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# ON A FORMULA OF BENIAMINO SEGRE

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**Abstract.** This formula concerns Jacobians of $n$ polynomials $f_1, \dots, f_n$ in $n$ variables having $\prod_{i=1}^{n} \deg f_i$ distinct zeros in affine $n$–space. It was originally proved for $n = 2$ in [**9**], no.20. The generalization we have in mind allows that some of the zeros are on the hyperplane at infinity. Moreover a whole series of formulas will be considered, the first one being a classical formula of Jacobi [**5**] and the second one the above mentioned formula of B. Segre. As many other geometric results also these formulas can be derived from the residue theorem for projective algebraic curves. It is well known that the formulas of Jacobi and B. Segre have numerous geometric applications including Pascal's theorem for conics, versions of the Cayley-Bacharach theorem and generalizations of the Reiss relation about the curvature of real plane algebraic curves (see [**8**], for instance).

**1. Assumptions and formulation of the theorem.** Let $K$ be an algebraically closed field of characteristic 0 and $H_1, \dots, H_n$ hypersurfaces in $\mathbb{P}^n_K = \operatorname{Proj} K[Y_0, \dots, Y_n]$ with $\deg H_i = d_i$ $(i = 1, \dots, n)$ such that $\Gamma := H_1 \cap \dots \cap H_{n-1}$ is a reduced curve and $\Gamma \cap H_n$ is a zero dimensional scheme. Clearly $\Gamma$ is locally everywhere a complete intersection. The hyperplane $Y_0 = 0$ will be denoted by $H_\infty$. Let $\mathbb{A}^n_K = D_+(Y_0) = \operatorname{Spec} K[X_1, \dots, X_n]$ with $X_i := \frac{Y_i}{Y_0}$ $(i = 1, \dots, n)$ and $C := \Gamma \cap \mathbb{A}^n_K$, $H := H_n \cap \mathbb{A}^n_K$ the affine parts of $\Gamma$ and $H_n$. We set $C_\infty := \Gamma \cap H_\infty$, and we assume that $\dim C_\infty = 0$ and $C \cap H$ is non-empty and is a reduced scheme.

Let $H_i := \mathcal{V}_+(F_i)$ with a homogeneous polynomial $F_i \in K[Y_0, \dots, Y_n]$ $(i = 1, \dots, n)$, and let $f_i \in K[X_1, \dots, X_n]$ be the dehomogenization of $F_i$ with respect to $Y_0$, i.e. $f_i(X_1, \dots, X_n) = F_i(1, X_1, \dots, X_n)$. Then $C = \mathcal{V}(f_1, \dots, f_{n-1})$, $H = \mathcal{V}(f_n)$. That $C \cap H$ be reduced is equivalent with the condition, that for every $P \in C \cap H$ the Jacobian determinant $J := \frac{\partial(f_1, \dots, f_n)}{\partial(X_1, \dots, X_n)}$ does not vanish at $P$. We have $J \neq 0$ since $C \cap H \neq \emptyset$.

When $\mathcal{V}(f_1,\dots,f_n)$ consists, as in the classical case, of $\prod_{i=1}^{n} d_i$ distinct points, i.e. $C_\infty \cap H_n = \emptyset$, then $\Gamma$ is automatically a reduced curve and $C \cap H$ a reduced scheme, so our assumptions are not stronger than the classical ones.

For a polynomial $h \in K[X_1,\dots,X_n]$ let $R_0 := J^{-1}\cdot h$. If $\rho > 0$ and $R_{\rho-1}$ is already defined, set

$$R_\rho := J^{-1}\cdot \frac{\partial(f_1,\dots,f_{n-1},R_{\rho-1})}{\partial(X_1,\dots,X_n)}.$$

In what follows we want to show that for $\rho \in \mathbb{N}$ we have

$$\sum_{P\in C\cap H} R_\rho(P) = 0$$

provided the degree of $h$ does not exceed a certain bound $N(\rho)$ depending on $\rho$.

The bound depends on the behavior of $\Gamma$ and $H_n$ at the points of $C_\infty$. For $Q \in C_\infty$ let $\widehat{\mathcal{O}_{\Gamma,Q}}$ be the completion of the local ring of $\Gamma$ at $Q$ and $\mathcal{Z}_k := \widehat{Q_{\Gamma,Q}}/\mathfrak{p}_k$ with the minimal primes $\mathfrak{p}_k$ of $\widehat{Q_{\Gamma,Q}}$ the "analytic branches" of $\Gamma$ at $Q$ $(k = 1,\dots,\eta_Q)$. Their integral closures $\overline{\mathcal{Z}_k}$ are power series algebras $\overline{\mathcal{Z}_k} = K[[t_k]]$ in a variable $t_k$ over $K$, and

$$\overline{\mathcal{Z}_1}\times\cdots\times\overline{\mathcal{Z}_{\eta_Q}} = K[[t_1]]\times\cdots\times K[[t_{\eta_Q}]]$$

is the integral closure $\overline{\widehat{\mathcal{O}_{\Gamma,Q}}}$ of $\widehat{\mathcal{O}_{\Gamma,Q}}$. The conductor $\mathcal{F}_Q$ of $\overline{\widehat{\mathcal{O}_{\Gamma,Q}}}/\widehat{\mathcal{O}_{\Gamma,Q}}$ is of the form

$$\mathcal{F}_Q = (t_1^{c_1},\dots,t_{\eta_Q}^{c_{\eta_Q}})\cdot\overline{\widehat{\mathcal{O}_{\Gamma,Q}}} \qquad (c_k \in \mathbb{N}).$$

Let $\nu_k := \mu_Q(\mathcal{Z}_k, H_\infty)$ and $\nu'_k := \mu_Q(\mathcal{Z}_k, H_n)$ be the intersection multiplicities of the branch $\mathcal{Z}_k$ with $H_\infty$ resp. $H_n$ $(k = 1,\dots,\eta_Q)$. Since $Q \in H_\infty$ we have $\nu_k > 0$ while $\nu'_k = 0$ in case $Q \notin H_n$ $(k = 1,\dots,\eta_Q)$. We introduce the rational number

$$\sigma_Q := \operatorname*{Max}_{k=1,\dots,\eta_Q}\{\frac{1}{\nu_k}(c_k + (\rho+1)\cdot\nu'_k)\}$$

and set

$$N(\rho) := \sum_{i=1}^{n} d_i + \rho\cdot d_n - n - 1 - \operatorname*{Max}_{Q\in C_\infty\cap H_n}\{\sigma_Q\}.$$

Theorem. *Given $\rho\in\mathbb{N}$ suppose that* $\deg h \le N(\rho)$. *Then*

$$\sum_{P\in C\cap H} R_\rho(P) = 0.$$

In case $C_\infty \cap H_n = \emptyset$ we have $N(\rho) = \sum d_i + \rho d_n - n - 1$. If $\Gamma$ is intersected transversally by $H_n$ at $Q \in C_\infty$, then $\eta_Q = 1$, $\nu'_1 = 1$ and $c_1 = 0$, therefore

$\sigma_Q = (\rho+1)\cdot\frac{1}{\nu_1}$. If in addition $H_\infty$ is not tangential to $\Gamma$ at $Q$, then $\nu_k = 1$ $(k = 1,\dots,\eta_Q)$, and $\sigma_Q = \rho+1$. Hence we have

Corollary. *Assume $\Gamma\cap H_n$ is a reduced zero-dimensional scheme and $H_\infty$ is nowhere tangential to $\Gamma$. If $\deg h \le \sum_{i=1}^{n} d_i + \rho\cdot d_n - n - \rho - 2$, then*

$$\sum_{P\in C\cap H} R_\rho(P) = 0.$$

For $K = \mathbb{C}$ a generalization of the Jacobian formula ($\rho = 0$) allowing zeros at infinity was proved by Biernat [**1**] with different methods. See also the work of Berenstein-Yger [**2**] on related questions. The case of quasihomogeneous polynomials with no common zeros at infinity has been discussed in [**8**], and Jacobian formulas for Laurent polynomials are derived in [**3**] and [**6**]. The case $\rho = 1$ is the one considered by B. Segre.

**2. Reduction to the residue theorem on the curve $\Gamma$.** In order to prove the theorem we may assume that no point of $C_\infty$ is contained in the hyperplane $Y_n = 0$. In fact, a suitable projective coordinate transformation leaving $H_\infty$ invariant achieves this goal and, by the chain rule for Jacobians, also leaves the $R_\rho$ invariant.

Let $K[C] = K[x_1,\dots,x_n]$ be the affine coordinate ring of $C$ with the images $x_i$ of the variables $X_i$ $(i = 1,\dots,n)$ mod $(f_1,\dots,f_{n-1})$. Let $K(C) := Q(K[C]) = K(\Gamma)$ be the ring of rational functions of $C$ and of $\Gamma$. The above choice of coordinates guarantees that $K[C]$ is a finite module over $K[x_n]$. For the module of differentials we then have

$$\Omega^1_{K(C)/K} = K(C)dx_n.$$

In the sequel for an $h \in K[X_1,\dots,X_n]$ its image in $K[C]$ is also denoted by the same letter $h$, when it is clear from the context what we mean.

From the equations $f_i(x_1,\dots,x_n) = 0$ $(i = 1,\dots,n-1)$ we obtain the systems

$$\sum_{k=1}^{n} (f_i)_{X_k} dx_k = 0 \tag{1}$$

and

$$\sum_{k=1}^{n} (f_i)_{X_k}\cdot\Delta_k = 0, \tag{2}$$

$(f_i)_{X_k}$ denoting $\frac{\partial f_i}{\partial X_k}(x_1,\dots,x_n)$ and

$$\Delta_k := \begin{vmatrix} 0 & \cdots & 1 & \cdots & 0 \\ (f_1)_{X_1} & \cdots & (f_1)_{X_k} & \cdots & (f_1)_{X_n} \\ \vdots & & \vdots & & \vdots \\ (f_{n-1})_{X_1} & \cdots & (f_{n-1})_{X_k} & \cdots & (f_{n-1})_{X_n} \end{vmatrix} \qquad (k=1,\dots,n).$$

Since $C$ is reduced the Jacobian $[(f_i)_{X_k}]_{\substack{i=1,\dots,n-1\\ k=1,\dots,n}}$ has rank $n-1$, hence the solution space of (2) is $K(C)\cdot(\Delta_1,\dots,\Delta_n)$, a free $K(C)$–module of rank 1. If we write $dx_k = \tau_k dx_n$ ($\tau_k \in K(C), k=1,\dots,n-1$), then $(\tau_1,\dots,\tau_{n-1},1)$ is also a solution of (2), in particular $\Delta_n$ is a unit of $K(C)$. As a polynomial in $K[X_1,\dots,X_n]$ the determinant $\Delta_n$ has degree $\alpha := \sum\limits_{i=1}^{n-1}(d_i-1)$. In fact, by our assumptions, $K[Y_0,\dots,Y_n]/(F_1,\dots,F_{n-1},Y_0,Y_n)$ is a zero-dimensional graded $K$–algebra whose socle is generated by the image of $\left.\frac{\partial(F_1,\dots,F_{n-1})}{\partial(Y_1,\dots,Y_{n-1}}\right|_{Y_0=0}$ which is therefore a non-zero polynomial of degree $\alpha$. Replacing $Y_i$ by $X_i$ ($i=1,\dots,n$) gives the degree form of $\Delta_n$ which also has degree $\alpha$.

Moreover $f_n$ is a unit in $K(C)$, since $H=\mathcal{V}(f_n)$ does not contain an irreducible component of $C$. Therefore the differential

$$\omega_\rho := \frac{h}{f_n^{\rho+1}}\cdot\frac{1}{\Delta_n}\cdot dx_n \in \Omega^1_{K(\Gamma)/K} \qquad (\rho\in\mathbb{N})$$

is well-defined. We say that this is the expression for $\omega_\rho$ in $X$–coordinates.

Add the equation

$$\sum_{k=1}^{n}(f_n)_{X_k}dx_k = df_n$$

at the end of system (1). Cramer's rule then implies that

(3) $$\frac{dx_n}{\Delta_n} = \frac{df_n}{J}.$$

We consider now the situation in the affine space $D_+(Y_n)$ and dehomogenize with respect to $Y_n$. Set $Z_i := \frac{Y_i}{Y_n}$, and let $z_i$ be the image of $Z_i$ in $K(C)$ ($i=0,\dots,n-1$). In $Z$–coordinates $\omega_\rho$ can be written as

(4) $$\omega_\rho = -\frac{\tilde h}{\tilde f_n^{\rho+1}\cdot\widetilde{\Delta_n}}\cdot\frac{dz_0}{z_0^{m+2}}.$$

Here the number $m$ is defined by

(5) $$m := \deg h - (\rho+1)d_n - \deg\Delta_n = \deg h - \sum_{i=1}^{n} d_i - \rho d_n + n - 1.$$

Further $\tilde{h}$ denotes the element obtained from $h$ by first homogenizing with respect to $Y_0$, then dehomogenizing with respect to $Y_n$ and taking the image in $K(C)$, similarly for $\tilde{f}_n$ and $\widetilde{\Delta_n}$.

Our assumptions imply that $\Gamma \subset D_+(Y_0) \cup D_+(Y_n)$, hence the residue theorem for $\Gamma$ states that

$$\sum_{P\in C} \mathrm{Res}_P(\frac{h}{f_n^{\rho+1}} \cdot \frac{1}{\Delta_n} \cdot dx_n) = \sum_{Q\in C_\infty} \mathrm{Res}_Q(\frac{\tilde{h}}{\tilde{f}_n^{\rho+1}} \cdot \frac{1}{\widetilde{\Delta_n}} \cdot \frac{dz_0}{z_0^{m+2}}). \tag{6}$$

The formula of the theorem will be obtained by a calculation of these residues. See [**4**] for this method of deriving geometric results from the residue theorem.

**3. Calculation of residues.**

a) Let $P \in C \cap H$. Without loss of generality we may assume that $P = 0$ is the origin of $\mathbb{A}^n_K$. Since $J(P) \neq 0$ we have $K[[X_1,\dots,X_n]] = K[[f_1,\dots,f_n]]$, hence

$$\widehat{\mathcal{O}_{C,P}} = K[[X_1,\dots,X_n]]/(f_1,\dots,f_{n-1}) = K[[t]],$$

where $t$ is the image of $f_n \mod (f_1,\dots,f_{n-1})$. Due to (3) the image of $\omega_\rho$ in $K((t))dt$ has the form

$$\omega_\rho = \frac{h(t)}{J(t)} \cdot \frac{1}{t^{\rho+1}} \cdot dt$$

with the images $h(t)$ and $J(t)$ of $h$ and $J$ in $K((t))$, hence

$$\mathrm{Res}_P(\omega_\rho) = \frac{1}{\rho!}\frac{\partial^\rho}{\partial t^\rho}\left(\frac{h(t)}{J(t)}\right)\bigg|_{t=0}.$$

For each $r \in K(X_1,\dots,X_n)$ the chain rule for differentiation yields

$$\frac{\partial r}{\partial f_n} = J^{-1} \cdot \frac{\partial(f_1,\dots,f_{n-1},r)}{\partial(X_1,\dots,X_n)},$$

therefore

$$\begin{aligned}\mathrm{Res}_P(\omega_\rho) &= \frac{1}{\rho!}\frac{\partial^{\rho-1}}{\partial t^{\rho-1}}\left(\frac{\partial}{\partial t}\left(\frac{h(t)}{J(t)}\right)\right)\bigg|_{t=0}\\ &= \frac{1}{\rho!}\cdot\frac{\partial^{\rho-1}}{\partial t^{\rho-1}}\Big(R_1(t)\Big)\bigg|_{t=0}\end{aligned}$$

and inductively

$$\mathrm{Res}_P(\omega_\rho) = \frac{1}{\rho!}R_\rho(P).$$

b) Let $P \in C$, $P \notin H$, and assume again that $P = 0$. As $x_n$ is contained in the maximal ideal of $\widehat{\mathcal{O}_{C,P}}$ and is not a zero-divisor of this ring, the extension $\widehat{\mathcal{O}_{C,P}}/K[[x_n]]$ is finite. Since $\widehat{\mathcal{O}_{C,P}} = K[[X_1,\dots,X_n]]/(f_1,\dots,f_{n-1})$ is a complete intersection the Dedekind complementary module of $\widehat{\mathcal{O}_{C,P}}/K[[x_n]]$ is

$$\frac{1}{\Delta_n}\cdot\widehat{\mathcal{O}_{C,P}},$$

i.e. $\frac{1}{\Delta_n}\cdot\widehat{\mathcal{O}_{C,P}} = \{z \in Q(\widehat{\mathcal{O}_{C,P}}) \mid \mathrm{Tr}(r\cdot z) \subset K[[x_n]] \text{ for all } r \in \widehat{\mathcal{O}_{C,P}}\}$ where $\mathrm{Tr} : Q(\widehat{\mathcal{O}_{C,P}}) \to K((X_n))$ is the standard trace, see [**7**], G.12 for instance. By the trace formula for residues (see [**4**], (2))

$$\mathrm{Res}_P(\omega_\rho) = \mathrm{Res}_{x_n}(\mathrm{Tr}(\frac{1}{\Delta_n}\cdot\frac{h}{f_n^{\rho+1}})dx_n) = 0$$

since $f_n$ is a unit in $\mathcal{O}_{C,P}$ and $\mathrm{Tr}(\frac{1}{\Delta_n}\frac{h}{f_n^{\rho+1}}) \in K[[x_n]]$ by what was said above. Here $\mathrm{Res}_{x_n}$ denotes the residue with respect to the parameter $x_n$ of $K[[x_n]]$.

c) We turn now to the residues at $Q \in C_\infty$, $Q \notin H_n$. An easy calculation shows that $\widetilde{\Delta_n}$ can be expressed as

$$\widetilde{\Delta_n} = (-1)^{n+1}\det[(\widetilde{f_i})_{Z_k}]_{i,k=1,\dots,n-1} = (-1)^{n+1}\cdot(\tilde{\Delta})_0,$$

where $\tilde{\Delta}$ is the Jacobian matrix of $\tilde{f}_1,\dots,\tilde{f}_{n-1}$ with respect to $Z_0,\dots,Z_{n-1}$ and $(\tilde{\Delta})_0$ is the minor obtained by deleting its first column. By our assumptions on the curve $\Gamma$ the ring $\widehat{\mathcal{O}_{\Gamma,Q}}$ is a complete intersection of the form $\widehat{\mathcal{O}_{\Gamma,Q}} = K[[Z_0,\dots,Z_{n-1}]]/(\tilde{f}_1,\dots,\tilde{f}_{n-1})$. The element $z_0 = \frac{1}{x_n}$ is a non-unit of $\widehat{\mathcal{O}_{\Gamma,Q}}$ and not a zerodivisor, hence $\widehat{\mathcal{O}_{\Gamma,Q}}/K[[z_0]]$ is a finite extension with complementary module

$$\frac{1}{(\tilde{\Delta})_0}\cdot\widehat{\mathcal{O}_{\Gamma,Q}}.$$

As $Q \notin H_n$, we have $\tilde{f}_n(Q) \neq 0$, hence $\tilde{f}_n$ is a unit in $\widehat{\mathcal{O}_{\Gamma,Q}}$. Since

$$\deg h \le \sum d_i + \rho\cdot d_n - n - 1$$

we have by (5) that $m+2 \le 0$, and with the trace $\mathrm{Tr} : Q(\widehat{\mathcal{O}_{\Gamma,Q}}) \to K((z_0))$ we obtain similarly as above

$$\begin{aligned}\mathrm{Res}_Q(\omega_\rho) &= (-1)^{n+1}\,\mathrm{Res}_Q(\frac{1}{(\tilde{\Delta})_0}\cdot\frac{\tilde{h}\cdot z_0^{-m-2}}{\tilde{f}_n^{\rho+1}}dz_0)\\ &= (-1)^{n+1}\,\mathrm{Res}_{z_0}(\mathrm{Tr}(\frac{1}{(\tilde{\Delta})_0}\cdot\frac{\tilde{h}\cdot z_0^{-m-2}}{\tilde{f}_n^{\rho+1}})\cdot dz_0) = 0.\end{aligned}$$

d) Finally we show that the residues of $\omega_\rho$ at the points $Q \in C_\infty \cap H_n$ vanish. Consider as in section 1 the ring extensions

$$K[[z_0]] \hookrightarrow \widehat{\mathcal{O}_{\Gamma,Q}} \hookrightarrow \overline{\mathcal{Z}_1}\times\cdots\times\overline{\mathcal{Z}_{\eta_Q}} = K[[t_1]]\times\cdots\times K[[t_{\eta_Q}]].$$

As elements of $K[[t_1]] \times \cdots \times K[[t_{\eta_Q}]]$ we can write for $z_0$ and $\tilde{f}_n$

$$z_0 = (t_1^{\nu_1}, \dots, t_{\eta_Q}^{\nu_{\eta_Q}}) \cdot \varepsilon$$

$$\tilde{f}_n = (t_1^{\nu'_1}, \dots, t_{\eta_Q}^{\nu'_{\eta_Q}}) \cdot \eta$$

with units $\varepsilon$ and $\eta$ and $\nu_k = \mu_Q(\mathcal{Z}_k, H_\infty)$, $\nu'_k = \mu_Q(\mathcal{Z}_k, H_n)$ $(k = 1, \dots, \eta_Q)$. By the trace formula it suffices to show that the residue with respect to $z_0$ of the following differential vanishes:

$$\mathrm{Tr}\left(\frac{1}{(\tilde{\Delta})_0}\left[\frac{\tilde{h} \cdot \varepsilon \cdot (t_1^{(-m-2)\nu_1-(\rho+1)\nu'_1}, \dots, t_{\eta_Q}^{(-m-2)\nu_{\eta_Q}-(\rho+1)\nu'_{\eta_Q}})}{\eta^{\rho+1}}\right]\right) dz_0$$

where $\mathrm{Tr} : Q(\widehat{\mathcal{O}_{\Gamma,Q}}) \to K((z_0))$ is again the standard trace. If

$$(-m-2)\nu_k - (\rho+1)\nu'_k \geq c_k \qquad (k = 1, \dots, \eta_Q) \tag{7}$$

then the function inside the bracket [ ] belongs to $\widehat{\mathcal{O}_{\Gamma,Q}}$, and the residue vanishes by the arguments used previously. By (5) condition (7) is equivalent with

$$-\deg h - 2 + \sum_{i=1}^{n} d_i + \rho d_n - n + 1 - \frac{\nu'_k}{\nu_k}(\rho+1) - \frac{c_k}{\nu_k} \geq 0,$$

hence with

$$\deg h \leq \sum_{i=1}^{n} d_i + \rho d_n - n - 1 - \frac{1}{\nu_k}(c_k + \nu'_k(\rho+1)).$$

The degree assumption of the theorem guarantees that this is the case.

The proof of the theorem is completed by substituting the residues of a)–d) into formula (6).

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