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ON MINIMAL AND INVARIANT SETS IN SEMIDYNAMICAL SYSTEMS

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Abstract. We investigate the structure of non-trivial, weakly minimal and negatively strongly invariant sets in a semidynamical system on a locally compact metric space. For a negative prolongational limit set for a semidynamical system we present two different definition which appear to be equivalent. Certain properties of these sets are discussed.

1. Introduction. In a dynamical system a movement is defined for positive and negative values of time. In a semidynamical system the situation is different. Here, the movement is defined only for positive values of time. This leads to a main difference between dynamical systems and semidynamical systems.

We can ask about "the past" of a given point x. It depends on a negative semisolution σ through x. It is possible that such a negative semisolution σ through x does not exist, there can exist only one semisolution or more, even infinitely many. In a dynamical system we define positive and negative limit sets $L^+(x)$ and $L^-(x)$. In a semidynamical system a negative limit set $L^-_\sigma(x)$ depends on a negative semisolution σ through x. In a dynamical system as well as in a semidynamical system we define a positive prolongational limit set $J^+(x)$ in the same way. In the case of a negative prolongational limit set for a semidynamical system the situation is more complicated. We may state such a definition in different ways, which in each case is the generalization of the analogous definition for dynamical systems. Here we define two types of such sets and prove that they give the same sets. We also show certain properties of the negative prolongational limit set.

In the second part of this paper we define some kinds of invariance and minimality, as in semidynamical systems this may be done in different ways.

Next we investigate non-trivial, weakly minimal and negatively strongly invariant sets in a semidynamical system on a locally compact metric space. S.Kono ([**3**]) characterized a non-trivial, non-compact minimal set in a dynamical system on a locally compact metric space. He fully characterized the types of trajectories which can be contained in such set.

In this paper a similar problem is studied for semidynamical systems. We prove that a non-trivial, weakly minimal and negatively strongly invariant sets in a semidynamical system on a locally compact metric space consists of infinitely many trajectories similarly as in the case of a dynamical system. However, the difference appears in the types of trajectories which may be contained in such set. The main theorem of this paper states that this set can contain four types of trajectories. Also, it is shown that certain trajectories cannot be contained in such set.

We also give some examples of non-trivial, weakly minimal sets, which may appear in semidynamical systems, describing what phenomena may occur here.

2. Preliminaries. A *semidynamical system* on a metric space X with metric d is a triplet $(X, \mathbf{R}^+, \pi)$ where $\pi : X \times \mathbf{R}^+ \to X$ is a continuous mapping such that:

(i) $\pi(x, 0) = x$ for all $x \in X$

(ii) $\pi(\pi(x,t), s) = \pi(x, t+s)$ for all $x \in X$ and all $s, t \in \mathbf{R}^+$.

Replacing $\mathbf{R}^+$ by $\mathbf{R}$ we get a definition of dynamical system.

The *positive trajectory of* $x \in X$ is defined as $\{\pi(x,t)\colon\ t \in \mathbf{R}^+\}$ and denoted by $\pi^+(x)$.

In a dynamical system the *negative trajectory of* $x \in X$ is defined as $\{\pi(x,t) : t \in \mathbf{R}^-\}$ and denoted by $\pi^-(x)$, the *trajectory of* $x \in X$ is defined as $\pi(x) = \pi^-(x) \cup \pi^+(x)$.

A point $x \in X$ is called:

– *start point* if $x \neq \pi(y,t)$ for any $y \in X$ and any $t > 0$;
– *stationary point* if $\pi(x,t) = x$ for any $t \geq 0$;
– *periodic point* if there exists a $t > 0$ such that $\pi(x,t) = x$ and x is not a stationary point.

A function $\sigma : I \to X$ where I is a non-empty interval in $\mathbf{R}$ is called a *solution* if $\pi(\sigma(t), s) = \sigma(t+s)$ whenever $t \in I$, $t+s \in I$ and $s \in \mathbf{R}^+$. If $0 \in I$ and $\sigma(0) = x$ then a solution is called a *solution through x*. If a solution σ is maximal (relative to the property of being a solution, with respect to inclusion), then its image is called a *trajectory through x*. Note that in such case $[0, \infty)$ is contained in the domain of a solution.

Let X be locally compact and π have no start points. Then we may assume without loss of generality that any solution can be prolongated so as to have $(-\infty, 0]$ contained in its domain. This is because we can transform the system by a suitable isomorphism which does not change trajectories, but only changes the speed of movement along trajectories (see [**4**]).

In this paper by a *solution* (through x) we mean a solution with a domain equal to $\mathbf{R}$. By a *positive (negative) semisolution* through x we mean a suitable solution defined on $[0,\infty)$ ($(-\infty,0]$); their images are called *positive (negative) semitrajectories*. Note that for any x there is precisely one positive semisolution through x, however there may exist even infinitively many negative semisolutions through x. Throughout this paper by a *trajectory* $\pi_\sigma(x)$ we mean $\sigma((-\infty,0])\cup\pi^+(x)$ where σ is a negative semisolution through x. By $B(x,\varepsilon)$ we denote an open ball of radius ε centred in x.

By a *positive limit set* we mean

$$L^+(x)=\{y\in X:\ \text{there exists a sequence } \{t_n\} \text{ in } \mathbf{R} \\ \text{with } t_n\to+\infty \text{ and } \pi(x,t_n)\to y\}.$$

By a *negative limit set* $L^-_\sigma(x)$ we define $\{y\in X$: there exists a sequence $\{t_n\}$ in $\mathbf{R}$ with $t_n\to-\infty$ and $\sigma(t_n)\to y\}$ where σ is a negative semisolution through x.

A set $M\subset X$ is called:

- *positively invariant* if $\pi(x,t)\in M$ for any $x\in M$ and any $t\in\mathbf{R}^+$;
- *negatively strongly invariant* if $\sigma((-\infty,0])\subset M$ for any $x\in M$ and any negative semisolution σ through x;
- *negatively weakly invariant* if for every $x\in M$ there exists a negative semisolution σ through x such that $\sigma((-\infty,0])\subset M$.

A set $M\subset X$ is called *strongly (weakly) invariant* if it is positively invariant and negatively strongly (weakly) invariant.

A set $M\subset X$ is called *strongly (weakly) minimal* if it is non-empty, closed, strongly (weakly) invariant and no proper subset of M has all these properties.

It is easy to see that for any x the positive trajectory $\pi^+(x)$ is positively invariant, the set $\sigma((-\infty,0])$ for any solution σ through x is negatively weakly invariant and $\pi_\sigma(x)$ is weakly invariant.

A strongly (weakly) minimal set is called *trivial* if it consists only of one trajectory. A strongly (weakly) minimal set which is not trivial is called *non-trivial.*

A point $x\in X$ is called a *point of negative unicity* if $F(x,t)$ contains at most one element for any $t\in\mathbf{R}$, where

$$F(x,t)=\{y\in X:\pi(y,t)=x\}.$$

A point $x\in X$ is called a *point of negative unicity in M* if $F_M(x,t)$ contains at most one element for any $t\in\mathbf{R}$, where

$$F_M(x,t)=\{y\in M:\pi(y,t)=x\}.$$

A set $M \subset X$ is called a *set of negative unicity* if every point $x \in M$ is a point of negative unicity in set M.

3. Negative prolongational limit set. In a semidynamical system the movement is defined only for positive values of time t. However, we may ask about "the past" of a given point x.

We have

LEMMA 3.1. ([**1**, 5.15]) *A negative limit set $L^-_\sigma(x)$ is closed, positively invariant and if X is locally compact, then it is weakly invariant and contains no start points.*

The *positive prolongational limit set* is $J^+(x) = \{y \in X$: there are a sequence $\{x_n\}$ in X and a sequence $\{t_n\}$ in $\mathbf{R}^+$ such that $x_n \to x$, $t_n \to +\infty$ and $\pi(x_n, t_n) \to y\}$.

This is the same definition as in the case of a dynamical system. All properties of these sets in semidynamical systems are the same as in dynamical systems. The negative prolongational limit set must be defined in another way.

DEFINITION 3.2. We define:

$$\begin{aligned} j^-(x) = \{y \in X :\ & \text{there are a sequence } \{x_n\} \text{ in } X \text{ and a sequence } \{t_n\} \\ & \text{in } \mathbf{R}^- \text{ such that } x_n \to x, t_n \to -\infty \text{ and for each } x_n \text{ there exists} \\ & \text{a semisolution } \sigma_n \text{ through } x_n \text{ such that } \sigma_n(t_n) \to y\} \end{aligned}$$

$$\begin{aligned} J^-(x) = \{y \in X :\ & \text{there are a sequence } \{x_n\} \text{ in } X \text{ and a sequence } \{t_n\} \\ & \text{in } \mathbf{R}^- \text{ and there exists a semisolution } \sigma_x \text{ through } x \text{ and } t \le 0 \\ & \text{such that } x_n \to \sigma_x(t), t_n \to -\infty \text{ and for each } x_n \text{ there exists} \\ & \text{a semisolution } \sigma_n \text{ through } x_n \text{ such that } \sigma_n(t_n) \to y\}. \end{aligned}$$

It will proved that the sets $j^-(x)$ and $J^-(x)$ are equal. It is clear that $j^-(x) \subset J^-(x)$ and $L^-_\sigma(x) \subset j^-(x)$ for any semisolution σ through x. This is an immediate consequence of the definitions.

THEOREM 3.3. *Let x, $y \in X$. If $x \in J^-(y)$ then $y \in J^+(x)$.*

PROOF. Let $x \in J^-(y)$. It means that there are a sequence $\{y_n\}$ in X and a sequence $\{t_n\}$ in $\mathbf{R}^-$ and there exist a solution σ_y through y and $t \le 0$ such that $y_n \to \sigma_y(t)$, $t_n \to -\infty$ and for each y_n there exists a semisolution σ_n through y_n such that $\sigma_n(t_n) \to x$. Set $\sigma_n(t_n) = x_n$ and $\tau_n = -t_n - t$. Then $x_n \to x$, $\tau_n \to +\infty$ and

$$\begin{aligned} \pi(x_n, \tau_n) &= \pi(\sigma_n(t_n), -t_n - t) = \pi(\pi(\sigma_n(t_n), -t_n), -t) = \pi(\sigma_n(t_n - t_n), -t) \\ &= \pi(\sigma_n(0), -t) = \pi(y_n, -t) \to \pi(\sigma_y(t), -t) = \sigma_y(t - t) = \sigma_y(0) = y. \end{aligned}$$

Consequently, $y \in J^+(x)$. □

THEOREM 3.4. *Let x, $y \in X$. If $y \in J^+(x)$ then $x \in j^-(y)$.*

PROOF. Let $y \in J^+(x)$. It means that there are a sequence $\{x_n\}$ in X and a sequence $\{t_n\}$ in $\mathbf{R}^+$ such that $x_n \to x$, $t_n \to +\infty$ and $\pi(x_n, t_n) \to y$. Set $\pi(x_n, t_n) = y_n$ and $\tau_n = -t_n$. Then $y_n \to y$, $\tau_n \to -\infty$. We claim that for each y_n there exists a semisolution σ_n through y_n such that $\sigma_n(\tau_n) \to x$. Since $y_n = \pi(x_n, t_n)$, then there exists a solution through y_n which contains y_n and x_n in its image. We denote this solution by σ_n. Hence $\sigma_n(\tau_n) = x_n$, and we see that $\sigma_n(\tau_n) \to x$. Consequently, $x \in j^-(y)$. $\square$

THEOREM 3.5. *Let $x \in X$. Then $J^-(x) = j^-(x)$.*

PROOF. According to Theorems 3.3 and 3.4 and the property $j^-(x) \subset J^-(x)$ we have: $x \in J^-(y) \Leftrightarrow y \in J^+(x) \Leftrightarrow x \in j^-(y)$. $\square$

THEOREM 3.6. *For any $x \in X$ the set $J^-(x)$ is closed and positively invariant.*

PROOF. We first prove that $j^-(x)$ is closed. Let $\{y_k\}$ be a sequence in $j^-(x)$ with $y_k \to y$. For each integer k, there are sequences $\{x_n^k\}$ in X and $\{t_n^k\}$ in $\mathbf{R}^-$ with $x_n^k \to x$, $t_n^k \to -\infty$ and such that for each x_n^k there exists a semisolution σ_n^k through x_n^k with $\sigma_n^k(t_n^k) \to y_k$. We may assume (taking subsequences, if necessary) that $t_n^k < -k$, $d(x_n^k, x) \le \frac{1}{k}$ and $d(\sigma_n^k(t_n^k), y_k) \le \frac{1}{k}$ for $n \ge k$. Now consider the sequences $\{x_n^n\}$, $\{t_n^n\}$. Since $d(x_n^n, x) \le \frac{1}{n}$ and $t_n^n < -n$, we have $x_n^n \to x$, $t_n^n \to -\infty$ for $n \to +\infty$. For every x_n^n we choose a semisolution σ_n^n through x_n^n such that $d(\sigma_n^n(t_n^n), y_n) \le \frac{1}{n}$ (such σ_n^n exists because $y_n \in j^-(x)$). To see that $\sigma_n^n(t_n^n) \to y$ note that $d(\sigma_n^n(t_n^n), y) \le d(\sigma_n^n(t_n^n), y_n) + d(y_n, y) \le \frac{1}{n} + d(y_n, y)$. Thus $y \in j^-(x)$ and $j^-(x)$ is closed.

To see that $j^-(x)$ is positively invariant, let $y \in j^-(x)$ and $t \in \mathbf{R}^+$. There are a sequence $\{x_n\}$ in X and a sequence $\{t_n\}$ in $\mathbf{R}^-$ such that $x_n \to x$, $t_n \to -\infty$ and for each x_n there exists a semisolution σ_n through x_n with $\sigma_n(t_n) \to y$. Now consider the sequence $\{t_n + t\}$. Clearly $t_n + t \to -\infty$ and $\sigma_n(t_n + t) = \pi(\sigma_n(t_n), t) \to \pi(y, t)$. Since $x_n \to x$ we have $\pi(y, t) \in j^-(x)$.
In view of Theorem 3.5 $J^-(x)$ is closed and positively invariant. $\square$

In a dynamical system we know that a point $x \in X$ is said to be *non-wandering* if every neighbourhood U of x is self positively recursive (i.e. for each $T \in \mathbf{R}$ there are a $t > T$ and a $y \in U$ such that $\pi(y, t) \in U$). We know also ([**2**, 2.12]) that the following conditions are equivalent:

- *a point x is non-wandering,*
- *$x \in J^+(x)$,*
- *every neighbourhood of x is self negatively recursive,*
- *$x \in J^-(x)$.*

In a semidynamical system the definition of a self positively recursive neighbourhood U of x is introduced in the same way as in a dynamical system. According to Theorems 3.3 and 3.4, also in a semidynamical system the following conditions are equivalent:

– *every neighbourhood of x is self positively recursive,*
– $x \in J^+(x)$,
– $x \in J^-(x)$.

Now we will define negatively weakly recursiveness.

DEFINITION 3.7. A set $A \subset X$ is said to be self negatively weakly recursive if for each $T \in \mathbf{R}$ there are a $t < T$, an $x \in A$ and a semisolution σ_x through x such that $\sigma_x(t) \in A$.

THEOREM 3.8. *For any $x \in X$, the following conditions are equivalent:*

(i) *every neighbourhood U of x is self negatively weakly recursive,*
(ii) $x \in j^-(x)$.

PROOF. Assume (i). Consider a sequence $\{\varepsilon_n\}$, $0 < \varepsilon_n$, $\varepsilon_n \to 0$, and a sequence $\{t_n\}$ in $\mathbf{R}$ with $t_n \to -\infty$. Since each $B(x, \varepsilon_n)$ is self negatively weakly recursive, we can find an $x_n \in B(x, \varepsilon_n)$, a $\tau_n < t_n$ and a semisolution σ_n through x_n with $\sigma_n(\tau_n) \in B(x, \varepsilon_n)$. Since $\varepsilon_n \to 0$ we have $x_n \to x$ and $\sigma_n(\tau_n) \to x$ and we conclude that $x \in j^-(x)$ (as $\tau_n \to -\infty$). Thus (ii) holds.

Now assume (ii). Then there are a sequence $\{x_n\}$ in X and a sequence $\{t_n\}$ in $\mathbf{R}$ with $t_n \to -\infty$, $x_n \to x$ such that for each x_n there exists a semisolution σ_n through x_n with $\sigma_n(t_n) \to x$. Now for any neighbourhood U and $T < 0$ there is an N such that $t_n < T$, $x_n \in U$ and $\sigma_n(t_n) \in U$ for $n \geq N$. Thus U is self negatively weakly recursive. □

THEOREM 3.9. *Let $x \in X$. Every $y \in L^+(x)$ is non-wandering* (*i.e.* $y \in J^+(y)$ *and* $y \in J^-(y)$).

The proof of this theorem is analogous to the proof in the case of dynamical systems (see [**2**, 2.13]).

4. Trajectories in non-trivial, weakly minimal and negatively strongly invariant set. The structure of a trivial weakly minimal, negatively strongly invariant set is simple. It is either a periodic point or a rest point or a single trajectory $\pi_\sigma(x)$ with $L^+(x) = \emptyset$ and $L^-_\sigma(x) = \emptyset$; there exists precisely one maximal solution σ through x. Thus in this chapter we concentrate on the study of the properties of non-trivial weakly minimal set.

A point $x \in X$ (or the positive trajectory $\pi^+(x)$) is called *positively Poisson stable*, if $L^+(x) \cap \pi^+(x) \neq \emptyset$.

A point $x \in X$ is called:

– *negatively strongly Poisson stable* if $L^-_\sigma(x) \cap \sigma((-\infty, 0]) \neq \emptyset$ for any negative semisolution σ through x;

– *negatively weakly Poisson stable* if there exists a negative semisolution σ through x such that $L^-_\sigma(x) \cap \sigma((-\infty, 0]) \neq \emptyset$;
– *strongly Poisson stable* if x is both positively and negatively strongly Poisson stable;
– *weakly Poisson stable* if x is both positively and negatively weakly Poisson stable.

LEMMA 4.1. *If M is strongly invariant then it is also weakly invariant. If M is strongly minimal then it must not be a weakly minimal set.*

This lemma is an immediate consequence of the definitions.

THEOREM 4.2. *If M is weakly minimal and negatively strongly invariant then M is also strongly minimal.*

PROOF. We know that M is non-empty, closed, positively invariant and negatively strongly invariant. Suppose that there exists a proper subset $B \subset M$ having these properties. Since B is negatively strongly invariant it is also negatively weakly invariant so it is weakly minimal. This contradicts the weakly minimality of M. □

In the sequel we consider a semidynamical system $(X, \mathbf{R}^+, \pi)$ on a locally compact metric space X and we assume that π has no start points.

THEOREM 4.3. *Let $M \subset X$ be a non-trivial, weakly minimal and negatively strongly invariant set. If a trajectory through x is contained in M then it does not fulfill any of the following properties:*

(i) $L^+(x) = \emptyset$ *and there exists a negative semisolution σ through x such that* $L^-_\sigma(x) = \emptyset$
(ii) $L^+(x) \neq \emptyset$ *and* $L^+(x) \cap \pi^+(x) = \emptyset$
(iii) *there exists a negative semisolution σ through x such that* $L^-_\sigma(x) \neq \emptyset$ *and* $L^-_\sigma(x) \cap \sigma((-\infty, 0]) = \emptyset$.

PROOF.
(i) Suppose that $L^+(x) = \emptyset$ and $L^-_\sigma(x) = \emptyset$. Then $\pi^+(x)$ is closed and positively invariant. The trajectory $\pi_\sigma(x)$ is contained in M because M is negatively strongly invariant. Then $\pi_\sigma(x)$ itself is a closed, weakly invariant set. It means that $\pi_\sigma(x)$ is weakly minimal. Hence $\pi_\sigma(x) = M$, which contradictis the non-triviality of M.
(ii) Suppose that $L^+(x) \neq \emptyset$ and $L^+(x) \cap \pi^+(x) = \emptyset$. Then $L^+(x) \subset M$ and $x \notin L^+(x)$, that is $L^+(x) \neq M$. The set $L^+(x)$ is closed and weakly invariant, which contradicts the weakly minimality of M.
(iii) Suppose that there exist an $x \in M$ and a negative semisolution σ through x such that $L^-_\sigma(x) \neq \emptyset$ and $L^-_\sigma(x) \cap \sigma((-\infty, 0]) = \emptyset$. Then $\overline{\sigma((-\infty, 0])} \subset M$ for every σ, because M is negatively strongly invariant and closed. Hence

$L^-_\sigma(x) \subset M$ and $L^-_\sigma(x) \neq M$. Since $L^-_\sigma(x)$ is closed, negatively weakly invariant and positively invariant set, then it is a weakly minimal set. This contradicts the minimality of M. □

For a question if there exists a weakly minimal set which is not a set of negative unicity, the answer is positive.

EXAMPLE 4.4. Consider a dynamical system defined on a 2-dimensional torus T (where $T = [0,1]^2/\sim$, with a suitable identification of points) by a planar differential system

$$\frac{d\varphi}{dt} = f(\varphi,\theta), \qquad \frac{d\theta}{dt} = \alpha \cdot f(\varphi,\theta)$$

where $f(\varphi,\theta) = f(\varphi+1,\theta+1) = f(\varphi+1,\theta) = f(\varphi,\theta+1)$ and $f(\varphi,\theta) > 0$ for all φ,θ. If $\alpha > 0$ is irrational, then every trajectory is dense in the torus T. Thus dynamical system on the torus looks as in Figure 1.

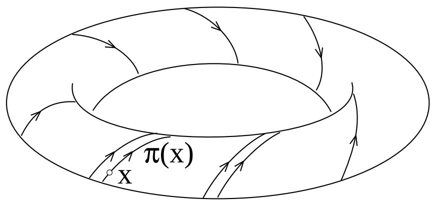

FIGURE 1.

We choose a point x and the trajectory $\pi(x)$ of x. To obtain a required semidymamical system (on another phase space, which is torus T') we change the trajectory $\pi(x)$ in the following way. For $t \geq 0$ the trajectory $\pi(x,t)$ does not change. For $t < 0$ we replace $\pi(x,t)$ by infinitely many negative semitrajectories σ such that: $\sigma(0) = x$, all negative semitrajectories σ ($\sigma \neq \sigma_1, \sigma \neq \sigma_2$) are between σ_1, σ_2 and $\sigma((-\infty,0)) \cap \pi(x) = \emptyset$. Also, $\sigma((-\infty,0])$ is disjoint from any other trajectory of π (see Figure 2).

The set A consisting of semitrajectories σ replaces the negative trajectory $\pi^-(x)$. The transformed torus T' is homeomorphic to torus T (see Figure 3).

In the obtained semidynamical system the point x is not a point of negative unicity. The trajectory $\pi_{\sigma_1}(x) = \pi^+(x) \cup \sigma_1((-\infty,0])$ is a weakly invariant set. The positive limit set $L^+(x)$ of x and $L^-_{\sigma_1}(x)$ are equal to $T \backslash intA$. It means that it contains negative semitrajectories σ_1, σ_2. Any other negative semitrajectory is not contained in any of limit sets. We denote this part of the torus as M.

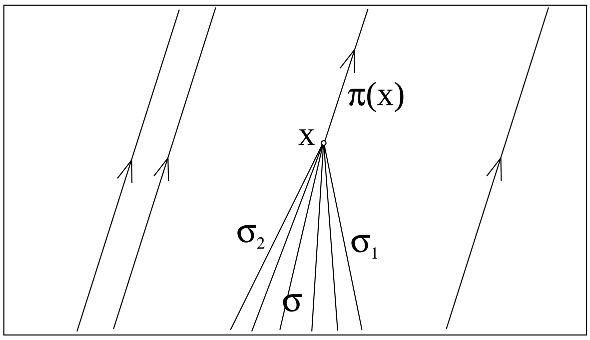

Figure 2.

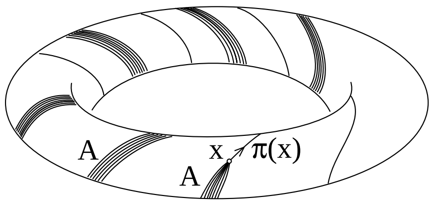

Figure 3.

The set M is weakly minimal and is not a set of negative unicity, but it is also not negatively strongly invariant.

Example 4.5. Now we restrict the semidynamical system from Example 4.4 to the phase space M. This semidynamical system has precisely one weakly minimal set, that is M. It is clear that M is non-trivial, negatively strongly invariant and is not a set of negative unicity.

Example 4.6. Consider a dynamical system defined on a 2-dimensional torus T from Example 4.4 with conditions

$$f(0,0) = f(0,1) = f(1,0) = f(1,1) = 0$$

and

$$f(\varphi, \theta) > 0 \text{ for } (\varphi, \theta) \notin \{(0,0), (1,0), (0,1), (1,1)\}.$$

This dynamical system has exactly one stationary point p, exactly one positively Poisson stable trajectory $\pi(x)$ (we have $L^-(x) = \{p\}$) and exactly one negatively Poisson stable trajectory $\pi(y)$ (we have $L^+(y) = \{p\}$). All other trajectories are Poisson stable. Restricting this dynamical system to $T \setminus \{p\}$, we obtain a dynamical system on a locally compact metric space. This dynamical system has exactly one positively Poisson stable trajectory $(\pi(x))$, $L^-(x) = \emptyset$ and exactly one negatively Poisson stable trajectory $(\pi(y))$, $L^+(y) = \emptyset$. All other trajectories are Poisson stable. We take the trajectory through x; we have $L^-(x) = \emptyset$. To obtain a suitable semidynamical system we change the trajectory $\pi(x)$ as in Example 4.4. In this case $L^-_\sigma(x) = \emptyset$ for any σ. The set $T' \setminus \{p\}$ is homeomorphic to $T \setminus \{p\}$ (see Figure 4.).

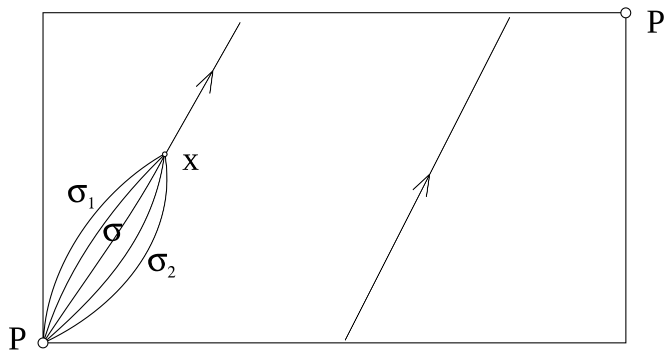

FIGURE 4.

We denote by M the positive limit set of x. It contains negative semitrajectories σ_1, σ_2. The negative semitrajectories, different from σ_1 and σ_2 are not contained in M. The set M is weakly minimal, is not a set of negative unicity, and it is not negatively strongly invariant. However, this semidynamical system restricted to M has only one weakly minimal set, that is M. It is clear that M is non-trivial, negatively strongly invariant and is not a set of negative unicity.

We have shown that there exists a weakly minimal, negatively strongly invariant set, which is not a set of negative unicity.

THEOREM 4.7. *Let $M \subset X$ be a non-trivial, weakly minimal and negatively strongly invariant set. Then each trajectory $\pi(x)$ contained in M has exactly one of the following properties:*

(i) *$\pi(x)$ is strongly Poisson stable.*

(ii) *$\pi(x)$ is positively Poisson stable and for any negative semisolution σ through x we have $L^-_\sigma(x) = \emptyset$.*

(iii) *$L^+(x) \cap \pi^+(x) \neq \emptyset$, there exists a negative semisolution σ_1 through x such that $L^-_{\sigma_1}(x) \cap \sigma_1((-\infty, 0]) \neq \emptyset$ and there exists a negative semisolution σ_2 through x such that $L^-_{\sigma_2}(x) = \emptyset$; moreover, each semisolution σ through x in M has the same properties.*

(iv) *$\pi(x)$ is negatively strongly Poisson stable and $L^+(x) = \emptyset$.*

PROOF. Let M be a non-trivial, weakly minimal and negatively strongly invariant set. Let $x \in M$. Consider an intersection of the positive limit set $L^+(x)$ with a positive semitrajectory $\pi^+(x)$.

According to Theorem 4.3 we have two cases:

- $L^+(x) \cap \pi^+(x) \neq \emptyset$
- $L^+(x) = \emptyset$ so $L^+(x) \cap \pi^+(x) = \emptyset$

The last case $L^+(x) \neq \emptyset$ and $L^+(x) \cap \pi^+(x) = \emptyset$ which contradicts (ii) from Theorem 4.3.

Consider now the intersection of the negative limit set $L^-_\sigma(x)$ with the image of a negative semisolution σ through x. Depending on the set $L^-_\sigma(x)$, this intersection is either empty or nonempty. Also, depending on σ the negative limit set $L^-_\sigma(x)$ is either empty or nonempty. Hence we have eight possibilities. Since we know that M contains no trajectories which fulfill property (iii) from Theorem 4.3, we have only three possible cases.

- $L^-_\sigma(x) \cap \sigma((-\infty, 0]) \neq \emptyset$ for any negative semisolution σ through x
- $L^-_\sigma(x) = \emptyset$ for any negative semisolution σ through x
- there exists a negative semisolution σ_1 through x such that $L^-_{\sigma_1}(x) \cap \sigma_1((-\infty, 0]) \neq \emptyset$ and there exists a negative semisolution σ_2 through x such that $L^-_{\sigma_2}(x) = \emptyset$; moreover, each semisolution σ through x in M has the same properties.

Considering now the sets $L^+(x) \cap \pi^+(x)$ and $L^-_\sigma(x) \cap \sigma((-\infty, 0])$ we obtain six cases:

(i) $L^+(x) \cap \pi^+(x) \neq \emptyset$ and for any negative semisolution σ through x we have $L^-_\sigma(x) \cap \sigma((-\infty, 0]) \neq \emptyset$.

(ii) $L^+(x) \cap \pi^+(x) \neq \emptyset$ and for any negative semisolution σ through x we have $L^-_\sigma(x) = \emptyset$.

(iii) $L^+(x) \cap \pi^+(x) \neq \emptyset$ and there exists a negative semisolution σ_1 through x such that $L^-_{\sigma_1}(x) \cap \sigma_1((-\infty, 0]) \neq \emptyset$ and there exists a negative semisolution σ_2 through x such that $L^-_{\sigma_2}(x) = \emptyset$; moreover, each semisolution σ through x in M has the same properties.

(iv) $L^+(x) = \emptyset$ and for any negative semisolution σ through x we have $L^-_\sigma(x) \cap \sigma((-\infty, 0]) \neq \emptyset$.

(v) $L^+(x) = \emptyset$ and for any negative semisolution σ through x we have $L^-_\sigma(x) = \emptyset$.

(vi) $L^+(x) = \emptyset$ and there exists a negative semisolution σ_1 through x such that $L^-_{\sigma_1}(x) \cap \sigma_1((-\infty, 0]) \neq \emptyset$ and there exists a negative semisolution σ_2 through x such that $L^-_{\sigma_2}(x) = \emptyset$; moreover, each semisolution σ through x in M has the same properties.

From (v) and (vi) we know that $L^+(x) = \emptyset$ and there exists a semisolution σ through x with $L^-_\sigma(x) = \emptyset$. According to (i) from Theorem 4.3 this is impossible. This means that any trajectory in M fulfills one of conditions (i)–(iv). It is easy to notice that a trajectory cannot fulfill any two of those conditions simultaneously. $\square$

In the same way as in Examples 4.5 and 4.6, for any property among (i)-(iv) we can construct a non-trivial, weakly minimal and negatively strongly invariant set containing a trajectory with this property.

LEMMA 4.8. *Let $M \subset X$ be a non-trivial, weakly minimal and negatively strongly invariant set and $x \in M$. Then either $M = L^+(x)$ or $M = \bigcup_\sigma L^-_\sigma(x)$.*

PROOF. We have four cases:

(1) $L^+(x) \neq \emptyset$ and $\bigcup\{L^-_\sigma(x) : \sigma$ is a semisolution through $x\} = \emptyset$
(2) $L^+(x) = \emptyset$ and $\bigcup\{L^-_\sigma(x) : \sigma$ is a semisolution through $x\} \neq \emptyset$
(3) $L^+(x) \neq \emptyset$ and $\bigcup\{L^-_\sigma(x) : \sigma$ is a semisolution through $x\} \neq \emptyset$
(4) $L^+(x) = \emptyset$ and $\bigcup\{L^-_\sigma(x) : \sigma$ is a semisolution through $x\} = \emptyset$

(1) If $L^+(x) \neq \emptyset$ then $L^+(x) \subset M$. Since $L^+(x)$ is closed, weakly invariant and M is weakly minimal, we have $M = L^+(x)$.
(2) If $\bigcup_\sigma L^-_\sigma(x) \neq \emptyset$ we know that there exists a solution σ_1 such that $L^-_{\sigma_1}(x) \neq \emptyset$ and $\overline{\sigma_1((-\infty, 0])} \subset M$, because M is negatively strongly invariant and closed. Hence $L^-_{\sigma_1}(x) \subset M$. Since $L^-_{\sigma_1}(x)$ is closed, weakly invariant and M is weakly minimal, we have $M = L^-_{\sigma_1}(x)$. In the same way we prove that $L^-_\sigma(x) = M$ for any $L^-_\sigma(x) \neq 0$. We have shown that $M = \bigcup_\sigma L^-_\sigma(x)$.
The proof of (3) is analogous to the proof of (1) and (2).
(4) If $L^+(x) = \emptyset$ and $\bigcup_\sigma L^-_\sigma(x) = \emptyset$ then there exists a solution σ_1 such that $L^-_{\sigma_1}(x) = \emptyset$. Hence $\pi_{\sigma_1}(x)$ is closed. The trajectory $\pi_{\sigma_1}(x)$ is weakly invariant and contained in M because M is negatively strongly invariant. Then $\pi_{\sigma_1}(x)$ is weakly minimal. Hence $\pi_{\sigma_1}(x) = M$, which contradicts the non-triviality of M. $\square$

Any trivial weakly minimal and negatively strongly invariant set contains only one trajectory. How many trajectories may a non-trivial weakly minimal and negatively strongly invariant set contain? The answer to this question is presented in the next theorem.

THEOREM 4.9. *Each non-trivial, weakly minimal and negatively strongly invariant set contains infinitely many trajectories.*

PROOF. Let X be a locally compact metric space. Let M be a non-trivial, weakly minimal and negatively strongly invariant set. Suppose that M consists of finitely many trajectories:

$$M = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m_i} \pi_{\sigma_j}(x_i)$$

where $n \in \mathbf{N}$ and $m_i \in \mathbf{N}$. Then we have

$$M = L^+(x_i) \cup \bigcup_{j=1}^{m_i} L^-_{\sigma_j}(x_i)$$

for every $i \in \{1, \dots, n\}$.

Choose any $p_0 \in M$. Then $p_0 \in L^+(x_1) \cup \bigcup_{j=1}^{m_1} L^-_{\sigma_j}(x_1)$. Take any $\varepsilon > 0$ such that $\overline{B(p_0, \varepsilon)}$ is compact. Then there exists a $t_1 \in \mathbf{R}$ such that $|t_1| > 1$ and if $t_1 > 1$ then $\pi(x_1, t_1) \in B(p_0, \varepsilon)$ and if $t_1 < -1$ then there exists a σ_j such that $\sigma_j(t_1) \in B(p_0, \varepsilon)$ for some $j \in \{1, \ldots, m_1\}$. Denote this σ_j as σ. Set

$$p_1 = \begin{cases} \pi(x_1, t_1) & \text{if } t_1 > 1, \\ \sigma(t_1) & \text{if } t_1 < -1. \end{cases}$$

Since M is non-trivial, weakly minimal and negatively strongly invariant it does not contain neither any periodic point nor any stationary point. Then x_1 is neither a periodic point nor a stationary point and we have $p_1 \notin \pi_\sigma(x_1, [-1, 1])$. It is easy to see that $p_1 \notin \pi_{\sigma_j}(x_1, [-1, 1])$ for $j \in \{1, \ldots, m_1\}$ and $\sigma_j \neq \sigma$. On the other hand, $p_1 \notin \pi_{\sigma_j}(x_i, [-1, 1])$ for every $i \neq 1$ and for every $j \in \{1, \ldots, m_i\}$. Thus we have

$$p_1 \notin \bigcup_{i=1}^{n} \bigcup_{j=1}^{m_i} \pi_{\sigma_j}(x_i, [-1, 1])$$

There exists an $\alpha > 0$ such that $\alpha < \frac{\varepsilon}{2}$ and $B(p_1, \alpha) \subset B(p_0, \varepsilon)$. On the other hand, there exists a $\beta > 0$ such that $\beta < \frac{\varepsilon}{2}$ and

$$\overline{B(p_1, \beta)} \cap \bigcup_{i=1}^{n} \bigcup_{j=1}^{m_i} \pi_{\sigma_j}(x_i, [-1, 1]) = \emptyset$$

Let $\varepsilon_1 = \min\{\alpha, \beta\}$. Then we obtain the following:

$$B(p_1, \varepsilon_1) \subset B(p_0, \varepsilon)$$

$$\overline{B(p_1, \varepsilon_1)} \cap (\bigcup_{i=1}^{n} \bigcup_{j=1}^{m_i} \pi_{\sigma_j}(x_i, [-1, 1])) = \emptyset$$

$$\varepsilon_1 < \frac{\varepsilon}{2}$$

We have

$$p_1 \in \pi_\sigma(x_1) \subset M = L^+(x_i) \cup \bigcup_{j=1}^{m_1} L^-_{\sigma_j}(x_1)$$

so there exists a $t_2 \in \mathbf{R}$ such that $|t_2| > 2$ and if $t_2 > 2$ then $\pi(x_1, t_2) \in B(p_1, \varepsilon_1)$ and if $t_2 < -2$ then there exists a σ_l such that $\sigma_l(t_2) \in B(p_1, \varepsilon_1)$ for some $l \in \{1, \ldots, m_1\}$. Denote

$$p_2 = \begin{cases} \pi(x_1, t_2) & \text{if } t_2 > 2, \\ \sigma_l(t_2) & \text{if } t_2 < -2. \end{cases}$$

So x_1 is neither periodic nor stationary and we have $p_2 \notin \pi_{\sigma_l}(x_1, [-2, 2])$ if $\sigma_l \neq \sigma$ and if $\sigma_l = \sigma$ then $p_2 \notin \pi_\sigma(x_1, [-2, 2])$.

In both cases it is easy to see that $p_2 \notin \pi_{\sigma_j}(x_1, [-2,2])$ for $j \in \{1,\dots,m_1\}$ and $\sigma_j \neq \sigma_l$ or $\sigma_j \neq \sigma$, respectively. On the other hand, $p_2 \notin \pi_{\sigma_j}(x_i, [-2,2])$ for every $i \neq 1$ and for every $j \in \{1,\dots,m_i\}$. Thus we have

$$p_1 \notin \bigcup_{i=1}^{n} \bigcup_{j=1}^{m_i} \pi_{\sigma_j}(x_i, [-2,2])$$

Repeating the above procedure, we obtain $\varepsilon_2 > 0$ such that:

$$B(p_2, \varepsilon_2) \subset B(p_1, \varepsilon_1)$$

$$\overline{B(p_2, \varepsilon_2)} \cap (\bigcup_{i=1}^{n} \bigcup_{j=1}^{m_i} \pi_{\sigma_j}(x_i, [-2,2])) = \emptyset$$

$$\varepsilon_2 < \frac{\varepsilon_1}{2}$$

In this way we obtain a sequence $\{\varepsilon_k\}$ in $\mathbf{R}$ such that for every $k \in \mathbf{N}$

$$B(p_k, \varepsilon_k) \subset B(p_{k-1}, \varepsilon_{k-1})$$

$$\overline{B(p_k, \varepsilon_k)} \cap (\bigcup_{i=1}^{n} \bigcup_{j=1}^{m_i} \pi_{\sigma_j}(x_i, [-k,k])) = \emptyset$$

$$\varepsilon_k < \frac{\varepsilon_{k-1}}{2}$$

where

$$p_k = \begin{cases} \pi(x_1, t_k) & \text{if } t_k > k, \\ \sigma_j(t_k) \text{ for } j \in \{1,\dots,m_1\} & \text{if } t_k < -k. \end{cases}$$

Thus we have a sequence

$$\overline{B(p_0, \varepsilon)} \supset \overline{B(p_1, \varepsilon_1)} \supset \dots \supset \overline{B(p_k, \varepsilon_k)} \supset \dots$$

where every $\overline{B(p_k, \varepsilon_k)}$ is non-empty and $\overline{B(p_0, \varepsilon)}$ is compact.
Hence $\bigcap_{k=1}^{\infty} \overline{B(p_k, \varepsilon_k)} \neq \emptyset$. Take any $q \in \bigcap_{k=1}^{\infty} \overline{B(p_k, \varepsilon_k)}$.
We have $d(p_k, q) \leq \varepsilon_k$ for every $k \in \mathbf{N}$. We know that $p_k = \pi(x_1, t_k)$ if $t_k > k$ and $p_k = \sigma_j(t_k)$ for $j \in \{1,\dots,m_1\}$ if $t_k < -k$ and $|t_k| > k$, for every $k \in \mathbf{N}$. Hence we have $|t_k| \to +\infty$ $(k \to +\infty)$. So, we can find a subsequence $\{t_{k_l}\}$ of $\{t_k\}$ such that $t_{k_l} \to +\infty$ or $t_{k_l} \to -\infty$.
Thus we have

$$\pi(x_1, t_{k_l}) \to q \text{ if } t_{k_l} \to +\infty \quad (l \to +\infty)$$

$$\sigma_j(t_{k_l}) \to q \text{ if } t_{k_l} \to -\infty \quad (l \to +\infty) \text{ for some } j \in \{1,\dots,m_1\}$$

(taking a subsequence, if necessary), so $q \in L^+(x_1) \cup \bigcup_{j=1}^{m_1} L^-_{\sigma_j}(x_1) = M$.
However, since

$$\overline{B(p_k, \varepsilon_k)} \cap (\bigcup_{i=1}^{n} \bigcup_{j=1}^{m_i} \pi_{\sigma_j}(x_i, [-k,k])) = \emptyset$$

holds for every $k \in \mathbf{N}$, we have

$$q \notin \bigcup_{i=1}^{n} \bigcup_{j=1}^{m_i} \pi_{\sigma_j}(x_i, [-k, k]) \text{ for every } k \in \mathbf{N}$$

which implies that $q \notin \pi_{\sigma_j}(x_i, [-k, k])$ for every $i \in \{1, \ldots, n\}$, for every $j \in \{1, \ldots, m_i\}$ and for every $k \in \mathbf{N}$. Thus $q \notin \pi_{\sigma_j}(x_i, \mathbf{R})$ for every $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m_i\}$. Hence we have

$$q \in M \setminus (\bigcup_{i=1}^{n} \bigcup_{j=1}^{m_i} \pi_{\sigma_j}(x_i, \mathbf{R}))$$

which is a contradiction. □

Theorem 4.10. *Let $M \subset X$ be a non-trivial, weakly minimal and negatively strongly invariant set. Then every point x in M is non-wandering in M.*

Proof. Let $x \in M$. Then $M = L^+(x)$ or $M = \bigcup_\sigma L^-_\sigma(x)$.
If $M = L^+(x)$ then the proof that x is non-wandering in M is the same as the proof of analogous theorem for dynamical systems ([**3**, Theorem 1]).
If $M = \bigcup_\sigma L^-_\sigma(x)$ then there exists a semisolution σ_1 through x such that $L^-_{\sigma_1}(x) \neq \emptyset$. Hence $L^-_{\sigma_1}(x) = M$. Put $\sigma_1 = \sigma$. Let π_M be the restriction of π to $M \times \mathbf{R}^+$. The map π_M defines a semidynamical system $(M, \mathbf{R}^+, \pi_M)$. By $L^-_{\sigma,M}(x)$ and $j^-_M(x)$ we denote the negative limit set and negative prolongational limit set of x in $(M, \mathbf{R}^+, \pi_M)$ respectively, i.e.,

$$\begin{aligned} L^-_{\sigma,M}(x) =& \{y \in M : \text{ there exists a sequence } \{t_n\} \text{ in } \mathbf{R} \\ & \text{with } t_n \to -\infty \text{ and } \sigma(t_n) \to y\} \\ j^-_M(x) =& \{y \in M : \text{ there is a sequence } \{x_n\} \text{ in } X \text{ and a sequence} \\ & \{t_n\} \text{ in } \mathbf{R}^- \text{ such that } x_n \to x, t_n \to -\infty \text{ and for each } x_n \\ & \text{there exists a } \sigma_n \text{ such that } \sigma_n(t_n) \to y\}. \end{aligned}$$

It is clear that

$$L^-_{\sigma,M}(x) \subset j^-_M(x) \subset M$$

On the other hand, we have $L^-_{\sigma,M}(x) = L^-_\sigma(x) = M$. Thus we see that

$$L^-_{\sigma,M}(x) = j^-_M(x) = M.$$

Hence $x \in j^-_M(x)$. This implies that x is non-wandering in $(M, \mathbf{R}^+, \pi_M)$. □

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