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FIRST-ORDER DIFFERENTIAL EQUATIONS OF THE HYPERBOLIC TYPE

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Abstract. By using the extrapolation spaces the existence and uniqueness of the solution of the semilinear first order equation in the “hyperbolic” case are studied.

1. Introduction. Let $(X, \|.\|)$ be a Banach space and for each $t \in [0, T]$ let $A(t)\colon X \supset D_t \to X$ be a linear closed densely defined operator, where D_t denotes domain of $A(t)$ depends on t. Let u be an unknown function from $[0, T]$ into X, f be a nonlinear function from $[0, T] \times X$ into X and $x_0 \in X$. We consider the abstract semilinear initial value problem

$$\begin{cases} u'(t) = A(t)u(t) + f(t, u(t)), & t \in (0, T] \\ u(0) = x_0 \in X. \end{cases} \tag{1}$$

Our purpose is to study the existence and uniqueness of solution of (1). We shall introduce the extrapolation space and reduce the problem (1) to the problem with operator whose domain does not depend on t.

2. Preliminaries. Let $A\colon X \supset D(A) \to X$ be a closed linear operator on a Banach space $(X, \|\cdot\|)$ with nonempty resolvent set $\rho(A)$. We do not assume that A is densely defined. For such an operator A we may define the extrapolation space X_{-1} which was introduced by R. Nagel ([**3**]). For details and proofs see, e.g. ([**4**], Chap.3).

For fixed $\mu \in \rho(A)$

$$|x| := \|R(\mu, A)x\|, \qquad x \in X \tag{2}$$

defines new norm on X.

It is easy to prove that

Proposition 2.1. *The space $(X, |\cdot|)$ is not a Banach space.*

This proposition motivates the following definition. We define the extrapolation space X_{-1} as the closure of X in the norm $|\cdot|$.

Next, we may extend the operator A. We denote by A_{-1} the extension of A with domain $X_0 := \overline{D(A)}^{\|\cdot\|}$. We collect some facts about A_{-1} in the following proposition.

PROPOSITION 2.2. *Let A be a closed operator and $\lambda \in \rho(A)$. Then*

(i) *the mapping $\lambda - A_{-1} \colon X_0 \to X_{-1}$ is an isomorphism*
(ii) *if $\lambda \in \rho(A)$, then $\lambda \in \rho(A_{-1})$ and $R(\lambda, A) = R(\lambda, A_{-1})|_X$*
(iii) $\|R(\lambda, A_{-1})\|_{X_{-1}} \le \|R(\lambda, A)\|$, $\lambda \in \rho(A)$
(iv) *A is the part of A_{-1} in X*
(v) *if there exist $M \ge 1$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$ and*

$$\|R(\lambda, A)^n\| \le M(\lambda - \omega)^{-n}, \quad \lambda > \omega, n = 1, 2, \dots$$

then A_{-1} generates a C_0-semigroup $T_{-1}(t)$ on X_{-1} such that $T_{-1}(t)|_{X_0} = T_0(t)$, where $T_0(t)$ is C_0-semigroup on X_0 whose generator is an operator $A_0 = A|_{\{x \in D(A)\colon Ax \in X_0\}}$.

From this proposition it follows that the norm on X_{-1} is given by

(3) $$\|x_{-1}\|_{X_{-1}} = |x_{-1}| = \|R(\mu, A_{-1})x_{-1}\| \qquad x_{-1} \in X_{-1}, \mu \in \rho(A).$$

In the sequel we shall need the following theorems.

THEOREM 2.3. ([**1**], *Th.1.47*). *Let for each $t \in [0,T]$, $A(t)$ be a linear closed densely defined operator, the domains D_t depend on t. For each $t \in [0,T]$ the operator $A(t)$ has the inverse operator $A^{-1}(t) \in \mathcal{B}(X)$, where $\mathcal{B}(X)$ denotes the Banach space of bounded linear operators from X into X. If for an arbitrary $s \in [0,T]$ the mapping $[0,T] \ni t \to \overline{A^{-1}(t)A(s)}$is continuous in $t = s$, then there exist $m > 0, M > 0$ such that for $t, r \in [0,T]$ and $x \in X$*

(4) $$m\|A^{-1}(t)x\| \le \|A^{-1}(r)x\| \le M\|A^{-1}(t)x\|.$$

THEOREM 2.4. ([**2**], *Lemma 3.8*). *If the operator $A(t) \in \mathcal{B}(X, Y)$ is strongly continuously differentiable on [0,T] and has an inverse operator $A^{-1}(t)$ uniformly bounded on that interval, then $A^{-1}(t)$ is also strongly continuously differentiable and the following formula holds*

$$[A^{-1}(t)]' = -A^{-1}(t)A'(t)A^{-1}(t).$$

THEOREM 2.5. ([**1**], *Th.1.52*). *Let $g \colon \Delta_T = \{(t,s) : 0 \le s \le t \le T\} \to X$ and suppose that*

(i) *for almost all $s \in [0,T]$ the function $[s,T] \ni t \to g(t,s)$ is continuous*
(ii) *for each $t \in [0,T], g(t, \cdot)$ is summable over $[0,t]$*

(iii) *there exists a function* $\varphi \in L^1(0,T;[0,\infty))$ *such that for* $(t,s) \in \Delta_T$, $\|g(t,s)\| \le \varphi(s)$.

Then the function $G\colon [0,T] \ni t \to \int_0^t g(t,s)ds \in X$ *is continuous.*

THEOREM 2.6. ([**6**], *Th.7.11, p.127*). *Let* $f_n\colon [0,T] \to X$ *and let for* $t \in [0,T]$, $\lim_{n\to\infty} f_n(t) = f(t)$. *Suppose that*

(i) $f_n \to f$ *uniformly on [0,T] as* $n \to \infty$, *i.e.*

$$sup\{\|f_n(t) - f(t)\| : t \in [0,T]\} \to 0, \quad n \to \infty$$

and let

(ii) $\lim_{t\to t_0} f_n(t) = A_n$, $n = 1,2,3,\dots$, $t \in [0,T]$.

Then $\{A_n\}$ *is convergent and*

$$\lim_{t\to t_0}(\lim_{n\to\infty} f_n(t)) = \lim_{n\to\infty}(\lim_{t\to t_0} f_n(t)).$$

3. The construction of a space $\hat{\mathbf{X}}_{\mathbf{0}}$. In this section we shall construct the extrapolation space $\hat{X}_0$ associated with the family $\{A(t)\}, t \in [0,T]$.

Let $(X, \|\cdot\|)$ be a Banach space. We make the following assumptions

(Z_1) Let for each $t \in [0,T], A(t)\colon X \supset D(A(t)) \to X$ be a closed densely defined linear operator; the domain $D(A(t)) = D_t$ of $A(t)$ depends on $t \in [0,T]$.

(Z_2) The resolvent set $\rho(A(t))$ does not depend on t and 0 belongs to $\rho(A(t))$.

(Z_3) For an arbitrary $s \in [0,T]$ the mapping $t \to \overline{A^{-1}(t)A(s)}$ is continuous in $t = s$ on [0,T] in the sense that $\lim_{t\to s} \|\overline{A^{-1}(t)A(s)} - I\| = 0$.

Analogously to the norm (2), for fixed $\mu \in \rho(A(t))$ and for each $t \in [0,T]$ define the new norm on X as

$$|x|_t := \|R(\mu, A(t))x\|, \qquad x \in X. \tag{5}$$

Applying Theorem 2.3 we can prove the following

THEOREM 3.1. *Let assumptions* $(Z_1) - (Z_3)$ *hold. For each* $t \in [0,T]$ *the norms* $|\cdot|_0$ *and* $|\cdot|_t$ *are equivalent.*

PROOF. It follows from (4) that there exist $m > 0$, $M > 0$ such that for $t \in [0,T]$ and $x \in X$

$$m\|A^{-1}(t)x\| \le \|A^{-1}(0)x\| \le M\|A^{-1}(t)x\|.$$

From this and from (5) we have

$$|x|_0 \le \|R(\mu, A(0))A(0)A^{-1}(0)A(t)R(\mu, A(t))x\| + M_2\|R(\mu, A(t))x\| \le M_3|x|_t.$$

Analogously $|x|_t \le m_3|x|_0$. □

We remark that from Theorem 3.1 it follows that we can for example choose the space $X_0 := (X, |\cdot|_0)$. By Proposition 2.1, X_0 is not a Banach space. Because X_0 is the normed space we can complete it in the sense of norm $|\cdot|_0$ to the complete space $\hat{X}_0$. The space $\hat{X}_0$ is the Banach space and does not depend on t.

Under assumptions $(Z_1) - (Z_3)$ we constructed the extrapolation space of X. Now, we shall extend the family of operators $\{A(t)\}, t \in [0, T]$.

4. The family of operators $\{\hat{\mathbf{A}}(\mathbf{t})\}, \mathbf{t} \in [\mathbf{0}, \mathbf{T}]$. Let assumptions $(Z_1) - (Z_3)$ hold.We remark that for each $t \in [0, T]$ the operator $A(t)$ is bounded as a map $A(t)\colon X \supset D_t \to X \subset \hat{X}_0$. In fact, from Theorem 3.1, for each $x \in D_t$, $t \in [0, T]$ we have

$$|A(t)x|_0 \leq M|A(t)x|_t = M\|A(t)R(\mu, A(t))x\| \leq \hat{M}\|x\|.$$

Hence we can extend it to a bounded linear operator on the all X. Consequently we obtain family of the closed linear operators

$$\hat{A}(t)\colon \hat{X}_0 \supset D(\hat{A}(t)) \to \hat{X}_0,$$

the domains $D(\hat{A}(t)) = X$ do not depend on t and X is dense in $\hat{X}_0$.

In the sequel we shall prove theorems about family $\{\hat{A}(t)\}$, $t \in [0, T]$, which we shall apply to the study of the existence and uniqueness of the solution of the Cauchy problem (1) with the operator $A(t)$, which domain D_t depends on $t \in [0, T]$.

Applying Proposition 2.2 to $\hat{A}(t)$ for each $t \in [0, T]$, we have the following theorem.

THEOREM 4.1. *Suppose that assumptions $(Z_1) - (Z_3)$ hold. Then*

(i) *for $\lambda \in \rho(A(t))$ and $t \in [0, T]$, $\lambda - \hat{A}(t)\colon X \to \hat{X}_0$ is an isomorphism,*
(ii) *if $\lambda \in \rho(A(t))$, then $\lambda \in \rho(\hat{A}(t))$ and $R(\lambda, A(t)) = R(\lambda, \hat{A}(t))|_X$, $t \in [0, T]$.*

Analogously to norm (3) we have the norm on $\hat{X}_0$ given by

$$\|\hat{x}\|_{\hat{X}_0} = |\hat{x}|_0 = \|R(\mu, \hat{A}(0))\hat{x}\|, \quad \hat{x} \in \hat{X}_0, \mu \in \rho(A(0)). \tag{6}$$

Our purpose is to study the existence and uniqueness of solution of (1) in the “hyperbolic” case. In this case we make the following assumptions on $\{A(t)\}$, $t \in [0, T]$.

(Z_4) For each $t \in [0,T]$, $A(t)$ is the generator of a C_0-semigroup on X.

(Z_5) The family $\{A(t)\}$, $t \in [0,T]$ is stable in the sense that there exist real numbers $M \geq 1$ and ω such that

$$\| \prod_{j=1}^{k} (\lambda - A(t_j))^{-1} \| \leq M(\lambda - \omega)^{-k}$$

for all $\lambda > \omega, 0 \leq t_1 \leq \cdots \leq t_k \leq T$, $k \in \mathbb{N}$.

We shall prove that family $\{\hat{A}(t)\}$, $t \in [0,T]$ has identical properties.

THEOREM 4.2. *Let assumptions $(Z_1) - (Z_4)$ hold.*
Then for each $t \in [0,T]$, $\hat{A}(t)$ is the generator of a C_0-semigroup on $\hat{X}_0$.

PROOF. Theorem 4.1 together with (Z_4) shows that there exist $\hat{M} \geq 1$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(\hat{A}(t))$ and

$$|R(\lambda, \hat{A}(t))^n|_0 \leq \hat{M}(\lambda - \omega)^{-n}, \quad \lambda > \omega, n = 1, 2, \ldots$$

Since $D(\hat{A}(t)) = X$ is dense in $\hat{X}_0$, it follows that for each $t \in [0,T]$, $\hat{A}(t)$ is the generator of a C_0-semigroup on $\hat{X}_0$ (by [**5**], Th.5.3. "Hille-Yosida"). □

Using the same method as in ([**4**], Prop.3.1.11, p.47) we prove

THEOREM 4.3. *Under the assumptions $(Z_1) - (Z_3)$ if $S_t(s), s \geq 0$ is a C_0-semigroup on X, whose generator is $A(t), t \in [0,T]$, then $S_t(s)$ extends to a C_0-semigroup $\hat{S}_t(s), s \geq 0$ on $\hat{X}_0$, whose generator is $\hat{A}(t), t \in [0,T]$.*

In the sequel we shall need the following

THEOREM 4.4. ([**8**], *Th. 5*). *Let assumptions $(Z_1) - (Z_5)$ hold. Then the family $\{\hat{A}(t)\}$, $t \in [0,T]$ is stable on $\hat{X}_0$.*

In the special case of problem (1) where $D(A(t)) = D$ is independent of t, it is usually assumed that for $x \in D$, $[0,T] \ni t \to A(t)x \in X$ is of class C^1. In our case, where $D(A(t)) = D_t$ depend on t, instead of the above condition, we assume

(Z_6) For each $x \in X$, $[0,T] \ni t \to R(\lambda, A(t))x$ is of class C^1.

THEOREM 4.5. *Under the assumptions $(Z_1) - (Z_6)$ for each $\hat{x} \in \hat{X}_0$,*
$[0,T] \ni t \to R(\lambda, \hat{A}(t))\hat{x}$ is of class C^1.

PROOF. First we show that for each $\hat{x} \in \hat{X}_0$, $[0,T] \ni t \to R(\lambda, \hat{A}(t))\hat{x}$ is continuous. Let $\hat{x} \in \hat{X}_0$. Since X is dense in $\hat{X}_0$, there is a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ such that $x_n \to \hat{x}$ in $\hat{X}_0$. Hence

$$R(\lambda, \hat{A}(t))x_n - R(\lambda, \hat{A}(t_0))x_n \to R(\lambda, \hat{A}(t))\hat{x} - R(\lambda, \hat{A}(t_0))\hat{x}, \quad n \to \infty$$

uniformly on [0,T].

Next, for arbitrary $x_n \in X$ from Theorem 4.1 and (Z_6) it follows that

$$\|R(\lambda, \hat{A}(t))x_n - R(\lambda, \hat{A}(t_0))x_n\| = \|R(\lambda, A(t))x_n - R(\lambda, A(t_0))x_n\| \to 0, \qquad t \to t_0.$$

Thus

$$R(\lambda, \hat{A}(t))x_n - R(\lambda, \hat{A}(t_0))x_n \to 0, \quad t \to t_0$$

for each fixed $x_n \in X$.

Therefore from Theorem 2.6

$$\|R(\lambda, \hat{A}(t))\hat{x} - R(\lambda, \hat{A}(t_0))\hat{x}\| \to 0, \quad t \to t_0, \hat{x} \in \hat{X}_0.$$

Secondly we show that $[0,T] \ni t \to R(\lambda, \hat{A}(t))\hat{x}$ is of class C^1. From (Z_6) it follows that $[0,T] \ni t \to \Phi(t)x \in X$, defined as

$$\Phi(t)x := \begin{cases} \dfrac{R(\lambda, A(t))x - R(\lambda, A(t_0))x}{t - t_0}, & t \neq t_0 \\ \dfrac{\partial}{\partial t} R(\lambda, A(t))x|_{t=t_0}, & t = t_0 \end{cases}$$

is continuous in $t_0 \in [0,T]$, $x \in X$.

The operator $\frac{\partial}{\partial t}R(\lambda, A(t)) \colon X \to X$ is bounded. Thus there is the bounded operator $B(t) \colon \hat{X}_0 \to \hat{X}_0$ which is the extension of $\frac{\partial}{\partial t}R(\lambda, A(t))$.
Let $\hat{\Phi}(t) \colon \hat{X}_0 \to \hat{X}_0$ be defined as follows

$$\hat{\Phi}(t)\hat{x} := \begin{cases} \dfrac{R(\lambda, \hat{A}(t))\hat{x} - R(\lambda, \hat{A}(t_0))\hat{x}}{t - t_0}, & t \neq t_0 \\ B(t)\hat{x}|_{t=t_0}, & t = t_0. \end{cases}$$

Analogously to the first part of proof we prove that for each $\hat{x} \in \hat{X}_0$, $[0,T] \ni t \to \hat{\Phi}(t)\hat{x}$ is continuous.

Consequently, there is $\frac{\partial}{\partial t}R(\lambda, \hat{A}(t))\hat{x}|_{t=t_0} = B(t)\hat{x}|_{t=t_0}$, for $t_0 \in [0,T]$, $\hat{x} \in \hat{X}_0$.

Since $[0,T] \ni t \to \frac{\partial}{\partial t}R(\lambda, A(t))x$, $x \in X$ is continuous, $[0,T] \ni t \to \frac{\partial}{\partial t}R(\lambda, \hat{A}(t))\hat{x}$, $\hat{x} \in \hat{X}_0$ is continuous (similarly to the first part of proof). □

The operator $R(\lambda, \hat{A}(t)) \colon \hat{X}_0 \to X$ has the inverse operator

$$\lambda - \hat{A}(t) \colon X \to \hat{X}_0, \lambda \in \rho(A(t)), t \in [0,T].$$

Applying Theorem 2.4 and Theorem 4.5 we have the following

Corollary 4.6. *The mapping* $[0,T] \ni t \to \hat{A}(t)x$, $x \in X$ *is of class* C^1.

5. An evolution system in $\hat{\mathbf{X}}_0$. In this section we consider the following initial value problem on $\hat{X}_0$

$$\begin{cases} u'(t) = \hat{A}(t)u(t), & t \in (0,T] \\ u(0) = x_0. \end{cases} \tag{7}$$

DEFINITION 5.1. ([**5**], Def.5.3, p.129). A two parameter family of bounded operators $\{\hat{U}(t,s)\}, 0 \le s \le t \le T$, on $\hat{X}_0$ is called an evolution system of (7) if the following two conditions are satisfied

(i) $\hat{U}(s,s) = I, \hat{U}(t,r)\hat{U}(r,s) = \hat{U}(t,s)$ for $0 \le s \le r \le t \le T$,
(ii) $(t,s) \to \hat{U}(t,s)$ is strongly continuous for $0 \le s \le t \le T$.

It is known that the following is true.

THEOREM 5.2. ([**5**], *Th.4.8, p.145*). *Let assumptions* $(Z_1) - (Z_6)$ *hold. Then there exists the unique evolution system of* (7) $\{\hat{U}(t,s)\}, 0 \le s \le t \le T$ *satisfying*

(i) $|\hat{U}(t,s)|_0 \le M\exp\{\omega(t-s)\}, \qquad 0 \le s \le t \le T$
(ii) $\frac{\partial^+}{\partial t}\hat{U}(t,s)x|_{t=s} = \hat{A}(s)x, \qquad x \in X, 0 \le s \le T$
(iii) $\frac{\partial}{\partial s}\hat{U}(t,s)x = -\hat{U}(t,s)\hat{A}(s)x, \qquad x \in X, 0 \le s \le t \le T$
(iv) $\hat{U}(t,s)X \subset X, \qquad 0 \le s \le t \le T$
(v) *for* $x \in X, \hat{U}(t,s)x$ *is continuous in* $(X, \|\cdot\|_{D(\hat{A}(0))})$ *for* $0 \le s \le t \le T$ *where*

$$\|x\|_{D(\hat{A}(0))} := |x|_0 + |\hat{A}(0)x|_0, \qquad x \in X. \tag{8}$$

PROPOSITION 5.3. *The norm* $\|\cdot\|_{D(\hat{A}(0))}$ *is equivalent to the norm on* X.

PROOF. From Theorem 4.1 it follows that for $x \in X$

$$\|x\|_{D(\hat{A}(0))} = \|R(\mu,\hat{A}(0))\hat{A}(0)x\| + \|R(\mu,\hat{A}(0))x\| \le \tilde{M}\|x\|.$$

On the other hand

$$\|x\| = \|R(\mu,\hat{A}(0))(\mu - \hat{A}(0))x\| \le |\mu|\|x\|_{D(\hat{A}(0))}.$$

$\square$

We remark that from Proposition 5.3 it follows

PROPOSITION 5.4. *For* $x \in X$, $\hat{U}(t,s)x$ *is continuous in* $(X, \|\cdot\|)$ *for* $0 \le s \le t \le T$.

Using the same construction as in the proof of ([**5**], Th.3.1, p.135), we have the following proposition

PROPOSITION 5.5. *Let assume that for each* $t \in [0,T], A(t)$ *is the generator of a* C_0*-semigroup. Let* $\{A(t)\}, t \in [0,T]$ *be a stable family. If* $D_t = D$ *is independent of* t *and for* $x \in D, [0,T] \ni t \to A(t)x \in X$ *is of class* C^1 *then* $\hat{U}(t,s)|_X = U(t,s)$.

PROOF. Using the same method as in the proof of Theorem 3.1, [**5**], p.135, we construct the evolution system $U(t,s)$ in the following way

$$U(t,s)x = \lim_{n\to\infty} U_n(t,s)x, \quad x \in X$$

where

$$U_n(t,s)x := \begin{cases} S_{t_j^n}(t-s)x, \quad t_j^n \le s \le t \le t_{j+1}^n \\ S_{t_k^n}(t-t_k^n)[\prod_{j=l+1}^{k-1} S_{t_j^n}(\frac{T}{n})]S_{t_l^n}(t_{l+1}^n - s)x, \\ k > l, t_k^n \le t \le t_{k+1}^n, t_l^n \le s \le t_{l+1}^n. \end{cases}$$

$S_{t_k^n}(s), s \ge 0$ is C_0-semigroup, generated by a operator $A(t_k^n)$ for $k = 0, 1, \ldots, n$, $t_k^n := k\frac{T}{n}$, $n = 1, 2, \ldots$

Analogously we may construct $\hat{U}(t,s)$ and from Theorem 4.3 we obtain the proposition. $\square$

6. The linear case. In this section we consider the following linear problem

$$\begin{cases} u'(t) = A(t)u(t) + f(t), \quad t \in (0,T] \\ u(0) = x_0 \end{cases} \tag{9}$$

where $\{A(t)\}$, $t \in [0,T]$ satisfies the assumptions

($Z_1{}'$) Let for each $t \in [0,T]$, $A(t)\colon X \supset D(A(t)) \to X$ be a closed densely defined linear operator; the domain $D(A(t)) = D$ of $A(t)$ does not depend on $t \in [0,T]$.

($Z_6{}'$) The mapping $[0,T] \ni t \to A(t)x$, $x \in D$ is of class C^1.

and $(Z_2), (Z_4), (Z_5)$ from section 3 (see Prop.5.5).

From Theorem 4.8, [**5**], p.145, (see Th.5.2) it follows that under assumptions $(Z_1{}'), (Z_2), (Z_4), (Z_5), (Z_6{}')$ there exists the unique evolution system of (9) $\{U(t,s)\}, 0 \le s \le t \le T$ satisfying

(i) $\|U(t,s)\| \le M\exp\{\omega(t-s)\}, \qquad 0 \le s \le t \le T$
(ii) $\frac{\partial^+}{\partial t}U(t,s)x|_{t=s} = A(s)x, \qquad x \in Y, 0 \le s \le T$
(iii) $\frac{\partial}{\partial s}U(t,s)x = -U(t,s)A(s)x, \qquad x \in Y, 0 \le s \le t \le T$
(iv) $U(t,s)Y \subset Y, \qquad 0 \le s \le t \le T$
(v) for $x \in Y, U(t,s)x$ is continuous for $0 \le s \le t \le T$, where $Y = D$ equipped with the norm $\|x\|_Y = \|x\| + \|A(0)x\|$ for $x \in Y = D$.

Now we shall prove a theorem which is a slight generalization of ([**5**], Th.5.2, p.146).

THEOREM 6.1. *Let assumptions* $(Z_1{}'), (Z_2), (Z_4), (Z_5), (Z_6{}')$ *hold. If* $f \in L^1(0,T;Y) \cap C([0,T],X)$ *then for every* $x_0 \in Y$ *the initial value problem* (9) *possesses the unique solution* u *given by*

$$u(t) = U(t,0)x_0 + \int_0^t U(t,s)f(s)ds, \quad t \in [0,T], \tag{10}$$

such that $u \in C([0,T],Y) \cap C^1((0,T],X)$.

PROOF. From (Theorem 4.3, [**5**], p.141) it follows that the function $v\colon [0,T] \to Y$ given by $v(t) = U(t,0)x_0$ is a solution of the problem

$$\begin{cases} u'(t) = A(t)u(t), & t \in (0,T] \\ u(0) = x_0 \end{cases}$$

such that $v \in C([0,T],Y) \cap C^1((0,T],X)$.

We need only show that the function $w\colon [0,T] \to X$ given by $w(t) = \int_0^t U(t,s)f(s)ds$ is:
1) such that $w \in C([0,T],Y) \cap C^1((0,T],X)$,
2) a solution of problem (9) with the initial value $w(0) = 0$.
Ad 1) By Theorem 2.5 $w \in C([0,T],Y)$. Next, we shall show that $w\colon [0,T] \to X$ is C^1. We remark that function $t \to U(t,s)f(s)$ is differentiable for almost all $s \in [0,T]$ and

$$\frac{\partial}{\partial t}U(t,s)f(s) = A(t)U(t,s)f(s).$$

Use once again Theorem 2.5 to $g(t,s) = A(t)U(t,s)f(s)$. We see that $g(\cdot,s)$ is continuous and

$$\|g(t,s)\| = \|A(t)U(t,s)A^{-1}(s)A(s)f(s)\| \le$$

$$\|A(t)U(t,s)A^{-1}(s)\|\|A(s)A^{-1}(0)\|\|A(0)f(s)\| \le C\|f(s)\|_Y.$$

Thus function $t \to \int_0^t A(t)U(t,s)f(s)ds$ is continuous. Therefore

$$w'(t) = f(t) + \int_0^t A(t)U(t,s)f(s)ds$$

is continuous on X.
Applying the same method as in the proof of ([**5**], Th.5.2, p.146) we prove that w is the solution of problem (9) with the initial value $w(0) = 0$. □

COROLLARY 6.2. *Let the assumptions* $(Z_1) - (Z_6)$ *hold.*
If $f \in L^1(0,T;X) \cap C([0,T], \hat{X}_0)$ *then for every* $x_0 \in X$ *the initial value problem*

$$\begin{cases} u'(t) = \hat{A}(t)u(t) + f(t), & t \in (0,T] \\ u(0) = x_0 \end{cases} \tag{11}$$

possesses the unique solution u *given by*

$$u(t) = \hat{U}(t,0)x_0 + \int_0^t \hat{U}(t,s)f(s)ds, \quad t \in [0,T], \tag{12}$$

such that $u \in C([0,T],X) \cap C^1((0,T],\hat{X}_0)$ *and* $\{\hat{U}(t,s)\}$, $0 \le s \le t \le T$ *is the evolution system given by Theorem 5.2.*

We remark that from Corollary 6.2 it follows that if $\{A(t)\}$, $t \in [0,T]$ satisfies the conditions of Proposition 5.5, and function $f \in L^1(0,T;X)$, $x_0 \in D$, then u given by (12) is the "mild solution" of problem (9) ([**5**], Def.5.1, p.146).

Motivated by this remark we make the following definition. Let the assumptions $(Z_1) - (Z_6)$ hold.

DEFINITION 6.3. A function $u \in C([0,T],X)$ given by

$$u(t) = \hat{U}(t,0)x_0 + \int_0^t \hat{U}(t,s)f(s)ds, \quad t \in [0,T],$$

is called the "mild solution" of linear problem (1).

To prove that the "mild solution" of linear problem (1) exists, it is enough to show that $f \in L^1(0,T;X)$ and $x_0 \in X$.

7. The semilinear case. In this section we consider semilinear problem (1), mentioned in the introduction.

Analogously as in the linear case, first we consider the abstract semilinear initial value problem

$$\begin{cases} u'(t) = A(t)u(t) + f(t,u(t)), & t \in (0,T] \\ u(0) = x_0 \end{cases} \tag{13}$$

where $\{A(t)\}, t \in [0,T]$ satisfies $({Z_1}'), (Z_2), (Z_4), (Z_5), ({Z_6}')$ and next we return to problem (1).

We have the following

THEOREM 7.1. *If* $f\colon [0,T]\times Y \to Y$ *is continuous and* u *is a solution of the problem* (13) *such that* $u \in C([0,T],Y)\cap C^1((0,T],X)$ *then* u *satisfies the integral equation*

$$u(t) = U(t,0)x_0 + \int_0^t U(t,s)f(s,u(s))ds, \quad t\in[0,T], \tag{14}$$

where $\{U(t,s)\}, 0\le s\le t\le T$ *is the evolution system from section 6.*

THEOREM 7.2. *Suppose that*

(i) $f\colon [0,T]\times Y\to Y$ *is such that* $f(\cdot,x)\in L^1(0,T;Y)$*;*
(ii) *there exists* $L>0$ *such that* $\|f(t,u)-f(t,v)\|_Y \le L\|u-v\|_Y$ *for* $t\in[0,T], u,v\in Y$.

Then for every $x_0\in Y$ *there exists exactly one continuous solution of* (14) *in* $(Y,\|\cdot\|_Y)$.

PROOF. Let

$$(Gu)(t) := U(t,0)x_0 + \int_0^t U(t,s)f(s,u(s))ds, \quad t\in[0,T]. \tag{15}$$

The operator G is a mapping from $C([0,T],Y)$ into itself. Indeed, let $u\in C([0,T],Y)$. Then function $\{f(\cdot,u(\cdot))\colon [0,T]\to Y\}\in L^1(0,T;Y)$. Similarly to the proof of Theorem 6.1 we prove that

$$\int_0^t U(t,s)f(s,u(s))ds \in C([0,T],Y).$$

From (15) it follows that Gu is continuous.

Let $C := sup\{\|U(t,s)\|_Y : 0\le s\le t\le T\}$. In the space $C([0,T],Y)$ consider the two equivalent norms

$$\|u\|_Y := \sup\{\|u(t)\|_Y : 0\le t\le T\}, \tag{16}$$

$$|u|_Y := \sup\{e^{-CLt}\|u(t)\|_Y : 0\le t\le T\}. \tag{17}$$

Analogously to the proof of ([**7**], Th.4.5, p.67) we can prove that G is a contraction under the norm (17). By Banach's contraction principle, this implies that (14) has exactly one solution $u\in C([0,T],Y)$. □

THEOREM 7.3. *Under assumptions* $(i),(ii)$ *of Theorem 7.2 if* $f(\cdot,x)\in C([0,T];X)$*,* $x_0\in D$ *and* u *is the solution of* (14) *then* $u\in C^1((0,T];X)$.

PROOF. Consider the abstract linear initial value problem

$$\begin{cases} z'(t) = A(t)z(t) + f(t,u(t)), & t\in(0,T] \\ z(0) = x_0 \end{cases} \tag{18}$$

where u satisfies (14). The existence and uniqueness of u follows from Theorem 7.2. In the proof of the previous theorem we showed that $f(\cdot, u(\cdot)) \in L^1(0,T;Y)$. Therefore applying Theorem 6.1 to the problem (18) we prove that this problem has exactly one solution $z \in C([0,T],Y) \cap C^1((0,T],X)$. From the uniqueness of the solution of (14) we have: $z(t) = u(t), t \in [0,T]$. Theorem 7.3 is proved. □

Analogously to the linear case we apply Theorems 7.1–7.3 to the problem with the operator $\hat{A}(t), t \in [0,T]$ and define the "mild solution" of problem (1).

Definition 7.4. A function $u \in C([0,T],X)$ given by

$$u(t) = \hat{U}(t,0)x_0 + \int_0^t \hat{U}(t,s)f(s,u(s))ds, \quad t \in [0,T],$$

is called the "mild solution" of initial value problem (1).

From the above theorems it follows that the "mild solution" of initial value problem (1) exists if

(i) $f\colon [0,T] \times X \to X$ is such that $f(\cdot, x) \in L^1(0,T;X)$
(ii) there exists $L > 0$ such that $\|f(t,u) - f(t,v)\| \le L\|u - v\|$ for $t \in [0,T]$, $u, v \in X$.

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