ISSN 1472-2739 (on-line) 1472-2747 (printed)

Algebraic & Geometric Topology
Volume 3 (2003) 873–904
Published: 25 September 2003

Finite subset spaces of graphs and punctured surfaces

Christopher Tuffley

Abstract The kth finite subset space of a topological space X is the space $\exp_k X$ of non-empty finite subsets of X of size at most k, topologised as a quotient of X^k. The construction is a homotopy functor and may be regarded as a union of configuration spaces of distinct unordered points in X. We calculate the homology of the finite subset spaces of a connected graph Γ, and study the maps $(\exp_k \phi)_*$ induced by a map $\phi\colon \Gamma \to \Gamma'$ between two such graphs. By homotopy functoriality the results apply to punctured surfaces also. The braid group B_n may be regarded as the mapping class group of an n–punctured disc D_n, and as such it acts on $H_*(\exp_k D_n)$. We prove a structure theorem for this action, showing that the image of the pure braid group is nilpotent of class at most $\lfloor (n-1)/2 \rfloor$.

AMS Classification 54B20; 05C10, 20F36, 55Q52

Keywords Configuration spaces, finite subset spaces, symmetric product, graphs, braid groups

1 Introduction

1.1 Finite subset spaces

Let X be a topological space and k a positive integer. The kth finite subset space of X is the space $\exp_k X$ of nonempty subsets of X of size at most k, topologised as a quotient of X^k via the map

$$(x_1, \dots, x_k) \mapsto \{x_1\} \cup \cdots \cup \{x_k\}.$$

The construction is functorial: given a map $f\colon X \to Y$ we obtain a map $\exp_k f\colon \exp_k X \to \exp_k Y$ by sending $S \subseteq X$ to $f(S) \subseteq Y$. Moreover, if $\{h_t\}$ is a homotopy between f and g then $\{\exp_k h_t\}$ is a homotopy between $\exp_k f$ and $\exp_k g$, so that $\exp_k$ is in fact a homotopy functor.

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The first finite subset space is of course simply X, and the second finite subset space co-incides with the second symmetric product $\mathrm{Sym}^2 X = X^2/S_2$. However, for $k \geq 3$ we have a proper quotient of the symmetric product as $\exp_k X$ is unable to record multiplicities: both (a,a,b) and (a,b,b) in X^3 map to $\{a,b\}$ in $\exp_3 X$. As a result there are natural inclusion maps

$$\exp_j X \hookrightarrow \exp_k X : S \mapsto S \tag{1.1}$$

whenever $j \leq k$, stratifying $\exp_k X$. We define the full finite subset space $\exp X$ to be the direct limit of this system of inclusions,

$$\exp X = \varinjlim \exp_k X.$$

If X is Hausdorff then the subspace topology on $\exp_j X \subseteq \exp_k X$ co-incides with the quotient topology it receives from X^j [11]. In this case each stratum $\exp_j X \setminus \exp_{j-1} X$ is homeomorphic to the configuration space of sets of j distinct unordered points in X, so that $\exp_k X$ may be regarded as a union over $1 \leq j \leq k$ of these spaces. Moreover $\exp_k X$ is compact whenever X is, in which case it gives a compactification of the corresponding configuration space. Such spaces and their compactifications have been of considerable interest recently in algebraic topology: see, for example, Fulton and MacPherson [10] and Ulyanov [22].

For each k and ℓ the isomorphism $X^k \times X^\ell \to X^{k+\ell}$ descends to a map

$$\cup\colon \exp_k X \times \exp_\ell X \to \exp_{k+\ell} X$$

sending (S,T) to $S \cup T$. This leads to a form of product on maps $g\colon Y \to \exp_k X$, $h\colon Z \to \exp_\ell X$, and we define $g \cup h\colon Y \times Z \to \exp_{k+\ell} X$ to be the composition

$$Y \times Z \xrightarrow{g \times h} \exp_k X \times \exp_\ell X \xrightarrow{\cup} \exp_{k+\ell} X.$$

Clearly $(f \cup g) \cup h = f \cup (g \cup h)$. Given a point $x_0 \in X$ we obtain as a special case a map

$$\cup\{x_0\}\colon \exp_k X \to \exp_{k+1} X$$

taking $S \subseteq X$ to $S \cup \{x_0\}$. The image of $\cup\{x_0\}$ is the subspace $\exp_{k+1}(X, x_0)$ consisting of the $k+1$ or fewer element subsets of X that contain x_0. In contrast to the symmetric product, where the analogous map plays the role of (1.1), the spaces $\exp_k X$ and $\exp_{k+1}(X, x_0)$ are in general topologically different. The map $\cup\{x_0\}$ is one-to-one at the point $\{x_0\} \in \exp_1 X$ and on the top level stratum $\exp_k X \setminus \exp_{k-1} X$, but is two-to-one elsewhere, as S and $S \cup \{x_0\}$ have the same image for $|S| < k$, $x_0 \notin S$. Nevertheless the based finite subset spaces $\exp_k(X, x_0)$ frequently act as a stepping stone in understanding $\exp_k X$, often being topologically simpler.

1.2 History

The space $\exp_k X$ was introduced by Borsuk and Ulam [6] in 1931 as the symmetric product, and since then appears to have been studied at irregular intervals, under various notations, and principally from the perspective of general topology. In their original paper Borsuk and Ulam showed that $\exp_k I = I^k$ for $k = 1, 2, 3$, but that $\exp_k I$ cannot be embedded in $\mathbb{R}^k$ for $k \geq 4$. In 1957 Molski [15] proved similar results for I^2 and I^n, namely that $\exp_2 I^2 = I^4$ but that neither $\exp_k I^2$ nor $\exp_2 I^k$ can be embedded in $\mathbb{R}^{2k}$ for any $k \geq 3$. The last was done by showing that $\exp_2 I^k$ contains a copy of $S^k \times \mathbb{R}P^{k-1}$.

Other authors including Curtis [8], Curtis and To Nhu [9], Handel [11], Illanes [12] and Macías [14] have established general topological and homotopy-theoretic properties of $\exp_k X$ and $\exp X$, and Beilinson and Drinfeld [2, sec. 3.5.1] and Ran [17] have used these spaces in the context of mathematical physics and algebraic geometry. The set $\exp X$ has also been studied extensively under a different topology as the Pixley-Roy hyperspace of finite subsets of X; the two topologies are surveyed in Bell [3]. We mention some results on $\exp_k X$ of a homotopy-theoretic nature. In 1999 Macías showed that for compact connected metric X the first Čech cohomology group $H^1(\exp_k X; \mathbb{Z})$ vanishes for $k \geq 3$, and in 2000 Handel proved that for closed connected n–manifolds, $n \geq 2$, the singular cohomology group $H^i(\exp_k M^n; \mathbb{Z}/2\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ for $i = nk$, and 0 for $i > nk$. In addition, Handel showed that the inclusion maps $\exp_k(X, x_0) \hookrightarrow \exp_{2k-1}(X, x_0)$ and $\exp_k X \hookrightarrow \exp_{2k+1} X$ induce the zero map on all homotopy groups for path connected Hausdorff X.

However, although these and other properties of $\exp_k$ have been established, it appears that until recently the only homotopically non-trivial space for which $\exp_k X$ was at all well understood for $k \geq 3$ was the circle. In 1952 Bott [7] proved the surprising result that $\exp_3 S^1$ is homeomorphic to the three-sphere, correcting Borsuk's 1949 paper [5], and Shchepin (unpublished; for three different proofs see [16] and [19]) later proved the even more striking result that $\exp_1 S^1$ inside $\exp_3 S^1$ is a trefoil knot. An elegant geometric construction due to Mostovoy [16] in 1999 connects both of these results with known facts about the space of lattices in the plane, and in our previous paper [19] we showed that Bott's and Shchepin's results can be viewed as part of a larger pattern: $\exp_k S^1$ has the homotopy type of an odd dimensional sphere, and $\exp_k S^1 \setminus \exp_{k-2} S^1$ that of a $(k-1, k)$–torus knot complement. This paper aims to increase the list of spaces for which $\exp_k$ is understood by using the techniques of [19] to study the finite subset spaces of connected graphs. The results apply to punctured surfaces too, by homotopy equivalence, and represent a step towards understanding finite subset spaces of closed surfaces, as they may be used to study these via Mayer-Vietoris type arguments. Further steps towards this goal are taken in our dissertation [20], in which this paper also appears.

Various different notations have been used for $\exp_k X$, including $X(k)$, $X^{(k)}$, $F_k(X)$ and $Sub(X,k)$. Our notation follows that used by Mostovoy [16] and reflects the idea that we are truncating the (suitably interpreted) series

$$\exp X = \emptyset \cup X \cup \frac{X^2}{2!} \cup \frac{X^3}{3!} \cup \cdots$$

at the $X^k/k! = X^k/S_k$ term. The name, however, is our own. There does not seem to be a satisfactory name in use among geometric topologists — indeed, recent authors Mostovoy and Handel do not use any name at all — and while symmetric product has stayed in use among general topologists we prefer to use this for X^k/S_k. We therefore propose the descriptive name "kth finite subset space" used here and in our previous paper.

1.3 Summary of results

We study the finite subset spaces of a connected graph Γ using techniques from our previous paper [19] on $\exp_k S^1$. Since $\exp_k$ is a homotopy functor we may reduce to the case where Γ has a single vertex, and accordingly define Γ_n to be the graph with one vertex v and n edges $e_1, \ldots, e_n$. Our first result is a complete calculation of the homology of $\exp_k(\Gamma_n, v)$ and $\exp_k \Gamma_n$ for each k and n:

Theorem 1 *The reduced homology groups of* $\exp_k(\Gamma_n, v)$ *vanish outside dimension* $k-1$ *and those of* $\exp_k \Gamma_n$ *vanish outside dimensions* $k-1$ *and* k. *The non-vanishing groups are free. The maps*

$$i\colon \exp_k(\Gamma_n, v) \hookrightarrow \exp_k \Gamma_n$$

and

$$\cup\{v\}\colon \exp_k \Gamma_n \to \exp_{k+1}(\Gamma_n, v)$$

induce isomorphisms on H_{k-1} *and* H_k *respectively while*

$$\exp_k \Gamma_n \hookrightarrow \exp_{k+1} \Gamma_n$$

is twice $(i \cup \{v\})$ *on* H_k. *The common rank of*

$$H_k(\exp_k \Gamma_n) = H_k(\exp_{k+1} \Gamma_n) = H_k(\exp_{k+1}(\Gamma_n, v))$$

is given by

$$
\begin{aligned}
b_k(\exp_k\Gamma_n) &= \sum_{j=1}^{k} (-1)^{j-k}\binom{n+j-1}{n-1} \\
&= \begin{cases} \sum_{j=1}^{\ell}\binom{n+2j-2}{n-2} & \text{if } k = 2\ell \text{ is even,} \\ n + \sum_{j=1}^{\ell}\binom{n+2j-1}{n-2} & \text{if } k = 2\ell+1 \text{ is odd.} \end{cases}
\end{aligned}
\tag{1.2}
$$

A list of Betti numbers $b_k(\exp_k\Gamma_n)$ for $1 \le k \le 20$ and $1 \le n \le 10$ appears as table 1 in the appendix on page 903.

In the case of a circle the homology and fundamental group of $\exp_k S^1$ were enough to determine its homotopy type completely. The argument no longer applies to $\exp_k\Gamma_n$, $n \ge 2$, and its applicability for $n = 1$ is perhaps properly regarded as being due to a "small numbers co-incidence", the vanishing of $H_{2\ell}(\exp_{2\ell}\Gamma_1)$. However, the argument does apply to $\exp_k(\Gamma_n; v)$, and for $k \ge 2$ we have the following:

Theorem 2 *For $k \ge 2$ the space $\exp_k(\Gamma_n; v)$ has the homotopy type of a wedge of $b_{k-1}(\exp_k(\Gamma_n; v))$ $(k-1)$–spheres.*

Having calculated the homology of $\exp_k\Gamma_n$ we turn our attention to the maps $(\exp_k\phi)_*$ induced by maps $\phi\colon \Gamma_n \to \Gamma_m$. Our main result is to reduce the problem of calculating such maps to one of finding images of chains under maps

$$\exp_1 S^1 \to \exp_1 S^1$$

and

$$\exp_2 S^1 \to \exp_2\Gamma_2$$

induced by maps $S^1 \to S^1$ and $S^1 \to \Gamma_2$ respectively. The reduction is achieved by defining a ring without unity structure on a subgroup C of the cellular chain complex of $\exp\Gamma_n$. The subgroup carries the top homology of $\exp_k\Gamma_n$ and is preserved by chain maps of the form $(\exp\phi)_J$, and the ring structure is defined in such a way that these chain maps are ring homomorphisms. The ring $C \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated over $\mathbb{Q}$ by cells in dimensions one and two, leaving a mere $2n$ cells whose images must be found directly.

As an application of these results and as an illustration of how much $(\exp_k\phi)_*$ remembers about ϕ we study the action of the braid group B_n on $H_k(\exp_k\Gamma_n)$. The braid group may be regarded as the mapping class group of a punctured disc and as such it acts on the graph Γ_n via homotopy equivalence. We show that, for a suitable choice of basis, the braid group acts by block upper-triangular matrices whose diagonal blocks are representations of B_n that factor through S_n. Consequently, the image of the pure braid group consists of upper-triangular matrices with ones on the diagonal and is therefore nilpotent. The number of blocks depends mildly on k and n but is no more than about $n/2$, and this leads to a bound on the length of the lower central series.

We remark that the main results of this paper may be used to study the finite subset spaces of a closed surface Σ via Mayer-Vietoris type arguments. This may be done by constructing a cover of $\exp_k \Sigma$ such that each element of the cover and each m–fold intersection is a finite subset space of a punctured surface, as follows. Choose $k+1$ distinct points $p_1, \ldots, p_{k+1}$ in Σ and let $U_i = \exp_k\left(\Sigma \setminus \{p_i\}\right)$. The U_i form an open cover of $\exp_k \Sigma$, since each $\xi \in \exp_k \Sigma$ must omit at least one of the p_i, and moreover each m–fold intersection has the form

$$\bigcap_{j=1}^{m} U_{i_j} = \exp_k\left(\Sigma \setminus \{p_{i_1}, \ldots, p_{i_m}\}\right),$$

a finite subset space of a punctured surface as desired. The results of this paper may then be used to calculate the homology of each intersection and the maps induced by inclusion, leading to a spectral sequence for $H_*(\exp_k \Sigma)$.

In [21] this idea is used to prove two vanishing theorems for the homotopy and homology groups of the finite subset spaces of a connected cell complex.

1.4 Outline of the paper

The calculation of the homology of $\exp_k \Gamma_n$ and $\exp_k(\Gamma_n; v)$ is the main topic of section 2. We find explicit cell structures for these spaces in section 2.2 and use them to calculate their fundamental groups. We then show that the reduced chain complex of $\exp_*(\Gamma_n; v)$ is exact in section 2.3, and use this to prove Theorems 1 and 2 in section 2.4. We give an explicit basis for $H_k(\exp_k \Gamma_n)$ in section 2.5 and close with generating functions for the Betti numbers $b_k(\exp_k \Gamma_n)$ in section 2.6. A table of Betti numbers for $1 \le k \le 20$ and $1 \le n \le 10$ appears in the appendix on page 903.

We then turn to the calculation of induced maps in section 3. The ring structure on $\mathcal{C}$ is motivated and defined in section 3.2 and we show that maps $\phi\colon \Gamma_n \to \Gamma_m$ induce ring homomorphisms in section 3.3. As illustration of the ideas we calculate two examples in section 3.4, the first reproducing a result from [19] and the second relating to the generators of the braid groups. We then state and prove the structure theorem for the braid group action in sections 4.1 and 4.2, and conclude by looking at the action of B_3 on $H_3(\exp_3\Gamma_3)$ in some detail in section 4.3.

1.5 Notation and terminology

We take a moment to fix some language and notation that will be used throughout.

We will work exclusively with graphs having just one vertex, so as above we define Γ_n to be the graph with one vertex v and n edges $e_1,\ldots,e_n$. Write I for the interval $[0,1]$, and for each non-negative integer m let $[m] = \{i \in \mathbb{Z} \mid 1 \le i \le m\}$. We parameterise Γ_n as the quotient of $I \times [n]$ by the subset $\{0,1\} \times [n]$, sending $\{0,1\} \times [n]$ to v and $[0,1] \times \{i\}$ to e_i. This directs each edge, allowing us to order any subset of its interior, and we will use this extensively.

Associated to a finite subset Λ of Γ_n is an n–tuple $J(\Lambda) = (j_1,\ldots,j_n)$ of non-negative integers $j_i = |\Lambda \cap \operatorname{int} e_i|$. Given an n–tuple $J = (j_1,\ldots,j_n)$ we define its support $\operatorname{supp} J$ to be

$$\operatorname{supp} J = \{i \in [n] \mid j_i \neq 0\}$$

and its norm $|J|$ by

$$|J| = \sum_{i=1}^{n} j_i.$$

Note that

$$|J(\Lambda)| = \begin{cases} |\Lambda| & \text{if } v \notin \Lambda; \\ |\Lambda| - 1 & \text{if } v \in \Lambda. \end{cases}$$

In addition we define the mod 2 support and norm by

$$\operatorname{supp}_2(J) = \{i \in [n] \mid j_i \not\equiv 0 \bmod 2\}$$

and

$$|J|_2 = |\operatorname{supp}_2(J)|.$$

Bringing two points together in the interior of e_i or moving a point to v decreases $J(\Lambda)_i$ by one. It will be convenient to have some notation for this, so we define

$$\partial_i(J) = (j_1,\ldots,j_i - 1,\ldots,j_n)$$

provided $j_i \ge 1$. Lastly, for each subset S of $[n]$ and n–tuple J we write $J|_S$ for the $|S|$–tuple obtained by restricting the index set to S.

2 The homology of finite subset spaces of graphs

2.1 Introduction

Our first step in calculating the homology of a finite subset space of a connected graph Γ is to find explicit cell structures for $\exp_k(\Gamma_n, v)$ and $\exp_k\Gamma_n$. The approach will be similar to that taken in [19], and we will make use of the boundary map calculated there. However, we will adopt a different orientation convention, with the result that some signs will be changed.

Our cell structure for $\exp_k(\Gamma_n, v)$ will consist of one j-cell σ^J for each n-tuple J such that $|J| = j \leq k-1$, the interior of σ^J containing those $\xi \in \exp_k(\Gamma_n, v)$ such that $J(\xi) = J$. A cell structure for $\exp_k\Gamma_n$ will be obtained by adding additional cells $\tilde\sigma^J$ for each J with $|J| = j \leq k$; the interior of $\tilde\sigma^J$ will contain those $\xi \subseteq \Gamma_n \setminus \{v\}$ such that $J(\xi) = J$. By a "stars and bars" argument there are $\binom{n+j-1}{n-1}$ solutions to

$$j_1 + \cdots + j_n = j$$

in non-negative integers (count the arrangements of j ones and $n-1$ pluses), so that

$$\chi(\exp_k(\Gamma_n, v)) = \sum_{j=0}^{k-1} (-1)^j \binom{n+j-1}{n-1} \tag{2.1}$$

and

$$\chi(\exp_k\Gamma_n) = 1 + 2\sum_{j=1}^{k-1} (-1)^j \binom{n+j-1}{n-1} + (-1)^k \binom{n+k-1}{n-1}.$$

A cell structure may be found in a similar way for an arbitrary connected graph Γ, with up to $2^{|V(\Gamma)|}$ j-cells for each $|E(\Gamma)|$-tuple J with $|J| = j$.

In these cell structures the spaces $\exp_{k+1}(\Gamma_n, v)$ and $\exp_{k+1}\Gamma_n$ are obtained from $\exp_k(\Gamma_n, v)$ and $\exp_k\Gamma_n$ by adding cells in dimensions k and $k+1$. This has the following consequence for their homotopy groups. The $(k-1)$-skeleta of $\exp_k(\Gamma_n, v)$ and $\exp_\ell(\Gamma_n, v)$ co-incide for $\ell \geq k$, and this means that the map on π_i induced by the inclusion $\exp_k(\Gamma_n, v) \hookrightarrow \exp_\ell(\Gamma_n, v)$ is an isomorphism for $i \leq k-2$. By Handel [11] this map is zero for $\ell = 2k-1$, implying that $\exp_k(\Gamma_n, v)$ (and by a similar argument $\exp_k\Gamma_n$) is $(k-2)$-connected. It follows immediately that the augmented chain complex of $\exp_k(\Gamma_n, v)$ is exact, and in conjunction with the Euler characteristic (2.1) and the boundary maps (2.2) and (2.3) this is enough to prove Theorem 1. We nevertheless show directly that this chain complex is exact in section 2.3 in order to find bases for the homology groups in section 2.5.

The fact that the kth finite subset space of a connected graph is $(k-2)$–connected can be used to show that the same conclusion holds for the kth finite subset space of a connected cell complex [21].

2.2 Cell structures for $\exp_k\Gamma_n$ and $\exp_k(\Gamma_n, v)$

We now proceed more concretely. Following the strategy of [19], each element $\sigma \in \exp_j e_i$ has at least one representative $(x_1, \ldots, x_j) \in [0,1]^j$ such that $x_1 \le \cdots \le x_j$. Define simplices

$$\Delta_j = \{(x_1, \ldots, x_{j+1}) \mid 0 \le x_1 \le \cdots \le x_j \le x_{j+1} = 1\}$$

for each $j \ge 0$, and

$$\tilde{\Delta}_j = \{(x_1, \ldots, x_j) \mid 0 \le x_1 \le \cdots \le x_j \le 1\}$$

for each $j \ge 1$. There are surjections $\Delta_j \to \exp_{j+1}(e_i, v)$, $\tilde{\Delta}_j \to \exp_j e_i$, and we let σ_i^j be the composition

$$\Delta_j \to \exp_{j+1}(e_i, v) \hookrightarrow \exp_{j+1}(\Gamma_n, v);$$

$\tilde{\sigma}_i^j$ the composition

$$\tilde{\Delta}_j \to \exp_j e_i \hookrightarrow \exp_j \Gamma_n.$$

We give Δ_j and $\tilde{\Delta}_j$ each the orientation $[x_1, \ldots, x_j]$, a convention that disagrees with the one used in [19] for some j. There $\tilde{\Delta}_j$ was oriented by letting its ith vertex be

$$v_i = (\underbrace{0, \ldots, 0}_{j-i}, \underbrace{1, \ldots, 1}_{i})$$

for $i = 0, \ldots, j$, and the sign of this orientation relative to the standard one on $\mathbb{R}^j$ is given by

$$\det[(v_1 - v_0)^T, \ldots, (v_j - v_0)^T] = \begin{cases} +1 & j \equiv 0, 1 \bmod 4, \\ -1 & j \equiv 2, 3 \bmod 4. \end{cases}$$

A similar calculation shows the same conclusion holds for Δ_j. To account for this difference we should insert a minus sign in the boundary map calculated in [19] precisely when it is applied to $\sigma = \sigma_i^j$ or $\tilde{\sigma}_i^j$ for j even, since then exactly one of σ and $\partial\sigma$ has been given the opposite orientation. Note however that $\partial\tilde{\sigma}_i^j = \partial\sigma_i^j = 0$ for j odd, simplifying the matter and allowing us to simply insert a minus everywhere.

Returning to the discussion at hand, given an n–tuple J let

$$\sigma_J = \sigma_{j_1} \times \cdots \times \sigma_{j_n},$$
$$\tilde\sigma_J = \tilde\sigma_{j_1} \times \cdots \times \tilde\sigma_{j_n},$$

omitting any empty factor $\tilde\sigma_0$ from this last product. Finally let

$$\phi^J = \phi_1^{j_1} \cup \cdots \cup \phi_n^{j_n} : \sigma_J \to \exp_{|J|+1}(\Gamma_n, v),$$
$$\tilde\phi^J = \tilde\phi_1^{j_1} \cup \cdots \cup \tilde\phi_n^{j_n} : \tilde\sigma_J \to \exp_{|J|}\Gamma_n,$$

again omitting any factor with $j_i = 0$ from $\tilde\phi^J$. Each of $\phi^J|_{\operatorname{int}\sigma_J}$, $\tilde\phi^J|_{\operatorname{int}\tilde\sigma_J}$ is a homeomorphism of an open $|J|$–ball onto its image, and we claim:

Lemma 1 *The spaces* $\exp_k(\Gamma_n, v)$ *and* $\exp_k\Gamma_n$ *have cell structures consisting respectively of* $\{\phi^J \mid |J| \le k-1\}$ *and of* $\{\phi^J \mid |J| \le k-1\} \cup \{\tilde\phi^J \mid 1 \le |J| \le k\}$*. The boundary maps are given by*

$$\partial\phi^J = -\sum_{i \in \operatorname{supp} J} \frac{1+(-1)^{j_i}}{2}(-1)^{|J_{[i-1]}|}\phi^{\partial_i(J)} \tag{2.2}$$

and

$$\partial\tilde\phi^J = \sum_{i \in \operatorname{supp} J} \frac{1+(-1)^{j_i}}{2}(-1)^{|J_{[i-1]}|}\left(\tilde\phi^{\partial_i(J)} - 2\phi^{\partial_i(J)}\right). \tag{2.3}$$

Notice that the behaviour of a cell under the boundary map depends only on the support and parity pattern of J. This fact will be of importance in understanding the chain complexes in section 2.3.

Proof Each element $\xi \in \exp_k\Gamma_n$ lies in the interior of the image of precisely one cell ϕ^J or $\tilde\phi^J$, namely $\phi^{J(\xi)}$ if $v \in \xi$ and $\tilde\phi^{J(\xi)}$ if $v \notin \xi$. The image of ϕ^J is contained in $\exp_{|J|+1}(\Gamma_n, v)$ and that of $\tilde\phi^J$ in $\exp_{|J|}\Gamma_n$, so we may set the j–skeleton of $\exp_k(\Gamma_n, v)$ equal to $\exp_{j+1}(\Gamma_n, v)$ and the j–skeleton of $\exp_k\Gamma_n$ equal to $(\exp_j\Gamma_n) \cup (\exp_{j+1}(\Gamma_n, v))$ for $j < k$ and $\exp_k\Gamma_n$ for $j = k$. The boundary of $\tilde\sigma_j$ is found by replacing one or more inequalities in $0 \le x_1 \le \cdots \le x_j \le 1$ with equalities, resulting in fewer points in the interior of e_i; thus the image of the boundary of $\tilde\sigma_J$ under $\tilde\phi^J$ is contained in $\exp_{|J|-1}\Gamma_n \cup \exp_{|J|}(\Gamma_n, v)$. Similarly, ϕ^J maps the boundary of σ_J into $\exp_{|J|}(\Gamma_n, v)$. So the boundary of a j–cell is contained in the $(j-1)$–skeleton, and the ϕ^J, $\tilde\phi^J$ form cell structures as claimed.

To calculate the boundary map we use Lemma 1 of [19], which with our present notation and orientation convention says

$$\begin{aligned} \partial \sigma_i^j &= -\frac{1+(-1)^j}{2}\,\sigma_i^{j-1}; \\ \partial \tilde{\sigma}_i^j &= \frac{1+(-1)^j}{2}\left(\tilde{\sigma}_i^{j-1} - 2\,\sigma_i^{j-1}\right); \end{aligned} \tag{2.4}$$

together with the relation $\partial(\sigma \times \tau) = (\partial\sigma)\times\tau + (-1)^{\dim\sigma}\sigma\times(\partial\tau)$. Calculating the boundary of $\tilde{\sigma}_1^{j_1}\times\cdots\times\tilde{\sigma}_1^{j_n}$ and then applying $\cup$ it follows that

$$\partial\tilde{\sigma}^J = \sum_{i\in\operatorname{supp} J} (-1)^{|J_{[i-1]}|}\,\tilde{\sigma}_1^{j_1}\cup\cdots\cup\partial\tilde{\sigma}_i^{j_i}\cup\cdots\cup\tilde{\sigma}_n^{j_n}.$$

Substituting (2.4) and observing that $\tilde{\sigma}_1^{j_1}\cup\cdots\cup\sigma_i^{j_i}\cup\cdots\cup\tilde{\sigma}_n^{j_n} = \sigma^J$ gives (2.3), and (2.2) follows by a similar argument or by using $\sigma^J = (\cup\{v\})_*\tilde{\sigma}^J$. □

Let C_* be the free abelian group generated by the σ^J, $0 \le |J| < \infty$, and $\tilde{C}_*$ the free abelian group generated by the $\tilde{\sigma}^J$, $1 \le |J| < \infty$, each graded by degree. Then

$$H_*(\exp_k(\Gamma_n; v)) = H_*(C_{*\le k-1})$$

and

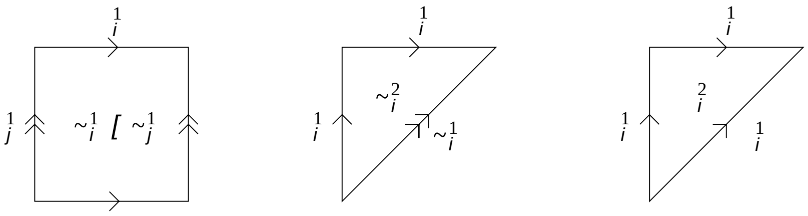
$$H_*(\exp_k\Gamma_n) = H_*(C_{*\le k-1} \oplus \tilde{C}_{*\le k}).$$

As discussed at the end of section 2.1 we know a priori that $(C_*;\partial)$ is exact except at C_0. We nevertheless give a direct proof of this in section 2.3, with a view to constructing explicit bases for the homology groups in section 2.5 after calculating their ranks in section 2.4. Before doing so however we use Lemma 1 to calculate the fundamental groups of $\exp_k\Gamma_n$ and $\exp_k(\Gamma_n; v)$ for each k and n, showing directly that $\exp_k\Gamma_n$ and $\exp_k(\Gamma_n; v)$ are simply connected for $k \ge 3$.

Theorem 3 *The fundamental group of $\exp_k\Gamma_n$ is*

1. *free of rank n if $k = 1$;*
2. *free abelian of rank n, containing $\pi_1(\exp_1\Gamma_n)$ as a subgroup of index 2^n, if $k = 2$; and*
3. *trivial if $k \ge 3$.*

The fundamental group of $\exp_k(\Gamma_n; v)$ is free of rank n if $k = 2$ and trivial otherwise.



(a) (b) (c)

Figure 1: Relations in $_1(\exp_k \Gamma_n)$ arising from the 2{cells. The boundary of a cell is found by moving a point in the interior of an edge to v, or bringing two points in the interior into co-incidence. The rst gives an untilded cell and the second a cell of the same kind as the interior. In (a) we see a torus killing the commutator of $[\ _i^1]$ and $[\ _j^1]$; in (b) a Möbius strip with fundamental group generated by $[\ _i^1]$ and boundary $[\tilde{}_i^1]$; and in (c) a dunce cap killing $[\ _i^1]$.

Proof In the unbased case $\exp_k \Gamma_n$ the result is obvious for $k = 1$ so consider $k = 2$. The group $_1(\exp_2 \Gamma_n)$ is generated by $[\ _i^1]$, $[\tilde{}_i^1]$, $1 \quad i \quad n$, with relations arising from the $\tilde{}_i^2$ and $\tilde{}_i^1 [\tilde{}_j^1$, $i \neq j$. The image of $\tilde{}_i^1 [\tilde{}_j^1$ is a torus with meridian $[\ _i^1]$ and longitude $[\ _j^1]$, while the image of $\tilde{}_i^2$ is a Möbius strip that imposes the relation $[\tilde{}_i^1] = [\ _i^1]^2$ (see gures 1(a) and (b)). It follows that $_1(\exp_2 \Gamma_n)$ is free abelian with generators $[\ _i^1]$, $1 \quad i \quad n$, and that $i \ _1(\exp_1 \Gamma_n) = h[\tilde{}_1^1]; : : : ; [\tilde{}_n^1]i$ has index 2^n. When $k \quad 3$ there are no new generators and additional relations $[\ _i^1] = 1$ from each $\ _i^2$ (see gure 1(c)), so that $_1(\exp_k \Gamma_n) = f1g$.

In the based case $\exp_1(\Gamma_n; v) = ffvgg$, the map $[fvg \colon \exp_1 \Gamma_n \ ! \ \exp_2(\Gamma_n; v)$ is a homeomorphism, and for $k \quad 3$ the relations $[\ _i^1] = 1$ from the $\ _i^2$ apply as above. □

2.3 Direct proof of the exactness of $C \ _1$

We show directly that C is exact at each $` > 0$ by expressing it as a sum of nite subcomplexes and showing that each summand is exact. This decomposition will be used to construct explicit bases for the homology in section 2.5.

As a rst reduction, for each subset S of $[n]$ let C^S be the free abelian group generated by $f \ ^J j \operatorname{supp} J = Sg$. Since $@ \ ^J$ is a linear combination of the cells $\ ^{@_i(J)}$ with $i \ 2 \operatorname{supp} J$ and $j_i \quad 0 \bmod 2$, each C^S is a subcomplex and we have

$$C = \bigoplus_{S \ [n]} C^S :$$

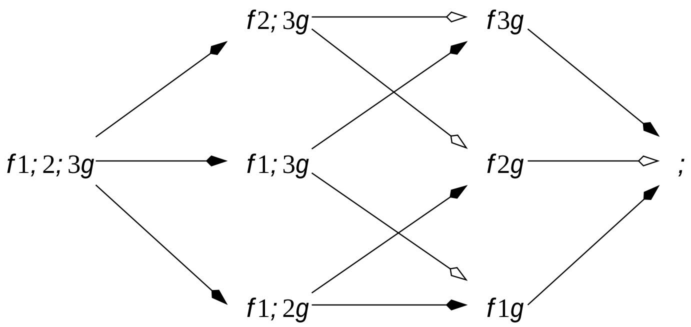


Figure 2: The 3–cube complex $\mathcal{C}^3$. The lattice of subsets of $\{1,2,3\}$ forms a 3–dimensional cube and ∂S is a signed sum of the neighbours of S of smaller cardinality. In the diagram positive terms are indicated by solid arrowheads, negative terms by empty arrowheads.

Note that $C^{\emptyset} = C_0$. Clearly $C^S \cong C^T$ if $|S| = |T|$ so we will show $C^{[m]}$ is exact for each $m > 0$.

We claim that $C^{[m]}$ may be regarded as a sum of many copies of a single finite complex, the m–cube complex. For each m–tuple L with all entries odd let $C^{[m]}(L)$ be the subgroup of $C^{[m]}$ generated by $\left\{ J \;\middle|\; j_i - \ell_i \in \{0,1\} \right\}$. Again the fact that ∂J is a linear combination of $\{\partial_i(J) \mid i \in \operatorname{supp} J, j_i \equiv 0 \bmod 2\}$ implies $C^{[m]}(L)$ is a subcomplex, and moreover that

$$C^{[m]} = \bigoplus_{L : |L|_2 = m} C^{[m]}(L).$$

Further, on translating each m–tuple by $(\ell_1 - 1, \ldots, \ell_m - 1)$ each $C^{[m]}(L)$ can be seen to be isomorphic to $C^{[m]}((1, \ldots, 1))$ with its grading shifted by $|L| - m$. We call this common isomorphism class of complex the m–cube complex $\mathcal{C}^m$, and, replacing J with the set of indices of its even entries, will take the free abelian group generated by the power set of $[m]$, graded by cardinality and with boundary map

$$\partial S = \sum_{i \in S} (-1)^{|[i-1] \setminus S|} S \setminus \{i\}$$

to be its canonical representative; for aesthetic purposes we are dropping the minus sign outside the sum. The name m-cube complex comes from the fact that the lattice of subsets of $[m]$ forms an m–dimensional cube, and that ∂S is a signed sum of the neighbours of S of smaller degree. See figure 2 for the case $m = 3$.

Let
$$V_j = \{S \subseteq [m] \mid |S| = j \text{ and } 1 \in S\}.$$

The exactness of $C^{[m]}$ follows from the first statement of the following lemma; the second statement will be used in section 2.5 to construct explicit bases for $H_*(\exp_k\Gamma_n)$.

Lemma 2 *The m–cube complex $\mathcal{C}^m$ is exact. The homology of the truncated complex $\mathcal{C}^m_{\geq j}$ is free of rank $\binom{m-1}{j}$ in dimension j, with basis $\{\partial S \mid S \in V_{j+1}\}$, and zero otherwise.*

Proof We claim that $V_j \cup \partial V_{j+1}$ forms a basis for $\mathcal{C}^m_j$, from which the lemma follows. Since $V_j \cup \partial V_{j+1}$ has at most $\binom{m-1}{j-1} + \binom{m-1}{j} = \binom{m}{j} = \operatorname{rank} \mathcal{C}^m_j$ elements we simply check that $V_j \cup \partial V_{j+1}$ spans $\mathcal{C}^m_j$. It suffices to show that $S \in \operatorname{span} V_j \cup \partial V_{j+1}$ for each subset S of size j not containing 1; this follows from
$$\partial(S \cup \{1\}) = S - \sum_{i \in S} (-1)^{|[i-1]\setminus S|} S \cup \{1\} \setminus \{i\}$$
if $1 \notin S$. ∎

2.4 The homology groups of $\exp_k(\Gamma_n; v)$ and $\exp_k\Gamma_n$

We calculate the homology groups of $\exp_k(\Gamma_n; v)$ and $\exp_k\Gamma_n$ using the exactness of $C_{\geq 1}$, the Euler characteristic (2.1), and the boundary maps (2.2) and (2.3). Explicit bases are found in section 2.5 using the decomposition of C into subcomplexes.

Proof of Theorem 1 Since $\exp_k(\Gamma_n; v)$ is path connected with homology equal to that of $C_{\geq k-1}$, its reduced homology vanishes except perhaps in dimension $k-1$, by the exactness of $C_{\geq 1}$. Moreover H_{k-1} is equal to $\ker \partial_{k-1}$ and is therefore free; its rank may be found using $\chi(\exp_k(\Gamma_n; v)) = b_0 + (-1)^{k-1} b_{k-1}$ and equation (2.1), yielding
$$b_{k-1}(\exp_k(\Gamma_n; v)) = \sum_{j=1}^{k-1} (-1)^{k-j-1} \binom{n+j-1}{n-1}.$$

This may be expressed as a sum of purely positive terms by grouping the summands in pairs, starting with the largest, and using $\binom{p}{q} - \binom{p-1}{q} = \binom{p-1}{q-1}$. Doing this for $b_k(\exp_{k+1}(\Gamma_n; v))$ gives the expression in equation (1.2).

Now consider $H_*(\exp_k\Gamma_n) = H_*(C_*^{k-1} \oplus C_*^{k})$. Write C_j for the jth chain group of $C_*^{k-1} \oplus \tilde{C}_*^{k}$, Z_j for the j–cycles in C_j and $\tilde{Z}_j$ for the j–cycles in $\tilde{C}_j$. Extending $\sigma^J \mapsto \tilde{\sigma}^J$ linearly to a group isomorphism from C_j to $\tilde{C}_j$ for each $j \geq 1$, the boundary maps (2.2) and (2.3) give

$$\partial\tilde{c} = 2\partial c - \widetilde{\partial c}$$

for each chain $c \in C_{\geq 1}$. It follows that $Z_j = Z_j \oplus \tilde{Z}_j$ for $1 \leq j \leq k-1$ and that $H_k(\exp_k\Gamma_n) = Z_k$ is equal to $\tilde{Z}_k$. Moreover

$$\partial C_k = \{2z - \tilde{z} \mid z \in Z_{k-1}\}$$

by the exactness of C_* at C_{k-1}, so that $H_{k-1}(\exp_k\Gamma_n) = Z_{k-1}$. We show that the remaining reduced homology groups vanish.

Let $z = z_1 \oplus \tilde{z}_2 \in Z_j$ for some $1 \leq j \leq k-2$. By the exactness of $C_{\geq 1}$ there are $w_1, w_2 \in C_{j+1}$ such that

$$\begin{aligned} \partial w_1 &= z_1 + 2z_2, \\ \partial w_2 &= z_2. \end{aligned}$$

Since $j \leq k-2$ we have $w_1 - \tilde{w}_2 \in C_{j+1}$, and

$$\begin{aligned} \partial(w_1 - \tilde{w}_2) &= \partial w_1 - \partial\tilde{w}_2 \\ &= z_1 + 2z_2 - 2\partial w_2 + \widetilde{\partial w_2} \\ &= z_1 + \tilde{z}_2, \end{aligned}$$

so that C_* is exact at C_j as claimed.

It remains to determine the maps induced by

$$\begin{aligned} i\colon\ & \exp_k(\Gamma_n; v) \hookrightarrow \exp_k\Gamma_n, \\ \cup\{v\}\colon\ & \exp_k\Gamma_n \to \exp_{k+1}(\Gamma_n; v) \end{aligned}$$

and

$$\exp_k\Gamma_n \hookrightarrow \exp_{k+1}\Gamma_n$$

on homology. In each case there is only one dimension in which the induced map is not trivially zero. We have $H_{k-1}(\exp_k(\Gamma_n; v)) = Z_{k-1} = H_{k-1}(\exp_k\Gamma_n)$, so that i_* is an isomorphism on H_{k-1}, and adding v to each element of $\exp_k\Gamma_n$ sends $\tilde{z}$ to z for each $z \in Z_k$, inducing an isomorphism on H_k. Lastly, $\exp_k\Gamma_n \hookrightarrow \exp_{k+1}\Gamma_n$ sends $[\tilde{z}]$ to $[\tilde{z}] = 2[z] = 2(i_* \circ \cup\{v\}_*)[\tilde{z}]$ for each $z \in Z_k$, inducing two times $(i_* \circ \cup\{v\}_*)$ as claimed. □

The homology and fundamental group of $\exp_k(\Gamma_n; v)$ are enough to determine its homotopy type completely. When $k = 1$ it is a single point $\{\{v\}\}$, and when $k \geq 2$ we have:

Corollary 1 (Theorem 2) *For $k \geq 2$ the space $\exp_k(\Gamma_n; v)$ has the homotopy type of a wedge of $b_{k-1}(\exp_k(\Gamma_n; v))$ $(k-1)$–spheres.*

Proof Since $\exp_2(\Gamma_n; v)$ is homeomorphic to Γ_n we may assume $k \geq 3$. But then $\exp_k(\Gamma_n; v)$ is a simply connected Moore space $M(\mathbb{Z}^{b_{k-1}}; k-1)$ and the result follows from the Hurewicz and Whitehead theorems. □

2.5 A basis for $H_k(\exp_k \Gamma_n)$

We use the decomposition of C_* as a sum of subcomplexes to give an explicit basis for $H_k(\exp_k \Gamma_n)$.

Theorem 4 *The set*

$$B(k;n) = \left\{ \partial g_J \;\middle|\; |J| = k+1 \text{ and } j_i \not\equiv 0 \bmod 2 \text{ for } i = \min(\operatorname{supp} J) \right\}$$

is a basis for $H_k(\exp_k \Gamma_n)$.

Proof It suffices to find a basis for Z_k and map it across to $\tilde{Z}_k$. Extending notation in obvious ways we have

$$\begin{aligned} Z_k &= \bigoplus_{S \subseteq [n]} Z_k^S \\ &= \bigoplus_{S \subseteq [n]} \bigoplus_{\substack{L : \operatorname{supp} L = S, \\ |L|_2 = |S|}} Z_k^S(L). \end{aligned}$$

Each $Z_k^S(L)$ in this sum is isomorphic to $Z_j^{|S|}$ for some j, and tracing back through this isomorphism we see that V_{j+1} is carried up to sign to

$$V_{k+1}^S(L) = \{ J \mid |J| = k+1;\ j_i - \ell_i \in \{0,1\};\ j_i \not\equiv 0 \bmod 2 \text{ for } i = \min(\operatorname{supp} J) \}.$$

By Lemma 2 $\{\partial g_J \mid J \in V_{k+1}^S(L)\}$ is a basis for $Z_k^S(L)$, and taking the union over S and L completes the proof. □

As an exercise in counting we check that $B(k;n)$ has the right cardinality. This is equivalent to showing that the number $s(k;n)$ of non-negative integer solutions to

$$j_1 + \cdots + j_n = k$$

in which the first non-zero summand is odd is given by equation (1.2). We do this by induction on k, inducting separately over the even and odd integers.

In the base cases $k = 1$ and 2 there are clearly n and $\binom{n}{2}$ solutions respectively. It therefore suffices to show that $s(k; n) - s(k-2; n) = \binom{n+k-2}{n-2}$. Adding two to the first non-zero summand gives an injection from solutions with $k = \ell - 2$ to solutions with $k = \ell$, hitting all solutions except those for which the first non-zero summand is one. If $j_{n-i} = 1$ is the first non-zero summand then what is left is an unconstrained non-negative integer solution to

$$j_{n-i+1} + \cdots + j_n = \ell - 1,$$

of which there are $\binom{\ell+i-2}{\ell-1}$, so that

$$s(k; n) - s(k-2; n) = \sum_{i=1}^{n-1} \binom{k+i-2}{k-1}.$$

This is a sum down a diagonal of Pascal's triangle and as such is easily seen to equal $\binom{k+n-2}{k} = \binom{k+n-2}{n-2}$.

2.6 Generating functions for $b_k(\exp_k \Gamma_n)$

We conclude this section by giving generating functions for the Betti numbers $b_k(\exp_k \Gamma_n)$.

Theorem 5 *The Betti number $b_k(\exp_k \Gamma_n)$ is the coefficient of x^k in*

$$\frac{1 - (1-x)^n}{(1+x)(1-x)^n}. \tag{2.5}$$

Proof The coefficient of x^j in

$$\frac{1}{(1-x)^n} = (1 + x + x^2 + x^3 + \cdots)^n$$

is the number of non-negative solutions to $j_1 + \cdots + j_n = j$, in other words $\binom{n+j-1}{n-1}$. Multiplication by $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$ has the effect of taking alternating sums of coefficients, so we subtract 1 first to remove the unwanted degree zero term from $(1-x)^{-n}$, arriving at (2.5). □

3 The calculation of induced maps

3.1 Introduction

Given a pointed map $\phi : (\Gamma_n; v) \to (\Gamma_m; v)$ there are induced maps

$$(\exp_k \phi)_* : H_{k-1}(\exp_k(\Gamma_n; v)) \to H_{k-1}(\exp_k(\Gamma_m; v))$$

and

$$(\exp_k \phi)_* : H_p(\exp_k \Gamma_n) \to H_p(\exp_k \Gamma_m)$$

for $p = k-1, k$. In view of the commutative diagrams

$$\begin{array}{ccc} \exp_k(\Gamma_n; v) & \xrightarrow{\exp_k \phi} & \exp_k(\Gamma_m; v) \\ \Big\downarrow i & & \Big\downarrow i \\ \exp_k \Gamma_n & \xrightarrow{\exp_k \phi} & \exp_k \Gamma_m \end{array}$$

and

$$\begin{array}{ccc} \exp_{k-1} \Gamma_n & \xrightarrow{\exp_{k-1} \phi} & \exp_{k-1} \Gamma_m \\ \Big\downarrow \cup \{v\} & & \Big\downarrow \cup \{v\} \\ \exp_k(\Gamma_n; v) & \xrightarrow{\exp_k \phi} & \exp_k(\Gamma_m; v) \end{array}$$

and the isomorphisms induced by i and $\cup\{v\}$ on H_{k-1} and H_k respectively it suffices to understand just one of these, and we will focus our attention on $H_k(\exp_k \Gamma_n) \to H_k(\exp_k \Gamma_m)$. The purpose of this section is to reduce the problem of calculating this map to the problem of finding the images of the basic cells σ_i^1, σ_i^2 under the chain map $(\exp \phi)_J$. The reduction will be done by defining a multiplication on C, giving it the structure of a ring without unity generated by the σ_i^j. The multiplication will be defined in such a way that the cellular chain map $(\exp \phi)_J|_C$ is a ring homomorphism, reducing calculating $(\exp_k \phi)_*$ to calculations in the chain ring once the $(\exp \phi)_J \sigma_i^j$ are found. The reduction to just the cells σ_i^1, σ_i^2 is achieved by working over the rationals, as $C \otimes_{\mathbb{Z}} \mathbb{Q}$ will be generated over $\mathbb{Q}$ by just these $2n$ cells.

In what follows we will assume that ϕ is smooth, in the sense that ϕ is smooth on the open set $\phi^{-1}(\Gamma_m \setminus \{v\})$. This ensures that $\exp_j \phi$ is smooth off the preimage of the $(j-1)$–skeleton $(\exp_j \Gamma_n)^{j-1} = (\exp_{j-1} \Gamma_n) \cup (\exp_j(\Gamma_n; v))$, allowing us to use smooth techniques on the manifold $\exp_j \Gamma_n \setminus (\exp_j \Gamma_n)^{j-1}$. Smoothness of ϕ may be ensured by homotoping it to a standard form defined as follows. The restriction of ϕ to each edge e_i is an element of $\pi_1(\Gamma_m; v)$, and as such is equivalent to a reduced word w_i in the $\{e_a\} \cup \{\bar{e}_a\}$. We consider ϕ to be in standard form if $\phi|_{e_i}$ traverses each letter of w_i at constant speed.

3.2 The chain ring

We observe that the operation $(g,h)\mapsto g\cup h$ suggests a natural way of multiplying cells and we study it with an eye to applying the results to maps of the form $(\exp_k\gamma)\circ\tilde\sigma^J$.

A map of pairs $g\colon (B^j,\partial B^j)\to\left(\exp_j\Gamma_n,(\exp_j\Gamma_n)^{j-1}\right)$ induces a map

$$g_*\colon H_j(B^j,\partial B^j)\to H_j(\exp_j\Gamma_n,(\exp_j\Gamma_n)^{j-1}),$$

and the homology group on the right is canonically isomorphic to the cellular chain group C_j. Writing β^j for the positive generator of $H_j(B^j,\partial B^j)$, if g is smooth on the open set $g^{-1}(\exp_j\Gamma_n\setminus(\exp_j\Gamma_n)^{j-1})$ then this map is given by

$$g_*\beta^j=\sum_{|J|=j}\langle g,\tilde\sigma^J\rangle\tilde\sigma^J,$$

in which $\langle g,\tilde\sigma^J\rangle$ is the signed sum of preimages of a generic point in the interior of $\tilde\sigma^J$. If h is a second map of pairs $(B^\ell,\partial B^\ell)\to\left(\exp_\ell\Gamma_n,(\exp_\ell\Gamma_n)^{\ell-1}\right)$ then $g\cup h$ is a map of pairs

$$(B^{j+\ell},\partial B^{j+\ell})\to\left(\exp_{j+\ell}\Gamma_n,(\exp_{j+\ell}\Gamma_n)^{j+\ell-1}\right)$$

also. The following lemma shows that $\cup$ behaves as might be hoped on the chain level.

Lemma 3 *Given two maps of pairs* $g\colon (B^j,\partial B^j)\to(\exp_j\Gamma_n,(\exp_j\Gamma_n)^{j-1})$, $h\colon (B^\ell,\partial B^\ell)\to(\exp_\ell\Gamma_n,(\exp_\ell\Gamma_n)^{\ell-1})$, *each smooth off the preimage of the codimension one skeleton, we have*

$$(g\cup h)_*\beta^{j+\ell}=\sum_{|J|=j,|L|=\ell}\langle g,\tilde\sigma^J\rangle\langle h,\tilde\sigma^L\rangle(\tilde\sigma^J\cup\tilde\sigma^L)\,. \qquad (3.1)$$

Proof Fix an n-tuple M such that $|M|=j+\ell$ and let ξ be a generic point in the interior of $\tilde\sigma^M$. It suffices to check that $g\times h$ and $\sum_{J,L}\langle g,\tilde\sigma^J\rangle\langle h,\tilde\sigma^L\rangle\tilde\sigma^J\times\tilde\sigma^L$ have the same signed sum of preimages at each point $(\xi',\xi'')\in\exp_{|J|}\Gamma_n\times\exp_{|L|}\Gamma_n$ such that $\xi'\cup\xi''=\xi$; note that for cardinality reasons (ξ',ξ'')

forms a partition of Δ. For $g \times h$ this signed sum is given by

$$
\begin{aligned}
\langle g \times h; \tilde\epsilon^{J(\Delta')} \times \tilde\epsilon^{J(\Delta'')} \rangle &= \sum_{p \in g^{-1}(\Delta')} \sum_{q \in h^{-1}(\Delta'')} \operatorname{sign}(\det D(g \times h)(p,q)) \\
&= \sum_{p \in g^{-1}(\Delta')} \sum_{q \in h^{-1}(\Delta'')} \operatorname{sign}\left(\det Dg(p)\right) \operatorname{sign}\left(\det Dh(q)\right) \\
&= \sum_{g^{-1}(\Delta')} \operatorname{sign}(\det Dg(p)) \sum_{h^{-1}(\Delta'')} \operatorname{sign}(\det Dh(q)) \\
&= \langle g; \tilde\epsilon^{J(\Delta')} \rangle \langle h; \tilde\epsilon^{J(\Delta'')} \rangle .
\end{aligned}
$$

The lemma follows from the fact that $\langle \tilde\epsilon^{J} \times \tilde\epsilon^{L}; \tilde\epsilon^{J(\Delta')} \times \tilde\epsilon^{J(\Delta'')} \rangle$ is zero unless $J = J(\Delta')$ and $L = J(\Delta'')$, in which case it is one. ∎

Since $(\tilde\epsilon^{J} \cup \tilde\epsilon^{L})^{j+\ell}$ is a multiple of $\tilde\epsilon^{J+L}$ the next step is to understand the pairings $\langle \tilde\epsilon^{J} \cup \tilde\epsilon^{L}; \tilde\epsilon^{J+L} \rangle$. Interchanging adjacent factors $\tilde\epsilon^{r}_{a}$ and $\tilde\epsilon^{s}_{b}$ in the product $\tilde\epsilon^{J} \cup \tilde\epsilon^{L}$ simply introduces a sign $(-1)^{rs}$, so we may gather basic cells from the same edge together and consider pairings of the form

$$
\langle (\tilde\epsilon^{j_1}_{1} \cup \tilde\epsilon^{\ell_1}_{1}) \cup \cdots \cup (\tilde\epsilon^{j_n}_{n} \cup \tilde\epsilon^{\ell_n}_{n}); \tilde\epsilon^{J+L} \rangle = \prod_{i=1}^{n} \langle \tilde\epsilon^{j_i}_{i} \cup \tilde\epsilon^{\ell_i}_{i}; \tilde\epsilon^{j_i+\ell_i}_{i} \rangle .
$$

The quantity $\langle \tilde\epsilon^{r}_{a} \cup \tilde\epsilon^{s}_{a}; \tilde\epsilon^{r+s}_{a} \rangle$ is equal to $\binom{r+s}{r}_{-1}$, the q-binomial coefficient $\binom{r+s}{r}$ specialised to $q = -1$. The correspondence can be seen as follows. Take $r+s$ objects, numbered from 1 to $r+s$ and laid out in order, and paint r of them blue and the rest red. Shuffle them so that the blue ones are at the front in ascending order, followed by the red ones in ascending order, giving an element of the symmetric group S_{r+s}. Then $\binom{r+s}{r}_{-1}$ is the number of ways of choosing r objects from $r+s$ in this way, counted with the sign of the associated permutation, and is equal to $\langle \tilde\epsilon^{r}_{a} \cup \tilde\epsilon^{s}_{a}; \tilde\epsilon^{r+s}_{a} \rangle$: the blue and red points represent the elements of a generic point in $\exp_{r+s} e_a$ coming from $\tilde\epsilon^{r}_{a}$ and $\tilde\epsilon^{s}_{a}$ respectively, and the derivative at this preimage is the matrix of the associated permutation.

The calculation of $\binom{r+s}{r}_{-1}$ is the subject of the following lemma. The result, which we might call "Pascal's other triangle" — being a much less popular model than the one we know and love — appears in figure 3. For further information on q-binomial coefficients and related topics see for example Stanley [18], Kac and Cheung [13], and Baez [1, weeks 183–188].

$$
\begin{array}{c}
1\\
1 \quad 1\\
1 \quad 0 \quad 1\\
1 \quad 1 \quad 1 \quad 1\\
1 \quad 0 \quad 2 \quad 0 \quad 1\\
1 \quad 1 \quad 2 \quad 2 \quad 1 \quad 1\\
1 \quad 0 \quad 3 \quad 0 \quad 3 \quad 0 \quad 1\\
1 \quad 1 \quad 3 \quad 3 \quad 3 \quad 3 \quad 1 \quad 1\\
1 \quad 0 \quad 4 \quad 0 \quad 6 \quad 0 \quad 4 \quad 0 \quad 1\\
1 \quad 1 \quad 4 \quad 4 \quad 6 \quad 6 \quad 4 \quad 4 \quad 1 \quad 1
\end{array}
$$

Figure 3: The first ten rows of Pascal's other triangle, enough so that the pattern is clear. Each row of Pascal's triangle appears twice; on the first occurrence zeros are inserted between each entry, and on the second each entry appears twice.

Lemma 4 *The value of the signed binomial coefficient* $\binom{m}{r}_{-1}$ *is given by*

$$\binom{m}{r}_{-1} = \frac{1 + (-1)^{r(m-r)}}{2} \binom{\lfloor m/2 \rfloor}{\lfloor r/2 \rfloor}. \tag{3.2}$$

Proof The q–binomial coefficient $\binom{m}{r}$ satisfies $\binom{m}{0} = \binom{m}{m} = 1$ and

$$\binom{m}{r} = \binom{m-1}{r-1} + q^r \binom{m-1}{r},$$

both easily seen for $q = -1$ from the definition above: the recurrence relation is proved analogously to the familiar one for Pascal's triangle, the sign $(-1)^r$ arising from shuffling the first object into place if it is red instead of blue. The equality (3.2) is then readily proved by induction on m, by considering in turn the four possibilities for the parities of m and r. □

We remark that the equality $\binom{r+s}{r} = \binom{r+s}{s}$ can be seen for $q = -1$ from the fact that $\lfloor r/2 \rfloor + \lfloor s/2 \rfloor = \lfloor (r+s)/2 \rfloor$ unless both r and s are odd, in which case the coefficient $\binom{r+s}{r}_{-1}$ vanishes. Additionally $\binom{r+s}{r}_{-1}\binom{r+s+t}{r+s}_{-1}$ and $\binom{r+s+t}{r}_{-1}\binom{s+t}{s}_{-1}$ are both equal to

$$\frac{\lfloor (r+s+t)/2 \rfloor !}{\lfloor r/2 \rfloor ! \lfloor s/2 \rfloor ! \lfloor t/2 \rfloor !}$$

if no more than one of $r; s; t$ is odd, and zero otherwise.

We now de ne the chain ring of exp Γ_n to be the ring without unity generated over Z by the set $f\sim^j_i$ 1 i $n; j$ 1g, with relations

$$\sim^j_a \sim^{\ell}_b = (-1)^{j\ell} \sim^{\ell}_b \sim^j_a$$
$$\sim^j_a \sim^{\ell}_a = \ {}^{j+\ell}_{\ \ j} \ {}_{-1} \sim^{j+\ell}_a$$

for all 1 $a; b$ n and $j; \ell$ 1. This de nition is again a specialisation to $q = -1$ of a construction that applies more generally, and is chosen so that the conclusion (3.1) of Lemma 3 may be rewritten as

$$(g\ [\ h) \quad {}^{j+\ell} = (g \quad {}^{j})(h \quad {}^{\ell}); \qquad (3.3)$$

in which the multiplication on the right hand side takes place in the chain ring. The lack of an identity could be easily remedied but we have chosen not to so that all elements of the ring are chains. We shall denote the chain ring simply by C , or C (Γ_n) in case of ambiguity.

3.3 Calculating induced maps

We now have all the machinery required to state and prove our main result on the calculation of $(\exp_k \ )$, namely that the cellular chain map $(\exp \ )_] j_C$ is a ring homomorphism. This reduces the calculation of the chain map to the calculation of the images of the basic cells $\sim^j_i$, each an exercise in counting points with signs, and multiplication and addition in the chain ring.

Theorem 6 *If* : $(\Gamma_n; v)$ $!$ $(\Gamma_m; v)$ *is smooth on* $^{-1}(\Gamma_m\ n\ fvg)$ *then the cellular chain map* $(\exp \ )_]$: C (Γ_n) $!$ C (Γ_m) *is a ring homomorphism.*

Proof If is smooth o $^{-1}(v)$ then as noted at the end of section 3.1 the map $\exp_j \ = \exp \ j_{\exp_j \Gamma_n}$ is smooth o the codimension one skeleton for each j, so we may use Lemma 3. Thus

$$
\begin{aligned}
(\exp \ )_](\sim^J \sim^L) &= (\exp_{j+\ell} \ ) ((\sim^J\ [\ \sim^L) \quad {}^{j+\ell}) && \text{by (3.3)}\\
&= (\exp_{j+\ell} \ ) \ (\sim^J\ [\ \sim^L) \quad {}^{j+\ell} \\
&= ((\exp_j \ ) \ \sim^J)\ [\ ((\exp_\ell \ ) \ \sim^L) \quad {}^{j+\ell} \\
&= ((\exp_j \ ) \ \sim^J) \ {}^{j} \ ((\exp_\ell \ ) \ \sim^L) \ {}^{\ell} && \text{by (3.3)}\\
&= (\exp_j \ ) \sim^J {}^{j} \ (\exp_\ell \ ) \sim^L {}^{\ell} \\
&= (\exp \ )_] \sim^J \ (\exp \ )_] \sim^L ;
\end{aligned}
$$

from which the result follows. □

As an immediate consequence we have

$$(\exp\phi)_\sharp \tilde\sigma^J = \prod_{i=1}^{n} (\exp\phi)_\sharp \tilde\sigma_i^{j_i};$$

so that $(\exp\phi)_\sharp \tilde\sigma^J$ may be found knowing just the images of the basic cells $\tilde\sigma_i^j$ as claimed. To reduce the number of cells $\tilde\sigma$ for which $(\exp\phi)_\sharp\tilde\sigma$ must be calculated directly even further, observe that

$$\tilde\sigma_i^j = \begin{cases} \frac{1}{\ell!}\left(\tilde\sigma_i^2\right)^{\ell}, & \text{if } j = 2\ell; \\ \frac{1}{\ell!}\tilde\sigma_i^1\left(\tilde\sigma_i^2\right)^{\ell}, & \text{if } j = 2\ell + 1; \end{cases} \tag{3.4}$$

so that $C_* \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated over $\mathbb{Q}$ by the set $\{\tilde\sigma_i^j \mid 1 \le i \le n; j = 1, 2\}$, subject only to the relations that the $\tilde\sigma_i^1$ anti-commute with each other and the $\tilde\sigma_i^2$ commute with everything. In essence this reduces calculating $(\exp\phi)_\sharp$ to understanding the behaviour of chains under maps

$$\exp_1 S^1 \to \exp_1 S^1$$

and

$$\exp_2 S^1 \to \exp_2 \Gamma_2$$

induced by maps $S^1 \to S^1$ and $S^1 \to \Gamma_2$ respectively. These are both simple exercises in counting points with signs and we give the answers, which are easily checked. Let w be a reduced word in $\{e_1, e_2, \bar e_1, \bar e_2\}$, and let ϕ from $S^1 = \Gamma_1$ to Γ_2 send e_1 to w. Then $\langle h(\exp\phi)\tilde\sigma_1^1, \tilde\sigma_i^1\rangle$ and $\langle h(\exp\phi)\tilde\sigma_1^2, \tilde\sigma_i^2\rangle$ are both given by the winding number of w around e_i, and $\langle h(\exp\phi)\tilde\sigma_1^2, \tilde\sigma^{(1,1)}\rangle$ is the number of pairs of letters (a_1, a_2) in w, $a_i \in \{e_i, \bar e_i\}$, counted with the product of a minus one for each bar and a further minus one if a_2 occurs before a_1.

3.4 Examples

As an illustration of the ideas in this section we calculate two examples. The first reproduces a result from [19] on maps $S^1 \to S^1$, and the second will be useful in understanding the action of the braid group.

Let $\phi\colon S^1 \to S^1$ be a degree d map. By Theorem 1 of this paper or Theorem 4 of [19] we have

$$H_k(\exp_k S^1) = \begin{cases} 0 & k \text{ even}; \\ \mathbb{Z} & k \text{ odd}; \end{cases}$$

so the only map of interest is $(\exp_{2\ell-1} \ )_*$ on $H_{2\ell-1}$. The homology group $H_{2\ell-1}(\exp_{2\ell-1} S^1)$ is generated by $\sim_1^{2\ell-1}$, and by (3.4) and the discussion at the end of section 3.3 we have

$$\begin{aligned}(\exp_{2\ell-1} \ )_J \sim_1^{2\ell-1} &= \frac{1}{(\ell-1)!}(\exp_{2\ell-1} \ )_J \left(\sim_1^1\right)\left(\sim_1^2\right)^{\ell-1} \\ &= \frac{1}{(\ell-1)!}(d\sim_1^1)(d\sim_1^2)^{\ell-1} \\ &= \frac{d^\ell}{(\ell-1)!}\sim_1^1\left(\sim_1^2\right)^{\ell-1} \\ &= d^\ell \sim_1^{2\ell-1}.\end{aligned}$$

Thus $\exp_{2\ell-1}$ is a degree d^ℓ map, as found in Theorem 7 of [19].

For the second example consider the map $\ : \Gamma_2 \to \Gamma_2$ sending e_1 to e_2 and e_2 to $e_2 e_1 e_2$. We shall compare this with the map that simply switches e_1 and e_2. These two maps are homotopic through a homotopy that drags v around e_2, so we expect the same induced map once we pass to homology, but an understanding of the chain maps will be useful in section 4 when we study the action of the braid group. Clearly $(\exp \ )_J$ simply interchanges $\sim_1^j$ and $\sim_2^j$, and likewise $(\exp \ )_J \sim_1^j = \sim_2^j$ for each j. The only difficulty therefore is in finding $(\exp \ )_J \sim_2^j$, and we shall do this in two ways, first by calculating it directly and then by working over $\mathbb{Q}$ using (3.4).

To calculate $(\exp \ )_J \sim_2^j$ directly let $p = (x_1, \ldots, x_\ell, y_1, \ldots, y_m)$ be a generic point in $\ _\ell \ _m$, $\ell + m = j$, $m \ \ 1$. Preimages of p come in pairs that differ only in whether the preimage of y_i comes from the letter e_2 or e_2 of w. In the first case the $(\ell+1)$th row of the derivative has a 3 in the first column and zeros elsewhere, and in the second it has a -3 in the last column and zeros elsewhere. Thus each pair contributes $(-1)^\ell + (-1)^m$ times the sign of the corresponding preimage in $\sim_2^{j-1}$ of $(x_1, \ldots, x_\ell, y_2, \ldots, y_m) \in \ _\ell \ _{m-1}$, and consequently

$$\langle(\exp \ ) \ \sim_2^j, \sim^{(\ell,m)}\rangle = \left((-1)^\ell + (-1)^m\right)\langle(\exp \ ) \ \sim_2^{j-1}, \sim^{(\ell,m-1)}\rangle.$$

This recurrence relation is easily solved to give

$$\langle(\exp \ ) \ \sim_2^j, \sim^{(\ell,m)}\rangle = \begin{cases} 1 & \text{if } m = 0, \\ -2 & \text{if } \ell \text{ odd}, m = 1, \\ 0 & \text{otherwise}, \end{cases}$$

so that

$$(\exp \ )_J \sim_2^j = \begin{cases} \sim_1^j & \text{if } j \text{ is odd}, \\ \sim_1^j - 2\sim^{(j-1,1)} & \text{if } j > 0 \text{ is even}. \end{cases} \qquad (3.5)$$

To find $(\exp\ )_J\tilde{\sigma}_2^j$ using (3.4) we first need $(\exp\ )_J\tilde{\sigma}_2^1 = \tilde{\sigma}_1^1$ and $(\exp\ )_J\tilde{\sigma}_2^2 = \tilde{\sigma}_1^2 - 2\tilde{\sigma}_1^1\tilde{\sigma}_2^1$, each easily found directly. Then

$$(\exp\ )_J\tilde{\sigma}_2^{2\ell} = \frac{1}{\ell!}(\tilde{\sigma}_1^2 - 2\tilde{\sigma}_1^1\tilde{\sigma}_2^1)^{\ell}.$$

The cell $\tilde{\sigma}_1^2$ commutes with everything, so the binomial theorem applies, but $\tilde{\sigma}_1^1$ and $\tilde{\sigma}_2^1$ square to zero, so only two terms are nonzero. We get

$$\begin{aligned}(\exp\ )_J\tilde{\sigma}_2^{2\ell} &= \frac{1}{\ell!}\left(\tilde{\sigma}_1^2\right)^{\ell} - \frac{2}{\ell!}\ell\tilde{\sigma}_1^1\tilde{\sigma}_2^1\left(\tilde{\sigma}_1^2\right)^{\ell-1}\\ &= \tilde{\sigma}_1^{2\ell} - 2\tilde{\sigma}_1^{2\ell-1}\tilde{\sigma}_2^1,\end{aligned}$$

the even case of (3.5). Multiplying by $(\exp\ )_J\tilde{\sigma}_2^1 = \tilde{\sigma}_1^1$ kills the second term and we get the odd case also.

To complete the calculation of $(\exp\ )_J$ we find the image of the cells $\tilde{\sigma}^{(\ell;m)}$. If $m = 2p+1$ is odd we have simply

$$(\exp\ )_J\tilde{\sigma}^{(\ell;2p+1)} = \tilde{\sigma}_2^{\ell}\tilde{\sigma}_1^{2p+1} = (\exp\ )_J\tilde{\sigma}^{(\ell;2p+1)},$$

while if $m = 2p > 0$ is even we get

$$\begin{aligned}(\exp\ )_J\tilde{\sigma}^{(\ell;2p)} &= \tilde{\sigma}_2^{\ell}\left(\tilde{\sigma}_1^{2p} - 2\tilde{\sigma}_1^{2p-1}\tilde{\sigma}_2^1\right)\\ &= \tilde{\sigma}_2^{\ell}\tilde{\sigma}_1^{2p} - 2(-1)^{\ell}\binom{\ell+1}{1}_{-1}\tilde{\sigma}_1^{2p-1}\tilde{\sigma}_2^{\ell+1}\\ &= \begin{cases}\tilde{\sigma}_2^{\ell}\tilde{\sigma}_1^{2p} & \text{if } \ell \text{ is odd};\\ \tilde{\sigma}_2^{\ell}\tilde{\sigma}_1^{2p} - 2\tilde{\sigma}^{(2p-1;\ell+1)} & \text{if } \ell \text{ is even}.\end{cases}\end{aligned}$$

Thus

$$(\exp\ )_J\tilde{\sigma}^{(\ell;m)} = \begin{cases}(\exp\ )_J\tilde{\sigma}^{(\ell;m)} - 2\tilde{\sigma}^{(m-1;\ell+1)} & \ell, m \text{ both even, } m > 0;\\ (\exp\ )_J\tilde{\sigma}^{(\ell;m)} & \text{otherwise}.\end{cases} \tag{3.6}$$

Since elements of homology are linear combinations of cells each having at least one odd index we have $(\exp_k\ ) = (\exp_k\ )$ as expected.

4 The action of the braid group

4.1 Introduction

The braid group on n strands B_n may be defined as the mapping class group of an n–punctured disc D_n, or more precisely as the group of homeomorphisms of D_n that fix ∂D_n pointwise, modulo those isotopic to the identity rel ∂D_n.

As such it acts on $H_k(\exp_k D_n)$, and since $D_n \simeq \Gamma_n$ we may regard this as an action on $H_k(\exp_k \Gamma_n)$. We prove the following structure theorem for this action.

Theorem 7 *The image of the pure braid group P_n under the action of B_n on $H_k(\exp_k \Gamma_n)$ is nilpotent of class at most $\min\{\lfloor (k-1)/2 \rfloor, \lfloor (n-1)/2 \rfloor\}$ if k is odd, or $\min\{\lfloor (k-2)/2 \rfloor, \lfloor (n-2)/2 \rfloor\}$ if k is even.*

For the above and other definitions of the braid group see Birman [4].

Recall that the pure braid group P_n is the kernel of the map $B_n \to S_n$ sending each braid β to the induced permutation $\bar\beta$ of the punctures. Consider the subgroup of P_n consisting of braids whose first $n-1$ strands form the trivial braid. The nth strand of such a braid may be regarded as an element of $\pi_1(D_{n-1})$, and doing so gives an isomorphism from this subgroup to the free group F_{n-1}. This shows that P_n is not nilpotent for $n \geq 3$. The group P_2 inside B_2 is isomorphic to $2\mathbb{Z}$ inside $\mathbb{Z}$, and is therefore nilpotent of class 1; however the bound for $n = 2$ in Theorem 7 is zero, implying P_2 acts trivially, and in fact this follows from the second example of section 3.4. Thus we have in particular that the action of B_n on $H_k(\exp_k \Gamma_n)$ is unfaithful for all k and $n \geq 2$.

There is an obvious action of S_n on $H_k(\exp_k \Gamma_n)$, induced by permuting the edges of Γ_n. The theorem will be proved by relating the action of each braid β to that of $\bar\beta$. Note that there is again nothing lost by considering only the action on $H_k(\exp_k \Gamma_n)$, because of the isomorphisms induced by i and $\cup\{v\}$. For brevity, in what follows we shall simply write H_k for $H_k(\exp_k \Gamma_n)$.

4.2 Proof of the structure theorem

We fix a representation of D_n and a homotopy equivalence $\Gamma_n \to D_n$, the embedding shown in figure 4(a). B_n is generated by the "half Dehn twists" $\sigma_1, \ldots, \sigma_{n-1}$, where σ_i interchanges the ith and $(i+1)$th punctures with an anti-clockwise twist. The effect of σ_i on the embedded graph is shown in figure 4(b), and we see that it induces the map

$$e_a \mapsto \begin{cases} e_{i+1} & \text{if } a = i, \\ e_{i+1}e_ie_{i+1} & \text{if } a = i+1, \\ e_a & \text{if } a \neq i, i+1, \end{cases}$$

on Γ_n; regarding the e_i as generators of the free group F_n this is the standard embedding $B_n \hookrightarrow \mathrm{Aut}(F_n)$. We will call the induced map $\Gamma_n \to \Gamma_n$ σ_i also,

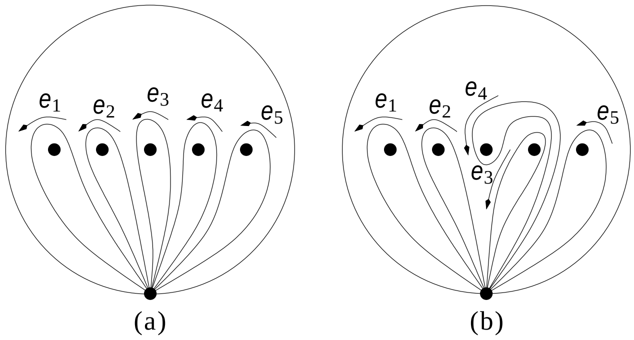


Figure 4: The punctured disc D_n and the generators σ_i for $n = 5$ and $i = 3$. We embed Γ_5 in D_5 as shown in (a); the effect of σ_3 on the embedding is shown in (b).

and our goal is to understand the $(\exp_k \sigma_i)_*$ well enough to show that the pure braid group acts by upper-triangular matrices.

To this end we define a filtration of H_k. Each homology class has precisely one representative in $\tilde{C}_k$, since there are no boundaries, so we may regard H_k as a subspace of $\tilde{C}_k$ and work unambiguously on the level of chains. Let

$$F_j = \left\{ \textstyle\sum_J c_J \tilde{J} \in H_k \;\middle|\; c_J = 0 \text{ if } |J|_2 < n - j \right\};$$

so that F_j is the subspace spanned by cells having no more than j even indices. Clearly $\{0\} \subseteq F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = H_k$, so the F_j form a filtration. Moreover F_j has $F_j \cap B(k;n)$ as a basis: each element of $B(k;n)$ has the form

$$\partial g_J = \sum_{i : j_i > 0 \text{ even}} c_i \widetilde{\partial_i(J)}$$

for some J with $|J| = k + 1$, and the mod 2 norm of every term in this sum is $|J|_2 + 1$.

We now describe when $F_i = F_j$ for $i \neq j$, as the indexing has been chosen to be uniform and meaningful at the expense of certain a priori isomorphisms among the F_j, arising from the parity and size of k. Since $|J| = k$ for each k–cell $\tilde{J}$ we have in particular $|J|_2 \equiv k \bmod 2$. Thus $F_{j+1} = F_j$ if $n - j$ and k have the same parity. Next, $|J|_2 \leq |J| = k$, so $F_j = \{0\}$ if $n - j > k$, or in other words if $j < n - k$. Lastly, no cell for which $|J|_2 = 0$ is a summand of an element of H_k (recall that C_* is exact, and that such a cell sits at the top of an m–cube complex and so is not part of any boundary), so that either F_{n-2} or F_{n-1} is all of H_k, depending on the parity of k. To see that these are the only circumstances in which F_i and F_j can coincide suppose that $i < j \leq n - 1$,

$n-i$ and $n-j$ have the same parity as k, $n-j \quad k$, and let

$$J = (\underbrace{0,\ldots,0}_{j},\underbrace{1,\ldots,1}_{n-j-1},k+j+1-n).$$

Every non-zero index of J is odd so $\tilde{\ }^J$ is a cycle. It has $j>i$ even indices and is therefore non-trivial in the quotient F_j/F_i.

The signi cance of the ltration is given by the following lemma, which is the main step in proving the structure theorem for the action.

Lemma 5 *Each F_j is an invariant subspace for the action of B_n on H_k. Moreover, the action on F_j/F_{j-1} factors through the symmetric group S_n.*

Proof The lemma is proved by relating the action of B_n to that of S_n induced by permuting the edges of Γ_n. Much of the work has been done already in section 3.4, since $\ _i$ and $\ _i$ act as the maps and considered there on $e_i \cup e_{i+1}$, and as the identity on the remaining edges.

For each n–tuple J let

$$\ _i(J) = (j_1,\ldots,j_i+1,\ldots,j_n)$$

so that

$$\partial_i \ _{i+1} \ _i(J) = (j_1,\ldots,j_{i-1},j_{i+1}-1,j_i+1,j_{i+2},\ldots,j_n).$$

By (3.6) we may write

$$(\exp\ _i)_J \tilde{\ }^J = \begin{cases} (\exp\ _i)_J \tilde{\ }^J - 2\tilde{\ }^{\partial_i \ _{i+1} \ _i(J)} & j_i, j_{i+1} \text{ both even, } j_{i+1}>0, \\ (\exp\ _i)_J \tilde{\ }^J & \text{otherwise.} \end{cases} \tag{4.1}$$

If j_i and j_{i+1} are both even then

$$|\partial_i \ _{i+1} \ _i(J)|_2 = |J|_2 + 2$$

and it follows that

$$(\exp_k\ _i)\, c \in (\exp_k\ _i)\, c + F_{j-1}$$

for each $c \in F_j$ and $i \in [n-1]$. Since the $\ _i$ generate B_n we get

$$(\exp_k\ )\, c \in (\exp_k\ )\, c + F_{j-1}$$

for all braids and $c \in F_j$, and the lemma follows immediately. □

Proof of Theorem 7 By Lemma 5, with respect to a suitable ordering of $B(k;n)$ the braid group acts by block upper-triangular matrices. The diagonal blocks are the matrices of the action on F_j/F_{j-1}, and since this factors through S_n we have that the pure braids act by upper-triangular matrices with ones on the diagonal. It follows immediately that the image of P_n is nilpotent.

To bound the length of the lower central series we count the number of nontrivial blocks, as the class of the image is at most one less than this. By the discussion following the definition of $\{F_j\}$ this is the number of $0 \leq j \leq n-1$ such that $n-j \leq k$ and $n-j \equiv k \bmod 2$; letting $\ell = n-j$ this is the number of $1 \leq \ell \leq \min\{n,k\}$ such that $\ell \equiv k \bmod 2$. There are $\lfloor (m+1)/2 \rfloor$ positive odd integers and $\lfloor m/2 \rfloor$ positive even integers less than or equal to a positive integer m, and the given bounds follow. □

4.3 The action of B_3 on $H_3(\exp_3\Gamma_3)$

We study the action of B_3 on $H_3(\exp_3\Gamma_3)$, being the smallest non-trivial example, and show that P_3 acts as a free abelian group of rank two.

For simplicity we will simply write J for the cell $\tilde{\ }^J$. From Theorem 4 we obtain the basis

$$
\begin{aligned}
u_1 &= (3;0;0) & w_1 &= (0;1;2) + (0;2;1)\\
u_2 &= (0;3;0) & w_2 &= (1;0;2) + (2;0;1)\\
u_3 &= (0;0;3) & w_3 &= (1;2;0) + (2;1;0)\\
v &= (1;1;1) & &
\end{aligned}
$$

for $H_3(\exp_3\Gamma_3)$ (in fact this is the negative of the basis given there). Let U be the span of $\{u_1;u_2;u_3\}$, V the span of $\{v\}$, and W the span of $\{w_1;w_2;w_3\}$. The subspaces U, V and $V \oplus W$ are easily seen to be invariant using equation (4.1), and moreover the action on U is simply the permutation representation of S_3. We therefore restrict our attention to $V \oplus W$. The actions on V and $(V \oplus W)/V$ are the sign and permutation representations of S_3 respectively, and with respect to the basis $\{v, w_1, w_2, w_3\}$ we find that

$$(\exp_3 \sigma_1)_{V \oplus W} = T_1 = \begin{bmatrix} -1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$(\exp_3 \sigma_2)_{V \oplus W} = T_2 = \begin{bmatrix} -1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

In each case the inverse is obtained by moving the -2 one place to the right.

A product of T_1, T_2 and their inverses has the form

$$P = \begin{pmatrix} \det \bar{P} & p \\ 0 & \bar{P} \end{pmatrix}$$

where $\bar{P}$ is a permutation matrix and $p = (p_1, p_2, p_3)$ is a vector of even integers. Consider $\varepsilon(P) = p_1 + p_2 + p_3$. Multiplying P by $T_i^{\pm 1}$ we see that $\varepsilon(T_i^{\pm 1} P) = -\varepsilon(P) - 2$, and it follows that $\varepsilon(P)$ is zero if $\bar{P}$ is even and -2 if $\bar{P}$ is odd. In particular $\varepsilon(P)$ is zero if P is the image of a pure braid.

If $P = QR$ where $\bar{Q} = \bar{R} = I$ then $p = q + r$ and it follows that the pure braid group P_3 acts as a free abelian group of rank at most two. That the rank is in fact two can be verified by calculating T_1^2 and T_2^2 and checking that the corresponding vectors are independent.

A Table of Betti numbers

Table 1 lists the Betti numbers

$$\begin{aligned} b_k(\exp_k \Gamma_n) &= \sum_{j=1}^{k} (-1)^{j-k} \binom{n+j-1}{n-1} \\ &= \begin{cases} \sum_{j=1}^{\ell} \binom{n+2j-2}{n-2} & \text{if } k = 2\ell \text{ is even,} \\ n + \sum_{j=1}^{\ell} \binom{n+2j-1}{n-2} & \text{if } k = 2\ell + 1 \text{ is odd,} \end{cases} \end{aligned}$$

for $1 \le k \le 20$, $1 \le n \le 10$. To find the other non-vanishing Betti numbers recall that $b_{k-1}(\exp_k \Gamma_n) = b_{k-1}(\exp_k(\Gamma_n, v)) = b_{k-1}(\exp_{k-1} \Gamma_n)$ for $k \ge 2$.

		n									
		1	2	3	4	5	6	7	8	9	10
	1	1	2	3	4	5	6	7	8	9	10
	2	0	1	3	6	10	15	21	28	36	45
	3	1	3	7	14	25	41	63	92	129	175
	4	0	2	8	21	45	85	147	238	366	540
	5	1	4	13	35	81	167	315	554	921	1462
	6	0	3	15	49	129	295	609	1162	2082	3543
	7	1	5	21	71	201	497	1107	2270	4353	7897
	8	0	4	24	94	294	790	1896	4165	8517	16413
	9	1	6	31	126	421	1212	3109	7275	15793	32207
k	10	0	5	35	160	580	1791	4899	12173	27965	60171
	11	1	7	43	204	785	2577	7477	19651	47617	107789
	12	0	6	48	251	1035	3611	11087	30737	78353	186141
	13	1	8	57	309	1345	4957	16045	46783	125137	311279
	14	0	7	63	371	1715	6671	22715	69497	194633	505911
	15	1	9	73	445	2161	8833	31549	101047	295681	801593
	16	0	8	80	524	2684	11516	43064	144110	439790	1241382
	17	1	10	91	616	3301	14818	57883	201994	641785	1883168
	18	0	9	99	714	4014	18831	76713	278706	920490	2803657
	19	1	11	111	826	4841	23673	100387	379094	1299585	4103243
	20	0	10	120	945	5785	29457	129843	508936	1808520	5911762

Table 1: Betti numbers $b_k(\exp_k \Gamma_n)$ for $1 \leq k \leq 20$ and $1 \leq n \leq 10$.

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Department of Mathematics, University of California at Davis
One Shields Avenue, Davis, CA 95616-8633, USA

Email: `tuffley@math.ucdavis.edu`

Received: 21 February 2003 Revised: 16 September 2003