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IMPLICIT DIFFERENCE METHODS FOR NONLINEAR FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. Classical solutions of initial boundary value problems for nonlinear equations are approximated with solutions of quasilinear systems of implicit difference equations. The proof of the convergence of the method is based on a comparison technique with nonlinear estimates of the Perron type for given functions.

This new approach to implicit difference methods for nonlinear equations is based on a quasilinearization method and theory of bicharacteristics.

In our considerations it is important that the Courant–Friedrichs–Levy condition is not need in convergence theorems for implicit difference methods.

Numerical examples are presented.

1. Introduction. For any metric spaces X and Y, by $C(X,Y)$ we denote the class of all continuous functions from X into Y. We will use vectorial inequalities meant component-wise
For $x, y \in R^n$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, we put

$$x \diamond y = (x_1 y_1, \ldots, x_n y_n) \quad \text{and} \quad \| x \| = \sum_{i=1}^{n} |x_i|.$$

Let $E = [0, a] \times [-b, b]$, where $a > 0$, $b = (b_1, \ldots, b_n)$ and $b_i > 0$ for $1 \le i \le n$. Suppose that $\kappa, 0 \le \kappa \le n$ is a fixed integer. We define the sets

$$\partial^+ E_i = \{(t, x) \in E : x_i = b_i\}, \quad 1 \le i \le \kappa,$$

$$\partial^- E_i = \{(t, x) \in E : x_i = -b_i\}, \quad \kappa + 1 \le i \le n$$

and

$$\partial_0 E = \bigcup_{i=1}^{\kappa} \partial^+ E_i \cup \bigcup_{i=\kappa+1}^{n} \partial^- E_i, \quad E_0 = \{0\} \times [-b, b], \quad \Omega = E \times R \times R^n.$$

Suppose that $F : \Omega \to R$, $\varphi : E_0 \cup \partial_0 E \to R$ are given functions. We consider the problem consisting of the differential equation

$$\partial_t z(t, x) = F(t, x, z(t, x), \partial_x z(t, x)) \tag{1}$$

and the initial boundary condition

$$z(t, x) = \varphi(t, x) \quad \text{for} \quad (t, x) \in E_0 \cup \partial_0 E, \tag{2}$$

where $\partial_x z = (\partial_{x_1} z, \dots, \partial_{x_n} z)$. We are interested in the construction of a method for the approximation of classical solutions to problem (1), (2) with solutions of associated implicit difference schemes and in the estimation of the difference between these solutions.

We define a mesh on the set E in the following way. Let $\mathbf{N}$ and $\mathbf{Z}$ be the sets of natural numbers and integers, respectively. Let (h_0, h'), $h' = (h_1, \dots, h_n)$, stand for steps of the mesh. For $h = (h_0, h')$ and $(r, m) \in \mathbf{Z}^{1+n}$, $m = (m_1, \dots, m_n)$, we define nodal points as follows

$$t^{(r)} = rh_0, \qquad x^{(m)} = m \diamond h', \qquad x^{(m)} = (x_1^{(m_1)}, \dots, x_n^{(m_n)}).$$

By H we will denote the set of all $h = (h_0, h')$ such that there is $N = (N_1, \dots, N_n)$, $N \in \mathbf{N}^n$ with $N \diamond h' = b$. Let $K \in \mathbf{N}$ be defined by the relations $Kh_0 \le a < (K+1)h_0$. We define the sets

$$R_h^{1+n} = \{(t^{(r)}, x^{(m)}) : (r, m) \in \mathbf{Z}^{1+n}\},$$

$$E_h = E \cap R_h^{1+n}, \qquad E_{0.h} = E_0 \times R_h^{1+n}, \qquad \partial_0 E_h = \partial_0 E \cap R_h^{1+n}$$

and

$$I_h = \{t^{(r)} : 0 \le r \le K\}.$$

For functions $w : I_h \to R$ and $z : E_h \to R$, $u : E_h \to R^n$, $u = (u_1, \dots, u_n)$, we write

$$w^{(r)} = w(t^{(r)}), \quad z^{(r,m)} = z(t^{(r)}, x^{(m)}), \quad u^{(r,m)} = u(t^{(r)}, x^{(m)}).$$

For $h \in H$ we put $\| h \| = h_0 + h_1 + \dots + h_n$. Let $e_i = (0, \dots, 0, 1, 0, \dots, 0) \in R^n$, 1 standing in the i-th place, $1 \le i \le n$. By δ_0 and $\delta = (\delta_1, \dots, \delta_n)$, we will denote the difference operators defined by

$$\delta_0 z^{(r,m)} = \frac{1}{h_0}\Big(z^{(r+1,m)} - z^{(r,m)}\Big), \tag{3}$$

$$\delta_i z^{(r,m)} = \frac{1}{h_i}\Big(z^{(r,m+e_i)} - z^{(r,m)}\Big) \quad \text{for} \quad 1 \le i \le \kappa, \tag{4}$$

$$\delta_i z^{(r,m)} = \frac{1}{h_i}\Big(z^{(r,m)} - z^{(r,m-e_i)}\Big) \quad \text{for} \quad \kappa+1 \le i \le n. \tag{5}$$

If $\kappa = 0$, then δ is given by (5); for $\kappa = n$, δ is defined by (4).
Write

$$\begin{aligned} \theta = (\theta_1, \ldots, \theta_n) \in R^n \quad \text{where} \quad \theta_i = 1 \quad \text{for} \quad 1 \le i \le \kappa \\ \text{and} \quad \theta_i = -1 \quad \text{for} \quad \kappa+1 \le i \le n. \end{aligned} \tag{6}$$

Suppose that we approximate solutions of (1), (2) by means of solutions of the difference equation

$$\delta_0 z^{(r,m)} = F(t^{(r)}, x^{(m)}, z^{(r,m)}, \delta z^{(r,m)}) \tag{7}$$

with the initial boundary condition

$$z^{(r,m)} = \varphi_h^{(r,m)} \quad \text{on} \quad E_{0.h} \cup \partial_0 E_h, \tag{8}$$

where $\varphi_h : E_{0.h} \cup \partial_0 E_h \to R$ is a given function. Problem (7), (8) is called the Euler method for (1), (2). We formulate sufficient conditions for the convergence of method (7), (8). We need the following assumption on F.

Assumption $H_0[F]$**.** Suppose that the function $F : \Omega \to R$ in the variables (t, x, p, q), $q = (q_1, \ldots, q_n)$, is continuous and

1) the partial derivatives $(\partial_{q_1} F, \ldots, \partial_{q_n} F) = \partial_q F$ exist on Ω, $\partial_q F \in C(\Omega, R^n)$ and

$$\partial_q F(t, x, p, q) \diamond \theta \ge 0 \quad \text{on} \quad \Omega, \tag{9}$$

2) there is $\sigma : [0, a] \times R_+ \to R_+$, $R_+ = [0, +\infty)$ such that
   (i) σ is continuous and it is nondecreasing with respect to both variables,
   (ii) $\sigma(t, 0) = 0$ for $t \in [0, a]$ and the maximal solution of the Cauchy problem

   $$w'(t) = \sigma(t, w(t)), \qquad w(0) = 0,$$

   is $\overline{w}(t) = 0$ for $t \in [0, a]$,
   (iii) the estimate

   $$|F(t, x, p, q) - F(t, x, \overline{p}, q)| \le \sigma(t, |p - \overline{p}|)$$

   is satisfied on Ω.

Theorem 1.1. *Suppose that Assumption* $H_0[F]$ *is satisfied and*

1) $h \in H$ *and for* $P = (t, x, p, q) \in \Omega$, *there is*

$$1 - h_0 \sum_{i=1}^{n} \frac{1}{h_i} |\partial_{q_i} F(P)| \ge 0, \tag{10}$$

2) $v : E \to R$ *is a solution of (1), (2) and* v *is of class* C^1,

3) $\widetilde{z}_h : E_h \to R$ *is a solution of (7), (8) and there is* $\alpha_0 : H \to R_+$ *such that*

$$|\varphi^{(r,m)} - \varphi_h^{(r,m)}| \le \alpha_0(h) \quad on \quad E_{0.h} \cup \partial_0 E_h \quad and \quad \lim_{h\to 0} \alpha_0(h) = 0.$$

Then there exist $\widetilde{\varepsilon} > 0$ *and* $\alpha : H \to R_+$ *such that for* $\| h \| < \widetilde{\varepsilon}$ *there is*

$$|\widetilde{z}_h^{(r,m)} - v_h^{(r,m)}| \le \alpha(h) \quad on \quad E_h \quad and \quad \lim_{h\to 0} \alpha(h) = 0,$$

where v_h *is the restriction of* v *to the set* E_h.

The above theorem may be proved by a method used in [**6**]–[**9**]; see also [**5**] Chapter 5.

In this paper we consider the following modifications of the classical Euler method.

We first approximate solutions of (1), (2) by means of solutions of the difference equation

$$\delta_0 z^{(r,m)} = F(t^{(r)}, x^{(m)}, z^{(r,m)}, \delta z^{(r+1,m)}) \tag{11}$$

with the initial boundary condition (8). The numerical method consisting of (8) and (11) is called the implicit Euler method for (1), (2). In Section 2, we prove that under natural assumptions on the given functions and on the mesh, there exists exactly one solution of (11), (8). We also give sufficient conditions for the convergence of the implicit Euler method.

Note that Theorem 1.1 does not apply to quasilinear equations. Neither does a general result on implicit method (8), (11), presented in Section 2, apply to quasilinear problems. But in a separate theorem in Section 3, we give sufficient conditions for the convergence of implicit difference methods generated by quasilinear problems.

We wish to emphasize that the main difficulty in carrying out the implicit Euler method for nonlinear equations is the problem of solving equation (11) numerically. For this reason, we separate a new class of difference problems corresponding to (1), (2). We transform nonlinear equation (1) into a quasi-linear system of difference equations. The method thus obtained is implicit and it is linear with respect to the difference operator δ for spatial variables. A convergence theorem and an error estimate for the method are presented in Section 4. It is the main part of the paper. Numerical examples are given in the last section.

2. Convergence of implicit Euler methods. Write

$$E'_h = \{(t^{(r)}, x^{(m)}) \in E_h \setminus \partial_0 E_h : 0 \le r \le K-1\}.$$

We formulate the main result on method (8), (11).

Theorem 2.1. *Suppose that Assumption* $H_0[F]$ *is satisfied and*

1) *$v : E \to R$ is a solution of (1), (2) and v is of class C^1,*
2) *there is $\alpha_0 : H \to R_+$ such that*

$$|\varphi^{(r,m)} - \varphi_h^{(r,m)}| \le \alpha_0(h) \quad on \quad E_{0.h} \cup \partial_0 E_h$$

and

$$\lim_{h\to 0} \alpha_0(h) = 0.$$

Then there exists exactly one solution $z_h : E_h \to R$, $h \in H$, of problem (8), (11) and there exist $\alpha : H \to R_+$ and $\widetilde{\varepsilon} > 0$ such that for $\| h \| < \widetilde{\varepsilon}$ there holds

$$|v_h^{(r,m)} - z_h^{(r,m)}| \le \alpha(h) \quad on \quad E_h \tag{12}$$

and

$$\lim_{h\to 0} \alpha(h) = 0, \tag{13}$$

where v_h is the restriction of v to the set E_h.

Proof. We first prove that there exists exactly one solution $z_h : E_h \to R$ of problem (8), (11). The proof will be divided into three steps.

(I) Suppose that $0 \le r \le K - 1$ and $m \in \mathbf{Z}^n$ are fixed and

$$-N_i \le m_i \le N_i - 1 \quad \text{for} \quad 1 \le i \le \kappa,$$

$$-N_i + 1 \le m_i \le N_i \quad \text{for} \quad \kappa + 1 \le i \le n.$$

Assume also that the numbers $z_h^{(r,m)}$, $z_h^{(r+1,m+e_i)}$ for $1 \le i \le \kappa$ and $z_h^{(r+1,m-e_i)}$ for $\kappa + 1 \le i \le n$ are known. Write

$$Q^{(r+1,m)}(y) = \Big(\frac{1}{h_1}\big(z_h^{(r+1,m+e_1)} - y\big), \dots, \frac{1}{h_\kappa}\big(z_h^{(r+1,m+e_\kappa)} - y\big),$$
$$\frac{1}{h_{\kappa+1}}\big(y - z_h^{(r+1,m-e_{\kappa+1})}\big), \dots, \frac{1}{h_n}\big(y - z_h^{(r+1,m-e_n)}\big)\Big)$$

and

$$\Psi(y) = z_h^{(r,m)} + h_0 F(t^{(r)}, x^{(m)}, z_h^{(r,m)}, Q^{(r+1,m)}(y)), \tag{14}$$

where $y \in R$. Then $\Psi : R \to R$ is of class C^1. It follows from assumption (9) that

$$\Psi'(y) = -h_0 \sum_{j=1}^{n} \frac{1}{h_j} \Big| \partial_{q_j} F(t^{(r)}, x^{(m)}, z_h^{(r,m)}, Q^{(r+1,m)}(y)) \Big| \le 0.$$

for $y \in R$. Therefore, the equation

$$y = \Psi(y) \tag{15}$$

has exactly one solution $\widetilde{y} \in R$.

(II) Suppose that $0 \le r \le K-1$ is fixed and that the numbers $z_h^{(r,m)}$, $-N \le m \le N$, are known. Consider equation (15) with Ψ given by (14) and

$$m = (N_1 - 1, N_2 - 1, \dots, N_\kappa - 1, -N_{\kappa+1} + 1, \dots, -N_n + 1). \tag{16}$$

It follows from (8) that the numbers

$$z_h^{(r+1,m+e_i)} \quad \text{for} \quad 1 \le i \le \kappa \quad \text{and} \quad z_h^{(r+1,m-e_i)} \quad \text{for} \quad \kappa+1 \le i \le n$$

are known. We conclude from (I) that there exists exactly one number $z_h^{(r+1,m)}$ for m given by (16). In the same manner, we can prove that there exists exactly one number $z_h^{(r+1,m)}$ for

$$m = (j, N_2 - 1, \dots, N_\kappa - 1, -N_{\kappa+1} + 1, \dots, -N_n + 1)$$

and $j = N_1 - 2, N_1 - 3, \dots, -N_1$. Suppose now that $-N_1 \le m_1 \le N_1 - 1$ is fixed and

$$m = (m_1, j, N_3 - 1, \dots, N_\kappa - 1, -N_{\kappa+1} + 1, \dots, -N_n + 1). \tag{17}$$

Repeated applications of (I) enable us to calculate the numbers $z_h^{(r+1,m)}$ for m given by (17) and for $j = N_2 - 1, N_2 - 2 \dots, -N_2$.

Now suppose that we have calculated the numbers

$$z_h^{(r+1,m_1,\dots,m_\kappa,-N_{\kappa+1}+1,\dots,-N_n+1)},$$

where $-N_i \le m_i \le N_i - 1$ for $i = 1, \dots, \kappa$. Put

$$m = (m_1, \dots, m_\kappa, j, -N_{\kappa+2} + 1, \dots, -N_n + 1).$$

We again apply (I) for $j = -N_{\kappa+1} + 1, -N_{\kappa+1} + 2, \dots, N_{\kappa+1}$.

In the same manner we can see that the numbers $z_h^{(r+1,m)}$ exit and they are unique for $-N_i + 1 \le m_i \le N_i$, $i = \kappa + 1, \dots, N$.

(III) It follows from initial boundary condition (8) and from (II) that the proof of the existence and uniqueness of a solution of (8), (11) may be completed by induction with respect to r.

We next show (12), (13). Let the function $\Gamma_h : E_h' \to R$ be defined by

$$\delta_0 v_h^{(r,m)} = F(t^{(r)}, x^{(m)}, v_h^{(r,m)}, \delta v_h^{(r+1,m)}) + \Gamma_h^{(r,m)}.$$

It follows that there exists $\gamma : H \to R_+$ such that

$$|\Gamma_h^{(r,m)}| \le \gamma(h) \quad \text{on} \quad E_h' \quad \text{and} \quad \lim_{h \to 0} \gamma(h) = 0.$$

Write $w_h = z_h - v_h$. An easy computation shows that

$$(18) \qquad w_h^{(r+1,m)}\left[1 + h_0 \sum_{i=1}^{n} \frac{1}{h_i}\theta_i \partial_{q_i} F(Q)\right]$$

$$= h_0 \sum_{i=1}^{\kappa} \frac{1}{h_i}\partial_{q_i} F(Q)\; w_h^{(r+1,m+e_i)} - h_0 \sum_{i=\kappa+1}^{n} \frac{1}{h_i}\partial_{q_i} F(Q)\; w_h^{(r+1,m-e_i)} + w_h^{(r,m)}$$

$$+h_0\Big[F(t^{(r)}, x^{(m)}, z_h^{(r,m)}, \delta v_h^{(r+1,m)}) - F(t^{(r)}, x^{(m)}, v_h^{(r,m)}, \delta v_h^{(r+1,m)})\Big] - h_0\Gamma_h^{(r,m)},$$

$(t^{(r)}, x^{(m)}) \in E'_h$, where $Q \in \Omega$ is an intermediate point and $(\theta_1, \ldots, \theta_n)$ is given by (6).
Let

$$\varepsilon_h^{(r)} = \max\{|w_h^{(r,m)}| : -N \le m \le N\}, \qquad 0 \le r \le K.$$

It follows from condition 2) of Assumption $H_0[F]$ and from (9), (18) that ε_h satisfies the recurrent inequality

$$(19) \quad \varepsilon_h^{(r+1)} \le \max\{\varepsilon_h^{(r)} + h_0\sigma(t^{(r)}, \varepsilon_h^{(r)}) + h_0\gamma(h), \alpha_0(h)\}, \quad 0 \le r \le K-1,$$

and $\varepsilon_h^{(0)} \le \alpha_0(h)$. Consider the Cauchy problem

$$(20) \qquad w'(t) = \sigma(t, w(t)) + \gamma(h), \qquad w(0) = \alpha_0(h).$$

It follows from condition 2) of Assumption $H_0[F]$ that there is $\widetilde{\varepsilon} > 0$ such that, for $\| h \| < \widetilde{\varepsilon}$, there exists the maximal solution $\widetilde{w}_h$ of (20) and $\widetilde{w}_h$ is defined on $[0, a]$. Moreover, there is

$$\lim_{h \to 0} \widetilde{w}_h(t) = 0 \quad \text{uniformly on} \quad [0, a].$$

The function $\widetilde{w}_h$ is convex and then it satisfies the recurrent inequality

$$\widetilde{w}_h^{(r+1)} \ge \widetilde{w}_h^{(r)} + h_0\sigma(t^{(r)}, \widetilde{w}_h^{(r)}) + h_0\gamma(h), \quad 0 \le r \le K-1.$$

From the above inequality and (19) we derive $\varepsilon_h^{(r)} \le \widetilde{w}_h^{(r)}$ for $0 \le r \le K$. Then we get (12) with $\alpha(h) = \widetilde{w}_h(a)$. This proves the theorem. □

Remark 2.1. Note that condition (10) is omitted in the theorem on the convergence of the implicit difference method for nonlinear equations. Thus the class of implicit difference method is larger than the set of classical difference schemes for (1), (2).

3. Implicit Euler method for quasilinear equations. Suppose that

$$f : E \times R \to R^n, \quad f = (f_1, \ldots, f_n), \quad g : E \times R \to R, \quad \varphi : E_0 \cup \partial_0 E \to R$$

are given functions. We consider the quasilinear differential equation

$$\partial_t z(t,x) = \sum_{i=1}^{n} f_i(t,x,z(t,x))\partial_{x_i} z(t,x) + g(t,x,z(t,x)) \tag{21}$$

and initial boundary condition (2).

Suppose that we approximate solutions of (2), (21) by means of solutions of the classical difference equation

$$\delta_0 z^{(r,m)} = \sum_{i=1}^{n} f_i(t^{(r)}, x^{(m)}, z^{(r,m)})\delta_i z^{(r,m)} + g(t^{(r)}, x^{(m)}, z^{(r,m)}) \tag{22}$$

with initial boundary condition (8), where $\varphi_h : E_{0.h} \cup \partial_0 E_h \to R$ is a given function. We formulate sufficient conditions for the convergence of method (8), (22).

Assumption $H[f,g]$. Suppose that the functions f and g are such that

1) $f \in C(E \times R, R^n)$, $g \in C(E \times R, R)$ and

$$f(t,x,p) \diamond \theta \geq 0 \quad \text{on} \quad E \times R, \tag{23}$$

2) there is $\sigma : [0,a] \times R_+ \to R_+$ such that
   (i) σ is continuous, nondecreasing with respect to both variables, $\sigma(t,0) = 0$ for $t \in [0,a]$ and for each $d \geq 1$ the maximal solution of the Cauchy problem

$$w'(t) = d\sigma(t, w(t)), \quad w(0) = 0,$$

   is $\overline{w}(t) = 0$ for $t \in [0,a]$,
   (ii) the estimates

$$\| f(t,x,p) - f(t,x,\overline{p}) \| \leq \sigma(t, |p - \overline{p}|),$$

$$|g(t,x,p) - g(t,x,\overline{p})| \leq \sigma(t, |p - \overline{p}|)$$

   are satisfied on $E \times R$.

LEMMA 3.1. *Suppose that Assumption $H[f,g]$ is satisfied and*

1) *$v : E \to R$ is a solution of (2), (21) and v is of class C^1 on E,*
2) *$h \in H$ and*

$$1 - h_0 \sum_{i=1}^{n} \frac{1}{h_i} \Big| f_i(t,x,p) \Big| \geq 0 \quad on \quad E \times R,$$

3) $\widetilde{z}_h : E_h \to R$ *is a solution of (8), (22) and there is* $\alpha_0 : H \to R_+$ *such that*

$$|\varphi^{(r,m)} - \varphi_h^{(r,m)}| \leq \alpha_0(h) \quad \text{on} \quad E_{0.h} \cup \partial_0 E_h \quad \text{and} \quad \lim_{h \to 0} \alpha_0(h) = 0.$$

Then there exists $\alpha : H \to R_+$ *such that*

$$|v_h^{(r,m)} - \widetilde{z}_h^{(r,m)}| \leq \alpha(h) \quad \text{on} \quad E_h \quad \text{and} \quad \lim_{h \to 0} \alpha(h) = 0,$$

where v_h *is the restriction of* v *to the set* E_h.

The above Lemma may be proved by the method used in [**6**]–[**8**]; see also [**5**], Chapter 5.

In this paper we will approximate classical solution of problem (2), (21) with solutions of the implicit difference equation

$$\delta_0 z^{(r,m)} = \sum_{i=1}^{n} f_i(t^{(r)}, x^{(m)}, z^{(r,m)})\, \delta_i z^{(r+1,m)} + g(t^{(r)}, x^{(m)}, z^{(r,m)}), \tag{24}$$

with initial boundary condition (8).

Theorem 3.1. *Suppose that Assumption* $H[f,g]$ *is satisfied and*

1) $v : E \to R$ *is a solution of (2), (21) and* v *is of class* C^1 *on* E,
2) $h \in H$ *and there exists* $\alpha_0 : H \to R_+$ *such that*

$$|\varphi^{(r,m)} - \varphi_h^{(r,m)}| \leq \alpha_0(h) \quad \text{on} \quad E_{0.h} \cup \partial_0 E_h \quad \text{and} \quad \lim_{h \to 0} \alpha_0(h) = 0.$$

Then

1) *there exists exactly one solution* $z_h : E_h \to R$ *of problem (8), (24),*
2) *there exist* $\widetilde{\varepsilon} > 0$ *and* $\alpha : H \to R_+$ *such that for* $\| h \| < \widetilde{\varepsilon}$ *there is*

$$|v_h^{(r,m)} - z_h^{(r,m)}| \leq \alpha(h) \quad \text{on} \quad E_h \quad \text{and} \quad \lim_{h \to 0} \alpha(h) = 0, \tag{25}$$

where v_h *is the restriction of* v *to the set* E_h.

Proof. We first prove that there exists exactly one solution $z_h : E_h \to R$ of (8), (24). Suppose that $0 \leq r \leq K-1$ is fixed and that z_h is defined on $E_h \cap ([0, t^{(r)}] \times R^n)$. From (23) we conclude that equations for $z_h^{(r+1,m)}$ have the form

$$z^{(r+1,m)}\left[1+h_0\sum_{i=1}^{n}\frac{1}{h_i}|f_i(t^{(r)},x^{(m)},z_h^{(r,m)})|\right]$$

$$(26)\qquad = z_h^{(r,m)}+h_0\sum_{i=1}^{\kappa}\frac{1}{h_i}f_i(t^{(r)},x^{(m)},z_h^{(r,m)})z^{(r+1,m+e_i)}$$

$$-\,h_0\sum_{i=\kappa+1}^{n}\frac{1}{h_i}f_i(t^{(r)},x^{(m)},z_h^{(r,m)})z^{(r+1,m-e_i)}+h_0 g(t^{(r)},x^{(m)},z_h^{(r,m)}).$$

From (26) we deduce that the numbers $z_h^{(r+1,m)}$ may be computed for

$$m=(j,N_2-1,\dots,N_\kappa-1,-N_{\kappa+1}+1,\dots,-N_n+1),$$

where $j=N_1-1,N_1-2,\dots,-N_1$. Our next goal is to determine the numbers $z_h^{(r+1,m)}$, where

$$m=(m_1,j,N_3-1,\dots,N_\kappa-1,-N_{\kappa+1}+1,\dots,-N_n+1)$$

and $-N_1\le m_1\le N_1-1$ is fixed and $j=N_2-1,N_2-2,\dots,-N_2$. From (26) we conclude that, for the above m, the numbers $z_h^{(r+1,m)}$ exist and are unique.

Suppose that the numbers $z_h^{(r+1,m)}$ are computed for $-N_i\le m_i\le N_i-1$, $i=1,\dots,\kappa$. Then we consider formula (26) for

$$m=(m_1,\dots,m_\kappa,j,-N_{\kappa+2}+1,\dots,-N_n+1),$$

where $(m_1,\dots,m_\kappa)$ is fixed and we put $j=-N_{\kappa+1}+1,-N_{\kappa+1}+2,\dots,N_{\kappa+1}$.

Repeated applications of (26) enable us to compute $z_h^{(r+1,m)}$ for $(t^{(r+1)},x^{(m)})\in E_h\setminus\partial_0 E_h$. It follows from (8) and from the above considerations that the proof may be completed by induction with respect to r.

We next show (25). Let $\Gamma_h,\Lambda_h : E'_h\to R$ be the functions defined by

$$\Gamma_h^{(r,m)}=\delta_0 v_h^{(r,m)}-\partial_t v^{(r,m)}$$

$$+\sum_{j=1}^{n}f_j(t^{(r)},x^{(m)},v_h^{(r,m)})(\partial_{x_j}v^{(r,m)}-\delta_j v_h^{(r+1,m)})$$

and

$$\Lambda_h^{(r,m)}=g(t^{(r)},x^{(m)},v_h^{(r,m)})-g(t^{(r)},x^{(m)},z_h^{(r,m)})$$

$$+\sum_{j=1}^{n}\left[f_j(t^{(r)},x^{(m)},v_h^{(r,m)})-f_j(t^{(r)},x^{(m)},z_h^{(r,m)})\right]\delta_j v_h^{(r+1,m)}.$$

Write $w_h=v_h-z_h$ and

$$P^{(r,m)}[z_h]=(t^{(r)},x^{(m)},z_h^{(r,m)}).$$

Then w_h satisfies the difference equation

$$\delta_0 w_h^{(r,m)} = \sum_{j=1}^{n} f_j(P^{(r,m)}[z_h])\delta_j w_h^{(r+1,m)} + \Gamma_h^{(r,m)} + \Lambda_h^{(r,m)}, \quad (t^{(r)}, x^{(m)}) \in E'_h,$$

and, consequently,

$$w_h^{(r+1,m)} + h_0 \sum_{j=1}^{n} \frac{1}{h_j}\theta_j f_j(P^{(r,m)}[z_h]) w_h^{(r+1,m)} \tag{27}$$

$$= w_h^{(r,m)} + h_0 \sum_{j=1}^{n} \frac{1}{h_j}\theta_j f_j(P^{(r,m)}[z_h]) w_h^{(r+1,m+\theta_j e_j)} + h_0\Big[\Gamma_h^{(r,m)} + \Lambda_h^{(r,m)}\Big],$$

where $\theta = (\theta_1, \ldots, \theta_n)$ is given by (6). Write

$$\varepsilon_h^{(r)} = \max\{|w_h^{(r,m)}| : -N \le m \le N\}, \quad 0 \le r \le K.$$

Let us denote by $c_0 \in R_+$ such a constant that

$$|\partial_{x_j} v(t,x)| \le c_0 \quad \text{for} \quad (t,x) \in E, \quad 1 \le j \le n.$$

It follows from Assumption $H[f,g]$ that

$$|\Lambda_h^{(r,m)}| \le (1 + c_0)\sigma(t^{(r)}, \varepsilon_h^{(r)}) \quad \text{for} \quad (t^{(r)}, x^{(m)}) \in E'_h. \tag{28}$$

There exists $\gamma : H \to R_+$ such that

$$|\Gamma_h^{(r,m)}| \le \gamma(h) \quad \text{on} \quad E'_h \quad \text{and} \quad \lim_{h\to 0} \gamma(h) = 0. \tag{29}$$

From (23), (27)–(29) and condition 2) of Assumption $H[f,g]$ we conclude that ε_h satisfies the recurrent inequality

$$\varepsilon_h^{(r+1)} \le \max\{\varepsilon_h^{(r)} + h_0(1+c_0)\sigma(t^{(r)}, \varepsilon_h^{(r)}) + h_0\gamma(h),\ \alpha_0(h)\}, \tag{30}$$

where $0 \le r \le K-1$ and $\varepsilon_h^{(0)} \le \alpha_0(h)$. Let $\widetilde{w}_h$ be the maximal solution of the Cauchy problem

$$w'(t) = (1+c_0)\sigma(t, w(t)) + \gamma(h), \quad w(0) = \alpha_0(h).$$

It follows that there is $\widetilde{\varepsilon} > 0$ such that for $\| h \| < \widetilde{\varepsilon}$ the solution $\widetilde{w}_h$ is defined on $[0,a]$. Moreover, there is

$$\lim_{h\to 0} \widetilde{w}_h(t) = 0 \quad \text{uniformly on} \quad [0,a].$$

The function $\widetilde{w}_h$ is convex; whence, recurrently

$$\widetilde{w}_h^{(r+1)} \ge \widetilde{w}_h^{(r)} + h_0(1+c_0)\sigma(t^{(r)}, \widetilde{w}_h^{(r)}) + h_0\gamma(h), \quad 0 \le r \le K-1.$$

The above relation and (30) imply $\varepsilon_h^{(r)} \le \widetilde{w}_h^{(r)}$ for $0 \le r \le K$. Then we get (25) for $\alpha(h) = \widetilde{w}_h(a)$. This completes the proof. □

Remark 3.1. Note that condition (10) is omitted in the theorem on the convergence of the implicit difference method for quasilinear equations. Thus the class of implicit difference method is larger than the set of classical difference schemes for (2), (21).

4. Generalized implicit Euler method for nonlinear equations. Now we define a new class of difference problems corresponding to (1), (2). We transform the nonlinear differential equation into a quasilinear system of difference equations. We consider implicit difference methods of the Euler type. In our considerations, it is important that condition (10) is omitted in a theorem on the convergence of an implicit difference method for nonlinear equation (1).

By $M_{n\times n}$ we will denote the class of all $n \times n$ matrices with real elements. For $X \in M_{n\times n}$ we put

$$\| X \| = \max\{\sum_{j=1}^{n} |x_{ij}| : 1 \le i \le n\},$$

where

$$X = [x_{ij}]_{i,j=1,\dots n}.$$

The product of two matrices is denoted by " $\star$ ". If $X \in M_{n\times n}$, then X^T is the transposed matrix. We use the symbol " $\circ$ " to denote the scalar product in R^n.

We need the following assumption on F.

Assumption $H[F]$**.** Suppose that the function $F : \Omega \to R$ is such that

1) $F \in C(\Omega, R)$ and there exist the partial derivatives

$$\partial_x F = (\partial_{x_1} F, \dots, \partial_{x_n} F), \quad \partial_p F, \quad \partial_q F = (\partial_{q_1} F, \dots, \partial_{q_n} F),$$

and $\partial_x F, \partial_q F \in C(\Omega, R^n)$, $\partial_p F \in C(\Omega, R)$,

2) for $P = (t, x, p, q) \in \Omega$ there is

$$\partial_q F(P) \diamond \theta \ge 0. \tag{31}$$

Now we formulate a difference problem corresponding to (1), (2). For $u = (u_1, \dots, u_n)$, let us denote by (z, u) the unknown functions of the variables $(t^{(r)}, x^{(m)})$. Write

$$P^{(r,m)}[z, u] = (t^{(r)}, x^{(m)}, z^{(r,m)}, u^{(r,m)})$$

and

$$\delta_0 u^{(r,m)} = (\delta_0 u_1^{(r,m)}, \dots, \delta_0 u_n^{(r,m)}),$$

$$\delta u^{(r,m)} = \left[\delta_j u_i^{(r,m)}\right]_{i,j=1,\dots,n}.$$

We consider the system of difference equations

$$\delta_0 z^{(r,m)} = F(P^{(r,m)}[z,u]) + \partial_q F(P^{(r,m)}[z,u]) \circ \Big(\delta z^{(r+1,m)} - u^{(r,m)}\Big), \tag{32}$$

$$\begin{aligned}\delta_0 u^{(r,m)} &= \partial_x F(P^{(r,m)}[z,u]) + \partial_p F(P^{(r,m)}[z,u]) u^{(r,m)} \\ &\quad + \partial_q F(P^{(r,m)}[z,u]) * \Big[\delta u^{(r+1,m)}\Big]^T\end{aligned} \tag{33}$$

with the initial condition

$$z^{(r,m)} = \varphi_h^{(r,m)}, \quad u^{(r,m)} = \psi_h^{(r,m)} \quad \text{on} \quad E_{0.h} \cup \partial_0 E_h, \tag{34}$$

where $\varphi_h : E_{0.h} \cup \partial_0 E_h \to R$, $\psi_h : E_{0.h} \cup \partial_0 E_h \to R^n$ are given functions.

The numerical method consisting of (32)–(34) is called the generalized implicit Euler method for (1), (2).

Difference problem (32), (34) is obtained in the following way. Suppose that Assumption $H[F]$ is satisfied and that the derivatives $(\partial_{x_1}\varphi, \dots, \partial_{x_n}\varphi) = \partial_x \varphi$ exist on $E_0 \cup \partial_0 E$. The method of quasilinearization for nonlinear equations consists in replacing problem (1), (2) with the following one. Let (z,u) be unknown functions in the variable $(t,x) \in E$. First we introduce an additional unknown function $u = \partial_x z$ in (1). Then we consider the following linearization of (1) with respect to u:

$$\begin{aligned}\partial_t z(t,x) =& F(t,x,z(t,x),u(t,x)) \\ &+ \partial_q F(t,x,z(t,x),u(t,x)) \circ (\partial_x z(t,x) - u(t,x)).\end{aligned} \tag{35}$$

We get differential equations for u by differentiating equation (1), resulting in the following:

$$\begin{aligned}\partial_t u(t,x) &= \partial_x F(t,x,z(t,x),u(t,x)) \\ &+ \partial_p F(t,x,z(t,x),u(t,x)) u(t,x) + \partial_q F(t,x,z(t,x),u(t,x)) * \Big[\partial_x u(t,x)\Big]^T.\end{aligned} \tag{36}$$

It is natural to consider the following initial boundary condition for (35), (36):

$$z(t,x) = \varphi(t,x), \quad u(t,x) = \partial_x \varphi(t,x) \quad \text{on} \quad E_0 \cup \partial_0 E. \tag{37}$$

Difference problem (32)–(34) is a discretization of (35)–(37).

The above method of quasilinearization and the theory of bicharacteristics were first considered by S. Cinquini [**2**] and M. Cinquini Cibrario [**3**]. Existence results for generalized or classical solutions for nonlinear systems with initial or initial boundary conditions are based on this process.

The method of quasilinearization is used in [**1**] for numerical solving of an initial problem on the Haar pyramid.

We formulate next assumptions on given functions.

Assumption $H[\sigma]$**.** Suppose that the function $\sigma : [0,a] \times R_+ \to R_+$ is continuous and

1) σ is nondecreasing with respect to both variables and $\sigma(t,0) = 0$ for $t \in [0,a]$,

2) for each $c \in R_+$ and $d \geq 1$, the maximal solution of the Cauchy problem

$$w'(t) = cw(t) + d\sigma(t, w(t)), \quad w(0) = 0,$$

is $\overline{w}(t) = 0$ for $t \in [0,a]$.

Assumption $H[F,\varphi]$**.** Suppose that Assumption $H[F]$ is satisfied and

1) there is $L \in R_+$ such that

$$|\partial_p F(t,x,p,q)|, \quad \| \partial_q F(t,x,p,q) \| \leq L \quad \text{on} \quad \Omega,$$

2) there is $\sigma : [0,a] \times R_+ \to R_+$ such that Assumption $H[\sigma]$ is satisfied and the terms

$$\| \partial_x F(t,x,p,q) - \partial_x F(t,x,\overline{p},\overline{q}) \|, \qquad |\partial_p F(t,x,p,q) - \partial_p F(t,x,\overline{p},\overline{q})|,$$

$$\| \partial_q F(t,x,p,q) - \partial_q F(t,x,\overline{p},\overline{q}) \|$$

are bounded from above by $\sigma(t, |p-\overline{p}| + \| q - \overline{q} \|)$,

3) $\varphi : E_0 \cup \partial_0 E \to R$ is of class C^1.

We formulate the main result on the implicit difference method for nonlinear equations.

Theorem 4.1. *Suppose that Assumption $H[F,\varphi]$ is satisfied and*

1) *$v : E \to R$ is a solution of (1), (2) and v is of class C^2 on E,*

2) *there exists $\alpha_0 : H \to R_+$ such that*

$$|\varphi^{(r,m)} - \varphi_h^{(r,m)}| + \| \partial_x \varphi^{(r,m)} - \psi_h^{(r,m)} \| \leq \alpha_0(h) \quad \text{on} \quad E_{0.h} \cup \partial_0 E_h$$

and

$$\lim_{h \to 0} \alpha_0(h) = 0.$$

Then there exists exactly one solution $(z_h, u_h) : E_h \to R^{1+n}$, $u_h = (u_{h.1}, \ldots, u_{h.n})$, of difference problem (32)–(34) and there exist a number $\widetilde{\varepsilon} > 0$ and a function $\alpha : H \to R_+$ such that, for $\| h \| < \widetilde{\varepsilon}$, there hold

$$|v^{(r,m)} - z_h^{(r,m)}| + \| \partial_x v^{(r,m)} - u_h^{(r,m)} \| \leq \alpha(h) \quad \text{on} \quad E_h \tag{38}$$

and

$$\lim_{h \to 0} \alpha(h) = 0.$$

Proof. We first show that there exists exactly one solution $(z_h, u_h) : E_h \to R^{1+n}$ of (32)–(34). We deduce from assumption (31) that system (32), (33) is equivalent to the following one:

$$
\begin{aligned}
z^{(r+1,m)}&\Big[1 + h_0 \sum_{i=1}^{n} \frac{1}{h_i}\Big|\partial_{q_i} F(P^{(r,m)}[z,u])\Big|\Big] \\
&= z^{(r,m)} + h_0 \sum_{i=1}^{\kappa} \frac{1}{h_i}\partial_{q_i} F(P^{(r,m)}[z,u]) z^{(r+1,m+e_i)} \\
&\quad - h_0 \sum_{i=\kappa+1}^{n} \frac{1}{h_i}\partial_{q_i} F(P^{(r,m)}[z,u]) z^{(r+1,m-e_i)} \\
&\quad + h_0 F(P^{(r,m)}[z,u]) - h_0 \partial_q F(P^{(r,m)}[z,u]) \circ u^{(r,m)}
\end{aligned}
$$

and

$$
\begin{aligned}
u_j^{(r+1,m)}&\Big[1 + h_0 \sum_{i=1}^{n} \frac{1}{h_i}\Big|\partial_{q_i} F(P^{(r,m)}[z,u])\Big|\Big] \\
&= u_j^{(r,m)} + h_0 \sum_{i=1}^{\kappa} \frac{1}{h_i}\partial_{q_i} F(P^{(r,m)}[z,u]) u_j^{(r+1,m+e_i)} \\
&\quad - h_0 \sum_{j=\kappa+1}^{n} \frac{1}{h_i}\partial_{q_i} F(P^{(r,m)}[z,u]) u_j^{(r+1,m-e_i)} + h_0 \partial_{x_j} F(P^{(r,m)}[z,u]) \\
&\quad + h_0 \partial_p F(P^{(r,m)}[z,u]) u_j^{(r,m)}, \qquad j = 1, \ldots, n.
\end{aligned}
$$

It is clear that the existence and uniqueness of a solution of the above system may be deduced by the method used in the proof of Theorem 3.1. Details are omitted.

We next show (38). Write $w = \partial_x v$, $w = (w_1, \ldots, w_n)$, and

$$v_h = v|_{E_h}, \quad w_h = w|_{E_h}, \quad w_h = (w_{1.h}, \ldots, w_{n.h}).$$

Let us consider the errors

$$
\begin{aligned}
\lambda_{h.0}^{(r)} &= \max\{|(z_h - v_h)^{(r,m)}| : -N \le m \le N\}, \\
\lambda_{h.1}^{(r)} &= \max\{\| (u_h - w_h)^{(r,m)} \| : -N \le m \le N\},
\end{aligned}
$$

where $0 \le r \le K$, and $\lambda_h^{(r)} = \lambda_{h.0}^{(r)} + \lambda_{h.1}^{(r)}$ for $0 \le r \le K$. We will write a difference inequality for the function λ_h.

We first examine $\lambda_{h.0}$. Let the functions $\Gamma_{h.0}, \Lambda_{h.0} : E'_h \to R$ be defined by

$$
\begin{aligned}
\Gamma_{h.0}^{(r,m)} &= \delta_0 v_h^{(r,m)} - \partial_t v^{(r,m)} \\
&+ \partial_q F\Big(P^{(r,m)}[v_h, w_h]\Big) \circ \Big[\partial_x v^{(r,m)} - \delta v_h^{(r+1,m)}\Big]
\end{aligned}
$$

and

$$\Lambda_{h.0}^{(r,m)} = F\Big(P^{(r,m)}[v_h, w_h]\Big) - F\Big(P^{(r,m)}[z_h, u_h]\Big)$$

$$-\partial_q F\Big(P^{(r,m)}[v_h, w_h]\Big) \circ w_h^{(r,m)} + \partial_q F\Big(P^{(r,m)}[z_h, u_h]\Big) \circ u_h^{(r,m)}$$

$$+\Big[\partial_q F\Big(P^{(r,m)}[v_h, w_h]\Big) - \partial_q F\Big(P^{(r,m)}[z_h, u_h]\Big)\Big] \circ \delta v_h^{(r+1,m)}.$$

It follows easily that the function (v, w) satisfies (35), (36). We thus get

$$\delta_0(v_h - z_h)^{(r,m)} = \partial_q F\Big(P^{(r,m)}[z_h, u_h]\Big) \circ \delta(v_h - z_h)^{(r+1,m)} \tag{39}$$
$$+ \Lambda_{h.0}^{(r,m)} + \Gamma_{h.0}^{(r,m)}, \qquad (t^{(r)}, x^{(m)}) \in E'_h$$

and, consequently,

$$(v_h - z_h)^{(r+1,m)}\Big[1 + h_0 \sum_{j=1}^{n} \frac{1}{h_j}\theta_j \partial_{q_j} F\Big(P^{(r,m)}[z_h, u_h]\Big)\Big] \tag{40}$$

$$= (v_h - z_h)^{(r,m)} + h_0 \sum_{j=1}^{\kappa} \frac{1}{h_j}\partial_{q_j} F\Big(P^{(r,m)}[z_h, u_h]\Big)(v_h - z_h)^{(r+1,m+e_j)}$$

$$-h_0 \sum_{j=\kappa+1}^{n} \frac{1}{h_j}\partial_{q_j} F\Big(P^{(r,m)}[z_h, u_h]\Big)(v_h - z_h)^{(r+1,m-e_j)}$$

$$+h_0\Big[\Lambda_{h.0}^{(r,m)} + \Gamma_{h.0}^{(r,m)}\Big], \quad (t^{(r)}, x^{(m)}) \in E'_h.$$

It follows easily that there is $\gamma_0 : H \to R_+$ such that

$$|\Gamma_{h.0}^{(r,m)}| \le \gamma_0(h) \quad \text{on} \quad E'_h \quad \text{and} \quad \lim_{h\to 0} \gamma_0(h) = 0. \tag{41}$$

Let $c_0 \in R_+$ be such a constant that

$$\| \partial_x v(t, x) \| \le c_0 \quad \text{and} \quad \| \partial_{xx} v(t, x) \| \le c_0 \quad \text{on} \quad E.$$

It follows from Assumption $H[f, \varphi]$ that

$$|\Lambda_{h.0}^{(r,m)}| \le 2\Big[L\lambda_h^{(r)} + c_0\sigma(t^{(r)}, \lambda_h^{(r)})\Big], \quad (t^{(r)}, x^{(m)}) \in E'_h. \tag{42}$$

According to the above estimates and (40), (41), there is

$$|(z_h - v_h)^{(r+1,m)}| \le \lambda_{h.0}^{(r)} + 2h_0\Big[L\lambda_h^{(r)} + c_0\sigma(t^{(r)}, \lambda_h^{(r)})\Big] + h_0\gamma_0(h), \tag{43}$$

where $(t^{(r)}, x^{(m)}) \in E'_h$. Now we write a difference inequality for $\lambda_{h.1}$. Let the functions

$$\Lambda_h = (\Lambda_{h.1}, \ldots, \Lambda_{h.n}) : E'_h \to R^n, \quad \Gamma_h = (\Gamma_{h.1}, \ldots, \Gamma_{h.n}) : E'_h \to R^n,$$

be defined by

$$\Gamma_h^{(r,m)} = \delta_0 w_h^{(r,m)} - \partial_t w^{(r,m)} + \partial_q F\Big(P^{(r,m)}[v_h, w_h]\Big) \star \Big[\partial_x w^{(r,m)} - \delta w_h^{(r+1,m)}\Big]^T$$

and

$$\begin{aligned}\Lambda_h^{(r,m)} &= \partial_x F\Big(P^{(r,m)}[v_h, w_h]\Big) - \partial_x F\Big(P^{(r,m)}[z_h, u_h]\Big)\\ &+\partial_p F\Big(P^{(r,m)}[v_h, w_h]\Big) w_h^{(r,m)} - \partial_p F\Big(P^{(r,m)}[z_h, u_h]\Big) u_h^{(r,m)}\\ &+\Big[\partial_q F\Big(P^{(r,m)}[v_h, w_h]\Big) - \partial_q F\Big(P^{(r,m)}[z_h, u_h]\Big)\Big] \star \Big[\delta w_h^{(r+1,m)}\Big]^T.\end{aligned}$$

Then the function $w_h - u_h$ satisfies the difference equation

$$\begin{aligned}\delta_0 (w_h - u_h)^{(r,m)} &= \partial_q F\Big(P^{(r,m)}[z_h, u_h]\Big) \star \Big[\delta (w_h - u_h)^{(r+1,m)}\Big]^T\\ &+\Lambda_h^{(r,m)} + \Gamma_h^{(r,m)}, \quad (t^{(r)}, x^{(m)}) \in E'_h.\end{aligned} \tag{44}$$

This gives

$$\begin{aligned}&(w_{h.i} - u_{h.i})^{(r+1,m)} \Big[1 + h_0 \sum_{j=1}^{n} \frac{1}{h_j} \theta_j \partial_{q_j} F\Big(P^{(r,m)}[z_h, u_h]\Big)\Big]\\ &= (w_{h.i} - u_{h.i})^{(r,m)} + h_0 \sum_{j=1}^{\kappa} \frac{1}{h_j} \partial_{q_j} F\Big(P^{(r,m)}[z_h, u_h]\Big) (w_{h.i} - u_{h.i})^{(r+1,m+e_j)}\\ &-h_0 \sum_{j=\kappa+1}^{n} \frac{1}{h_j} \partial_{q_j} F\Big(P^{(r,m)}[z_h, u_h]\Big) (w_{h.i} - u_{h.i})^{(r+1,m-e_j)}\\ &+h_0 \Big[\Lambda_{h.i}^{(r,m)} + \Gamma_{h.i}^{(r,m)}\Big], \quad 1 \le i \le n, \quad (t^{(r)}, x^{(m)}) \in E'_h.\end{aligned}$$

According to assumption (31),

$$\begin{aligned}&\| (w_h - u_h)^{(r+1,m)} \| \Big[1 + h_0 \sum_{j=1}^{n} \frac{1}{h_j} \theta_j \partial_{q_j} F\Big(P^{(r,m)}[z_h, u_h]\Big)\Big]\\ &\le \| (w_h - u_h)^{(r,m)} \| + h_0 \sum_{j=1}^{\kappa} \frac{1}{h_j} \partial_{q_j} F\Big(P^{(r,m)}[z_h, u_h]\Big) \| (w_h - u_h)^{(r+1,m+e_j)} \|\\ &-h_0 \sum_{j=\kappa+1}^{n} \frac{1}{h_j} \partial_{q_j} F\Big(P^{(r,m)}[z_h, u_h]\Big) \| (w_h - u_h)^{(r+1,m-e_j)} \|\\ &+h_0 \Big[\| \Lambda_h^{(r,m)} \| + \| \Gamma_h^{(r,m)} \| \Big], \quad (t^{(r)}, x^{(r,m)}) \in E'_h.\end{aligned} \tag{45}$$

It follows from Assumption $H[F, \varphi]$ that

$$\| \Lambda_h^{(r,m)} \| \le (1 + 2c_0) \sigma(t^{(r)}, \lambda_h^{(r)}) + L \lambda_h^{(r)}, \quad (t^{(r)}, x^{(m)}) \in E'_h, \tag{46}$$

and there is $\gamma : H \to R_+$ such that

$$\| \Gamma_h^{(r,m)} \| \leq \gamma(h) \quad \text{on} \quad E'_h \quad \text{and} \quad \lim_{h\to 0} \gamma(h) = 0. \tag{47}$$

From (45)–(47) we conclude that

$$\| (u_h - w_h)^{(r+1,m)} \| \leq \lambda_{h.1}^{(r)} + h_0(1+2c_0)\sigma(t^{(r)}, \lambda_h^{(r)}) + Lh_0\lambda_h^{(r)} + h_0\gamma(h), \tag{48}$$

where $(t^{(r)}, x^{(m)}) \in E'_h$. Adding inequalities (43) and (48) we get

$$\lambda_h^{(r+1)} \leq \max\{\alpha_0(h), U_h^{(r)}[\lambda_h]\}, \quad 0 \leq r \leq K-1, \tag{49}$$

where

$$U_h^{(r)}[\lambda_h] = \lambda_h^{(r)} + h_0(1+4c_0)\sigma(t^{(r)}, \lambda_h^{(r)}) + 3Lh_0\lambda_h^{(r)} + h_0(\gamma_0(h) + \gamma(h)).$$

Consider the Cauchy problem

$$w'(t) = (1+4c_0)\sigma(t, w(t)) + 3Lw(t) + \gamma_0(h) + \gamma(h), \tag{50}$$

$$w(0) = \alpha_0(h).$$

It follows from condition 2) of Assumption $H[\sigma]$ that there is $\widetilde{\varepsilon} > 0$ such that for $\| h \| < \widetilde{\varepsilon}$ there exists the maximal solution $\widetilde{w}_h$ of (50) and $\widetilde{w}_h$ is defined on $[0, a]$. Moreover,

$$\lim_{h\to 0} \widetilde{w}_h(t) = 0 \quad \text{uniformly on} \quad [0, a].$$

It is easily seen that $\widetilde{w}_h$ satisfies the recurrent inequality

$$\widetilde{w}_h^{(r+1)} \geq \widetilde{w}_h^{(r)} + h_0(1+4c_0)\sigma(t^{(r)}, \widetilde{w}_h^{(r)})$$

$$+3h_0L\widetilde{w}_h^{(r)} + h_0(\gamma_0(h) + \gamma(h)), \quad 0 \leq r \leq K-1.$$

By the above relation and (49), there is

$$\lambda_h^{(r)} \leq \widetilde{w}_h^{(r)} \quad \text{for} \quad 0 \leq r \leq K.$$

Thus we get (38) for $\alpha(h) = \widetilde{w}_h(a)$. This proves the theorem. □

REMARK 4.1. Suppose that all the assumptions of Theorem 4.1 are satisfied with
$\sigma(t, p) = L_0 p$ for $(t, p) \in [0, a] \times R_+$, where $L_0 \in R_+$ and

1) the solution $v : E \to R$ of (1), (2) is of class C^3,
2) there is $\widetilde{C} > 0$ such that for $h \in H$ there is

$$h_i \leq \widetilde{C} h_j, \quad i, j = 0, 1, \ldots, n.$$

Then there exist $C_0, C_1 \in R_+$ such that we have the following error estimate holds:

$$|v^{(r,m)} - z_h^{(r,m)}| + \| \partial_x v^{(r,m)} - u_h^{(r,m)} \| \leq C_0 \alpha_0(h) + C_1 \| h \| \quad \text{on} \quad E_h.$$

We obtain the above inequality by solving problem (50) and using the estimate

$$\gamma_0(h) + \gamma(h) \leq \overline{C} \| h \|$$

with some $\overline{C} \in R_+$.

In the above result the error estimate we need estimates for the derivatives of the solution of problem (1), (2). One may obtain them by the method of differential inequalities. Comparison results for initial problem presented in [**10**], Chapter 7, can be extended on the initial boundary value problem.

Remark 4.2. The stability of difference equations generated by quasilinear first order partial differential equations or systems is strictly connected with Courant–Friedrichs–Levy (CFL) condition ([**4**], Chapter III). Assumption (10) can be considered as the (CFL) condition for nonlinear equations. In our considerations, it is important that we have omitted the (CFL) condition for implicit difference methods generated by quasilinear equations. Note also that we do not need the (CFL) condition for nonlinear problems and generalized implicit Euler method.

5. Numerical examples.

Example 1. For $n = 1$ we put

$$E = [0,1] \times [-1,1],\ E_0 = \{0\} \times [-1,1],\ \partial_0 E = [0,1] \times \{-1\}. \tag{51}$$

Consider the quasilinear differential equation

$$\partial_t z(t,x) = \Big[-1 + x \sin\left(z(t,x)\right) \Big] \partial_x z(t,x) + f(t,x), \tag{52}$$

where

$$f(t,x) = 2e^{2t} \Big[x^2 - 1 + x - x^2 \sin\Big(e^{2t}(x^2-1) \Big) \Big],$$

with the initial boundary condition

$$\begin{aligned} z(0,x) &= x^2 - 1, \qquad x \in [-1,1], \\ z(t,-1) &= 0, \qquad t \in [0,1]. \end{aligned} \tag{53}$$

The solution of the above problem is given by $v(t,x) = e^{2t}(x^2-1)$. Write $t^{(r)} = rh_0$, $0 \leq r \leq K$, and $x^{(m)} = mh_1$, $-N \leq m \leq N$, where $Kh_0 = 1$ and $Nh_1 = 1$. Let us denote by $z_h : E_h \to R$ the solution of the implicit difference problem corresponding to (52), (53). We also consider the function $\widetilde{z}_h : E_h \to R$ which is the solution of a classical difference equation corresponding to (52), (53). It follows from Lemma 3.1 that the classical difference method is stable for $2h_0 \leq h_1$. We consider the implicit difference method and the classical difference scheme with $2h_0 > h_1$. Below we give information on errors of the methods. Write

$$\eta_h^{(r)} = \frac{1}{2N} \sum_{m=-N+1}^{N} |z_h^{(r,m)} - v^{(r,m)}|, \tag{54}$$

$$\widetilde{\eta}_h^{(r)} = \frac{1}{2N} \sum_{m=-N+1}^{N} |\widetilde{z}_h^{(r,m)} - v^{(r,m)}|. \tag{55}$$

The numbers $\eta_h^{(r)}$ and $\widetilde{\eta}_h^{(r)}$ are the arithmetical means of the errors with fixed $t^{(r)}$. The values of the functions η_h and $\widetilde{\eta}_h$ are listed in the table. We write " $\times$ " for $\widetilde{\eta}_h^{(r)} > 100$.

	$h_0 = 0.01$,	$h_1 = 0.01$	$h_0 = 0.001$,	$h_1 = 0.001$
$t = 0.20$	0.021965	0.000374	$\times$	0.000036
$t = 0.40$	0.084962	0.001622	$\times$	0.000164
$t = 0.60$	$\times$	0.003584	$\times$	0.000364
$t = 0.80$	$\times$	0.005868	$\times$	0.000587
$t = 1.00$	$\times$	0.009583	$\times$	0.000929

Table 1. Table of errors ($\widetilde{\eta}_h$, η_h)

	$h_0 = 0.002$,	$h_1 = 0.001$
$t = 0.20$	$\times$	0.000044
$t = 0.40$	$\times$	0.000211
$t = 0.60$	$\times$	0.000516
$t = 0.80$	$\times$	0.000948
$t = 1.00$	$\times$	0.001512

Table 2. Table of errors ($\widetilde{\eta}_h$, η_h)

Note that $\eta^{(r)} < \widetilde{\eta}^{(r)}$ for all values $t^{(r)}$. Thus the class of implicit difference method is larger than the set of classical difference schemes.

Example 2. Suppose that $E, E_0, \partial_0 E$ are given by (51). Consider the nonlinear differential equation

$$\partial_t z(t,x) = -\partial_x z(t,x) + x \sin\Big(\partial_x z(t,x)\Big) + z(t,x) + f(t,x), \tag{56}$$

where

$$f(t,x) = xe^t - x\sin(e^t x),$$

with the initial boundary condition

$$\begin{aligned} z(0,x) &= \tfrac{1}{2}x^2, \qquad x \in [-1,1], \\ z(t,-1) &= \tfrac{1}{2}e^t, \qquad t \in [0,1]. \end{aligned} \tag{57}$$

The solution of the above problem is given by $v(t,x) = 0.5e^t x^2$. Let us denote by $z_h : E_h \to R$ the solution obtained by the generalized implicit Euler method corresponding to (56), (57). We also consider the solution $\widetilde{z}_h : E_h \to R$ of a classical difference scheme for the above problem. The numbers $\eta_h^{(r)}$ and $\widetilde{\eta}_h^{(r)}$ are the arithmetical mean of the errors defined by (54) and (55), respectively. It follows from Theorem 1.1 that the classical difference method is stable for $2h_0 \le h_1$. We consider the generalized Euler method and the classical difference scheme for $2h_0 > h_1$. The values of the functions η_h and $\widetilde{\eta}_h$ are listed in the table. We write " $\times$ " for $\widetilde{\eta}_h^{(r)} > 100$.

	$h_0 = 0.01,$	$h_1 = 0.01$	$h_0 = 0.002,$	$h_1 = 0.002$
$t = 0.20$	0.027676	0.001760	$\times$	0.000725
$t = 0.40$	1.237993	0.003941	$\times$	0.001667
$t = 0.60$	1.551493	0.006564	$\times$	0.002847
$t = 0.80$	2.635383	0.009701	$\times$	0.004288
$t = 1.00$	3.435612	0.013681	$\times$	0.006064

Table 3. Table of errors ($\widetilde{\eta}_h$, η_h)

Thus we see that the implicit difference method is stable with arbitrary steps.

Methods described in Theorems 3.1 and 4.1 have the potential for applications to solving of mixed problems for first order partial differential equations numerically. In our method we approximate the spatial derivatives of the unknown function in (1) by solutions of difference equations which are generated by the original problem. In the classical schemes we use previous values of an approximate solution to calculate the difference expressions corresponding to $\partial_x z$ in (1).

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