (M/\mathfrak{m}M) \cong \mathfrak{m}^r M/\mathfrak{m}^{r+1}M.$$

Let I be an index set for a vector space basis of $M/\mathfrak{m}M$, so that $M/\mathfrak{m}M \cong \bigoplus_I k$. Then there is a map

$$\bigoplus_I A \xrightarrow{\epsilon} M$$

which is an isomorphism when completed at $\mathfrak{m}$.

Proof We construct a map as follows. Let $\{\bar{x}_\alpha; \alpha \in I\}$ be a basis for $M/\mathfrak{m}M$ as a $A/\mathfrak{m}$–vector space. Let $\{x_\alpha\}$ be lifts to M. Then we map $\bigoplus_I A$ to M by $\epsilon(1_\alpha) = x_\alpha$.

Our hypotheses imply that this map gives an isomorphism on the associated graded with respect to the filtration induced by powers of $\mathfrak{m}$. This implies that the completion of the map is an isomorphism. □

Note that if the index set I in Lemma 3.1 is infinite, M will not generally be isomorphic to a free A–module. For example if $A = \mathbb{Z}_p$ and $M = (\bigoplus_{\mathbf{N}} \mathbb{Z}_p)^{\wedge}_p$ then M is not a free $\mathbb{Z}_p$–module. To see this, observe that no free $\mathbb{Z}_p$–module of infinite rank can be p–complete.

Next we recall the result proved in [13, Theorem 4.1]. The identification $BP_*BP = BP_*[t_1, t_2, \ldots]$ gives a splitting

$$\Psi : BP \wedge BP \simeq \bigvee_R \Sigma^{|R|} BP \tag{3.2}$$

where R ranges over multi-indices of non-negative integers (with only finitely many positive coordinates) $R = (r_1, r_2, r_3, \ldots)$, $t^R = t_1^{r_1} t_2^{r_2} \ldots$ and

$$|R| = |t^R| = 2(r_1(p-1) + r_2(p^2-1) + r_3(p^3-1) + \ldots).$$

To build a map from right to left of (3.2), take $\Sigma^{|R|}BP$ to $BP \wedge BP$ by using the homotopy class $t^R \in \pi_{|R|}(BP \wedge BP)$, smashing on the left with BP and then multiplying the left pair of BP's:

$$BP \wedge S^{|R|} \xrightarrow{1_{BP} \wedge t^R} BP \wedge BP \wedge BP \xrightarrow{\mu \wedge 1_{BP}} BP \wedge BP. \tag{3.3}$$

The wedge over all R of these maps is an isomorphism on homotopy groups, so is invertible, and Ψ is that inverse.

Theorem 4.1 of [13] states that the composite

$$BP \xrightarrow{\eta_R} BP \wedge BP \xrightarrow{\Psi} \bigvee_R \Sigma^{|R|} BP \xrightarrow{\pi} \bigvee_{R \in \mathcal{R}_j} \Sigma^{|\Delta R|} BP \xrightarrow{\rho} \bigvee_{R \in \mathcal{R}_j} \Sigma^{|\Delta R|} BP\langle j \rangle \tag{3.4}$$

is a homotopy equivalence after smashing with a type j spectrum and inverting v_j. The map π is induced by leaving out wedge summands, and ρ by the usual reduction of ring spectra $BP \to BP\langle j \rangle$. We use the notation of [13], derived from [15]: $\mathcal{R}_j$ is the set of multi-indices with the first $j-1$ indices 0, and $\Delta R = (p^j e_j, p^j e_{j+1}, \ldots)$.

We shall use the following facts about MU–module spectra M:

$$M \wedge L_j Z = v_j^{-1} M \wedge Z = M \wedge v_j^{-1} Z \tag{3.5}$$

when Z is a finite type j spectrum (which follows from [25, Theorem 1]);

$$M^{\wedge}_{I_{j-1}} = \lim_{\overleftarrow{Z}} M \wedge Z \tag{3.6}$$

where Z runs through finite type j spectra under S^0. Use of the nilpotence theorem, [4], is required to produce a sufficient supply of such Z and to ensure there are enough maps between them, as in [21, Proposition 3.7]. For our purposes equation (3.6) can be taken to be a definition, but see the remark below. Inverting v_j, we get

$$L_{K(j)}M = \varprojlim_{Z} (M \wedge v_j^{-1}Z) = (v_j^{-1}M)^{\wedge}_{I_{j-1}}.$$

This last equality is by equation (3.5) combined with, say, the proof of [10, Proposition 7.4] which verifies that $L_{K(n)}X = \varprojlim_Z (X \wedge L_n Z)$ as Z runs over finite type n spectra under S^0.

So if v_j is already a unit on M then

$$L_{K(j)}M = M^{\wedge}_{I_{j-1}}$$

and we will generally use the first notation rather than the second below.

Remark Although we use (3.6) as the definition of $M^{\wedge}_{I_{j-1}}$, there are other approaches that can be taken for specific M. For certain MU–modules M, one can define a spectrum $\widehat{M}$ with homotopy groups $(\pi_* M)^{\wedge}_{I_{j-1}}$ by using Landweber exactness. In those cases, also by using Landweber exactness, one can prove that $\widehat{M} \simeq M^{\wedge}_{I_{j-1}}$ as given in (3.6). If M has enough structure, one may also be able to define a completion of M using either Baas–Sullivan bordism with singularities [1], or structured ring spectra using the techniques of [5]. In either case, one can use the nilpotence theorem to verify that the construction is homotopy equivalent to the one in (3.6).

Each map in (3.4) except η_R is a map of left BP–modules. Recall I_j is the ideal $(p, \ldots, v_{j-1}) \subset BP_*$. Since I_j is invariant, $\eta_R(I_j) = \eta_L(I_j)$, and thus each map in (3.4) is compatible on homotopy with the I_j–adic filtration. Let Z be a finite type j spectrum; then smashing (3.4) with $L_j Z$ gives an equivalence, and thus an $E(j)$–module structure on $BP \wedge v_j^{-1}Z = BP \wedge L_j Z$ or, by taking inverse limits, a (possibly not unique) module structure on $L_{K(j)}BP = \varprojlim_Z BP \wedge v_j^{-1}Z$.

We prove the following proposition as a warm-up to our additive splitting of the Tate cohomology spectrum. The construction is a general method for splitting $L_{K(j)}F$ when F is a nice enough BP–module, given the splitting of $L_{K(j)}BP$. A theorem equivalent to Proposition 3.7 is proved as [13, Theorem 4.7].

Proposition 3.7 *If $n > j$, there is a map*

$$s\colon \bigvee_{v} \Sigma^{|v|} E(j) \longrightarrow L_{K(j)}E(n) \tag{3.8}$$

which gives an equivalence after completing with respect to $(p, \ldots, v_{j-1})$*, or equivalently, after localizing with respect to* $K(j)$*. The index* V *runs through the monomials in*

$$\mathbb{F}_p[v_{j+1}, \ldots, v_{n-1}, v_n^{\pm 1}].$$

Proof $L_{K(j)}E(n)$ is an $L_{K(j)}BP$–module, and thus an $E(j)$–module. Let $\mathfrak{m} = I_j = (p, \ldots, v_{j-1})$. Since

$$L_{K(j)}E(n) = \varprojlim_{Z} (E(n) \wedge v_j^{-1} Z) \tag{3.9}$$

where Z runs through finite type j spectra,

$$\pi_* L_{K(j)}E(n) = \mathbb{Z}_p[[v_1, \ldots, v_{j-1}]][v_j^{\pm 1}, v_{j+1}, \ldots, v_{n-1}, v_n^{\pm 1}].$$

Now since $\mathfrak{m}$ is an invariant ideal $\mathfrak{m}\, \pi_* L_{K(j)}BP$ is well-defined, whether we think of $\mathfrak{m}$ as an ideal in $E(j)_*$ acting on $M = \pi_* L_{K(j)}E(n)$ via the $E(j)$–module action on $L_{K(j)}BP$ or as an ideal in BP_* acting via the associated localization map to $\pi_* L_{K(j)}BP$.

We calculate that the associated graded to the $\mathfrak{m}$–adic filtration on M is

$$E_0 M = \bigoplus_V \Sigma^{|V|} \mathbb{F}_p[[v_0, \ldots, v_{j-1}]][v_j^{\pm 1}]$$

where V runs through monomials in $\mathbb{F}_p[v_{j+1}, \ldots, v_{n-1}, v_n^{\pm 1}]$.

M satisfies the hypotheses of Lemma 3.1 (for the left $\mathfrak{m}$–structure which comes from the map $BP \to E(n) \to L_{K(j)}E(n)$), where

$$A = \pi_* \hat{E}(j) = \mathbb{Z}_p[[v_1, \ldots, v_{j-1}]][v_j^{\pm 1}].$$

Now the splitting of $L_{K(j)}BP$ gives $L_{K(j)}E(n)$ an $\hat{E}(j)$–structure, and thus an associated A–action; the $\mathfrak{m}$–adic filtration is the same, so M also satisfies the hypotheses of Lemma 3.1 for that $\mathfrak{m}$–adic filtration.

We can now make the usual homotopy theoretic argument: take generators of $M/\mathfrak{m}M$ and lift them to elements of M. Use the $E(j)$ structure of $L_{K(j)}E(n)$ to make maps $E(j) \to L_{K(j)}E(n)$ realizing these generators on the unit of the ring spectrum $E(j)$. This gives a map

$$\bigvee_V \Sigma^{|V|} E(j) \to L_{K(j)}E(n).$$

By Lemma 3.1, this map induces an isomorphism on homotopy groups after completion with respect to $(p, \ldots, v_{j-1})$, ie after applying $L_{K(j)}$, which leaves the right hand side unchanged. □

We apply techniques similar to those used in Proposition 3.7 to prove the following.

Theorem 3.10 *There is a map of spectra*

$$\varprojlim_{i\in\mathbf{N}} \bigvee_{j \geq i} \Sigma^{2i} E(n-1) \to tE(n)^{\wedge}_{I_{n-1}}$$

that becomes an isomorphism on homotopy groups after completion at I_{n-1} (*or equivalently after localization with respect to* $K(n-1)$).

Proof We proceed as in the proof of Proposition 3.7. We have given $M = tE(n)^{\wedge}_{I_{n-1}}$ as a BP_*–module in Proposition 2.11. It satisfies the hypotheses of Lemma 3.1 with $\mathfrak{m} = (p, \ldots, v_{n-2}) \subset BP_*$, $A = E^{\wedge}(n-1)_*$. Since $tE(n)^{\wedge}_{I_{n-1}} = L_{K(n-1)}tE(n)$, we have an $L_{K(n-1)}BP$–action and hence an $E(n-1)$–action. As above, the two available $\mathfrak{m}$–adic filtrations on $\pi_* tE(n)^{\wedge}_{I_{n-1}}$ (one from BP_*, the other from $E(n-1)_*$ via the $E(n-1)$–structure on $L_{K(n-1)}BP$) are the same since $\mathfrak{m}$ is an invariant ideal. We now proceed in a slightly different manner. Note that

$$M/\mathfrak{m} = K(n-1)_*((x)) = \bigoplus_{i\in I} \Sigma^{c_i} K(n-1)_* \tag{3.11}$$

for some indexing set I, since $K(n-1)_*$ is a graded field. We could apply Lemma 3.1 and proceed as before, but we would actually like better control over our expression for $tE(n)^{\wedge}_{I_{n-1}}$. In particular, the index set I in equation (3.11) must be uncountable, but we would like to find a countable set of topological generators for $tE(n)^{\wedge}_{I_{n-1}}$. In fact, we would like these generators to correspond to the (positive and negative) powers of x.

To accomplish this, we first recall [7, Theorem 16.1] which states that $tE = \varprojlim_i [(B\mathbf{Z}/(p))^{-i\xi} \wedge \Sigma E]$, where ξ is the usual complex line bundle. Recall also that the Thom class of $(B\mathbf{Z}/(p))^{-i\xi}$ is in dimension $-2i$, and is not torsion. In fact the spectrum $(B\mathbf{Z}/(p))^{-i\xi}$ has a CW–structure with exactly one cell in each dimension greater than or equal to $-2i$. Since the $-2i$–cell generates a non-torsion element of homology, the attaching map of the $-2i+1$ cell to the $-2i$ skeleton is null. So the cell in dimension $-2i+1$ is spherical, and the inclusion of that $-2i+1$ cell, smashed with the unit of E when E is a ring spectrum, gives

$$x^{-i+1} \in \pi_{-2i+2}[(B\mathbf{Z}/(p))^{-i\xi} \wedge \Sigma E].$$

Now, we take $x^j \in \pi_{2j} tE(n)^{\wedge}_{I_{n-1}}$, and use the $\hat{E}(n-1)$–structure to construct a sequence of maps

$$\bigvee_{j \geq -i} \Sigma^{2j} \hat{E}(n-1) \longrightarrow tE(n)^{\wedge}_{I_{n-1}}. \tag{3.12}$$

We make a map ϕ_{-i} by composing the map of (3.12) with the map

$$tE(n)^{\wedge}_{I_{n-1}} \to (B\mathbb{Z}/(p))^{-(i+1)} \wedge E^{\wedge}_{I_{n-1}}$$

given by [7, Theorem 16.1].

Taking inverse limits of the maps ϕ_{-i} gives a map

$$\varprojlim_i \Big(\bigvee_{j \geq i} \Sigma^{2j} \hat{E}(n-1)\Big) \xrightarrow{f} tE(n)^{\wedge}_{I_{n-1}}.$$

This map defines an isomorphism on the associated graded modules with respect to $\mathfrak{m}$. It follows that f is an equivalence after completion, that is

$$\varprojlim_i \Big(\bigvee_{j \geq i} \Sigma^{2j} \hat{E}(n-1)\Big)^{\wedge}_{I_{n-1}} \simeq tE(n)^{\wedge}_{I_{n-1}}. \qquad \square$$

By a very similar argument one can prove the analog to Proposition 2.8:

Proposition 3.13 *There is an equivalence of spectra*

$$[v_{n-1}^{-1} \varprojlim_i \Big(\bigvee_{j \geq i} \Sigma^{2j} BP\langle n-1\rangle\Big)]^{\wedge}_{I_{n-1}} \to (v_n^{-1} tBP\langle n\rangle)^{\wedge}_{I_{n-1}}.$$

We leave the proof to the interested reader.

Given all the completions that occur in this section and in section 2, one might hope that by using some other, already complete theory like $\hat{E}(n)$ or E_n, we could prove a theorem with a simpler statement. This is unfortunately not the case. There are similar results for these theories, but even if E is complete with respect to $(p, v_1, \ldots, v_{n-2})$, tE will generally not be, and so will need to be completed again. There are variants of Theorem 3.10 for these other spectra as well, but the statement is not simpler.

4 A Honda coordinate on the formal group over tE

In this section we shall take p to be an odd prime and $n > 1$ to be an integer. It will ease the superabundance of superscripts to use the abbreviation $q = p^{n-1}$.

We define a number of formal group laws in this section that are used in the remainder of the paper. For reference, we list them here.

$G(n)$: a homogeneous formal group law of degree 2 on $E(n)^*$ induced by the usual orientation of $E(n)$.

G_n: a twist of the pushforward of $G(n)$ to E_n by the element u. This is homogeneous of degree 0.

F: a twist of the pushforward of $G(n)$ to $E(n)[w]$ of $G(n)$ by w. This is also homogeneous of degree 0.

F_0: the pushforward of F to the residue field of $\pi_0 TE$; TE is defined in equation (4.3).

H: the Honda formal group law of height $n-1$ over $\mathbb{F}_p$.

$\overline{F}$: A formal group law introduced in the proof of Proposition 4.15 that is shown in that proof to be the same as H.

We use the canonical orientation of $E(n)$, which provides a coordinate so that

$$E(n)^*(\mathbb{C}P^\infty) = E(n)^*[\![x]\!];$$

and as usual, if μ is the multiplication on $\mathbb{C}P^\infty$,

$$\mu^*(x) \in E(n)^*[\![x,y]\!] = E(n)^*[\![x]\!]\hat{\otimes} E(n)^*[\![y]\!] = E(n)^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$$

is a formal group law which we will denote $G(n)(x,y)$.

This formal group law has the feature that its p-series is given by $\sum^{G(n)}_{i\le n} v_i x^{p^i}$. Recall that

$$E_n = \mathbb{Z}_p[\![u_1,\dots,u_{n-1}]\!][u,u^{-1}] \tag{4.1}$$

with $|u_i| = 0$ and $|u| = 2$. There is an isomorphism

$$E_n^0(\mathbb{C}P^\infty) \cong \pi_0 E_n[\![t]\!]$$

in terms of which the coproduct on $E_n^0(\mathbb{C}P^\infty)$ is determined by the formula

$$t \mapsto G_n(s,t)$$

where G_n is the group law

$$G_n(s,t) \overset{\text{def}}{=} uG(n)(u^{-1}s,u^{-1}t)$$

over $\pi_0 E_n$.

Since v_{n-1} is a unit in $\pi_*\left(tE(n)\right)^{\wedge}_{I_{n-1}}$, we shall also consider the theory $E(n)[w]$ obtained by adjoining an element w of degree 2 such that

$$w^{q-1} = v_{n-1} \tag{4.2}$$

and then completing with respect to the ideal $I_{n-1} = (p, \dots, v_{n-2})$. We prefer this choice to the usual parameter $u = v_n^{1/(p^n-1)}$ (which gives E_n) because the functor t will emphasize height $n-1$ behavior instead of height n, and the normalization we choose to make things 2–periodic leads to simpler statements in section 4 and this section.

Equation (2.2) shows that there is an isomorphism

$$tE(n)[w] \cong E(n)[w]((x))/[p]_{G(n)}(x).$$

Proposition 2.11 shows that v_{n-1}, and so also w, is a unit in the homotopy of the I_{n-1}–adic completion

$$TE \overset{\text{def}}{=} \left(tE(n)[w] \right)^{\wedge}_{I_{n-1}} \tag{4.3}$$

which is thus 2–periodic. If F denotes the formal group law

$$F(s,t) \overset{\text{def}}{=} wG(n)(w^{-1}s, w^{-1}t) \tag{4.4}$$

(which implies $[p]_F(s) = w[p]_{G(n)}(w^{-1}s)$) and we introduce elements

$$\begin{aligned} w_i &= v_i w^{-(p^i-1)} \\ y &= wx \end{aligned}$$

of degree zero, then the argument of Proposition 2.11 shows that there are isomorphisms

$$\begin{aligned} {}_0TE &\cong \left[\mathbb{Z}_p[w_1, \dots, w_{n-2}, w_n^{\pm 1}]((y))/([p]_F(y)) \right]^{\wedge}_{I_{n-1}} \\ &\cong \left[\mathbb{Z}_p[w_1, \dots, w_{n-2}]((y)) \right]^{\wedge}_{I_{n-1}} \\ TE &\cong \left[\mathbb{Z}_p[w_1, \dots, w_{n-2}]((y)) \right]^{\wedge}_{I_{n-1}} [w, w^{-1}]. \end{aligned} \tag{4.5}$$

The group law F is defined over ${}_0E(n)[w]$ and hence over ${}_0TE$; it is p–typical, and its p–series satisfies the functional equation

$$[p]_F(t) = pt \mathop{+}_F w_1 t^p \mathop{+}_F \cdots \mathop{+}_F w_{n-2} t^{p^{n-2}} \mathop{+}_F t^q \mathop{+}_F w_n t^{pq}. \tag{4.6}$$

We denote by F_0 the image of the group law F in the residue field of ${}_0TE$.

Proposition 4.7 *The residue field of ${}_0TE$ is $\mathbb{F}_p((y))$. The element w_n of ${}_0TE$ maps to $-y^{(1-p)q}$ in the residue field. The formal group law F_0 has coefficients in the subring $\mathbb{F}_p[y^{-1}]$, and its p–series satisfies the functional equation*

$$[p]_{F_0}(t) = t^q \mathop{-}_{F_0} y^{(1-p)q} t^{pq}. \tag{4.8}$$

Proof The statement about the residue field follows from equation (4.5). Before we proceed further, recall that since F_0 is p–typical and p is odd, $[-1]_{F_0}(t) = -t$. Note that the image of equation (4.6) in the residue field is

$$[p]_{F_0}(t) = t^q \underset{F_0}{+} w_n t^{pq};$$

so equation (4.8) follows from the assertion that w_n maps to $-y^{(1-p)q}$.

Since $[p]_{F_0}(y) = 0$ we have

$$0 = y^q \underset{F_0}{+} w_n y^{pq}$$

and so

$$\begin{aligned} y^q &= [-1]_{F_0}(w_n y^{pq}) = -w_n y^{pq} \\ w_n &= -y^{(1-p)q}. \end{aligned}$$

Finally, F_0 is defined over the subring $\mathbb{F}_p[y^{-1}]$ since F is actually defined over the polynomial ring in $w_1, \ldots, w_n$, and $(1-p)q$ is negative. □

So F_0 is a p–typical formal group law over $\mathbb{F}_p[y^{-1}]$, of height $n-1$ in the field $\mathbb{F}_p(y)$ or $\mathbb{F}_p((y))$. On the other hand, let H be the Honda law of height $n-1$ over $\mathbb{F}_p$, characterized by the fact that it is p–typical with p–series

$$[p]_H(t) = t^q.$$

Comparison with equation (4.8) shows that

$$F_0 \equiv H \mod y^{-1}. \tag{4.9}$$

Now Lazard [16, 6] proves that over the separable closure $\mathbb{F}_p(y)^{\text{sep}}$, there is an isomorphism of formal group laws

$$F_0 \cong H;$$

which is in general not at all canonical. In this section we show that there is a unique isomorphism $\theta: F_0 \cong H$ which preserves equation (4.9) in a suitable sense.

To make this precise, note that expanding a rational function as a power series at infinity gives a map of fields

$$\mathbb{F}_p(y) \to \mathbb{F}_p((y^{-1}))$$

which extends to

$$\mathbb{F}_p(y)^{\text{sep}} \to \mathbb{F}_p((y^{-1}))^{\text{sep}}.$$

Theorem 4.10 *There is a unique isomorphism* $: F_0 \to H$ *of formal group laws over* $\mathbb{F}_p(y)^{\text{sep}}$ *such that the image of* *in* $\mathbb{F}_p((y^{-1}))^{\text{sep}}$ *has coe cients in the image of*

$$\mathbb{F}_p[[y^{-1}]] \to \mathbb{F}_p((y^{-1}))^{\text{sep}}.$$

gives an isomorphism over $\mathbb{F}_p[[y^{-1}]]$ *satisfying*

$$(t) \quad t \mod y^{-1}.$$

We build up to the proof gradually; the proof itself appears after Proposition 4.15.

Lemma 4.11 *There is a unique series* $(t) \in \mathbb{F}_p[y^{-1}][[t]]$ *such that*

$$[p]_{F_0} = [p]_H \quad .$$

This power series has the properties that

$$(t) \quad t \mod t^2$$
$$(t) \quad t \mod y^{-1}.$$

Proof The equation which must satisfy is

$$[p]_{F_0}(t) = (t)^q.$$

If

$$F_0(s;t) = \sum_{i,j} b_{ij} s^i t^j$$

then the functional equation (4.8) becomes

$$[p]_{F_0}(t) = \sum_{i,j} b_{ij} (t^q)^i (-y^{(1-p)q} t^{pq})^j$$
$$= \sum_{i,j} b_{ij} (-y^{1-p})^{jq} t^{(i+pj)q}.$$

So we must show that b_{ij} has a unique q^{th} root. If it does then

$$(t) = \sum_{i,j} b_{ij}^{1/q} (-y^{1-p})^j t^{(i+pj)}$$

which shows that $(t) \quad t \mod t^2$.

If

$$G(n)(s;t) = \sum_{i,j} a_{ij} s^i t^j$$

then by definition

$$F(s,t) = \sum_{i,j} a_{ij} w^{1-i-j} s^i t^j,$$

with $b_{ij} = a_{ij} w^{1-i-j}$ homogeneous of degree zero. The $(p, v_1, \dots, v_{n-2})$ reduction of a_{ij} is of the form

$$a_{ij} = \sum_{a,b} c_{ab} v_{n-1}^a v_n^b \in \mathbb{F}_p[v_{n-1}, v_n] \tag{4.12}$$

for coefficients c_{ab} depending on i, j. Substituting

$$\begin{aligned} v_{n-1} &= w^{q-1} \\ v_n &= w^{pq-1} w_n \\ &= -w^{pq-1} y^{(1-p)q}, \end{aligned}$$

into equation (4.12), one has

$$b_{ij} = w^{1-i-j} \sum_{a,b} c_{ab} w^{a(q-1)} (-w^{pq-1} y^{(1-p)q})^b.$$

As b_{ij} is homogeneous of degree zero, the exponent of w in each term must add to zero, and so

$$\begin{aligned} b_{ij} &= \sum_{a,b} c_{ab} (-y^{1-p})^{bq} \\ b_{ij}^{1/q} &= \sum_{a,b} c_{ab} (-y^{1-p})^b \end{aligned}$$

since $c_{ab} \in \mathbb{F}_p$ so $c_{ab}^{1/q} = c_{ab}$. Thus one has

$$\begin{aligned} (t) &= \sum_{i,j} r_{ij} t^{(i+pj)}, \text{ with} \\ r_{ij} &= \sum_{a,b} c_{ab} (-y^{1-p})^{b+j} \end{aligned}$$

(recall the c_{ab} depend on i and j), which shows that $(t) \quad t \mod y^{-1}$ as well. □

Proposition 4.13 *There is a unique power series* $(t) \in \mathbb{F}_p[[y^{-1}]][[t]]$ *such that*

$$\begin{aligned} (t) &\quad t \mod t^2; \\ (t) &\quad t \mod y^{-1}; \text{ and} \\ [p]_H &= [p]_{F_0} \end{aligned}$$

in $\mathbb{F}_p[[y^{-1}]][[t]]$.

Proof If f in $A[[t]]$ is a series with coe cients in a ring A of characteristic p, let f be the corresponding series with coe cients those of f, raised to the q^{th} power; thus

$$f\ (t^q) = (f(t))^q:$$

In this notation, the equation supposedly satis ed by takes the form

$$=$$

where $2\ \mathbb{F}_p[y^{-1}][[t]]$ is the power series constructed in Lemma 4.11. If $^{-1}$ denotes the compositional inverse (not reciprocal) of , and similarly $^{-}$ denotes $(\ ^{-1})$, then this equation can be rewritten in the form

$$= \quad ^{-}$$

$$= [\quad ^{-}\ ]\quad ^{-} = ::::$$

If is to be of the form

$$(t) \quad t \mod y^{-1};$$

then as r grows, ^{r} will converge to the identity in the (y^{-1}){adic topology, and we must have

$$= \lim_{r\ !\ 1}\ ^{-\ r}\quad ^{-\ r-1}\quad ^{-}: \tag{4.14}$$

On the one hand, if the limit exists, then it certainly intertwines the p{series. On the other hand, the limit exists: Lemma 4.11 asserts that

$$(t)\ 2\ t + y^{-1}\mathbb{F}_p[y^{-1}][[t]];$$

so $^{-\ r}$ converges to the identity in the (y^{-1}){adic topology. The group of formal power series under composition is complete with respect to the non-archimedean norm de ned by the degree of the leading term, so the in nite composite (4.14) converges because the sequence of composita converges to the identity. It is easy to see in addition that inherits the property

$$(t) \quad t \mod t^2$$

from . □

Proposition 4.15 *The power series is the unique strict isomorphism*

$$F_0 -!\ H$$

of formal group laws over $\mathbb{F}_p[[y^{-1}]]$ *such that*

$$(t) \quad t \mod y^{-1}:$$

Proof Let us write $\overline{F}$ for the formal group law F_0 where

$$\overline{F}_0(x, y) = \ (F_0(\ ^{-1}(x),\ ^{-1}(y))).$$

Since $[p]_{\overline{F}}(t) = t^q$, the uniqueness of follows from Proposition 4.13.

We need to show $\overline{F} = H$. There is a canonical strict isomorphism

$$G \longrightarrow \overline{F},$$

from a p–typical formal group law G, de ned over $\mathbb{F}_p[\![y^{-1}]\!]$. Indeed is given by the formula [24, A2.1.23]

$$(t) = \sum_{p \nmid r \geq 1}^{\overline{F}} [\ (r)]_{\overline{F}}[\tfrac{1}{r}]_{\overline{F}} \sum_{i=1}^{r}{}^{\overline{F}} \ ^{i}t, \tag{4.16}$$

where

$$(r) = \begin{cases} 0 & r \text{ is divisible by a square} \\ (-1)^k & r \text{ is the product of } k \text{ distinct primes} \end{cases}$$

and is a primitive rth root of 1. We rst claim that $G = H$; then we shall show that is the identity.

As G and H are both are p–typical, it su ces to show that $[p]_G = [p]_H$. Since is a homomorphism of groups, one has

$$[p]_{\overline{F}} \quad = \quad [p]_G \tag{4.17}$$

Using equation (4.16), one has

$$\begin{aligned} [p]_{\overline{F}} \quad (t) &= \sum_{p \nmid r \geq 1}^{\overline{F}} [\tfrac{(r)}{r}]_{\overline{F}} \sum_{i=1}^{r}{}^{\overline{F}} [p]_{\overline{F}}(\ ^{i}t) \\ &= \sum_{p \nmid r \geq 1}^{\overline{F}} [\tfrac{(r)}{r}]_{\overline{F}} \sum_{i=1}^{r}{}^{\overline{F}} \ ^{iq}t^q \\ &= \ (t^q) \end{aligned}$$

since $q = p^{n-1}$ and $[p]_{\overline{F}}(t) = t^q$. Thus

$$([p]_G(t)) = \ (t^q) = \ ([p]_H(t)).$$

Next we must show that is the identity. As

$$(t) \quad t \quad \bmod y^{-1}$$

we have

$$F_0 \quad \overline{F} \quad \bmod y^{-1}.$$

But F_0 is p-typical, so

$$[\tfrac{1}{r}]_{F_0} \sum_{i=1}^{r}{}^{F_0} \; {}^{i}t = 0$$

for $p \nmid r > 1$; it follows from equation (4.16) that

$$(t) \quad t \quad \bmod y^{-1}. \tag{4.18}$$

But equation (4.17) and $[p]_{\overline{F}}(t) = t^q$ imply that

$$(t^q) = \; (t)^q;$$

if $\; = \sum_{i \; 1} \; {}_i t^i$ then we must have

$$_i = \; ^q_i.$$

Together with equation (4.18) this implies $\;_1 = 1$. For $i > 1$ we have $\;_i \in y^{-1}\mathbb{F}_p[\![y^{-1}]\!]$, so $\;_i = 0$. □

Proof of Theorem 4.10 If

$$\Psi(t) = \sum_{i \; 0} \Psi_i t^{i+1}$$

is *any* isomorphism from the group law F_0 to the group law H over $\mathbb{F}_p(y)^{\mathrm{sep}}$, then it must satisfy the equation

$$[p]_H \quad \Psi = \Psi \quad [p]_{F_0}.$$

By Lemma 4.11, we must have

$$\Psi \quad = \Psi \qquad . \tag{4.19}$$

Because

$$(t) \quad t \quad \bmod t^2,$$

the intertwining equation (4.19) can be rewritten inductively as a sequence of generalized Artin-Schreier equations

$$\Psi_i^q - \Psi_i = \text{a polynomial in } \Psi_j\text{'s with } j < i, \tag{4.20}$$

beginning with

$$\Psi_0^q - \Psi_0 = 0.$$

Because of (4.20), the Ψ_i are all algebraic, and the Galois group of the extension they generate acts by translating the solutions by an element of the field with q elements.

The coefficients of ψ satisfy the same equations in $\mathbb{F}_p((y^{-1}))$. Starting with $\Psi_0 = 1$, we may adjust each Ψ_i by a Galois transformation so that its image in $\mathbb{F}_p((y^{-1}))^{\mathrm{sep}}$ is ψ_i. The resulting power series Ψ is an isomorphism of formal group laws in $\mathbb{F}_p(y)^{\mathrm{sep}}$, since it becomes one in $\mathbb{F}_p((y^{-1}))^{\mathrm{sep}}$.

The uniqueness of Ψ satisfying the hypotheses is a trivial consequence of the uniqueness of ψ in Proposition 4.13. □

It is the field $\mathbb{F}_p((y))$ rather than $\mathbb{F}_p((y^{-1}))$ which appears in the Tate homology calculations. Expanding a rational function as a power series at zero gives an embedding $\mathbb{F}_p(y) \to \mathbb{F}_p((y))$, which extends to an embedding

$$\mathbb{F}_p(y)^{\mathrm{sep}} \to \mathbb{F}_p((y))^{\mathrm{sep}}.$$

Thus we have

Corollary 4.21 *There is a unique strict isomorphism $\psi\colon F_0 \to H$, satisfying*

(1) *the coefficients of ψ are in the subfield $\mathbb{F}_p(y)^{\mathrm{sep}}$ and*

(2) *the expansion of ψ at $y = \infty$ is a power series with coefficients in $\mathbb{F}_p[[y^{-1}]]$, congruent to the identity modulo y^{-1}.*

5 A map of ring spectra

Lubin and Tate's theory of lifts

We recall briefly the deformation theory of Lubin and Tate [19]. Suppose that $\mathbb{F}$ is a field of characteristic p.

Definition 5.1 A *lift* of $\mathbb{F}$ is a pair (A, i) consisting of

(1) a Noetherian complete local ring A with residue field A_0;

(2) a map of fields $i\colon \mathbb{F} \to A_0$.

A map $f\colon (A, i) \to (B, j)$ (or (A, i)–*algebra*) is a local homomorphism

$$A \xrightarrow{f} B$$

such that $j = f_0 \circ i$, where $f_0\colon A_0 \to B_0$ is induced by f.

We shall abbreviate (A, i) to A when i is clear from context.

Suppose that Γ is a formal group law of finite height n over a field $\mathbb{F}$ of characteristic p, that (A, i) is a lift of $\mathbb{F}$, and that (B, j) is an (A, i)–algebra.

Definition 5.2 A *deformation of* Γ *to* (B, j) is a pair (G, ι) consisting of

(1) a formal group law G over B;

(2) an isomorphism of group laws $\iota\colon j^*\Gamma \cong G_0$, where G_0 denotes the group law over B_0 induced by G.

Two deformations (G, ι) and (G', ι') are $\star$–isomorphic if there is an isomorphism $c\colon G \to G'$ such that

$$\iota' = c_0 \circ \iota\colon j^*\Gamma \to G'_0$$

in B_0.

The set of $\star$–isomorphism classes of deformations to (B, j) is a functor from (A, i)–algebras to sets; Lubin and Tate show that this functor is representable. Namely, let

$$R = A[\![u_1, \ldots, u_{n-1}]\!],$$
$$C_{p^i}(x, y) = \frac{1}{p}[(x+y)^{p^i} - x^{p^i} - y^{p^i}],$$

and let (G, ι) be any deformation of Γ to R such that

$$\begin{aligned} G_0 &= \Gamma \\ G(s, t) &\equiv s + t + u_i C_{p^i}(x, y) \quad \text{mod } u_1, \ldots, u_{i-1}, \text{ and degree } p^i + 1 \end{aligned} \tag{5.3}$$

for a unit of A_0 and $1 \leq i \leq n-1$. Lubin and Tate show that such deformations exist; we shall call such a group law a *Lubin-Tate lift* of Γ. Theorem 3.1 of [19] may be phrased as follows.

Theorem 5.4 *If* (G', ι') *is a deformation of* Γ *to an* (A, i)*–algebra* (B, j)*, then there is a unique map of* (A, i)*–algebras* $f\colon R \to (B, j)$ *such that there is a* $\star$*–isomorphism*

$$(f^*G, f_0 \circ \iota) \xrightarrow{\cong} (G', \iota').$$

Moreover the $\star$*–isomorphism is unique.*

The map of ring theories

A classical theorem of I Cohen (see [27]) asserts that for any field $\mathbf{F}$ of characteristic p, there is an essentially unique complete discrete valuation ring $C_{\mathbf{F}}$ with maximal ideal generated by p and residue field $\mathbf{F}$; in particular, $C_{\mathbf{F}}$ is Noetherian. If $\mathbf{F}$ is perfect, then $C_{\mathbf{F}}$ is just the ring $W_{\mathbf{F}}$ of Witt vectors of $\mathbf{F}$, but in general it is a subring of the Witt vectors, and (although it is a functor) it is not very easily described. However, when the degree $[\mathbf{F} : \mathbf{F}^p]$ is finite (instead of 1 as in the perfect case), eg if $\mathbf{F} = \mathbf{F}_p((y))$, then the Cohen ring is still relatively tractable. In this case, for example, we have

Lemma 5.5 *The Cohen ring of $\mathbf{F}_p((y))$ is isomorphic to the p–adic completion of $\mathbf{Z}_p((y))$. Any such isomorphism extends to an isomorphism*

$$C_{\mathbf{F}_p((y))}[[w_1, \dots, w_{n-2}]] \cong \pi_0 TE.$$

Proof The p–adic completion of $\mathbf{Z}_p((y))$ is a complete discrete valuation ring with maximal ideal generated by p and residue field $\mathbf{F}_p((y))$, so it is isomorphic to the Cohen ring of $\mathbf{F}_p((y))$. Indeed Schoeller [27, §8] shows that an isomorphism is given by a choice of p–base for $\mathbf{F}_p((y))$ and representatives of that p–base in $(\mathbf{Z}_p((y)))^\wedge_p$.

Equation (4.5) reduces the second part of the lemma to the observation that the rings

$$\left(\mathbf{Z}_p((y))\right)^\wedge_p[[w_1, \dots, w_{n-2}]] \cong \left(\mathbf{Z}_p[[w_1, \dots, w_{n-2}]]((y))\right)^\wedge_{I_{n-1}}$$

are isomorphic. □

Proposition 5.6 *The formal group law F over $\pi_0 TE$ is a Lubin–Tate lift of the group law F_0 of height $n-1$ over $\mathbf{F}_p((y))$ to $C_{\mathbf{F}_p((y))}$–algebras.*

Proof It is a standard fact about $G(n)$ that

$$G(s,t) \equiv s + t + v_i C_{p^i}(s,t) \quad \text{mod } v_1, \dots, v_{i-1}, \text{ and degree } p^i + 1$$

for $1 \le i \le n-1$. By the definition (4.4) of F, one then has

$$F(s,t) \equiv s + t + w_i C_{p^i}(s,t) \quad \text{mod } w_1, \dots, w_{i-1}, \text{ and degree } p^i + 1$$

for $1 \le i \le n-2$. It follows that F satisfies equations (5.3) for the group law $\Gamma = F_0$. □

The formal group law G_{n-1} over ${}_0E_{n-1}$ has image H in $\mathbf{F}_p$, so by Corollary 4.21 the pair $(C_{\mathbf{F}_p((y))^{\mathrm{sep}}} \wedge G_{n-1}, \ )$ is a deformation of F_0 to the $C_{\mathbf{F}_p((y))}$–algebra

$$C_{\mathbf{F}_p((y))^{\mathrm{sep}}} \underset{\mathbf{Z}_p}{\wedge} {}_0E_{n-1} = C_{\mathbf{F}_p((y))^{\mathrm{sep}}}[\![u_1, \ldots, u_{n-2}]\!].$$

Theorem 5.4 provides a ring homomorphism

$${}_0TE \xrightarrow{f} C_{\mathbf{F}_p((y))^{\mathrm{sep}}} \underset{\mathbf{Z}_p}{\wedge} {}_0E_{n-1} \tag{5.7}$$

and an isomorphism of formal group laws

$$f\ F \underset{\cong}{\xrightarrow{c}} C_{\mathbf{F}_p((y))^{\mathrm{sep}}} \underset{\mathbf{Z}_p}{\wedge} G_{n-1} \tag{5.8}$$

such that $c_0 = \ $.

Proposition 5.9 *The map*

$$C_{\mathbf{F}_p((y))^{\mathrm{sep}}} \wedge f \colon C_{\mathbf{F}_p((y))^{\mathrm{sep}}} \wedge_{C_{\mathbf{F}_p((y))}} {}_0TE \to C_{\mathbf{F}_p((y))^{\mathrm{sep}}} \underset{\mathbf{Z}_p}{\wedge} {}_0E_{n-1}$$

is an isomorphism.

Proof The ring on the left represents deformations of $\mathbf{F}_p((y))^{\mathrm{sep}} \wedge F_0$ to $C_{\mathbf{F}_p((y))^{\mathrm{sep}}}$–algebras, while the ring on the right represents deformations of $\mathbf{F}_p((y))^{\mathrm{sep}} \wedge H$ to $C_{\mathbf{F}_p((y))^{\mathrm{sep}}}$–algebras. The isomorphism of Corollary 4.21 induces an isomorphism between these functors. □

There are isomorphisms

$$TE^0(\mathbf{C}P^{\infty}) = TE^0[\![t_1]\!]$$
$$E^0_{n-1}(\mathbf{C}P^{\infty}) = E^0_{n-1}[\![t_2]\!]$$

such that the formal group law F expresses the coproduct on $TE^0(\mathbf{C}P^{\infty})$, and the formal group law G_{n-1} expresses the coproduct on E_{n-1}. A standard argument [23] using Landweber's exact functor theorem gives:

Theorem 5.10 *There is a canonical map of ring theories*

$$TE \longrightarrow C_{\mathbf{F}_p((y))^{\mathrm{sep}}} \underset{\mathbf{Z}_p}{\wedge} E_{n-1}$$

whose value on coefficients is f, and whose value on $TE^0(\mathbf{C}P^{\infty})$ is determined by the equation

$$(t_1) = c(t_2).$$

Here c is the isomorphism (5.8). □

Corollary 5.11 *$C_{\mathbf{F}_p((y))^{sep}} \wedge_{\mathbf{Z}_p}$ gives an equivalence of complex oriented ring spectra*

$$C_{\mathbf{F}_p((y))^{sep}} \wedge_{C_{\mathbf{F}_p((y))}} TE \to C_{\mathbf{F}_p((y))^{sep}} \wedge_{\mathbf{Z}_p} E_{n-1}.$$

Composing the map from Theorem 5.10 with the maps

$$E(n) \to E(n)[w] \to TE$$

gives a canonical map of ring theories

$$E(n) \to C_{\mathbf{F}_p((y))^{sep}} \wedge_{\mathbf{Z}_p} E_{n-1},$$

our generalized Chern character.

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