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INFINITE SYSTEMS OF STRONG PARABOLIC DIFFERENTIAL–FUNCTIONAL INEQUALITIES

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Abstract. We investigate a weakly coupled infinite system of nonlinear strong parabolic differential–functional inequalities of the following form

$$\partial_t z^i(t,x) < f^i(t,x,z(t,x),\partial_x z^i(t,x),\partial^2_{xx} z^i(t,x),z),\ \ i \in \mathcal{S}, \tag{1}$$

in an arbitrary domain D. The right-hand sides f^i of these inequalites are functionals of an unknown function z and Volterra functionals only will be regarded in this paper. We give a fundamental theorem on strong parabolic differential–functional inequalities, generalizing the well-known Nagumo–Westphal lemma to encompass the case of an infinite system. This paper continues and, in a way, concludes Szarski's research on various generalizations of the theorem on strong differential inequalities.

1. Introduction. We consider a weakly coupled[1] infinite system of nonlinear strong parabolic differential–functional inequalities of form (1), where $\mathcal{S}$ is a set of indices (finite or infinite), $(t,x) = (t,x_1,\dots,x_m) \in D \subset \mathbb{R}^{m+1}$, where D is an arbitrary open (bounded or unbounded) domain.

Now, let $\mathcal{S}$ be an infinite countable set of indices and $\mathcal{S} = \mathcal{N}$, where $\mathcal{N}$ is the set of natural numbers, and z stands for the mapping

$$z : \mathcal{N} \times \overline{D} \to \mathbb{R}, \qquad (i,t,x) \mapsto z^i(t,x)$$

composed of unknown functions z^i. The gradient and the Hessian (the matrix of second-order derivatives) of z^i , $i \in \mathcal{N}$, with respect to x are denoted by

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[1]This means that every equation contains all unknown functions and derivatives of only one unknown function.

$\partial_x z^i := grad_x z^i := (\partial_{x_1} z^i, \partial_{x_2} z^i, \ldots, \partial_{x_m} z^i)$ and $\partial^2_{xx} z^i := (\partial^2_{x_j x_k} z^i)$, for $j, k = 1, \ldots, m$.

Let $\mathcal{B}(\mathcal{N}) := l^\infty(\mathcal{N}) := l^\infty$ be the Banach space of mappings

$$w : \mathcal{N} \to \mathbb{R}, \; i \mapsto w(i) := w^i,$$

with the finite norm

$$\|w\|_{l^\infty} := \sup\{|w^i| : \; i \in \mathcal{N}\}.$$

If $w \in \mathcal{B}(\mathcal{N})$, then we write $w = \{w^i\}_{i \in \mathcal{N}}$.

By $\mathcal{C}_\mathcal{N}(\overline{D}, l^\infty)$ we denote the space of continuous mappings

$$w : \; \overline{D} \to l^\infty, \; (t, x) \mapsto w(t, x)$$

and

$$w(t, x) : \; \mathcal{N} \to \mathbb{R}, \; i \mapsto w^i(t, x),$$

equipped with the finite norm

$$\|w\|_0 := \sup\{|w^i(t, x)| : \; (t, x) \in \overline{D}, \; i \in \mathcal{N}\}.$$

As in the earlier papers [**3, 14, 15**], we denote by $C_\mathcal{N}(\overline{D})$ the Banach space of mappings

$$w : \; \overline{D} \to l^\infty, \; (t, x) \mapsto w(t, x)$$

and

$$w(t, x) : \; \mathcal{N} \to \mathbb{R}, \; i \mapsto w^i(t, x),$$

whose coordinates, i.e., functions w^i for all $i \in \mathcal{N}$, are continuous in $\overline{D}$.

In the present paper, we give a fundamental theorem on strong differential-functional inequalities of parabolic type for infinite systems, which generalizes the well-known classical Nagumo–Westphal lemma (see Lakshmikantham and Leela [**4**], Nickel [**8, 9, 10**], Szarski [**13**], Walter [**16**]) to encompass the case of an infinite system.

In the 1970s, theorems on weak differential inequalities of parabolic type were generalized by Szarski [**14, 15**] to include infinite systems; however, theorems on strong differential inequalities have not been generalized in this manner. The difficulty lies in a proof of the existence of so-called *Nagumo point.*[2]

[2]This fundamental lemma on parabolic differential inequalities was proved and the notions of a so-called "Nagumo point" and the "Nagumo method" of getting inequalities for solutions of parabolic inequalities were introduced by Nagumo in 1939 [**6**]. We remark (cf. Walter [**17**, p. 4699], [**18**, pp. 451–452]) that this work of Nagumo, being written in Japanese, had remained unknown until a follow-up paper by Nagumo and Simode appeared in 1951 [**7**]. This lemma was rediscovered by Westphal in 1949 [**19**] (see Redheffer and Walter [**12**, p. 285]). A similar result was obtained by Max Müller in 1927 [**5**] (cp. [**9**]). Therefore, this lemma is sometimes called the Max Müller–Nagumo–Westphal Lemma. Redheffer (1963) observed in [**11**] that the Nagumo procedure applies to equations containing functionals, provided these functionals are monotone and of Volterra type.

In the theorems on strong differential-functional inequalities, the fundamental assumption is some condition on monotonicity in y and s only, and in the theorems on weak differential inequalities, the fundamental assumptions are this monotonicity condition and the Lipschitz condition with respect to y and s. These theorems were proved (see [**3**], [**14**], [**15**]) in the Banach space $C_{\mathcal{N}}(\overline{D})$. It has unfortunately turned out, though, that no theorem on strong differential inequalities for infinite systems in the space $C_{\mathcal{N}}(\overline{D})$ can be proved. Precisely, the theorem fails to hold if the considered functions $u, v \in C_{\mathcal{N}}(\overline{D})$. A problem arises at the very beginning of the proof, when one attempts to use the Nagumo point method. Namely, if we deal with an infinitely countable system, that is if u, v are infinite sequences $u = \{u^k\}_{k\in\mathcal{N}}$ and $v = \{v^k\}_{k\in\mathcal{N}}$, then it may happen that

$$u^k(t,x) < v^k(t,x) \text{ for } (t,x) \in D,\ 0 < t < \tilde{t} = \frac{1}{k},\ k \in \mathcal{N}.$$

Then, though, the intersection of the intervals $(0, \frac{1}{k})$ for $k \in \mathcal{N}$ is empty and the inequality $u(t,x) < v(t,x)$ for $(t,x) \in D$, $0 < t < t^*$ is not true; hence the proof fails. Therefore, we introduce the space $\mathcal{C}_{\mathcal{N}}(\overline{D}, l^\infty)$ of all continuous functions from $\overline{D}$ into l^∞ equipped with the supremum norm from the space l^∞ and in this space we prove the theorem on strong differential inequalities. The gist of the idea thus consists in assuming the continuity of the functions considered and not only of the coordinates of these functions.

Main results are formulated in Theorem 1 and again in Theorem 2 with the boundary inequalities corresponding to a certain generalization of the first and the third classical Fourier problems.

2. Notations, definitions and assumptions. A function $w \in \mathcal{C}_{\mathcal{N}}(\overline{D}, l^\infty)$ will be called *regular* in domain $\overline{D}$ if functions w^i, $i \in \mathcal{N}$, have continuous derivatives $\partial_t w^i$, $\partial^2_{x_j x_k} w^i$ for $j,k = 1,\dots,m$, in D, i.e., $w \in \mathcal{C}^{reg}_{\mathcal{N}}(\overline{D}, l^\infty) := \mathcal{C}_{\mathcal{N}}(\overline{D}, l^\infty) \cap C^{1,2}_{\mathcal{N}}(D)$.

We say that a function $u \in C^*_{\mathcal{N}}(D)$ if it is continuous and possesses first derivatives $\partial_{x_j} u^i$ for $j = 1,\dots,m$, $i \in \mathcal{N}$ in D.

($\mathcal{O}$): In the space $\mathcal{C}_{\mathcal{N}}(\overline{D}, l^\infty)$, we introduce the order relations "$\leq$" and "$<$" defined by

$$u \leq v \iff u^i(t,x) \leq v^i(t,x) \text{ for arbitrary } (t,x) \in \overline{D} \text{ and all } i \in N,$$

$$u < v \iff u^i(t,x) < v^i(t,x) \text{ for arbitrary } (t,x) \in \overline{D} \text{ and all } i \in N.$$

We introduce the following notation: for every fixed t, $0 < t \leq T$ and for $s, \tilde{s} \in \mathcal{C}_N(\overline{D}, l^\infty)$

$$s \overset{t}{\leq} \tilde{s} \iff s^i(\bar{t},x) \leq \tilde{s}^i(\bar{t},x) \text{ for any } 0 < \bar{t} \leq t,\ (\bar{t},x) \in \overline{D} \text{ and all } i \in \mathcal{N}.$$

If $t \geq T$, then we simply write $s \leq \tilde{s}$ instead of $s \overset{t}{\leq} \tilde{s}$.

2.1. ***Property*** (**P**). We shall say that a set D (possibly unbounded) in the time-space $(t, x) = (t, x_1, \ldots, x_n)$ has property (P) if:

1^0 the projection of the interior of D on the t-axis is an interval $(0, T)$, where $T < \infty$;

2^0 for every $(\tilde{t}, \tilde{x}) \in D$, there is a positive number r such that the lower half neighbourhood is contained in D, i.e.,

$$\{(t, x) : (t - \tilde{t})^2 + \sum_{j=1}^{m} (x_i - \tilde{x}_i)^2 < r^2,\ t \leq \tilde{t}\} \subset D.$$

Let D be an open domain having property (P). We denote by σ the part of the boundary of D situated in the open zone $0 < t < T$, $S_0 := \{(t, x) \in D : t = 0\}$ be nonempty, $\Gamma := S_0 \cup \sigma$ be the parabolic boundary of the domain D and $\overline{D} := D \cup \Gamma$.

2.2. ***Definitions of the notion of parabolicity in Besala's, Szarski's and classical sense.*** (cp. [**1**], [**13**], [**14**])

According to the definition introduced by Besala, given a function $u = u(t, x)$ of class C^1 in the domain D, the function $f^i(t, x, y, p, q, s)$ is said to be *uniformly elliptic with respect to u in D* (we say: in *Besala's sense*) if there is a constant $\kappa > 0$ such that for all $(t, x) \in D$ and any two real square symmetric matrices $q, \tilde{q} \in \mathcal{M}_{m \times m}$, $q = (q_{jk})$ and $\tilde{q} = (\tilde{q}_{jk})$, $j, k = 1, \ldots, m$, there is

$$\tilde{q} \geq q \Rightarrow f^i(t, x, u(t, x), \partial_x u^i(t, x), \tilde{q}, u) - f^i(t, x, u(t, x), \partial_x u^i(t, x), q, u) \geq$$

$$\geq \kappa \sum_{j=1}^{m} (\tilde{q}_{jj} - q_{jj}), \tag{2}$$

where the inequality $\tilde{q} \geq q$ means that

$$\sum_{j,k=1}^{m} (\tilde{q}_{jk} - q_{jk}) \xi_j \xi_k \geq 0 \text{ for all } \xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m.$$

A system of differential equations (or inequalities)

$$\partial_t z^i(t, x) = f^i(t, x, z(t, x), \partial_x z^i(t, x), \partial^2_{xx} z^i(t, x), z),\ i \in \mathcal{N}, \tag{3}$$

is called *uniformly parabolic with respect to the function* $u \in C^*_{\mathcal{N}}(D)$ *in* D if every f^i is uniformly elliptic with respect to this function.

The solution z of system (3) is called a *regular parabolic solution of* (3) *in* $\overline{D}$ if z is a regular solution of (3) in $\overline{D}$ and if f^i, for $i \in \mathcal{N}$, are elliptic functions with respect to this solution in D.

In particular, if $\kappa = 0$ in (2), then f^i is called *parabolic with respect to* $u = u(t, x)$ *in* D; precisely: *parabolic in Szarski's sense.*

On the other hand, it can easily be shown that if f^i is of class C^1 with respect to $q = (q_{jk})$ then the condition

$$\sum_{j,k=1}^{m} \partial_{q_{jk}} f^i \xi_j \xi_k \geq \kappa \sum_{j=1}^{m} \xi_j^2 \text{ for all } \xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m,$$

where a constant $\kappa > 0$, usually called the *uniform parabolicity*, implies the uniform parabolicity in the sense of the above definition.

Let us consider the semilinear infinite system of strong parabolic inequalities of the following form

$$\mathcal{L}^i[z^i](t,x) := \partial_t z^i(t,x) - \sum_{j,k=1}^{m} a^i_{jk}(t,x) \partial^2_{x_j x_k} z^i(t,x) < \tag{4}$$

$$< f^i(t,x,z(t,x), \partial_x z^i(t,x), z), \ i \in \mathcal{N}.$$

If the operators $\mathcal{L}^i$ $(i \in \mathcal{N})$ are *uniformly parabolic in* D in the classical sense, i.e., there exists a constant $\nu > 0$ such that

(5)

$$\sum_{j,k=1}^{m} a^i_{jk}(t,x) \xi_j \xi_k \geq \nu \sum_{j=1}^{m} \xi_j^2 \text{ for all } \xi = (\xi_1, \ldots \xi_m) \in \mathbb{R}^m, \ (t,x) \in \overline{D}, \ i \in \mathcal{N},$$

then these operators are uniformly parabolic in D with respect to any function $u \in C^*_{\mathcal{N}}(\overline{D})$.

2.3. ***Monotonicity conditions and Volterra condition***. Let the real functions $f^i(t,x,y,p,q,s)$, $i \in \mathcal{N}$, be defined for $(t,x,y,p,q,s) \in \mathcal{K}$, where

$$\mathcal{K} := D \times l^\infty \times \mathbb{R}^m \times \mathcal{M}_{m\times m} \times \mathcal{C}_{\mathcal{N}}(\overline{D}, l^\infty)$$

and $\mathcal{M}_{m\times m}$ denote the set of all real square symmetric matrices $q = (q_{jk})$, $j,k = 1, \ldots, m$.

We say that the functions f^i, $i \in \mathcal{N}$, satisfy:

($\mathbf{W}_+$): Condition $(W+)$ with respect to y if for every fixed index i the function f^i is nondecreasing with respect to the arguments y^j for all $j \neq i$, $j \in \mathcal{N}$.

($\mathbf{W}$) : Condition (W) with respect to s if they are nondecreasing with respect to s.

($\mathbf{V}$) : The functions f^i, $i \in \mathcal{N}$, are *Volterra functionals* (or: the functions f^i, $i \in \mathcal{N}$, satisfy the so-called *Volterra condition*) with respect to the argument s if

$$f^i(t,x,y,p,q,s) = f^i(t,x,y,p,q,\tilde{s})$$

for all $(t, x, y, p, q) \in \overline{D} \times l^{\infty} \times \mathbb{R}^m \times \mathcal{M}_{m\times m}$, $i \in \mathcal{N}$ is true for all functions $s, \tilde{s} \in \mathcal{C}_{\mathcal{N}}(\overline{D}, l^{\infty})$ which satisfy the equality

$$s(\bar{t}, x) = \tilde{s}(\bar{t}, x) \text{ for all } 0 < \bar{t} \le t,\ (\bar{t}, x) \in \overline{D}.$$

This condition means that the value of the functions $f^i(t, x, y, p, q, s)$, $i \in \mathcal{N}$, depends on the past of the function s.

(**M**) : The functionals f^i, $i \in \mathcal{N}$, are *monotone nondecreasing* (or: the functions f^i, $i \in \mathcal{N}$, satisfy the *monotonicity condition* (M)) with respect to the argument s, if for every fixed t, $0 < t \le T$ and for all functions $s, \tilde{s} \in \mathcal{C}_{\mathcal{N}}(\overline{D}, l^{\infty})$

$$s \overset{t}{\le} \tilde{s} \Rightarrow f^i(t, x, y, p, q, r, s) \le f^i(t, x, y, p, q, \tilde{s}),\ i \in \mathcal{N}$$

(cf. [**8**, p. 167] and [**15**, p. 478]).

Remark 1. It is easy to see (cf. [**2**, p. 144]) that the functions f^i, $i \in \mathcal{N}$, satisfy Volterra condition (V) and condition (W) with respect to s if and only if they satisfy condition (M).

3. Main result.

Theorem 1 (on strong inequalities for infinite systems).

Let real functions $f^i(t, x, y, p, q, s)$, $i \in \mathcal{N}$, be defined for $(t, x, y, p, q, s) \in \mathcal{K}$ and the domain D has the property (P).

Assume that

1^0 *the functions f^i, $i \in \mathcal{N}$, satisfy condition $(W+)$ with respect to y, condition (W) with respect to s and condition (V) in the set $\mathcal{K}$;*

2^0 *the functions $u, v \in C^{reg}_{\mathcal{N}}(\overline{D}, l^{\infty})$;*

3^0 *every function f^i is elliptic with respect to the function u in Szarski's sense;*

4^0 *the infinite systems of inequalities*

$$\partial_t u^i(t, x) \le f^i(t, x, u(t, x), \partial_x u^i(t, x), \partial^2_{xx} u^i(t, x), u),\ (<), \tag{6}$$

$$\partial_t v^i(t, x) > f^i(t, x, v(t, x), \partial_x v^i(t, x), \partial^2_{xx} v^i(t, x), v)\ i \in \mathcal{N},\ (\ge), \tag{7}$$

hold for $(t, x) \in D$.

5^0 *Suppose finally that the initial inequality*

$$u(0, x) < v(0, x)\ for\ x \in S_0 \tag{8}$$

and the boundary inequality

$$u(t, x) < v(t, x)\ for\ (t, x) \in \sigma \tag{9}$$

hold true.

If one of the inequalities (6) *or* (7) *is strict, then under the above assumptions there is*

$$u(t,x) < v(t,x) \ for \ (t,x) \in \overline{D}. \tag{10}$$

Proof. Notice that the proof of our theorem is simple and similar to the proofs of Szarski's theorems on strong inequalities given in papers [**13**, pp. 135–136, 190–193] and [**14**, pp. 199–201] but it is based on the assumption that the functions $u, v \in \mathcal{C}_{\mathcal{N}}(\overline{D}, l^{\infty})$.

Since the set of points $(0,x)$ such that $x \in S_0$ is compact, there is, by (8) and by the continuity of functions u, $v \in \mathcal{C}_{\mathcal{N}}(\overline{D}, l^{\infty})$, a time $\tilde{t}$, $0 < \tilde{t} < T$, such that (10) holds true in the intersection of $\overline{D}$ with the zone $0 \le t < \tilde{t}$. Let t^* be the least upper bound for such $\tilde{t}$, or $+\infty$ if there is no such bound. The assertion of theorem is obviously equivalent to the equality $t^* = T$.

Suppose for the contrary that the conclusion is not true, i.e., $t^* < T$. Then by 2^0 there is $u - v \in \mathcal{C}_{\mathcal{N}}(\overline{D}, l^{\infty})$ and by the continuity of $u - v$ there would be

$$u(t,x) \le v(t,x) \text{ for } (t,x) \in \overline{D},\ 0 < t \le t^*. \tag{11}$$

This inequality means that

$$u \overset{t^*}{\le} v. \tag{12}$$

The domain D has property (P) and strong initial and boundary inequalities (8), (9) hold; therefore, by the definition of t^* and the definition of order "$<$", there exist an index $i^* \in \mathcal{N}$ and a point $x^* \in S_{t^*}$ (a Nagumo point) such that

$$u^{i^*}(t^*, x^*) = v^{i^*}(t^*, x^*) \tag{13}$$

and (t^*, x^*) is an interior point of D.

From (11) and (13) it follows that the function

$$V(x) := u^{i^*}(t^*, x) - v^{i^*}(t^*, x)$$

as the function in $x = (x_1, \ldots, x_m)$ attains its maximum in S_{t^*} at $x = x^*$, i.e.,

$$\max_{x \in S_{t^*}} V(x) = \max_{x \in S_{t^*}} [u^{i^*}(t^*, x) - v^{i^*}(t^*, x)] = u^{i^*}(t^*, x^*) - v^{i^*}(t^*, x^*) = 0. \tag{14}$$

Since S_{t^*} is open and the function $V(x)$ is of class C^2 in S_{t^*} and attains its maximum at an interior point x^* of S_{t^*}, there is

$$\partial_x u^{i^*}(t^*, x^*) = \partial_x v^{i^*}(t^*, x^*) \tag{15}$$

and

$$\partial^2_{xx} u^{i^*}(t^*, x^*) \le \partial^2_{xx} v^{i^*}(t^*, x^*). \tag{16}$$

Hence,

$$(17)\quad \sum_{j,k=1}^{m}[\partial^2_{x_jx_k}u^{i^*}(t^*,x^*)-\partial^2_{x_jx_k}v^{i^*}(t^*,x^*)]\xi_j\xi_k\le 0 \text{ for all } \xi=(\xi_1,\dots,\xi_m).$$

From assumptions 1^0–4^0 and (11)–(13), (15)–(17), conditions (V), (W) and by Remark 1 it follows that

$$(18)\quad \begin{aligned}\partial_t u^{i^*}(t^*,x^*) &\le f^{i^*}(t^*,x^*,u(t^*,x^*),\partial_x u^{i^*}(t^*,x^*),\partial^2_{xx}u^{i^*}(t^*,x^*),u)\le\\ &\le f^{i^*}(t^*,x^*,u(t^*,x^*),\partial_x u^{i^*}(t^*,x^*),\partial^2_{xx}v^{i^*}(t^*,x^*),v)\le\\ &\le f^{i^*}(t^*,x^*,v(t^*,x^*),\partial_x v^{i^*}(t^*,x^*),\partial^2_{xx}v^{i^*}(t^*,x^*),v)<\partial_t v^{i^*}(t^*,x^*).\end{aligned}$$

On the other hand, the function

$$W(t):=u^{i^*}(t,x^*)-v^{i^*}(t,x^*),$$

as the function in one variable t, defined for t in some interval $(0,t^*)$ attains, by (11) and (13), its maximum at the right-hand extremity t^* of the interval $[0,t^*]$. Hence, there is

$$(19)\quad \partial_t W(t^*)=\partial_t u^{i^*}(t^*,x^*)-\partial_t v^{i^*}(t^*,x^*)\ge 0,$$

which contradicts (18). This proves the theorem. □

Now we formulate a theorem with another, more general boundary inequality, to some exten corresponding to the first and the third classical Fourier problems. Let the functions

$$(20)\quad \alpha^i(t,x)\ge 0,\ \ \beta^i(t,x)>0 \text{ for } i\in\mathcal{N},$$

be defined on σ and suppose that for every point $(t,x)\in\Sigma_{\alpha^i}$ (by Σ_{α^i} we denote the subset of σ on which $\alpha^i(t,x)\ne 0$) the direction $l^i=l^i(t,x)$ orthogonal to the t-axis and penetrating into the closed domain D is given. We will assume that

$$(21)\quad \frac{d[u^i-v^i]}{dl^i}(t,x)=0 \text{ for } (t,x)\in\sigma-\Sigma_{\alpha^i},\ i\in\mathcal{N}.$$

The following theorem is true.

Theorem 2. *Under the all assumptions of Theorem 1, with the exception of inequality* (9)*, which is replaced with the following more general inequality*

$$(22)\quad \alpha(t,x)\frac{d[u-v]}{dl}(t,x)-\beta(t,x)[u(t,x)-v(t,x)]>0 \ for\ (t,x)\in\sigma,$$

where the functions α, β and the direction l satisfy (20) *and* (21)*, inequality* (10) *is true, i.e., there is*

$$(23)\quad u(t,x)<v(t,x) \ for\ (t,x)\in\overline{D}.$$

The proof is quite similar to that of Theorem 1. □

Remark 2. In the particular case of $\alpha = 0$ and $\beta = 1$, boundary inequality (23) is reduced to inequality (9). $\square$

Remark 3. Theorems 1 and 2 hold true for an arbitrary infinite system of inequalities (1) (the method used in the proof does not need the assumption that the system is countable). In this case we introduce the space $\mathcal{B}(\mathcal{S})$, where $\mathcal{S}$ is an arbitrary set of indices, and the space $\mathcal{C}_{\mathcal{S}}(\overline{D}, \mathcal{B}(\mathcal{S}))$ as the space of all continuous mappings from $\overline{D}$ into $\mathcal{B}(\mathcal{S})$, equipped with the supremum norm from the space $\mathcal{B}(\mathcal{S})$. $\square$

Remark 4. Theorems 1 and 2 hold true in unbounded domains (cp. [**3**],[**14**],[**15**]) for functions $h = h(t,x)$ which fullfil the growth condition $|h(t,x)| \le M \exp(K|x|^2)$ in D. $\square$

Remark 5. In a similar manner, theorems on strong differential inequalities of other type of infinite systems in the space $\mathcal{C}_{\mathcal{S}}(\overline{D}, \mathcal{B}(\mathcal{S}))$ can be proved. $\square$

References

1. Besala P., *An extension of the strong maximum principle for parabolic equations*, Bull. Acad. Polon. Sci., Série sci. math., astr. et phys., **19**, **11** (1971), 1003–1006.
2. Brzychczy S., Janus J., *Monotone iterative method for nonlinear hyperbolic differential–functional equations*, Univ. Iagel. Acta. Math., **38** (2000), 141–152.
3. Brzychczy S., *Existence and uniqueness of solutions of infinite systems of semilinear parabolic differential–functional equations in arbitrary domains in ordered Banach spaces*, Math. Comp. Modelling, **36** (2002), 1183–1192.
4. Lakshmikantham V., Leela S., *Differential and Integral Inequalities*, Vol. II, Academic Press, New York, 1968.
5. Müller M., *Über die Eindeutigkeit der Integrale eines Systems gewöhnlicher Differentialgleichungen und die Konvergenz einer Gattung von Verfahren zur Approximation dieser Integrale*, Sitz.-ber. Heidelberg, Akad. Wiss., Math.-Naturw. Kl., 9. Abh. 1927.
6. Nagumo M., *Note in "Kansũ-Hõteisiki"* [Japanese], No. 15 (1939).
7. Nagumo M., Simoda S., *Note sur l'inéqualité différentielle concernant les équations du type parabolique*, Proc. Japan. Acad., **27** (1951), 536–539.
8. Nickel K., *Fehlerabschätzungs- und Eindeutigkeitssätze für Integro–Differentialgleichungen*, Arch. Ration. Mech. Anal., **8** (1961), 159–180.
9. Nickel K., *Das Lemma von Max Müller–Nagumo–Westphal für stark gekoppelte Systeme parabolischer Funktional-Differentialgleichungen*, Math. Z., **161** (1978), 221–234.
10. Nickel K., *Bounds for the set of solutions of functional–differential equations*, Ann. Polon. Math., **52** (1983), 241–257.
11. Redheffer R., *Die Collatzsche Monotonie bei Aufangswertproblem*, Arch. Ration. Mech. Anal., **14** (1963), 196–212.
12. Redheffer R., Walter W., *Stability of the null solution of parabolic functional inequalities*, Trans. Amer. Math. Soc., **262**, **1** (1980), 285–302.
13. Szarski J., *Differential Inequalities*, Monografie Matematyczne, Vol. 43, PWN, Polish Sci. Publ., Warszawa, 1965.

14. Szarski J., *Strong maximum principle for nonlinear parabolic differential–functional inequalities in arbitrary domains*, Ann. Polon. Math., **31** (1975), 197–203.
15. Szarski J., *Infinite systems of parabolic differential–functional inequalities*, Bull. Acad. Pol. Sci., Série sci. math., **28**, 9–10 (1980), 477–481.
16. Walter W., *Differential and Integral Inequalities*, Berlin, Springer Verlag, 1970.
17. Walter W., *Differential inequalities and maximum principles: theory, new methods and applications*, Nonlinear Anal., **30**, **8** (1997), 4697–4711.
18. Walter W., *Nonlinear parabolic differential equations and inequalities*, Discrete Contin. Dynam. Systems, **8**, **2** (2002), 451–468.
19. Westphal H., *Zur Abschätzung der Lösungen nichtlinearer parabolischer Differentialgleichungen*, Math. Z., **51** (1949), 690–695.

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