

## HADAMARD'S INEQUALITY IN INNER PRODUCT SPACES

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**Abstract.** The aim of this paper is to prove a generalized version of Hadamard's inequality in inner product spaces with assumptions significantly weaker than the ones found in existing variants.

**1. Introduction.**

Several different results are known in the literature as Hadamard's inequality. The most basic version for real square matrices  $A = [a_{ij}]_{i,j=1,\dots,n}$  takes the following form:

$$(\det A)^2 \leq \prod_{j=1}^n \left( \sum_{i=1}^n a_{ij}^2 \right)$$

It is, though, but a simple corollary of a more advanced theorem.

From now on, given a unitary space, we shall consider norms and scalar products on the exterior products of this vector space to be given by a natural extrapolation of a canonical construction presented in [1]. This standard method is presented for real spaces only. It is possible, however, to modify it to fit the case of complex spaces. The following result is often cited as Hadamard's inequality.

**THEOREM 1.1.** *For each finite-dimensional real unitary space  $V$  and for each pair of vectors  $\xi \in \bigwedge_p V, \eta \in \bigwedge_q V$ , if at least one of the vectors  $\xi$  or  $\eta$  is simple, then*

$$|\xi \wedge \eta| \leq |\xi||\eta|$$

The simplicity of  $\xi \wedge \eta$  is also a sufficient condition for the above inequality to hold, as was shown by Anna Wach in [3].

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It is also known that a somewhat weaker inequality is true for any pair of vectors  $\xi \in \bigwedge_p V, \eta \in \bigwedge_q V$

$$|\xi \wedge \eta| \leq \binom{p+q}{p}^{\frac{1}{2}} |\xi| |\eta|.$$

See [1] for details.

Our objective is to prove the inequality in a much more general setting. The above results will turn out to be particular cases of the main result.

**2. Definitions and notation.** Define  $\binom{a}{b} = 0$  for  $b > a$ . For  $1 \leq p \leq n$  let

$$\Lambda(n, p) = \{ \lambda : \{1, \dots, p\} \rightarrow \{1, \dots, n\} \text{ strictly increasing} \}$$

Let  $V$  be a finite-dimensional inner product space. If  $e_1, \dots, e_n$  is a given orthonormal basis of  $V$ , then for every  $\lambda \in \Lambda(n, p)$  define  $e_\lambda \in \bigwedge_p V$  by

$$e_\lambda = e_{\lambda(1)} \wedge \dots \wedge e_{\lambda(p)}$$

For  $p \geq 0$ , we say that  $\eta \in \bigwedge_p V$  is of rank  $r$  iff there exist  $v_1, \dots, v_{p-r} \in V, \tilde{v} \in \bigwedge_r V$  such that  $\eta = v_1 \wedge \dots \wedge v_{p-r} \wedge \tilde{v}$  and  $r$  is the least number having this property.

REMARK 2.1. It is a simple modification of a known result that  $\{e_\lambda\}_{\lambda \in \Lambda(n, p)}$  is a basis of  $\bigwedge_p V$ .

DEFINITION 2.2. We introduce an inner product on  $\bigwedge_p V$  by defining the aforementioned basis to be an orthonormal basis for  $\bigwedge_p V$ .

REMARK 2.3. The above definition only depends on the scalar product on  $V$  rather than the original choice of the basis  $e_1, \dots, e_n$  for  $V$ . In fact, an equivalent construction can be done without need for a particular basis of  $V$  to be fixed first, thus guaranteeing the uniqueness of a product (see [1] for such construction for  $\mathbb{R}$  case). In the proof of theorem 3.1, however, it is more convenient to have the scalar product on  $\bigwedge_p V$  introduced in the above way.

**3. Hadamard's inequality.** The following theorem is the main result of this paper.

THEOREM 3.1. *For every pair of vectors  $\xi \in \bigwedge_p V, \eta \in \bigwedge_q V$  such that  $\xi$  is of rank not greater than  $r, \eta$  is of rank not greater than  $s$ , and  $\xi \wedge \eta$  is of rank not greater than  $t$  we have*

$$|\xi \wedge \eta| \leq \min\left\{ \binom{r+s}{r}^{\frac{1}{2}}, \binom{n+t-p-q}{t}^{\frac{1}{2}} \right\} \cdot |\xi| |\eta|$$

REMARK 3.2. Since a vector is simple iff it is of rank 0, all previously mentioned versions of Hadamard's inequality follow from the above result.

PROOF OF THEOREM 3.1. The cases  $p = 0, q = 0$  or  $p + q > n$  are elementary. Let  $p, q > 0, p + q \leq n$ . It is enough to prove

$$(A) \quad (\xi \wedge \eta) \bullet (\xi \wedge \eta) \leq \binom{r+s}{r} (\xi \bullet \xi) (\eta \bullet \eta)$$

$$(B) \quad (\xi \wedge \eta) \bullet (\xi \wedge \eta) \leq \binom{n+t-p-q}{t} (\xi \bullet \xi) (\eta \bullet \eta),$$

where  $\bullet$  is the scalar product on the exterior products.

*Ad A.* It is a known result that by the ordinary orthonormalization we can get two orthonormal bases of  $V$  :  $e_1, \dots, e_n$  and  $\tilde{e}_1, \dots, \tilde{e}_n$  such that

$$\begin{aligned} \xi &= e_1 \wedge \dots \wedge e_{p-r} \wedge \tilde{\xi}, & \tilde{\xi} &\in \bigwedge_r V, \\ \eta &= \tilde{e}_1 \wedge \dots \wedge \tilde{e}_{q-s} \wedge \tilde{\eta}, & \tilde{\eta} &\in \bigwedge_s V, \\ \tilde{\xi} \wedge \eta &= \sum_{\lambda \in \Lambda(n, r+q)} a_\lambda e_\lambda, & a_\lambda &\in \mathbb{C}, \\ \tilde{\xi} \wedge \tilde{\eta} &= \sum_{\lambda \in \Lambda(n, r+s)} b_\lambda \tilde{e}_\lambda, & b_\lambda &\in \mathbb{C}, \\ \tilde{\xi} &= \sum_{\lambda \in \Lambda(n, r)} c_\lambda e_\lambda, & c_\lambda &\in \mathbb{C}, \\ \tilde{\eta} &= \sum_{\lambda \in \Lambda(n, s)} d_\lambda e_\lambda, & d_\lambda &\in \mathbb{C}. \end{aligned}$$

Then

$$\begin{aligned} (\xi \wedge \eta) \bullet (\xi \wedge \eta) &= (e_1 \wedge \dots \wedge e_{p-r} \wedge \tilde{\xi} \wedge \eta) \bullet (e_1 \wedge \dots \wedge e_{p-r} \wedge \tilde{\xi} \wedge \eta) \\ &= (e_1 \wedge \dots \wedge e_{p-r} \wedge (\sum_{\lambda \in \Lambda(n, r+q)} a_\lambda e_\lambda)) \bullet (e_1 \wedge \dots \wedge e_{p-r} \wedge (\sum_{\lambda \in \Lambda(n, r+q)} a_\lambda e_\lambda)) \\ &\leq \sum_{\lambda \in \Lambda(n, r+q)} a_\lambda \bar{a}_\lambda = (\tilde{\xi} \wedge \eta) \bullet (\tilde{\xi} \wedge \eta) \\ &= ((-1)^{r(q-s)} \tilde{e}_1 \wedge \dots \wedge \tilde{e}_{q-s} \wedge \tilde{\xi} \wedge \tilde{\eta}) \bullet ((-1)^{r(q-s)} \tilde{e}_1 \wedge \dots \wedge \tilde{e}_{q-s} \wedge \tilde{\xi} \wedge \tilde{\eta}) \\ &= (\tilde{e}_1 \wedge \dots \wedge \tilde{e}_{q-s} \wedge (\sum_{\lambda \in \Lambda(n, r+s)} b_\lambda \tilde{e}_\lambda)) \bullet (\tilde{e}_1 \wedge \dots \wedge \tilde{e}_{q-s} \wedge (\sum_{\lambda \in \Lambda(n, r+s)} b_\lambda \tilde{e}_\lambda)) \\ &\leq \sum_{\lambda \in \Lambda(n, r+s)} b_\lambda \bar{b}_\lambda = (\tilde{\xi} \wedge \tilde{\eta}) \bullet (\tilde{\xi} \wedge \tilde{\eta}) = \sum_{\mu \in \Lambda(n, r+s)} (\sum_{e_\lambda \wedge e_\kappa = \pm e_\mu} \pm c_\lambda d_\kappa) (\sum_{e_\lambda \wedge e_\kappa = \pm e_\mu} \pm \bar{c}_\lambda \bar{d}_\kappa) \\ &\leq \sum_{\mu \in \Lambda(n, r+s)} (\sum_{e_\lambda \wedge e_\kappa = \pm e_\mu} |c_\lambda| |d_\kappa|)^2 \leq \sum_{\mu \in \Lambda(n, r+s)} \binom{r+s}{r} (\sum_{e_\lambda \wedge e_\kappa = \pm e_\mu} |c_\lambda|^2 |d_\kappa|^2) \\ &\leq \binom{r+s}{r} (\sum_{\lambda \in \Lambda(n, r)} |c_\lambda|^2) (\sum_{\kappa \in \Lambda(n, s)} |d_\kappa|^2) = \binom{r+s}{r} (\tilde{\xi} \bullet \tilde{\xi}) (\tilde{\eta} \bullet \tilde{\eta}) = \binom{r+s}{r} (\xi \bullet \xi) (\eta \bullet \eta). \end{aligned}$$

*Ad B.* It is again possible to obtain, by orthonormalization, an orthonormal

basis  $e_1, \dots, e_n$  of  $V$  such that

$$\begin{aligned}\xi \wedge \eta &= e_1 \wedge \dots \wedge e_{p+q-t} \wedge \tilde{e}, \quad \tilde{e} \in \bigwedge_t V, \\ \xi &= \sum_{\lambda \in \Lambda(n,p)} x_\lambda e_\lambda, \quad x_\lambda \in \mathbb{C}, \\ \eta &= \sum_{\lambda \in \Lambda(n,q)} y_\lambda e_\lambda, \quad y_\lambda \in \mathbb{C}.\end{aligned}$$

Define  $w = e_1 \wedge \dots \wedge e_{p+q-t}$ . Then

$$\begin{aligned}(\xi \wedge \eta) \bullet (\xi \wedge \eta) &= \sum_{\mu \in \Lambda(n+t-p-q,t)} \left( \sum_{e_\lambda \wedge e_\kappa = \pm w \wedge e_{\mu+p+q-t}} \pm x_\lambda y_\kappa \right) \left( \sum_{e_\lambda \wedge e_\kappa = \pm w \wedge e_{\mu+p+q-t}} \pm \overline{x_\lambda y_\kappa} \right) \\ &\leq \sum_{\mu \in \Lambda(n+t-p-q,t)} \left( \sum_{e_\lambda \wedge e_\kappa = \pm w \wedge e_{\mu+p+q-t}} |x_\lambda| |y_\kappa| \right)^2 \leq \text{by the Schwartz inequality} \\ &\leq \sum_{\mu \in \Lambda(n+t-p-q,t)} \left( \sum_{e_\lambda \wedge e_\kappa = \pm w \wedge e_{\mu+p+q-t}} x_\lambda \overline{x_\lambda} \right) \left( \sum_{e_\lambda \wedge e_\kappa = \pm w \wedge e_{\mu+p+q-t}} y_\kappa \overline{y_\kappa} \right) \\ &\leq \sum_{\mu \in \Lambda(n+t-p-q,t)} \left( \sum_{\lambda \in \Lambda(n,p)} x_\lambda \overline{x_\lambda} \right) \left( \sum_{\kappa \in \Lambda(n,q)} y_\kappa \overline{y_\kappa} \right) = \binom{n+t-p-q}{t} (\xi \bullet \xi) (\eta \bullet \eta).\end{aligned}$$

□

## References

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