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A NOTE ON ABHYANKAR–MOH'S APPROXIMATE ROOTS OF POLYNOMIALS

by Szymon Brzostowski

Abstract. We make a contribution to the Abhyankar–Moh's theory by studying approximate roots of non-characteristic degrees of an irreducible element of $\mathbb{K}((X))[Y]$.

Introduction. Let $\mathbb{K}((X))$ be the power series field in one variable X with coefficients in an algebraically closed field $\mathbb{K}$. Our aim is to examine approximate roots $\sqrt[l]{f}$ of an irreducible element f of the ring $\mathbb{K}((X))[Y]$ (as Abhyankar and Moh did in [**2**]), without assuming, though, that we deal with an approximate root of a 'characteristic degree' (see the end of Introduction for an explanation). For similar considerations in a more general setting see Moh [**6**].

Let us recall the basic notions and results of [**1, 2**]. For more information about approximate roots see also [**1, 2, 4**].

Let R be a commutative ring with unity, $f \in R[Y]$, $\deg_Y f = k$ be monic in Y and let $l|k$ be a divisor of k such that $1/l \in R$. A monic polynomial $g \in R[Y]$ satisfying the relation $\deg_Y(f - g^l) < k - k/l$ is called an *approximate l-th root of f*. We will denote it by $\sqrt[l]{f}$.

It is a well known fact that under above assumptions an l-th approximate root of f exists and is uniquely determined (cf. [**1, 4**]).

Now we pass to the classical situation. The following assumptions will be made in the sequel. We will call them the Basic Assumptions.

Let f be an irreducible and monic element of $\mathbb{K}((X))[Y]$, $\mathbb{K} = \overline{\mathbb{K}}$, $\operatorname{char} \mathbb{K} = 0$, $\deg_Y f = k$. Then, by Newton's theorem, $f\left(t^k, Y\right) = \prod_{\varepsilon \in U_k(\mathbb{K})} (Y - y(\varepsilon t))$ for some $y(t) \in \mathbb{K}((t))$, $y(t) = \sum_j y_j t^j$ (by $U_k(\mathbb{K})$ we denote the set $\{\varepsilon \in \mathbb{K} : \varepsilon^k = 1\}$).

Further, let $m = (m_0, \dots, m_h)$ be the characteristic of f (roughly speaking: $|m_0| = k$, $m_1 = \operatorname{ord}_t y(t)$ and $m_2, \dots, m_h$ are consecutive exponents of the Laurent expansion of $y(t)$ such that $\gcd(m_0, \dots, m_i) < \gcd(m_0, \dots, m_{i-1})$ for $2 \leqslant i \leqslant h$ and $\gcd(m_0, \dots, m_h) = 1$; for the definition see [**1**, Definition (6.8)]) and $d = (d_1, \dots, d_{h+1})$, where $d_{h+1} = 1$, be the divisor sequence defined by

$$d_i = \gcd(m_0, \dots, m_{i-1}) \text{ for } 1 \leqslant i \leqslant h+1.$$

It is also convenient to define the following derived sequences:

$$s = (s_0, \dots, s_{h+1}),$$

putting $s_0 := m_0$, $s_i := m_1 d_1 + \sum\limits_{2 \leqslant j \leqslant i} (m_j - m_{j-1})\, d_j$, for $1 \leqslant i \leqslant h$, and $s_{h+1} := +\infty$;

$$r = (r_0, \dots, r_{h+1}),$$

putting $r_0 := m_0$, $r_i := \frac{s_i}{d_i}$, for $1 \leqslant i \leqslant h$, and $r_{h+1} := +\infty$;

$$n = (n_1, \dots, n_h),$$

putting $n_i = \frac{d_i}{d_{i+1}}$, for $1 \leqslant i \leqslant h$.

Remark 1. *Under an additional assumption* $\operatorname{char} \mathbb{K} \nmid \deg_Y f$, *one can extend the results of this work to the case of a positive characteristic.*

We summarize basic properties of approximate roots in the following well-known theorem ([**1**, Theorem (13.2) (i) and (ii), Theorem (8.2)]).

Abhyankar–Moh Theorem. *If* $l = d_j$ *for some* $1 \leqslant j \leqslant h+1$ *then:*

1. $\sqrt[d_j]{f}$ *is irreducible in* $K((X))[Y]$,
2. *if* $2 \leqslant j \leqslant h+1$, *then for every Puiseux root* $z(t)$ *of* $\sqrt[d_j]{f}(t, Y)$ *there exists* $\varepsilon \in U_k(\mathbb{K})$ *such that*
$$\operatorname{ord}_t \left(y(\varepsilon t) - z\left(t^k\right) \right) = m_j,$$
3. $$\operatorname{ord}_t \left(\sqrt[d_j]{f}\left(t^k, y(t)\right) \right) = r_j.$$

In the sequel, we try to examine 'non-characteristic' approximate roots (i.e., we skip the assumption $l = d_j$) and to give some results similar to those stated in the above theorem. More specifically: property 1 is not true (Example 1), properties 2 and 3 carry over in the form of inequalities (Theorem 1, Corollary 1 and Theorem 3), which are then proved to be in fact equalities in some special case (Theorem 2 and Theorem 3).

Auxiliary results. Throughout the work we freely utilize the notations and results from [**1**]. We recall that the symbol θ stands for an unspecified, non-zero element of a field under consideration.

Under the Basic Assumptions, it is easy to prove the following lemma, which is a slight improvement of Lemma (7.16) in [**1**].

LEMMA 1. *Let e be an integer such that $1 \leqslant e \leqslant h$ and let $\mathbb{K}_0$ be a subfield of $\mathbb{K}$ such that the k-th primitive root of $1 \in \mathbb{K}$ belongs to $\mathbb{K}_0$. Assume that $\sum_{j<m_e} y_j t^j \in \mathbb{K}_0((t))$. Then for every (e, U)-deformation $y^*(t)$ of $y(t)$ (i.e., an element of $\mathbb{K}'((t))$, where $\mathbb{K}'$ is an overfield of $\mathbb{K}$, such that $\mathrm{info}_t(y^*(t) - \sum_{j<m_e} y_j t^j) = Ut^{m_e}$), there is*

$$\mathrm{info}_t\, f\left(t^k, y^*(t)\right) = \theta \left(U^{n_e} - y_{m_e}^{n_e}\right)^{d_{e+1}} t^{s_e} \text{ with } \theta \in \mathbb{K}_0.$$

PROOF. We repeat the proof of (7.16) in [**1**] to obtain the equality (7.16.2): $\mathrm{info}_t(y^*(t) - y(wt)) = \mathrm{info}_t(y(t) - y(wt))$ for a fixed $w \in Q(e) = \{\varepsilon \in U_k(\mathbb{K}_0) : \mathrm{ord}_t(y(t) - y(\varepsilon t)) < m_e\}$. Since, by assumption, $\mathrm{inco}_t(y(t) - y(wt)) \in \mathbb{K}_0$, then (7.16.3) takes the form

$$\mathrm{info}_t\left(\prod_{w \in Q(e)} (y^*(t) - y(wt))\right) = \begin{cases} \theta, & \text{if } e = 1 \\ \theta\, t^{s_{e-1} - m_{e-1} d_e}, & \text{if } e \geqslant 2 \end{cases} \quad \text{with } \theta \in \mathbb{K}_0.$$

The rest of the proof goes through without changes. □

Next we need a version of the Newton Polygon Method, which is a consequence of [**1**, Theorem (14.2)]. For a more explicit formulation see also § 2. in [**5**].

PROPOSITION 1. *Let g be an element of $\mathbb{K}((X))[Y]$ splitting into linear factors of the form $Y - z_j(X)$, where $z_j(X) \in \mathbb{K}\left(\left(X^{1/M}\right)\right)$ and $1 \leqslant j \leqslant \deg_Y g$. Let us consider an arbitrary $u(t) = \sum_{j \leqslant L} u_j t^{j/M} \in \mathbb{K}\left(\left(t^{1/M}\right)\right)$, for some $L \in \mathbb{Z}$. Then the following two conditions are equivalent:*

i) there exists $1 \leqslant j \leqslant \deg_Y g$ such that $\mathrm{ord}_t(u(t) - z_j(t)) > \frac{L}{M}$;

ii) the polynomial $h(U) := \mathrm{inco}_t\, g\left(t, u(t) + Ut^{L/M}\right) \in \mathbb{K}[U]$ is not constant and one of its roots is $U = 0$.

Furthermore, if $U = 0$ has multiplicity $l > 0$ as a root of $h(U)$, then there exist at least l different indices $j_1, \ldots, j_l \in \{1, \ldots, \deg_Y g\}$ such that $\mathrm{ord}_t(u(t) - z_{j_i}(t)) > \frac{L}{M}$ for $i = 1, \ldots, l$.

Now we can prove

LEMMA 2. *Let f fulfil the Basic Assumptions. Let l be an integer such that $l | d_i$ for some $1 \leqslant i \leqslant h$ and $l \notin \{d_1, \ldots, d_{h+1}\}$, and assume that there* *exists an integer m', $m_{i-1} < m' < m_i$ (in the case of $i = 1$, we only demand $m' < m_1$) such that* $\gcd(d_i, m') = l$. *Then for every Puiseux series $z(t)$ with $\sqrt[l]{f}(t, z(t)) = 0$ there exists $\varepsilon \in U_k(\mathbb{K})$ such that* $\operatorname{ord}_t\left(y(\varepsilon t) - z\left(t^k\right)\right) > m'$.

PROOF. Let Z be an indeterminate over $\mathbb{K}$ and consider $y^Z(t) = y(t) + Zt^{m'} \in \mathbb{K}[Z]((t))$. Put $f^Z\left(t^k, Y\right) = \prod_{\varepsilon^k = 1}\left(Y - y^Z(\varepsilon t)\right)$. Then $f^Z(X, Y) \in \mathbb{K}[Z]((X))[Y]$ has the characteristic sequence $m^Z = \left(m_0, \ldots, m_{i-1}, m', m'_{i+1}, \ldots\right)$ and the divisor sequence $d^Z = (d_1, \ldots, d_i, l, \ldots, 1)$. Notice that $l > 1$, because $l \neq d_{h+1}$. From the Abhyankar–Moh theory it follows that $\sqrt[l]{f^Z} \in \mathbb{K}[Z]((X))[Y]$ is irreducible in $\overline{\mathbb{K}(Z)}((X))[Y]$. Let

$$\sqrt[l]{f^Z}\left(t^{k/l}, Y\right) = \prod_{\varepsilon_1^{k/l}=1}\left(Y - \bar{z}(\varepsilon_1 t)\right),$$

where $\bar{z}(t) \in \overline{\mathbb{K}(Z)}((t))$ has the property that $\operatorname{ord}_t\left(\bar{z}\left(t^l\right) - y^Z(t)\right) = m'_{i+1} > m'$ ([**1**, Theorem (13.2) (ii)]).

Fix $\varepsilon_1 \in U_{k/l}(\mathbb{K})$ and consider $\bar{z}(\varepsilon_1 t)$. It follows that $z^*(t) = \sum_{j<m'} y_j (\varepsilon_1 t)^{j/l} + U(\varepsilon_1 t)^{m'/l} \in \mathbb{K}[U]((t))$ is (i, U)-deformation of $\bar{z}(\varepsilon_1 t)$. Applying Lemma 1 to $\sqrt[l]{f^Z}$ and $z^*(t)$, we get

$$\text{(1)} \qquad \operatorname{info}_t \sqrt[l]{f^Z}\left(t^{k/l}, z^*(t)\right) = \theta\left(U^{\bar{n}_i} - Z^{\bar{n}_i}\right)^{\bar{d}_{i+1}} t^{\bar{s}_i} \text{ with } \theta \in \mathbb{K}.$$

(Here the bar '$^-$' indicates characteristic sequences for $\sqrt[l]{f^Z}$.) From the definition of the approximate root we conclude that $\deg_Y(f^Z - \left(\sqrt[l]{f^Z}\right)^l) < k - \frac{k}{l}$. After the substitution $Z = 0$ in that inequality, we thus get $\deg_Y(f^Z_{Z=0} - \left(\sqrt[l]{f^Z}_{Z=0}\right)^l) < k - \frac{k}{l}$. But, obviously, $f^Z_{Z=0} = f$. This means that $\sqrt[l]{f^Z}_{Z=0} = \sqrt[l]{f}$. Since $\sqrt[l]{f^Z}\left(t^{k/l}, z^*(t)\right) \in \mathbb{K}[Z][U]((t))$, then substituting $Z = 0$ in (1) we get

$$\operatorname{info}_t \sqrt[l]{f}\left(t^{k/l}, z^*(t)\right) = \theta U^{\bar{n}_i \bar{d}_{i+1}} t^{\bar{s}_i} = \theta U^{\bar{d}_i} t^{\bar{s}_i} = \theta U^{d_i/l} t^{\bar{s}_i} \text{ with } \theta \in \mathbb{K}.$$

From Proposition 1 we conclude that there exist d_i/l Puiseux roots $z_{j_1}(t), \ldots, z_{j_{d_i/l}}(t)$ of $\sqrt[l]{f}(t, Y)$ such that $m'/k < \operatorname{ord}_t\left(\sum_{j \leqslant m'} y_j \varepsilon_1^{j/l} t^{j/k} - z_{j_p}(t)\right)$ and so

$$m' < \operatorname{ord}_t\left(y\left(\varepsilon_1^{1/l} t\right) - z_{j_p}\left(t^k\right)\right) \text{ for } p = 1, \ldots, d_i/l.$$

(Above, $\varepsilon_1^{1/l}$ denotes any of l-th roots of ε_1 in $\mathbb{K}$.) Since ε_1 was a fixed element of $U_{k/l}(\mathbb{K})$, then we have proven that for any $\varepsilon \in U_k(\mathbb{K})$ there exist d_i/l

Puiseux roots $z_{\varepsilon,1}(t),\ldots,z_{\varepsilon,d_i/l}(t)$ of $\sqrt[l]{f}(t,Y)$ such that

$$\mathrm{ord}_t\left(y(\varepsilon t)-z_{\varepsilon,p}\left(t^k\right)\right)>m' \text{ for } p=1,\ldots,d_i/l.$$

Now for $A=\left\{\varepsilon^{d_i}:\varepsilon\in U_k(\mathbb{K})\right\}$ there is $\mathrm{card}\,A=k/d_i$ and if $\sigma_1,\sigma_2\in U_k(\mathbb{K})$, $\sigma_1^{d_i}\neq\sigma_2^{d_i}$, then $\mathrm{ord}_t(y(\sigma_1 t)-y(\sigma_2 t))<m_{i-1}<m'$. Thus

$$\mathrm{ord}_t\left(z_{\sigma_1,p_1}\left(t^k\right)-z_{\sigma_2,p_2}\left(t^k\right)\right)<m' \text{ for } p_1,p_2=1,\ldots,d_i/l.$$

Since $\frac{k}{d_i}\frac{d_i}{l}=\frac{k}{l}=\deg_Y\sqrt[l]{f}$, the lemma is proved. □

Property 1. *Let $g\in K((X))[Y]$, $\deg_Y g=\overline{k}$, $g=g_1\cdot\ldots\cdot g_r$ be the decomposition of g into irreducible factors in $K((X))[Y]$. Let N be a positive integer such that $\gcd\left(N,\overline{k}!\right)=1$. Then $g\left(X^N,Y\right)=g_1\left(X^N,Y\right)\cdot\ldots\cdot g_r\left(X^N,Y\right)$ is the decomposition of $g\left(X^N,Y\right)$ into irreducible factors in $K((X))[Y]$. Furthermore, if $z_1(t),\ldots,z_{\overline{k}}(t)$ are all Puiseux roots of $g(t,Y)$, then $z_1\left(t^N\right),\ldots,z_{\overline{k}}\left(t^N\right)$ are all Puiseux roots of $g\left(t^N,Y\right)$.*

Proof. It is enough to prove the property under the assumption that g is irreducible in $\mathbb{K}((X))[Y]$. Let g have characteristic $\overline{m}=(\overline{m}_0,\ldots\overline{m}_{\overline{h}})$, $\deg_Y g=\overline{k}$ and $z(t)$ be any of Puiseux roots of $g(t,Y)=0$. Since $g\left(t^{\overline{k}},z\left(t^{\overline{k}}\right)\right)=0$, then $g\left(t^{\overline{k}N},z\left(t^{\overline{k}N}\right)\right)=0$ with $\gcd\left(\overline{k},\mathrm{Supp}\left(z\left(t^{\overline{k}N}\right)\right)\right)=1$. Thus $g\left(X^N,Y\right)$ is irreducible in $\mathbb{K}((X))[Y]$ and the characteristic sequence of $g\left(X^N,Y\right)$ is $(\overline{m}_0,N\overline{m}_1,\ldots,N\overline{m}_{\overline{h}})$. □

Main results. Our first theorem is an improvement of Lemma 2. It is a generalization of the item 2 in Abhyankar–Moh Theorem covering the non-characteristic case.

Theorem 1. *Let f fulfill the Basic Assumptions. Let l be such an integer that $l|d_i$ for some $1\leqslant i\leqslant h$ and $l\notin\{d_1,\ldots,d_{h+1}\}$. Then for every Puiseux series $z(t)$ with $\sqrt[l]{f}(t,z(t))=0$ there exists $\varepsilon\in U_k(\mathbb{K})$ such that*

$$\mathrm{ord}_t\left(y(\varepsilon t)-z\left(t^k\right)\right)\geqslant m_i. \tag{2}$$

Proof. Let $M=k!$ and let N be any positive integer such that $(N,M)=1$. Then by Property 1, $f\left(X^N,Y\right)$ is irreducible in $\mathbb{K}((X))[Y]$ and has the characteristic $(m_0,Nm_1,\ldots,Nm_h)$ (see the proof of Property 1). If N is large enough, then there exists such an integer m' that $Nm_{i-1}<m'<Nm_i$ and $\gcd(m_0,Nm_1,\ldots,Nm_{i-1},m')=l$ (if $i=1$, then we demand $m'<Nm_1$ and $\gcd(m_0,m')=l$). Consequently, $f_1:=f\left(X^N,Y\right)$ fulfills the assumptions of Lemma 2. We conclude that for every Puiseux series $\bar{z}(t)$ with $\sqrt[l]{f_1}(t,\bar{z}(t))=0$, there exists $\varepsilon\in U_k(\mathbb{K})$ such that $\mathrm{ord}_t\left(y\left(\varepsilon t^N\right)-\bar{z}\left(t^k\right)\right)>m'$.

But it is evident that $\sqrt[l]{f_1}(X,Y) = \sqrt[l]{f}\left(X^N,Y\right)$. And so, by Property 1, there exists a Puiseux root $z(t)$ of $\sqrt[l]{f}(t,Y)$ such that $z\left(t^N\right) = \bar{z}(t)$ and $\mathrm{ord}_t\left(y\left(\varepsilon t^N\right) - z\left(t^{kN}\right)\right) > m'$, or in other words

$$\mathrm{ord}_t\left(y(\varepsilon t) - z\left(t^k\right)\right) > m'/N. \tag{3}$$

From Property 1 it follows that every Puiseux root of $\sqrt[l]{f}(t,Y)$ satisfies the above inequality.

Choosing a suitable N tending to infinity, we will now improve (3) to obtain inequality (2). By Dirichlet's theorem, the sequence $\{1 + j \cdot d_i l\}_{j\in\mathbb{N}}$ contains infinitely many prime numbers. Let $\{N_p\}_{p\in\mathbb{N}} = \{1 + j_p \cdot d_i l\}_{p\in\mathbb{N}}$ be the sequence of primes. Let $A := m_i - l$. Now we can write

$$N_p m_i = m_i + j_p \cdot d_i l m_i = (m_i - l) + (d_i m_i j_p + 1)\, l = A + (d_i m_i j_p + 1)\, l.$$

Taking a large enough $r \in \mathbb{N}$ we define $B := A + j_r \cdot d_i l m_i$ (or, respectively, $B := A - j_r \cdot d_i l m_i$ if $m_i < 0$) with the property that $B > 0$. For $p > r$ we now obtain

$$N_p m_i = B + (d_i m_i (j_p \mp j_r) + 1)\, l = B + m'_p,$$

taking $m'_p = (d_i m_i (j_p \mp j_r) + 1)\, l$. Here $\gcd\left(d_i, m'_p\right) = \gcd(d_i, l) = l$ for $p > r$. Since $m'_p = N_p m_i - B$ and $B > 0$, then $m'_p < N_p m_i$, and for a p large enough, also $N_p m_{i-1} < m'_p$ if $i > 1$. Obviously, we can also assume that $\gcd(N_p, M) = 1$.

Fix a Puiseux series $z(t)$ satisfying $\sqrt[l]{f}(t, z(t)) = 0$. From the first part of the proof it follows that for every $N = N_p$, $p \gg 0$, there exists $\varepsilon_p \in U_k(\mathbb{K})$ such that

$$\mathrm{ord}_t\left(y(\varepsilon_p t) - z\left(t^k\right)\right) > m'_p/N_p = m_i - B/N_p.$$

We conclude that there exists an $\varepsilon \in U_k(\mathbb{K})$ such that

$$\mathrm{ord}_t\left(y(\varepsilon t) - z\left(t^k\right)\right) > m_i - B/N_p$$

for infinitely many $p \in \mathbb{N}$. Since B is constant and N_p tends to infinity with p, then it means that

$$\mathrm{ord}_t\left(y(\varepsilon t) - z\left(t^k\right)\right) \geqslant m_i.$$

Thus the theorem is proved. □

Remark 2. *The construction of the sequence $\{N_p\}$ can be simplified: demanding only that the sequence $\{N_p m_i - m'_p\}$ should be bounded, there is no need to use Dirichlet's theorem.*

COROLLARY 1. *For a given integer $l|d_1$, the above theorem is true with $i = \max\{1 \leqslant j \leqslant h+1 : l|d_j\}$. If, in addition, $l > d_{i+1}$, then for every Puiseux root $z(t)$ of $\sqrt[l]{f}(t, Y)$ and every $\sigma \in U_k(\mathbb{K})$ there holds*

$$\operatorname{ord}_t\left(y(\sigma t) - z\left(t^k\right)\right) \leqslant m_i.$$

PROOF. The first part of the corollary is obvious. As for the second one, if there were $\operatorname{ord}_t\left(y(\sigma t) - z\left(t^k\right)\right) > m_i$, for some Puiseux root $z(t)$ of $\sqrt[l]{f}(t, Y)$ and some $\sigma \in U_k(\mathbb{K})$, then

$$z(t) = \sum_{j \leqslant m_i} y_j(\sigma^j t^{j/k}) + \ldots = \sum_{j \leqslant m_i} y_j \sigma^j t^{\frac{j/d_{i+1}}{k/d_{i+1}}} + \ldots$$

and since $\gcd(k/d_{i+1}, m_1/d_{i+1}, \ldots, m_i/d_{i+1}) = 1$, there would also hold $\deg_Y \sqrt[l]{f} \geqslant k/d_{i+1}$ and so $k/l \geqslant k/d_{i+1}$, which is impossible by the assumption. □

Combining Theorem 1 and Corollary 1, we get

THEOREM 2. *Let f fulfill the Basic Assumptions. Let l be an integer such that $l|d_1$ and $l \notin \{d_1, \ldots, d_{h+1}\}$. Define $i = \max\{1 \leqslant j \leqslant h+1 : l|d_j\}$. If $l > d_{i+1}$, then for every Puiseux root $z(t)$ of $\sqrt[l]{f}(t, Y)$ and every $\sigma \in U_k(\mathbb{K})$,*

$$\operatorname{ord}_t\left(y(\sigma t) - z\left(t^k\right)\right) \leqslant m_i.$$

Furthermore, there exists an $\varepsilon \in U_k(\mathbb{K})$ such that

$$\operatorname{ord}_t\left(y(\varepsilon t) - z\left(t^k\right)\right) = m_i.$$

THEOREM 3. *Let f fulfill the Basic Assumptions. If l is an integer such that $l|d_1$ and $l \notin \{d_1, \ldots, d_{h+1}\}$, then for $i = \max\{1 \leqslant j \leqslant h+1 : l|d_j\}$:*

$$\operatorname{ord}_t\left(\sqrt[l]{f}\left(t^k, y(t)\right)\right) \geqslant r_i \frac{d_i}{l}.$$

If, in addition, $l > d_{i+1}$, then the equality holds.

PROOF. The concept of the proof is similar to that of the proofs of Lemma 2 and Theorem 1, so we will just sketch it, omitting the details.

1. First we return to the proof of Lemma 2. Assume accordingly, that there exists an integer m', $m_{i-1} < m' < m_i$ (or simply $m' < m_1$ for $i = 1$) such that $\gcd(d_i, m') = l$. Defining $y^Z(t)$, f^Z, $\bar{z}(t)$ as in that proof, we obtain $\sqrt[l]{f^Z}\left(t^{k/l}, Y\right) = \prod_{\varepsilon_1^{k/l}=1} (Y - \bar{z}(\varepsilon_1 t))$ and so $\left(\sqrt[l]{f^Z}\left(t^k, Y\right)\right)^l = \prod_{\varepsilon^k=1}\left(Y - \bar{z}\left((\varepsilon t)^l\right)\right)$. We put $z^V(t) = \bar{z}\left(t^l\right) + Vt^\alpha \in \overline{\mathbb{K}(Z)}[V]((t))$, where $\alpha > m_i$ is chosen in such a way that $\gcd\left(k, \operatorname{Supp}\left(\bar{z}\left(t^l\right)\right), \alpha\right) = 1$. Let $h^{ZV}\left(t^k, Y\right) = \prod_{\varepsilon^k=1}\left(Y - z^V(\varepsilon t)\right)$. Then

$h^{ZV}(X,Y) \in \mathbb{K}[Z,V]\left((X)\right)[Y]$ is an irreducible element of $\overline{\mathbb{K}\left(Z,V\right)}\left((X)\right)[Y]$. By [**1**, Theorem (13.2) (ii)], we can assume that $m_i = \mathrm{ord}_t(\bar{z}\left(t^l\right) - y^Z\left(t\right)) = \mathrm{ord}_t\left(z^V\left(t\right) - y^Z\left(t\right)\right)$, and so $\mathrm{ord}_t\left(z^V\left(t\right) - y\left(t\right)\right) = m'$. Since m' is the i-th characteristic exponent of $z^V\left(t\right)$, from Lemma 1 we get

$$\mathrm{info}_t\, h^{ZV}\left(t^k, y\left(t\right)\right) = \theta\left(0 - Z^{n_i^{ZV}}\right)^{d_{i+1}^{ZV}} t^{s_i^{ZV}} \text{ with } \theta \in \mathbb{K},$$

where the superscript '^{ZV}' indicates characteristic sequences for h^{ZV}. But this implies that also

$$\mathrm{info}_t\, h^{ZV}_{V=0}\left(t^k, y\left(t\right)\right) = \theta\, Z^{n_i^{ZV} d_{i+1}^{ZV}}\, t^{s_i^{ZV}} = \theta\, Z^{d_i^{ZV}}\, t^{s_i^{ZV}} \text{ with } \theta \in \mathbb{K}.$$

Since, obviously, $h^{ZV}_{V=0} = \left(\sqrt[l]{f^Z}\right)^l$, we thus get

$$\mathrm{info}_t \sqrt[l]{f^Z}\left(t^k, y\left(t\right)\right) = \theta\, Z^{d_i^{ZV}/l}\, t^{s_i^{ZV}/l} \text{ with } \theta \in \mathbb{K}. \tag{4}$$

We now notice that $s_i^{ZV} = s_i + \left(m' - m_i\right) d_i$ and so, having substituted $Z = 0$ in (4),

$$\mathrm{ord}_t \sqrt[l]{f}\left(t^k, y\left(t\right)\right) > \frac{d_i}{l} r_i + \frac{d_i}{l}\left(m' - m_i\right).$$

2. Now we return to the proof of Theorem 1. Constructing a sequence of primes $\{N_p\}_{p\in\mathbb{N}}$ as in that proof and applying Property 1, we improve the above inequality to

$$\mathrm{ord}_t \sqrt[l]{f}\left(t^k, y\left(t\right)\right) > \frac{d_i}{l} r_i - \frac{const}{N_p}$$

and so

$$\mathrm{ord}_t \sqrt[l]{f}\left(t^k, y\left(t\right)\right) \geqslant \frac{d_i}{l} r_i.$$

Finally, from [**1**, Theorem (8.5)] it follows that in the case of $d_{i+1} < l$, the equality has to hold in the above formula. Indeed, let $g_1 = Y$ and $g_j = \sqrt[d_j]{f}$ for $2 \leqslant j \leqslant h+1$. Then for $G = (g_1, \ldots, g_{h+1})$ we obtain the G-adic expansion of $\sqrt[l]{f}$ in the form $\sqrt[l]{f} = g_i^{d_i/l} + \ldots$, because $\frac{d_i}{l} < \frac{d_i}{d_{i+1}}$, and so, by [**1**, Theorem (8.5)], $\mathrm{ord}_t \sqrt[l]{f}\left(t^k, y\left(t\right)\right) \leqslant \frac{d_i}{l} r_i$. □

Example 1. *In general nothing can be said about the (ir)reducibility of non-characteristic approximate roots. Take the parametrization* $X = t^{48}$, $Y = 1/(t^{36}) + 1/(t^6) + 1/(t^5)$ *and let* $f \in \mathbb{C}\left((X)\right)[Y]$ *be its minimal monic polynomial. Then* $f = Y^{48} + \ldots$ *. It can be verified that for* $l = 2$*, there is* $\mathrm{inco}_t \sqrt[l]{f}\left(t^8, 1/t^6 + 1/t + U \cdot t\right) = 4096\left(-51 + 8U^3\right)$ *and so, by* [**1**, Theorem (14.2)], $\sqrt[l]{f}$ *splits into three irreducible factors in* $\mathbb{C}\left((X)\right)[Y]$ *each of them having a Puiseux root of the form* $t^{-3/4} + t^{-1/8} + \theta\, t^{1/8} + h.o.t.$ *It is worth* *noticing that the divisor $l = 2$ here is very 'regular', as $d_4 = 1|2|d_3 = 6$ and, despite of that, irreducibility does not follow.*

It is also easy to give examples in the other direction. Let $X = t^{18}$, $Y = t^{-12}+t^{-2}+t^{-1}$, $l = 3$ and let $f \in \mathbb{C}((X))[Y]$ be its minimal monic polynomial. Then $f = Y^{18} + \dots$. There is $\operatorname{inco}_t \sqrt[l]{f}\left(t^6, 1/t^4 + Ut\right) = 9U^2 - 6$, *so $\sqrt[l]{f}$ is irreducible.*

To end with, let us mention that the restriction $l > d_{i+1}$ made in Theorem 2 and Theorem 3 does not seem to be indispensable; in fact, we were not able to find any counterexample to their respective conclusions. An interesting insight gives the following example, which is a slight modification of the one above.

Example 2. *Let $X = t^{18}$, $Y = t^{-12}+at^{-3}+bt^{-1}$, where a, b are indeterminates over $\mathbb{C}$, $l = 2$. Then $l = 2 < d_{i+1} = 3$, so the assumption $l > d_{i+1}$ is not fulfilled. In spite of that* $\operatorname{inco}_t \sqrt[l]{f}\left(t^6, 1/t^4 + U/t\right) = -27/2 \cdot U(-2U^2+3a^2)$. *We conclude that $\sqrt[l]{f}$ has two non-conjugate Puiseux roots. One of them is of the form $z_1(t) = t^{-2/3} + \sqrt{6}/2 \cdot a \cdot t^{-1/6} + h.o.t.$, whereas $y(t) = t^{-12} + at^{-3} + bt^{-18}$, so still* $\operatorname{ord}_t\left(y(t) - z_1\left(t^{18}\right)\right) = -3 = m_2$. *Also* $\operatorname{ord}_t\left(\sqrt[l]{f}\left(t^{18}, y(t)\right)\right) = r_2\frac{d_2}{l} = -81$.

We state

Problem 1. *Can we drop the assumption $l > d_{i+1}$ from the formulation of Theorem 2 and Theorem 3?*

Problem 2. *If $\sqrt[l]{f}$ is reducible in $\mathbb{K}((X))[Y]$, do the degrees of the factors of $\sqrt[l]{f}$ divide k?*

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Faculty of Mathematics
University of Łódź
ul. Banacha 22, Łódź
Poland
e-mail: `brzosts@math.uni.lodz.pl`