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ON HYPERSURFACES IN $\mathbb{P}^3$ WITH FAT POINTS IN GENERAL POSITION

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Abstract. We prove that a linear system of hypersurfaces in $\mathbb{P}^3$ of degree d, $14 \leq d \leq 40$, with double, triple and quadruple points in general position is non-special. This solves the cases that have not been completed in a paper by E. Ballico and M. C. Brambilla.

1. Introduction. In what follows we assume that the ground field $\mathbb{K}$ is of characteristic zero. Let $d \in \mathbb{Z}$ and $m_1, \dots, m_r \in \mathbb{N}$. By $\mathcal{L}_3(d; m_1, \dots, m_r)$, we denote a linear system of hypersurfaces (in $\mathbb{P}^3 := \mathbb{P}^3(\mathbb{K})$) of degree d passing through r points $p_1, \dots, p_r$ in general position with multiplicities at least $m_1, \dots, m_r$ (a point with multiplicity m will often be called an m-point). The dimension of such system is denoted by $\dim \mathcal{L}_3(d; m_1, \dots, m_r)$.

We will use the following notation: $m^{\times k}$ denotes the sequence of m's taken k times,

$$m^{\times k} = \underbrace{(m, \dots, m)}_{k}.$$

Define the *virtual dimension of* $L = \mathcal{L}_3(d; m_1, \dots, m_r)$

$$\operatorname{vdim} L := \binom{d+3}{3} - \sum_{j=1}^{r} \binom{m_j+2}{3} - 1$$

and the *expected dimension of* L

$$\operatorname{edim} L := \max\{\operatorname{vdim} L, -1\}.$$

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Observe that $\dim L \geq \operatorname{edim} L$. L is called *special* if this inequality is strict, and *non-special* otherwise.

2. Results for multiplicities bounded by 4. In [**1**] the following Theorem 1 was proved:

THEOREM 1. *System* $L = \mathcal{L}_3(d; 4^{\times x}, 3^{\times y}, 2^{\times z})$ *is non-special for non-negative integers* d, x, y, z *satisfying* $d \geq 41$.

To consider all the cases with $d \leq 40$, in [**1**] th authors proposed the algorithm, which essentially runs as follows:

ALGORITHM A

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d=   -- degree should be chosen
N=binomial(d+3,3);
for z from 0 to ceiling(N/4) do
 for y from 0 to ceiling(N/10) do
  for x from 0 to ceiling(N/20) do (
   if ((20*x+10*y+4*z>N-4)and(20*x+10*y+4*z<N+20)) then (
    -- computation of the rank of the interpolation matrix
    -- the rank is maximal if and only if the system is non-special
   )
  )
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With aid of the program Macaulay2, the following was proved ([**1**], Theorem 14):

THEOREM 2. *System* $L = \mathcal{L}_3(d; 4^{\times x}, 3^{\times y}, 2^{\times z})$ *is non-special for non-negative integers* d, x, y, z *satisfying* $9 \leq d \leq 13$.

In order to prove Theorem 2, one has to check a large number of cases, each case being the computation of the rank of the interpolation matrix. This matrix is usually obtained in the following way: let $\varphi : V_d \longrightarrow \mathbb{K}^m$ be a linear mapping from the space V_d of all hypersurfaces of degree d in $\mathbb{P}^3$ to $\mathbb{K}^v$, $v = \sum_{j=1}^{r} \binom{m_j+2}{3}$, such that $\mathcal{L}_3(d; m_1, \ldots, m_r)$ is equal to $\ker \varphi$. Then the interpolation matrix is the matrix of φ in the basis

$$\{X^{\alpha_1} Y^{\alpha_2} Z^{\alpha_3} W^{\alpha_4} : \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = d\}$$

of V_d and standard basis of $\mathbb{K}^v$.

The computation of the rank of this matrix can be performed over $\mathbb{Z}_p$ for a small prime p, but it is still time-consuming. The case $d \geq 14$ has not been considered in [**1**] due to the length of computations. In this note we show how, using the glueing theorem from [**2**], one can reduce the number of cases. For example, one needs to consider 261 cases for $d = 14$, instead of 6816 (produced by Algorithm A). For greater d's, the difference is even more visible — originally, for $d = 40$, $2,294,011$ cases were needed, while in our approach it suffices to deal with 22 only.

The size of each matrix is at most (about) $12,341$, which is small enough to be managed by a fast computer.

REMARK 3. For each $d \leq 8$, the system $\mathcal{L}_3(d; 4^{\times x}, 3^{\times y}, 2^{\times z})$ appears to be special for a suitable choice of x, y and z. We can enumerate all special systems for $d \leq 8$ using an algorithm similar to Algorithm A.

3. Main result. Using a smarter version of Algorithm A, we will show the following:

THEOREM 4. *System* $L = \mathcal{L}_3(d; 4^{\times x}, 3^{\times y}, 2^{\times z})$ *is non-special for non-negative integers* d, x, y, z *satisfying* $14 \leq d \leq 40$.

We will use the following fact:

THEOREM 5. *Let* $\mathcal{L}_3(k; \ell_1^{\times s_1}, \ell_2^{\times s_2})$ *be non-special and*

$$L_1 = \mathcal{L}_3(d; m_1, \ldots, m_r, \ell_1^{\times s_1}, \ell_2^{\times s_2}),$$
$$L_2 = \mathcal{L}_3(d; m_1, \ldots, m_r, k+1).$$

If either $-1 \leq \operatorname{vdim} L_2 \leq \operatorname{vdim} L_1$ *or* $\operatorname{vdim} L_1 \leq \operatorname{vdim} L_2 \leq -1$ *then in order to show non-specialty of* L_1 *it is enough to show non-specialty of* L_2.

The above theorem follows immediately from the following

THEOREM 6 (Theorem 1 in [**2**]). *Let* $n \geq 2$, $d, k, m_1, \ldots, m_r, m_{r+1}, \ldots, m_s$ *be non-negative integers. If*

- $L_1 = \mathcal{L}_n(k; m_1, \ldots, m_s)$ *is non-special,*
- $L_2 = \mathcal{L}_n(d; m_{s+1}, \ldots, m_r, k+1)$ *is non-special,*
- $(\operatorname{vdim} L_1 + 1)(\operatorname{vdim} L_2 + 1) \geq 0$,

then the system $L = \mathcal{L}_n(d; m_1, \ldots, m_r)$ *is non-special.*

Theorem 5 allows to "glue" s_1 ℓ_1-points and s_2 ℓ_2-points to one $(k+1)$-point during computations (the glueing will be denoted by $\ell_1^{\times s_1}, \ell_2^{\times s_2} \longrightarrow k+1$). We will glue using the following

PROPOSITION 7. *Systems* $\mathcal{L}_3(3; 2^{\times 5})$ *and* $\mathcal{L}_3(9; 4^{\times a}, 3^{\times b})$ *for non-negative integers* a, b *satisfying* $2a + b = 22$ *are non-special, of virtual dimension equal to* -1.

PROOF. Non-specialty of $\mathcal{L}_3(3; 2^{\times 5})$ can be proved by matrix computation or, without it, by [**3**]. Non-specialty of $\mathcal{L}(9; 4^{\times a}, 3^{\times b})$ follows from Theorem 2.

The computation of vdim is straightforward:

$$\begin{aligned}\operatorname{vdim}\mathcal{L}_3(9;4^{\times a},3^{\times b}) &= \binom{13}{3} - a\binom{6}{3} - b\binom{5}{3} - 1\\ &= 220 - 10(2a+b) - 1 = -1,\\ \operatorname{vdim}\mathcal{L}_3(3;2^{\times 5}) &= \binom{6}{3} - 5\binom{4}{3} - 1 = -1.\end{aligned}$$

□

So the possible glueings are $2^{\times 5} \longrightarrow 4$ and $4^{\times a}, 3^{\times b} \longrightarrow 10$, whenever $2a+b=22$. Observe that the virtual dimension of a system does not change after glueing. Hence, it is enough to consider systems $\mathcal{L}_3(d;10^{\times q},4^{\times x},3^{\times y},2^{\times z})$ with $2x+y \leq 21$ and $z \leq 4$. We summarize the above in the following algorithm:

Algorithm B

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 d=   -- degree should be chosen
 N=binomial(d+3,3);
 for z from 0 to 4 do
  for y from 0 to 10 do
   for x from 0 to 21 do
(*) for q from 0 to ceilinq(N/220) do (
     if ((220*q+20*x+10*y+4*z>N-4)and(220*q+20*x+10*y+4*z<N+20)
         and(2*a+b<22)) then (
      -- computation of the rank of the interpolation matrix
      -- the rank is maximal if and only if the system is non-special
     )
    )
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The above algorithm works properly for $d \geq 22$. For lower values of d the maximal possible number of 10-points forces some systems to be special and our approach does not work. To avoid this, we can change one line in Algorithm B

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(*)   for q from 1 to 1 do (  -- 1 can be replaced by
                              -- arbitrary non-negative integer
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choosing the number of 10-points to be considered. It is reasonable to choose this number as big as possible to obtain less cases. The right choices are one 10-point for $d \in \{13,\ldots,18\}$, five 10-points for $d=19$, seven for $d=20$ and eight for $d=21$. For example, for $d=19$ we will use

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(*)   for q from 1 to 5 do (
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The above algorithm has been implemented in FreePascal (but it is easy to implement it in Singular or Macaulay2 or any other computer algebra program) and ran on several computers simultaneously. All systems turned out to be non-special.

Remark 8. By Theorem 1 ("splitting theorem") in [**2**] we can avoid some of painful computations. As an example, consider $L = \mathcal{L}_3(40; 10^{\times 56}, 4^{\times 2})$. This system appears in a list of systems to be checked for $d = 40$. By splitting, it is enough to check non-specialty of $\mathcal{L}_3(39; 10^{\times 52}, 4^{\times 2})$ and $\mathcal{L}_3(40; 40, 10^{\times 4})$. The first system is assumed to be examined during the previous stage (for $d = 39$), the second is non-special by [**3**]. Another possibility of making the computations faster is to assume that four points with multiplicities $m_1, \ldots, m_4$ are fundamental ones ($(1:0:0:0)$, $(0:1:0:0)$, $(0:0:1:0)$ and $(0:0:0:1)$). This is possible by using a linear automorphism of $\mathbb{P}^3$. Then we can see that the monomials

$$\{X^{\alpha_1}Y^{\alpha_2}Z^{\alpha_3}W^{\alpha_4} : \alpha_j > d - m_j \text{ for } j = 1, 2, 3, 4\}$$

will not appear in the equation of any hypersurface in $\mathcal{L}_3(d; m_1, \ldots, m_r)$. Consequently, the size of the non-trivial part of interpolation matrix is smaller by $\sum_{j=1}^{4} \binom{m_j+2}{3}$, so, in most cases, where at least four 10-points are present, the advantage is 880 rows and columns. For d large enough we can also assume the multiplicities $m_1, \ldots, m_4$ to be 14 (resp. 15, 18, 20), since the system $\mathcal{L}_3(13; 4^{\times a}, 3^{\times b})$ for $2a + b = 56$ (resp. $\mathcal{L}_3(14; 4^{\times a}, 3^{\times b})$ for $2a + b = 68$, $\mathcal{L}_3(17; 4^{\times a}, 3^{\times b})$ for $2a + b = 114$, $\mathcal{L}_3(19; 4^{\times a}, 3^{\times b})$ for $2a + b = 154$) is non-special, of virtual dimension equal to -1.

References

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