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Equivalences to the triangulation conjecture

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Abstract We utilize the obstruction theory of Galewski-Matumoto-Stern to derive equivalent formulations of the Triangulation Conjecture. For example, every closed topological manifold M^n with $n \geq 5$ can be simplicially triangulated if and only if the two distinct combinatorial triangulations of RP^5 are simplicially concordant.

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1 Introduction

The Triangulation Conjecture (TC) affirms that every closed topological manifold M^n of dimension $n \geq 5$ admits a simplicial triangulation. The vanishing of the Kirby-Siebenmann class $KS(M)$ in $H^4(M; Z/2)$ is both necessary and sufficient for the existence of a combinatorial triangulation of M^n for $n \geq 5$ by [7]. A combinatorial triangulation of a closed manifold M^n is a simplicial triangulation for which the link of every i-simplex is a combinatorial sphere of dimension $n-i-1$. Galewski and Stern [3, Theorem 5] and Matumoto [8] independently proved that a closed connected topological manifold M^n with $n \geq 5$ is simplicially triangulable if and only if

$$\delta_\alpha KS(M) = 0 \quad \text{in} \quad H^5(M; \ker\alpha) \tag{1.1}$$

where δ_α denotes the Bockstein operator associated to the exact sequence $0 \to \ker\alpha \to \theta_3 \xrightarrow{\alpha} Z/2 \to 0$ of abelian groups. Moreover, the Triangulation Conjecture is true if and only if this exact sequence splits by [3] or [11, page 26]. The Rochlin invariant morphism α is defined on the homology bordism group θ_3 of oriented homology 3-spheres modulo those which bound acyclic compact PL 4-manifolds. Fintushel and Stern [1] and Furuta [2] proved that θ_3 is infinitely generated.

We freely employ the notation and information given in Ranicki's excellent exposition [11]. The relative boundary version of the Galewski-Matumoto-Stern obstruction theory in [11] produces the following result. Given any homeomorphism $f : |K| \to |L|$ of the polyhedra of closed m-dimensional PL manifolds K and L with $m \geq 5$, f is homotopic to a PL homeomorphism if and only if $KS(f)$ vanishes in $H^3(L; Z/2)$. More generally, a homeomorphism $f : |K| \to |L|$ is homotopic to a PL map $F : K \to L$ with acyclic point inverses if and only if

$$\delta_\alpha(KS(f)) = 0 \quad \text{in} \quad H^4(L; \ker \alpha) \ . \tag{1.2}$$

Concordance classes of simplicial triangulations on M^n for $n \geq 5$ correspond bijectively to vertical homotopy classes of liftings of the stable topological tangent bundle $\tau : M \to \mathrm{BTOP}$ to BH by [3, Theorem 1] and so are enumerated by $H^4(M; \ker \alpha)$. The classifying space BH for the stable bundle theory associated to combinatorial homology manifolds in [11] is denoted by BTRI in [3] and by BHML in [8]. We employ obstruction theory to derive some known and new results and generalizations of [4] and [13] on the existence of simplicial triangulations in section 2 and to record some equivalent formulations of TC in section 3. Although some of these formulations may be known, they do not seem to be documented in the literature.

2 Simplicial Triangulations

Let δ^* denote the integral Bockstein operator associated to the exact sequence $0 \to Z \xrightarrow{\times 2} Z \xrightarrow{\rho} Z/2 \to 0$. We proceed to derive some consequences of the vanishing of δ^* on Kirby-Siebenmann classes. The coefficient group for cohomology is understood to be $Z/2$ whenever omitted. Matumoto knew in [8] that the vanishing of $\delta^* KS(M)$ implied the vanishing of $\delta_\alpha KS(M)$. Let ι_m denote the fundamental class of the Eilenberg-MacLane space $K(Z, m)$. Since $H^{m+1}(K(Z,m); G) = 0$ for all coefficient groups G, trivially $\delta_\alpha(\rho\iota_m) = 0$ in $H^{m+1}(K(Z,m); \ker \alpha)$. Thus δ_α vanishes on $KS(M)$ in (1.1) or $KS(f)$ in (1.2) whenever δ^* does. This observation together with (1.1) and (1.2) justifies the following well-known statements. Every closed connected topological manifold M^n with $n \geq 5$ and $\delta^* KS(M) = 0$ admits a simplicial triangulation. Let $f : |K| \to |L|$ be any homeomorphism of the polyhedra of closed m-dimensional PL manifolds K and L with $m \geq 5$. If $\delta^* KS(f) = 0$, then f is homotopic to a PL map $F : K \to L$ with acyclic point inverses.

Proposition 2.1 *All k-fold Cartesian products of closed 4-manifolds are simplicially triangulable for $k \geq 2$. All products $M^4 \times S^1$ with non-orientable closed*

4-manifolds M^4 are simplicially triangulable. Let N^4 be any simply connected closed 4-manifold with $KS(N)$ trivial and also $b =$ rank of $H_2(N;Z) \geq 1$. Let $f : |K| \to |L|$ be any homeomorphism with $KS(f)$ nontrivial and $|K| = |L| = N \times S^1$. Then f is homotopic to a PL map $F : K \to L$ with acyclic point inverses.

Proof of 2.1 Since $KS(\gamma)$ is a primitive cohomology class for the universal bundle γ on BTOP, we have $KS(M_1 \times M_2) = KS(M_1) \otimes 1 + 1 \otimes KS(M_2)$ in $H^4(M_1 \times M_2)$. Triviality of δ_α on $H^4(M^4)$ by dimensionality yields triangulability of all k-fold products of closed 4-manifolds for $k \geq 2$, and of $M^4 \times S^1$ by (1.1).

The product $N^4 \times S^1$ admits 2^b distinct combinatorial structures by [7]; moreover, for every non-zero class u in $H^3(N \times S^1)$, there is a homeomorphism of polyhedra with distinct combinatorial structures whose Casson-Sullivan invariant is u by [11, page 15]. The vanishing of $\delta^* KS(f)$ follows from the triviality of δ^* on $H^3(N \times S^1) = \rho(H^2(N;Z) \otimes H^1(S^1;Z))$. □

No closed 4-manifold M^4 with $KS(M)$ non-zero can be simplicially triangulated. Yet k-fold products of such manifolds M^4 by (2.1) and their products with spheres or tori produce infinitely many distinct non-combinatorial, yet simplicially triangulable closed manifolds in every dimension ≥ 5. In contrast, there are no known examples of non-smoothable closed 4-manifolds which can be simplicially triangulated, according to Problem 4.72 of [6, page 287].

Theorem 2.2 *Let M^n be any closed connected topological manifold with $n \geq 5$ such that the stable spherical fibration determined by the tangent bundle $\tau(M)$ has odd order in $[M, BSG]$. Suppose that either $H_2(M;Z)$ has no 2-torsion or else all 2-torsion in $H_4(M;Z)$ has order 2. Then M is simplicially triangulable.*

Proof The Stiefel-Whitney classes of M are trivial by the hypothesis of odd order. We first consider the special case that $\tau(M)$ is stably fiber homotopically trivial. Let $g : M \to SG/\mathrm{STOP}$ be any lifting of a classifying map $\tau(M) : M \to$ BSTOP in the fibration

$$SG/\mathrm{STOP} \xrightarrow{\mathrm{j}} \mathrm{BSTOP} \xrightarrow{\pi} \mathrm{BSG} \tag{2.3}$$

The Postnikov 4-stage of SG/STOP is $K(Z/2, 2) \times K(Z, 4)$. Now $j^* KS(\tilde{\gamma}) = \iota_2^2 + \rho(\iota_4)$ by Theorem 15.1 of [7, page 328] where $\tilde{\gamma}$ denotes the universal bundle over BSTOP. Clearly $\delta^*(j^* KS(\tilde{\gamma})) = \delta^*(\iota_2^2) = 2u$ where u generates $H^5(K(Z/2, 2); Z) \approx Z/4$. If all nonzero 2-torsion in $H_4(M;Z)$ has order 2, then $\delta^* KS(M) = 2g^* u = 0$. If $H_2(M;Z)$ has no 2-torsion, then $\delta^*(g^*\iota_2) = 0$ so again $\delta^* KS(M) = 0$. Thus $\delta_\alpha KS(M) = 0$.

We suppose now that the stable spherical fibration of $\tau(M)$ has order $2a+1$ in $[M, BSG]$ with $a \geq 1$. Let $s : M \to S(2a\tau(M))$ be a section to the sphere bundle projection $p : S(2a\tau(M)) \to M$ associated to $2a\tau(M)$. Now $S(2a\tau(M))$ is a stably fiber homotopically trivial manifold, since its stable tangent bundle is $(2a+1)p^*\tau(M)$. Since $KS(M) = (2a+1)KS(M) = s^*(KS(S(2a\tau(M))))$ we conclude that

$$\delta^* KS(M) = s^*(\delta^* KS(S(2a\tau(M)))) = s^* 0 = 0 \ . \tag{2.4}$$

We consider the following homotopy commutative diagram of principal fibrations.

$$\begin{array}{ccccccc}
 & & K(\ker\alpha, 4) & \xrightarrow{i} & (K(\ker\alpha,4), *) & = & (K(\ker\alpha,4),*) \\
 & & \downarrow & & \downarrow & & \downarrow i \\
 & & BH & \xrightarrow{i} & (BH, BPL) & \xrightarrow{t} & (K(\theta_3,4),*) \\
 & & \downarrow \pi & & \downarrow \hat{\pi} & & \downarrow \alpha \\
S^4 & \xrightarrow{ks} & BTOP & \xrightarrow{i} & (BTOP, BPL) & \xrightarrow{\widehat{KS}} & (K(Z/2,4),*) \\
 & & & & \downarrow \delta_\alpha \widehat{KS} & & \downarrow \delta_\alpha \iota \\
 & & & & (K(\ker\alpha,5),*) & = & (K(\ker\alpha,5),*)
\end{array} \tag{2.5}$$

The fiber map α is induced from the path-loop fibration on $K(\ker\alpha, 5)$ via the Bockstein operator $\delta_\alpha\iota$ on the fundamental class ι of $K(Z/2,4)$. The induced morphism α_* on π_4 is the Rochlin morphism $\alpha : \theta_3 \to Z/2$ by construction. The relative principal fibration $\hat{\pi}$ is induced from α via the map $\widehat{KS}$ classifying the relative universal Kirby-Siebenmann class. Thus $(\widehat{KS} \circ i)^*\iota = KS(\gamma)$. Inclusion maps are denoted by i in (2.5). The induced morphisms t_* and $(\widehat{KS})_*$ are isomorphisms on π_4. We employ (2.5) in the proof of Theorem 3.1.

3 Equivalent formulations to TC

Galewski and Stern constructed a non-orientable closed connected 5-manifold M^5 in [4] such that $Sq^1 KS(M)$ generates $H^5(M) \approx Z/2$. They also proved that any such M^5 is "universal" for TC. Moreover, Theorem 2.1 of [4] essentially affirms that either TC is true or else no closed connected topological n-manifold M^n with $Sq^1 KS(M) \neq 0$ and $n \geq 5$ can be simplicially triangulated.

Theorem 3.1

The following statements are equivalent to the Triangulation Conjecture.

(1) *Any (equivalently all) of the classes* $\delta_\alpha KS(\gamma)$, $\delta_\alpha \widehat{KS}$, *and* $\delta_\alpha \iota$ *in (2.5) is trivial if and only if any (equivalently all) of the fiber maps* π, $\hat{\pi}$, *and* α *in (2.5) admits a section.*

(2) *The essential map* $f : S^4 \cup_2 e^5 \to \mathrm{BTOP}$ *lifts to* BH *in (2.5).*

(3) $Sq^1 KS(\hat{\gamma}) \neq 0$ *in* $H^5(BH)$ *for the universal bundle* $\hat{\gamma} = \pi^*\gamma$ *on* BH.

(4) *Any closed connected topological manifold* M^n *with* $Sq^1 KS(M) \neq 0$ *and* $n \geq 5$ *admits a simplicial triangulation.*

(5) *Every homeomorphism* $f : |K| \to |L|$ *with* $KS(f)$ *non-trivial is homotopic to a PL map with acyclic point inverses where* K *and* L *are any combinatorially distinct polyhedra with* $|K| = |L| = N^4 \times RP^2$. *Here* N^4 *denotes any simply connected, closed 4-manifold with* $KS(N)$ *trivial and positive rank for* $H_2(N; Z)$.

(6) *All combinatorial triangulations of each closed connected PL manifold* M^n *with* $n \geq 5$ *are concordant as simplicial triangulations.*

(7) *The two distinct combinatorial triangulations of* RP^5 *are simplicially concordant.*

(8) *Every closed connected topological manifold* M^n *with* $n \geq 5$ *that is stably fiber homotopically trivial admits a simplicial triangulation.*

Proof $TC \Leftrightarrow (1)$ Statement (1) is equivalent to the splitting of the exact sequence $0 \to \ker\alpha \to \theta_3 \xrightarrow{\alpha} Z/2 \to 0$ through the induced morphisms on homotopy in dimension 4.

$TC \Leftrightarrow (2)$ Let $ks : S^4 \to \mathrm{BTOP}$ represent the Kirby-Siebenmann class in homotopy. That is, $[ks]$ has order 2 and is dual to $KS(\gamma)$ under the mod 2 Hurewicz morphism. Now ks admits an extension $f : S^4 \cup_2 e^5 \to \mathrm{BTOP}$, since the cofibration exact sequence

$$\pi_5(\mathrm{BTOP}) \longrightarrow [\mathrm{S}^4 \cup_2 \mathrm{e}^5, \mathrm{BTOP}] \to \pi_4(\mathrm{BTOP}) \xrightarrow{\times 2} \pi_4(\mathrm{BTOP}) \tag{3.2}$$

corresponds to $0 \longrightarrow Z/2 \longrightarrow Z \oplus Z/2 \xrightarrow{\times 2} Z \oplus Z/2$. If $g : S^4 \cup_2 e^5 \to BH$ is any lifting of f, the composite map using (2.5)

$$h : S^4 \subset S^4 \cup_2 e^5 \xrightarrow{g} BH \xrightarrow{i} (BH, BPL) \xrightarrow{t} (K(\theta_3, 4), *) \tag{3.3}$$

produces $u = [h]$ in θ_3 with $2u = 0$ and $\alpha(u) = 1$, since $\alpha(u) = [\alpha \circ h] = [\widehat{KS} \circ ks]$ generates $\pi_4(K(Z/2, 4))$. Thus TC is true. Conversely, if TC is true, a section $s : \mathrm{BTOP} \to \mathrm{BH}$ to π in (2.5) gives a lifting $s \circ f$ of f.

$TC \Leftrightarrow (3)$ Properties of $KS(\gamma)$ are enumerated in [9] and [10]. Since $Sq^1 KS(\gamma) \neq 0$, a section s to π in (2.5) gives $Sq^1(KS(\hat{\gamma}) \neq 0$ so TC implies 3. We now assume that TC is false and claim that the generator $Sq^1\iota$ for $H^5(K(Z/2,4)) \approx Z/2$ lies in the image of

$$H^5(K(\ker\alpha, 5)) \approx Hom(\pi_5(K(\ker_\alpha, 5)), Z/2) \approx Hom(\ker\alpha, Z/2).$$

The Serre exact sequence then gives $\alpha^*(Sq^1\iota) = 0$ in $H^5(K(\theta_3, 4))$ so

$$Sq^1 KS(\hat{\gamma}) = (t \circ i)^*(\alpha^* Sq^1 \iota) = 0.$$

Thus we must construct a morphism $\ker\alpha \to Z/2$ which does not extend to θ_3. We consider the sequence $\ker\alpha \xrightarrow{\times 2} \ker\alpha \xrightarrow{\rho} \ker\alpha \otimes Z/2$ and define $h : \ker\alpha \otimes Z/2 \to Z/2$ as follows. $h(v) = 1$ if and only if $v = \rho(2z)$ for some $z \in \theta_3$ with $\alpha(z) = 1$. Now h is a well-defined and non-trivial morphism, since θ_3 does not have an element u with $2u = 0$ and $\alpha(u) = 1$ by hypothesis. The composite morphism $h \circ \rho : \ker\alpha \to Z/2$ does not extend to θ_3.

$TC \Leftrightarrow (4)$ Suppose M^n with $Sq^1 KS(M) \neq 0$ admits a simplicial triangulation. Now $Sq^1 KS(M) = g^* Sq^1 KS(\hat{\gamma})$ for any lifting $g : M \to BH$ of $\tau : M \to \mathrm{BTOP}$. Since $Sq^1 KS(\hat{\gamma}) \neq 0$, TC holds by (3).

$TC \Leftrightarrow (5)$ Clearly triviality of $\delta_\alpha \widehat{KS}$ in (2.5) gives $\delta_\alpha KS(f) = 0$ via naturality for every f. Suppose that $\delta_\alpha KS(f) = 0$ for any such f in 5. Now $KS(f) = \rho(v) \otimes i^*a$ in $\rho(H^2(M;Z)) \otimes H^1(RP^2) \approx H^3(L)$. Here a generates $H^*(RP^\infty)$ and $i : RP^2 \subset RP^\infty$. Naturality via the universal example $CP^\infty \times RP^\infty$ for $\rho(v) \otimes i^*a$ gives $\delta_\alpha KS(f) = v \otimes \delta_\alpha(i^*a)$. Since $i^* : H^2(RP^\infty; \ker\alpha) \to H^2(RP^2; \ker\alpha)$ is a monomorphism, $\delta_\alpha(i^*a) = 0$ if and only if $\delta_\alpha(a) = 0$. Now $\delta_\alpha(a) = 0$ if and only if TC is true via the fibration

$$K(\ker\alpha, 1) \longrightarrow K(\theta_3, 1) \xrightarrow{\alpha} RP^\infty.$$

$TC \Leftrightarrow (6) \Leftrightarrow (7)$ TC holds if and only if $\delta_\alpha \iota = 0$ for the fundamental class ι of $K(Z/2, 3)$. Concordance classes of simplicial triangulations of M^n arising from combinatorial triangulations differ by classes in $\delta_\alpha H^3(M)$. This subgroup of $H^4(M, \ker\alpha)$ is trivial by naturality if $\delta_\alpha \iota = 0$. Conversely, $\delta_\alpha H^3(RP^5) = 0$ if the two distinct combinatorial triangulations of RP^5 given by Theorem 16.5 in [7, pages 332 and 337] are simplicially concordant. But $\delta_\alpha(a^3) = 0$ if and only if $\delta_\alpha \iota = 0$ via the skeletal inclusion $RP^5_3 \subset K(Z/2, 3)$ and naturality for $RP^5 \to RP^5_3$.

$TC \Leftrightarrow (8)$ Similar to Theorem 5.1 of [12], we consider a regular neighborhood of the 9-skeleton of SG/STOP embedded in R^m for some $m \geq 19$ in order to obtain a smoothly parallelizable manifold W with boundary and a map $g : W \to SG/\mathrm{STOP}$ which is a homotopy equivalence through dimension 7. The double DW is smoothly parallelizable and admits an extension $\widehat{g} : DW \to SG/\mathrm{STOP}$. Note that $(\widehat{g})^*$ is a monomorphism through dimension 7. Let $h : M \to DW$ be a degree one normal map. Now M is stably fiber homotopically trivial and h^* is a monomorphism in cohomology. In particular, $(\widehat{g} \circ h)^*$ is a monomorphism on $H^5(SG/\mathrm{STOP}; \ker\alpha)$. We conclude that $\delta_\alpha KS(M) = (\widehat{g} \circ h)^*(\delta_\alpha \iota_2^2) = 0$ if and only if $\delta_\alpha \iota_2^2 = 0$ for the fundamental class ι_2 of $K(Z/2, 2)$. So statement (8) yields $\delta_\alpha \iota_2^2 = 0$.

Let $f : K(Z/2, 2) \to K(Z/2, 4)$ classify ι_2^2. Since $\delta_\alpha \iota_2^2 = 0$ assuming statement (8), f admits a lifting $h : K(Z/2, 2) \to K(\theta_3, 4)$ in (2.5) such that $f = \alpha \circ h$. The diagram

$$\begin{array}{ccccccc} & & & & [CP^3, K(\theta_3, 4)] & \approx & \theta_3 \\ (3.4) & & & {\scriptstyle h_*}\nearrow & \big\downarrow {\scriptstyle \alpha_*} & & \big\downarrow \\ & Z/2 \approx [CP^3, K(Z/2, 2)] & & \xrightarrow{f_*} & [CP^3, K(Z/2, 4)] & \approx & Z/2 \end{array}$$

yields a splitting to the exact sequence $0 \to \ker\alpha \to \theta_3 \to Z/2 \to 0$ so TC holds.

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