Shrinking projection methods for a split equilibrium problem and a nonspreading-type multivalued mapping

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Abstract

We propose in this paper the shrinking projection method for finding common elements of the set of fixed points of a nonspreading-type multivalued mapping and the set of solutions of split equilibrium problems. We then prove strong convergence theorems in Hilbert spaces. Furthermore, we give an example and numerical results to illustrate our main theorem.

Keywords: Nonspreading-type multivalued mapping; Monotone hybrid method; Fixed point; Strong convergence; Hausdorff metric space.

AMS Subject Classification: 47H10, 47H09, 54H25.

1 Introduction

In what follows, let H_1 and H_2 be real Hilbert spaces with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Let C and Q be a nonempty convex subsets of H_1 and H_2, respectively. A subset $C \subset H_1$ is said to be *proximinal* if for each $x \in H_1$, there exists $y \in C$ such that

$$\|x - y\| = d(x, C) = \inf\{\|x - z\| : z \in C\}.$$

Let $CB(C), K(C)$ and $P(C)$ denote the families of nonempty closed bounded subsets, nonempty compact subsets and nonempty proximinal bounded subset of C, respectively. The *Hausdorff metric* on $CB(C)$ is defined by

$$H(A, B) = \max\left\{ \sup_{x\in A} d(x, B),\ \sup_{y\in B} d(y, A)\right\}$$

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for all $A, B \in CB(C)$ where $d(x, B) = \inf_{b\in B} \|x - b\|$. A singlevalued mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\|$$

for all $x, y \in C$. A multivalued mapping $T : C \to CB(C)$ is said to be *nonexpansive* if

$$H(Tx, Ty) \le \|x - y\|$$

for all $x, y \in C$. An element $z \in C$ is called a *fixed point* of $T : C \to C$ (resp., $T : C \to CB(C)$) if $z = Tz$ (resp., $z \in Tz$). The fixed point set of T is denoted by $F(T)$. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ implies that $\{x_n\}$ converges strongly to x.

Recent fixed point results for multivalued mappings can be found in [1, 7, 12, 14, 15, 16, 17, 21] and references therein.

A mapping $T : C \to CB(C)$ is said to be *demiclosed* at 0 if $\{x_n\} \subset C$ such that $x_n \rightharpoonup x$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ imply $x \in Tx$.

Let $F_1 : C \times C \to \mathbb{R}$ be a bifunction. The *equilibrium problem* is to find a point $\hat{x} \in C$ such that

$$F_1(\hat{x}, y) \ge 0 \tag{1.1}$$

for all $y \in C$, which has been introduced and studied by Blum and Oettli [2]. The solution set of the equilibrium problem (1.1) is denoted by $EP(F_1)$.

Recently, Combettes and Hirstoaga [4] introduced and studied an iterative method for finding the best approximation to the initial data when $EP(F_1) \ne \emptyset$ and prove a strong convergence theorem. Subsequently, Takahashi et al.[18] introduced a new projection method called the *shrinking projection method* for finding the common element of the set of solution of equilibriums and the set of fixed points for a nonexpansive singlevalued mapping in Hilbert spaces. They proved the following theorem:

Theorem 1.1. [18] *Let H_1 be a Hilbert space and C be a nonempty closed convex subset of H_1. Let $\{T_n\}$ and τ be a family of nonexpansive mappings of C into H such that $F := \cap_{n=1}^{\infty} F(T_n) = F(\tau) \ne \emptyset$ and let $x_0 \in H$. Suppose that $\{T_n\}$ satisfies the NST-condition (I) with τ. For $C_1 = C$ and $u_1 = P_{C_1} x_0$, define a sequence $\{u_n\}$ in C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1 - \alpha_n) T_n u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $0 \le \alpha_n \le a < 1$ for all $n \in \mathbb{N}$. Then the sequence $\{u_n\}$ converges strongly to a point $z_0 = P_F x_0$.

Very recently, Kazmi and Rizvi [8] introduced and studied the following split equilibrium problem which is a generalization of the equilibrium problem:

Let $C \subseteq H_1$ and $Q \subseteq H_2$. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be two bifunctions. Let $A : H_1 \to H_2$ be a bounded linear operator. The *split equilibrium problem* is to find $\hat{x} \in C$ such that

$$F_1(\hat{x}, x) \geq 0 \text{ for all } x \in C \tag{1.3}$$

and

$$\hat{y} = A\hat{x} \in Q \text{ solves } F_2(\hat{y}, y) \geq 0 \text{ for all } y \in Q. \tag{1.4}$$

Note that the inequality (1.3) is the classical equilibrium problem and we denote its solution set by $EP(F_1)$. The problems (1.3) and (1.4) constitute a pair of equilibrium problems which have to find the image $\hat{y} = A\hat{x}$, under a given bounded linear operator A, of the solution $\hat{x}$ of the problem (1.3) in H_1 which is the solution of the problem (1.4) in H_2. It's easy to see that the split equilibrium problem generalize an equilibrium problem. We denote the solution set of the problem (1.4) by $EP(F_2)$. The solution set of the split equilibrium (1.3) and (1.4) is denoted by $\Omega = \{z \in EP(F_1) : Az \in EP(F_2)\}$.

In the recent years, the problem of finding a common element of the set of solution of split equilibriums and the set of fixed points for a singlevalued mapping in the framework of Hilbert spaces and Banach spaces have been intensively studied by many authors, for instance, (see [5, 8, 19, 20]) and the references cited therein.

In 2008, Kohsaka and Takahashi [10] introduced a new class of mappings, which is called the class of *nonspreading mappings*.

Let H be a Hilbert space and C be nonempty closed convex subset of H. Then a mapping $T : C \to C$ is said to be nonspreading if

$$2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2$$

for all $x, y \in C$. Recently, Iemoto and Takahashi [6] showed that $T : C \to C$ is nonspreading if and only if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Ty, y - Ty\rangle, \ \forall x, y \in C.$$

Very recently, Liu [11] introduced the following class of multi-valued mappings:

A mapping $T : C \to CB(C)$ is called *nonspreading* if

$$2\|u_x - u_y\|^2 \leq \|u_x - y\|^2 + \|u_y - x\|^2, \ for \ u_x \in Tx, \ u_y \in Ty, \ \forall x, y \in C.$$

for all $u_x \in Tx$ and $u_y \in Ty$ for all $x, y \in C$. Also, he proved a weak convergence theorem for finding a common element of the set of solutions of an equilibrium problem and the set of common fixed points.

In this paper, inspired by Liu [11] and Takahashi et al.[18], we define and study a new multivalued mapping which is called nonspreading-type by using the Hausdorff metric. We then introduce an iterative method by using the shrinking projection method for finding the common element of the set of solutions of a split equilibrium problem and the set of fixed points of a nonspreading-type multivalued mapping, also, obtain strong convergence theorems in a Hilbert space. Furthermore, we give an example and numerical results for supporting our main theorem.

2 Preliminaries

We now provide some results for the main results. In a Hilbert space H_1, let C be a nonempty closed convex subset of H_1. For every point $x \in H_1$, there exists a unique nearest point of C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. Such a P_C is called the metric projection from H_1 on to C. Further, for any $x \in H_1$ and $z \in C$, $z = P_C x$ if and only if

$$\langle x - z, z - y \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.1. *Let H_1 be a real Hilbert space. Then the following equations hold:*

(1) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle$ *for all* $x, y \in H_1$*;*

(2) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle$ *for all* $x, y \in H_1$*;*

(3) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2$ *for all* $t \in [0, 1]$ *and* $x, y \in H_1$*;*

(4) *If* $\{x_n\}_{n=1}^{\infty}$ *is a sequence in* H_1 *which converges weakly to* $z \in H_1$*, then*

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - z\|^2 + \|z - y\|^2$$

for all $y \in H_1$*.*

Lemma 2.2. [13] *Let C be a nonempty, closed and convex subset of a real Hilbert space H_1 and $P_C : H_1 \to C$ be the metric projection from H_1 onto C. Then the following inequality holds:*

$$\|y - P_C x\|^2 + \|x - P_C x\|^2 \leq \|x - y\|^2, \quad \forall x \in H_1, \ \forall y \in C.$$

Lemma 2.3. [9] *Let C be a nonempty, closed and convex subset of a real Hilbert space H_1. Given $x, y, z \in H_1$ and also given $a \in \mathbb{R}$, the set*

$$\{v \in C : \ \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v \rangle + a\}$$

is convex and closed.

Assumption 2.4. [2] *Let $F_1 : C \times C \to \mathbb{R}$ be a bifunction satisfying the following assumptions:*

(1) $F_1(x, x) = 0$ *for all* $x \in C$*;*

(2) *F_1 is monotone, i.e., $F_1(x,y)+F_1(y,x)\le 0$ for all $x\in C$;*

(3) *For each $x,y,z\in C$, $\limsup_{t\to 0} F_1(tz+(1-t)x,y)\le F_1(x,y)$;*

(4) *For each $x\in C$, $y\to F_1(x,y)$ is convex and lower semi-continuous.*

Lemma 2.5. [4] *Let $F_1 : C\times C\to\mathbb{R}$ be a bifunction satisfying Assumption 2.4. For any $r>0$ and $x\in H_1$, define a mapping $T_r^{F_1} : H_1\to C$ as follows:*

$$T_r^{F_1}(x)=\Big\{z\in C : F_1(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\ge 0,\ \forall y\in C\Big\}.$$

Then we have the following:

(1) *$T_r^{F_1}$ is nonempty and single-value;*

(2) *$T_r^{F_1}$ is firmly nonexpansive, i.e., for any $x,y\in H_1$,*

$$\|T_r^{F_1}x-T_r^{F_1}y\|^2\le\langle T_r^{F_1}x-T_r^{F_1}y,x-y\rangle;$$

(3) *$F(T_r^{F_1})=EP(F_1)$;*

(4) *$EP(F_1)$ is closed and convex.*

Further, assume that $F_2 : Q\times Q\to\mathbb{R}$ satisfying Assumption 2.4. For each $s>0$ and $w\in H_2$, define a mapping $T_s^{F_2} : H_2\to Q$ as follows:

$$T_s^{F_2}(w)=\Big\{d\in Q : F_2(d,e)+\frac{1}{s}\langle e-d,d-w\rangle\ge 0,\ \forall e\in Q\Big\}.$$

Then we have the following:

(5) $T_s^{F_2}$ is nonempty and single-value;

(6) $T_s^{F_2}$ is firmly nonexpansive;

(7) $F(T_s^{F_2})=EP(F_2,Q)$;

(8) $EP(F_2,Q)$ is closed and convex.

Condition(A). Let H_1 be a Hilbert space and C be a subset of H_1. A multi-valued mapping $T : C\to CB(C)$ is said to satisfy *Condition (A)* if $\|x-p\|=d(x,Tp)$ for all $x\in H_1$ and $p\in F(T)$.

Remark 2.6. We see that T satisfies Condition (A) if and only if $Tp=\{p\}$ for all $p\in F(T)$. It is known that the best approximation operator P_T, which is defined by $P_Tx=\{y\in Tx : \|y-x\|=d(x,Tx)\}$, also satisfies Condition (A).

3 Main results

Let H_1 be a real Hilbert space and C be a nonempty convex subset of H_1. In this paper, we introduce, by using Hausdorff metric, the class of nonspreading multivalued mappings. We say that a mapping $T : C \to CB(C)$ is a *k-nonspreading multivalued mapping* if there exists $k > 0$ such that

$$H(Tx, Ty)^2 \leq k\big(d(Tx, y)^2 + d(x, Ty)^2\big) \tag{3.1}$$

for all $x, y \in C$.

It is easy to see that, if T is $\frac{1}{2}$-nonspreading, then T is nonspreading in the case of singlevalued mappings (see [10]). Moreover, if T is a $\frac{1}{2}$-nonspreading and $F(T) \neq \emptyset$, then T is quasi-nonexpansive. Indeed, for all $x \in C$ and $p \in F(T)$, we have

$$\begin{aligned} 2H(Tx, Tp)^2 &\leq d(Tx, p)^2 + d(x, Tp)^2 \\ &\leq H(Tx, Tp)^2 + \|x - p\|^2. \end{aligned}$$

It follows that

$$H(Tx, Tp) \leq \|x - p\|. \tag{3.2}$$

We say that a mapping $T : C \to CB(C)$ is a *nonspreading-type multivalued mapping* if T is $\frac{1}{2}$-nonspreading.

Now, we give an example of a nonspreading-type multivalued mapping which is not a nonexpansive multivalued mapping.

Example 3.1. Consider $C = [-3, 0]$ with the usual norm. Define a multivalued mapping $T : C \to CB(C)$ by

$$Tx = \begin{cases} \{0\}, & x \in [-2, 2]; \\ \big[-\exp\{x + 2\}, 0\big], & x \notin [-2, 2]. \end{cases}$$

To see that T is nonspreading-type, we observe the following cases:

Case 1: if $x, y \in [-2, 0]$, then $H(Tx, Tx) = 0$.

Case 2: if $x \in [-2, 0]$ and $y \notin [-2, 0]$, then $Tx = \{0\}$ and $Ty = \big[-\exp\{y + 2\}, 0\big]$. This implies that

$$2H(Tx, Ty)^2 = 2\Big(-\exp\{y + 2\}\Big)^2 < 2 < d(Tx, y)^2 + d(x, Ty)^2.$$

Case 3: if $x, y \notin [-2, 2]$, then $Tx = \big[-\exp\{x + 2\}, 0\big]$ and $Ty = \big[-\exp\{y + 2\}, 0\big]$. This implies that

$$2H(Tx, Ty)^2 = 2\Big(-\exp\{x + 2\} + \exp\{y + 2\}\Big)^2 < 2 < d(Tx, y)^2 + d(x, Ty)^2.$$

But T is not nonexpansive since for $x = -2$ and $y = -\frac{9}{4}$, we have $Tx = \{0\}$ and $Ty = \big[-\frac{1}{\exp\{1/4\}}, 0\big]$. This implies that $H(Tx, Ty) = \frac{1}{\exp\{1/4\}} > \frac{1}{4} = \big| -2 - \big(-\frac{9}{4}\big)\big| = \|x - y\|$.

Let C be a nonempty set in a Hilbert space H_1. We define $T(C) = \cup_{x \in C} Tx$ and $(ST)x = S(Tx)$ for all $x \in C$. Now, we are ready to prove some convergence theorem for a nonspreading-type multivalued mapping in Hilbert spaces. To this end, we need the following crucial results:

Lemma 3.2. *Let C be a closed convex subset of a real Hilbert space H_1. Let $T : C \to CB(C)$ be a nonspreading-type multivalued mapping and $F(T) \neq \emptyset$. Then the followings hold*

(i) *$F(T)$ is closed;*

(ii) *if T satisfies Condition (A), then $F(T)$ is convex.*

Proof. (i) Let $\{x_n\}$ be a sequence in $F(T)$ such that $x_n \to x$ as $n \to \infty$. We have

$$\begin{aligned} d(x, Tx) &\leq \|x - x_n\| + d(x_n, Tx) \\ &\leq \|x - x_n\| + H(Tx_n, Tx) \\ &\leq 2\|x - x_n\|. \end{aligned}$$

It follows that $d(x, Tx) = 0$. Hence $x \in F(T)$.
(ii) Let $p = tp_1 + (1-t)p_2$, where $p_1, p_2 \in F(T)$ and $t \in (0,1)$. Let $z \in Tp$. It follows from (3.2) that

$$\begin{aligned} \|p - z\|^2 &= \|t(z - p_1) + (1-t)(z - p_2)\|^2 \\ &= t\|z - p_1\|^2 + (1-t)\|z - p_2\|^2 - t(1-t)\|p_1 - p_2\|^2 \\ &= td(z, Tp_1)^2 + (1-t)d(z, Tp_2)^2 - t(1-t)\|p_1 - p_2\|^2 \\ &\leq tH(Tp, Tp_1)^2 + (1-t)H(Tp, Tp_2)^2 - t(1-t)\|p_1 - p_2\|^2 \\ &\leq t\|p - p_1\|^2 + (1-t)\|p - p_2\|^2 - t(1-t)\|p_1 - p_2\|^2 \\ &= t(1-t)^2\|p_1 - p_2\|^2 + (1-t)t^2\|p_1 - p_2\|^2 - t(1-t)\|p_1 - p_2\|^2 \\ &= 0 \end{aligned}$$

and hence $p = z$. Therefore, $p \in F(T)$. This completes the proof. □

Lemma 3.3. *Let C be a closed and convex subset of a real Hilbert space H_1 and $T : C \to K(C)$ be a k-nonspreading multivalued mapping such that $k \in (0, \frac{1}{2}]$. If $x, y \in C$ and $a \in Tx$, then there exists $b \in Ty$ such that*

$$\|a - b\|^2 \leq H(Tx, Ty)^2 \leq \frac{k}{1-k}\big(\|x - y\|^2 + 2\langle x - a, y - b\rangle\big).$$

Proof. Let $x, y \in C$ and $a \in Tx$. By Nadler's Theorem (see [12]), there exists $b \in Ty$ such that

$$\|a - b\|^2 \leq H(Tx, Ty)^2.$$

It follows that

$$\begin{aligned}
& \frac{1}{k}H(Tx,Ty)^2 \\
\leq\ & d(Tx,y)^2 + d(x,Ty)^2 \\
\leq\ & \|a-y\|^2 + \|x-b\|^2 \\
\leq\ & \|a-x\|^2 + 2\langle a-x, x-y\rangle + \|x-y\|^2 + \|x-a\|^2 + 2\langle x-a, a-b\rangle + \|a-b\|^2 \\
=\ & 2\|a-x\|^2 + \|x-y\|^2 + \|a-b\|^2 + 2\langle a-x, x-a-(y-b)\rangle \\
\leq\ & 2\|a-x\|^2 + \|x-y\|^2 + H(Tx,Ty)^2 + 2\langle a-x, x-a-(y-b)\rangle.
\end{aligned}$$

This implies that

$$H(Tx,Ty)^2 \leq \frac{k}{1-k}\big(\|x-y\|^2 + 2\langle x-a, y-b\rangle\big).$$

This completes the proof. □

Lemma 3.4. *Let C be a closed and convex subset of a real Hilbert space H_1 and $T : C \to K(C)$ be a k-nonspreading multivalued mapping such that $k \in (0, \frac{1}{2}]$. Let $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup p$ and $\lim_{n\to\infty}\|x_n - y_n\| = 0$ for some $y_n \in Tx_n$. Then $p \in Tp$.*

Proof. Let $\{x_n\}$ be a sequence in C which converges weakly to p and let $y_n \in Tx_n$ be such that $\|x_n - y_n\| \to 0$.

Now, we show that $p \in F(T)$. By Lemma 3.4, there exists $z_n \in Tp$ such that

$$\|y_n - z_n\|^2 \leq \frac{k}{1-k}\big(\|x_n - p\|^2 + 2\langle x_n - y_n, p - z_n\rangle\big).$$

Since Tp is compact and $z_n \in Tp$, there exists $\{z_{n_i}\} \subset \{z_n\}$ such that $z_{n_i} \to z \in Tp$. Since $\{x_n\}$ converges weakly, it is bounded. For each $x \in H_1$, define a function $f : H_1 \to [0,\infty)$ by

$$f(x) := \limsup_{i\to\infty} \frac{k}{1-k}\|x_{n_i} - x\|^2.$$

Then, by Lemma 2.1(4), we obtain

$$f(x) = \limsup_{i\to\infty} \frac{k}{1-k}\big(\|x_{n_i} - p\|^2 + \|p - x\|^2\big)$$

for all $x \in H_1$. Thus $f(x) = f(p) + \frac{k}{1-k}\|p - x\|^2$ for all $x \in H_1$. It follows that

$$f(z) = f(p) + \frac{k}{1-k}\|p - z\|^2. \tag{3.3}$$

We observe that

$$f(z) = \limsup_{i\to\infty} \frac{k}{1-k}\|x_{n_i} - z\|^2 = \limsup_{i\to\infty} \frac{k}{1-k}\|x_{n_i} - y_{n_i} + y_{n_i} - z\|^2 \leq \limsup_{i\to\infty} \frac{k}{1-k}\|y_{n_i} - z\|^2.$$

This implies that

$$\begin{aligned} f(z) &\leq \limsup_{i\to\infty} \frac{k}{1-k}\|y_{n_i} - z\|^2 \\ &= \limsup_{i\to\infty} \frac{k}{1-k}\big(\|y_{n_i} - z_{n_i} + z_{n_i} - z\|\big)^2 \\ &\leq \limsup_{i\to\infty} \frac{k}{1-k}\big(\|x_{n_i} - p\|^2 + 2\langle x_{n_i} - y_{n_i}, p - z_{n_i}\rangle\big) \\ &\leq \limsup_{i\to\infty} \frac{k}{1-k}\|x_{n_i} - p\|^2 \\ &= f(p). \end{aligned} \tag{3.4}$$

Hence it follows from (3.3) and (3.4) that $\|p - z\| = 0$. This completes the proof. □

Theorem 3.5. *Let H_1, H_2 be two real Hilbert spaces and $C \subset H_1$, $Q \subset H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and $T : C \to K(C)$ a nonspreading-type multivalued mapping. Let $F_1 : C \times C \to \mathbb{R}$, $F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumtion 2.4 and F_2 is upper semi-continuous in the first argument. Assume that $\Theta = F(T) \cap \Omega \neq \emptyset$, where $\Omega = \{z \in C : z \in EP(F_1)$ and $Az \in EP(F_2)\}$. For an initial point $x_1 \in H_1$ with $C_1 = C$, let $\{u_n\}$, $\{y_n\}$ and $\{x_n\}$ be sequences defined by*

$$\begin{cases} u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n, \\ y_n \in \alpha_n u_n + (1-\alpha_n)Tu_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \ \forall n \geq 1 \end{cases} \tag{3.5}$$

where $\{\alpha_n\} \subset (0,1)$, $r_n \subset (0,\infty)$ and $\gamma \in (0, 1/L)$ such that L is the spectral radius of A^*A and A^* is the adjoint of A. Assume that the following conditions hold:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(ii) $\liminf_{n\to\infty} r_n > 0$.

If T satisfies Condition (A), then the sequences $\{x_n\}$, $\{y_n\}$ and $\{x_n\}$ converge strongly to $P_\Theta x_1$.

Proof. We split the proof into six steps.

Step 1. Show that $P_{C_{n+1}}x_1$ is well-defined for every $x_1 \in H_1$.

By Lemma 3.2, we obtain that $F(T)$ is closed and convex. Since A is a bounded linear operator, it is easy to prove that Ω is closed and convex. So, $\Theta = F(T) \cap \Omega$ is also closed and convex. From the definition of C_{n+1}, it follows from Lemma 2.3 that C_{n+1} is closed and convex for each $n \geq 1$.

Since $T_{r_n}^{F_2}$ is firmly nonexpansive and $I - T_{r_n}^{F_2}$ is 1-inverse strongly monotone, we see that

$$\begin{aligned}
\|A^*(I - T_{r_n}^{F_2})Ax - A^*(I - T_{r_n}^{F_2})Ay\|^2 &= \langle A^*(I - T_{r_n}^{F_2})(Ax - Ay), A^*(I - T_{r_n}^{F_2})(Ax - Ay)\rangle \\
&= \langle (I - T_{r_n}^{F_2})(Ax - Ay), AA^*(I - T_{r_n}^{F_2})(Ax - Ay)\rangle \\
&\leq L\langle (I - T_{r_n}^{F_2})(Ax - Ay), (I - T_{r_n}^{F_2})(Ax - Ay)\rangle \\
&= L\|(I - T_{r_n}^{F_2})(Ax - Ay)\|^2 \\
&\leq L\langle Ax - Ay, (I - T_{r_n}^{F_2})(Ax - Ay)\rangle \\
&= L\langle x - y, A^*(I - T_{r_n}^{F_2})Ax - A^*(I - T_{r_n}^{F_2})Ay\rangle
\end{aligned}$$

for all $x, y \in H_1$. This implies that $A^*(I - T_{r_n}^{F_2})A$ is a $\frac{1}{L}$-inverse strongly monotone mapping. Since $\gamma \in (0, \frac{1}{L})$, it follows that $I - \gamma A^*(I - T_{r_n}^{F_2})A$ is nonexpansive. Let $p \in \Theta$. Then $p = T_{r_n}^{F_1}p$ and $(I - \gamma A^*(I - T_{r_n}^{F_2})A)p = p$. Thus, we have

$$\begin{aligned}
\|u_n - p\| &= \|T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n - T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)p\| \\
&\leq \|(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n - (I - \gamma A^*(I - T_{r_n}^{F_2})A)p\| \\
&\leq \|x_n - p\|. \qquad (3.6)
\end{aligned}$$

This implies that

$$\begin{aligned}
\|y_n - p\| &= \|\alpha_n u_n + (1 - \alpha_n)z_n - p\| \\
&\leq \alpha_n\|u_n - p\| + (1 - \alpha_n)\|z_n - p\| \\
&= \alpha_n\|u_n - p\| + (1 - \alpha_n)d(z_n, Tp) \\
&\leq \alpha_n\|u_n - p\| + (1 - \alpha_n)H(Tu_n, Tp) \\
&\leq \|u_n - p\|
\end{aligned}$$

for all $z_n \in Tu_n$. So, we have $p \in C_{n+1}$, thus $\Theta \subset C_{n+1}$. Therefore $P_{C_{n+1}}x_1$ is well defined.

Step 2. Show that $\lim_{n\to\infty} \|x_n - x_1\|$ exists.

Since Θ is a nonempty, closed and convex subset of H_1, there exists a unique $v \in \Theta$ such that

$$v = P_\Theta x_1.$$

From $x_n = P_{C_n}x_1$, $C_{n+1} \subset C_n$ and $x_{n+1} \in C_n$, $\forall n \geq 1$, we get

$$\|x_n - x_1\| \leq \|x_{n+1} - x_1\|, \quad \forall n \geq 1.$$

On the other hand, as $\Theta \subset C_n$, we obtain

$$\|x_n - x_1\| \leq \|v - x_1\|, \quad \forall n \geq 1.$$

It follows that the sequence $\{x_n\}$ is bounded and nondecreasing. Therefore $\lim_{n\to\infty} \|x_n - x_1\|$ exists.

Step 3. Show that $x_n \to w \in C$ as $n \to \infty$.

For $m > n$, by the definition of C_n, we see that $x_m = P_{C_m} x_1 \in C_m \subset C_n$. By Lemma 2.2, we get

$$\|x_m - x_n\|^2 \le \|x_m - x_1\|^2 - \|x_n - x_1\|^2.$$

From Step 2, we obtain that $\{x_n\}$ is Cauchy. Hence, there exists $w \in C$ such that $x_n \to w$ as $n \to \infty$.

Step 4. Show that $w \in F(T)$.

From Step 3, we get

$$\|x_{n+1} - x_n\| \to 0 \tag{3.7}$$

as $n \to \infty$. Since $x_{n+1} \in C_{n+1} \subset C_n$, we have

$$\|y_n - x_n\| \le \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \le 2\|x_{n+1} - x_n\| \to 0 \tag{3.8}$$

as $n \to \infty$. Hence, $y_n \to w$ as $n \to \infty$. For $p \in \Theta$, we estimate

$$\begin{aligned} \|u_n - p\|^2 &= \|T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n - p\|^2 \\ &= \|T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n - T_{r_n}^{F_1} p\|^2 \\ &\le \|x_n - \gamma A^*(I - T_{r_n}^{F_2})Ax_n - p\|^2 \\ &\le \|x_n - p\|^2 + \gamma^2 \|A^*(I - T_{r_n}^{F_2})Ax_n\|^2 + 2\gamma\langle p - x_n, A^*(I - T_{r_n}^{F_2})Ax_n\rangle. \end{aligned}$$

Thus we have

$$\begin{aligned} \|u_n - p\|^2 \le\; & \|x_n - p\|^2 + \gamma^2\langle Ax_n - T_{r_n}^{F_2}Ax_n, AA^*(I - T_{r_n}^{F_2})Ax_n\rangle \\ & +2\gamma\langle p - x_n, A^*(I - T_{r_n}^{F_2})Ax_n\rangle. \end{aligned} \tag{3.9}$$

On the other hand, we have

$$\begin{aligned} \gamma^2\langle Ax_n - T_{r_n}^{F_2}Ax_n, AA^*(I - T_{r_n}^{F_2})Ax_n\rangle &\le L\gamma^2\langle Ax_n - T_{r_n}^{F_2}Ax_n, Ax_n - T_{r_n}^{F_2}Ax_n\rangle \\ &= L\gamma^2\|Ax_n - T_{r_n}^{F_2}Ax_n\|^2 \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} 2\gamma\langle p - x_n, A^*(I - T_{r_n}^{F_2})Ax_n\rangle &= 2\gamma\langle A(p - x_n), Ax_n - T_{r_n}^{F_2}Ax_n\rangle \\ &= 2\gamma\langle A(p - x_n) + (Ax_n - T_{r_n}^{F_2}Ax_n) \\ &\quad -(Ax_n - T_{r_n}^{F_2}Ax_n), Ax_n - T_{r_n}^{F_2}Ax_n\rangle \\ &= 2\gamma\{\langle Ap - T_{r_n}^{F_2}Ax_n, Ax_n - T_{r_n}^{F_2}Ax_n\rangle - \|Ax_n - T_{r_n}^{F_2}Ax_n\|^2\} \\ &\le 2\gamma\{\frac{1}{2}\|Ax_n - T_{r_n}^{F_2}Ax_n\|^2 - \|Ax_n - T_{r_n}^{F_2}Ax_n\|^2\} \\ &= -\gamma\|Ax_n - T_{r_n}^{F_2}Ax_n\|^2. \end{aligned} \tag{3.11}$$

Using (3.9), (3.10) and (3.11), we have

$$\begin{aligned}\|u_n - p\|^2 &\leq \|x_n - p\|^2 + L\gamma^2\|Ax_n - T_{r_n}^{F_2}Ax_n\|^2 - \gamma\|Ax_n - T_{r_n}^{F_2}Ax_n\|^2 \\ &= \|x_n - p\|^2 + \gamma(L\gamma - 1)\|Ax_n - T_{r_n}^{F_2}Ax_n\|^2. \end{aligned} \tag{3.12}$$

It follows that, for all $z_n \in Tu_n$,

$$\begin{aligned}\|y_n - p\|^2 &= \|\alpha_n u_n + (1-\alpha_n)z_n - p\|^2 \\ &\leq \alpha_n\|u_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 \\ &= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)d(z_n, Tp)^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)H(Tu_n, Tp)^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|u_n - p\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)(\|x_n - p\|^2 + \gamma(L\gamma - 1)\|Ax_n - T_{r_n}^{F_2}Ax_n\|^2) \\ &\leq \|x_n - p\|^2 + \gamma(L\gamma - 1)\|Ax_n - T_{r_n}^{F_2}Ax_n\|^2. \end{aligned}$$

Therefore, we have

$$\begin{aligned} -\gamma(L\gamma - 1)\|Ax_n - T_{r_n}^{F_2}Ax_n\|^2 &\leq \|x_n - p\|^2 - \|y_n - p\|^2 \\ &\leq \big(\|x_n - p\| + \|y_n - p\|\big)\|x_n - y_n\|. \end{aligned}$$

It follows from $\gamma(L\gamma - 1) < 0$ and (3.8) that

$$\lim_{n\to\infty}\|Ax_n - T_{r_n}^{F_2}Ax_n\| = 0. \tag{3.13}$$

Since $T_{r_n}^{F_1}$ is firmly nonexpansive and $I - \gamma A^*(T_{r_n}^{F_2} - I)A$ is nonexpansive, it follows that

$$\begin{aligned} &\|u_n - p\|^2 \\ = \;& \|T_{r_n}^{F_1}(x_n - \gamma A^*(I - T_{r_n}^{F_2})Ax_n) - T_{r_n}^{F_1}p\|^2 \\ \leq \;& \langle T_{r_n}^{F_1}(x_n - \gamma A^*(I - T_{r_n}^{F_2})Ax_n) - T_{r_n}^{F_1}p, x_n - \gamma A^*(I - T_{r_n}^{F_2})Ax_n - p\rangle \\ = \;& \langle u_n - p, x_n - \gamma A^*(I - T_{r_n}^{F_2})Ax_n - p\rangle \\ = \;& \frac{1}{2}\{\|u_n - p\|^2 + \|x_n - \gamma A^*(I - T_{r_n}^{F_2})Ax_n - p\|^2 - \|u_n - x_n - \gamma A^*(I - T_{r_n}^{F_2})Ax_n\|^2\} \\ \leq \;& \frac{1}{2}\{\|u_n - p\|^2 + \|x_n - p\|^2 - \|u_n - x_n - \gamma A^*(I - T_{r_n}^{F_2})Ax_n\|^2\} \\ = \;& \frac{1}{2}\{\|u_n - p\|^2 + \|x_n - p\|^2 - (\|u_n - x_n\|^2 + \gamma^2\|A^*(I - T_{r_n}^{F_2})Ax_n\|^2 \\ - \;& 2\gamma\langle u_n - x_n, A^*(I - T_{r_n}^{F_2} - I)Ax_n\rangle)\}, \end{aligned}$$

which implies that

$$\begin{aligned}\|u_n - p\|^2 &\leq \|x_n - p\|^2 - \|u_n - x_n\|^2 + 2\gamma\langle u_n - x_n, A^*(I - T_{r_n}^{F_2})Ax_n\rangle \\ &\leq \|x_n - p\|^2 - \|u_n - x_n\|^2 + 2\gamma\|u_n - x_n\|\|A^*(I - T_{r_n}^{F_2})Ax_n\|. \end{aligned} \tag{3.14}$$

It follows from (3.6) that

$$\begin{aligned}\|y_n - p\|^2 &\le \alpha_n\|u_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)d(z_n, Tp)^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)H(Tu_n, Tp)^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|u_n - p\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)(\|x_n - p\|^2 \\ &\quad -\|u_n - x_n\|^2 + 2\gamma\|u_n - x_n\|\|A^*(I - T_{r_n}^{F_2})Ax_n\|)\end{aligned}$$

Therefore, we have

$$(1-\alpha_n)\|u_n - x_n\|^2 \le 2\gamma\|u_n - x_n\|\|A^*(I - T_{r_n}^{F_2})Ax_n\| + \|x_n - p\|^2 - \|y_n - p\|^2.$$

It follows from the condition (i), (3.8) and (3.13), we have

$$\lim_{n\to\infty}\|u_n - x_n\| = 0. \tag{3.15}$$

We know that $x_n \to w$ as $n \to \infty$, thus $u_n \to w$ as $n \to \infty$. It follows from Lemma 2.1 and (3.6), we have

$$\begin{aligned}\|y_n - p\|^2 &= \|\alpha_n u_n + (1-\alpha_n)z_n - p\|^2 \\ &\le \alpha_n\|u_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 - \alpha_n(1-\alpha_n)\|u_n - z_n\|^2 \\ &= \alpha_n\|u_n - p\|^2 + (1-\alpha_n)d(z_n, Tp)^2 - \alpha_n(1-\alpha_n)\|u_n - z_n\|^2 \\ &\le \alpha_n\|u_n - p\|^2 + (1-\alpha_n)H(Tu_n, Tp)^2 - \alpha_n(1-\alpha_n)\|u_n - z_n\|^2 \\ &\le \|u_n - p\|^2 - \alpha_n(1-\alpha_n)\|u_n - z_n\|^2 \\ &\le \|x_n - p\|^2 - \alpha_n(1-\alpha_n)\|u_n - z_n\|^2.\end{aligned}$$

This implies that

$$\begin{aligned}\alpha_n(1-\alpha_n)\|u_n - z_n\|^2 &\le \|x_n - p\|^2 - \|y_n - p\|^2 \\ &\le \big(\|x_n - p\| + \|y_n - p\|\big)\|x_n - y_n\|.\end{aligned}$$

It follows from the condition (i) and (3.8) that

$$\lim_{n\to\infty}\|u_n - z_n\| = 0. \tag{3.16}$$

By Lemma 3.4, we obtain $w \in F(T)$.

Step 5. Show that $w \in EP(F)$.

From $u_n = T_{r_n}^{F_1}(I + \gamma A^*(I - T_{r_n}^{F_2})A)x_n$, we have

$$F_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n - \gamma A^*(I - T_{r_n}^{F_2})Ax_n\rangle \;\ge\; 0$$

for all $y \in C$, which implies that

$$F_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle - \frac{1}{r_n}\langle y - u_n, \gamma A^*(I - T_{r_n}^{F_2})Ax_n\rangle \quad \geq \quad 0$$

for all $y \in C$. By Assumption 2.4 (2), we have

$$\frac{1}{r_{n_i}}\langle y - u_{n_i}, u_{n_i} - x_{n_i}\rangle - \frac{1}{r_{n_i}}\langle y - u_{n_i}, \gamma A^*(I - T_{r_{n_i}}^{F_1})Ax_{n_i}\rangle \quad \geq \quad F_1(y, u_{n_i})$$

for all $y \in C$. From $\liminf_{n\to\infty} r_n > 0$, from (3.12), (3.14) and the Assumption 2.4 (4), we obtain

$$F_1(y, w) \quad \leq \quad 0$$

for all $y \in C$. For any $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)w$. Since $y \in C$ and $w \in C$, $y_t \in C$ and hence $F_1(y_t, w) \leq 0$. So, by Assumption 2.4 (1) and (4), we have

$$0 = F_1(y_t, y_t) \leq tF_1(y_t, y) + (1-t)F_1(y_t, w) \leq tF_1(y_t, y)$$

and hence $F_1(y_t, y) \geq 0$. So $F_1(w, y) \geq 0$ for all $y \in C$ and hence $w \in EP(F_1)$. Since A is a bounded linear operator, $Ax_{n_i} \rightharpoonup Aw$. Then it follows from (3.13) that

$$T_{r_{n_i}}^{F_2}Ax_{n_i} \rightharpoonup Aw \tag{3.17}$$

as $i \to \infty$. By the definition of $T_{r_{n_i}}^{F_2}Ax_{n_i}$, we have

$$F_2(T_{r_{n_i}}^{F_2}Ax_{n_i}, y) + \frac{1}{r_{n_i}}\langle y - T_{r_{n_i}}^{F_2}Ax_{n_i}, T_{r_{n_i}}^{F_2}Ax_{n_i} - Ax_{n_i}\rangle \quad \geq \quad 0$$

for all $y \in C$. Since F_2 is upper semi-continuous in the first argument and (3.17), it follows that

$$F_2(Aw, y) \quad \geq \quad 0$$

for all $y \in C$. This shows that $Aw \in EP(F_2)$. Hence $w \in \Omega$.

Step 6. Show that $w = v = P_\Theta x_1$.

Since $x_n = P_{C_n}x_1$ and $\Theta \subset C_n$, we obtain

$$\big\langle x_1 - x_n, x_n - p\big\rangle \geq 0 \quad \forall p \in \Theta. \tag{3.18}$$

By taking the limit in (3.18), we obtain

$$\big\langle x_1 - w, w - p\big\rangle \geq 0 \quad \forall p \in \Theta.$$

This shows that $w = P_\Theta x_1 = v$.

From Step 4, we obtain that $\{x_n\}, \{y_n\}$ and $\{u_n\}$ converge strongly to $v = P_\Theta x_1$. This completes the proof. □

If $Tp = \{p\}$ for all $p \in F(T)$, then T satisfies Condition (A). We then obtain the following result:

Theorem 3.6. *Let H_1, H_2 be two real Hilbert space and $C \subset H_1$, $Q \subset H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and $T : C \to K(C)$ a nonspreading-type multivalued mapping. Let $F_1 : C \times C \to \mathbb{R}$, $F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumtion 2.4 and F_2 is upper semi-continuous in the first argument. Assume that $\Theta = F(T) \cap \Omega \neq \emptyset$, where $\Omega = \{z \in C : z \in EP(F_1)$ and $Az \in EP(F_2)\}$. For an initial point $x_1 \in H_1$ with $C_1 = C$, let $\{u_n\}$, $\{y_n\}$ and $\{x_n\}$ be sequences defined by*

$$\begin{cases} u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n, \\ y_n \in \alpha_n u_n + (1-\alpha_n)Tu_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \ \forall n \geq 1 \end{cases} \tag{3.19}$$

where $\{\alpha_n\} \subset (0,1)$, $r_n \subset (0,\infty)$ and $\gamma \in (0, 1/L)$ such that L is the spectral radius of A^*A and A^* is the adjoint of A. Assume that the following conditions hold:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$*;*

(ii) $\liminf_{n\to\infty} r_n > 0$.

If $Tp = \{p\}$ for all $p \in F(T)$, then the sequences $\{x_n\}$, $\{y_n\}$ and $\{x_n\}$ converge strongly to $P_\Theta x_1$.

Since P_T satisfies Condition (A), we also obtain the following result:

Theorem 3.7. *Let H_1, H_2 be two real Hilbert space and $C \subset H_1$, $Q \subset H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and $T : C \to K(C)$ a multivalued mapping with $I - T$ is demiclosed at 0. Let $F_1 : C \times C \to \mathbb{R}$, $F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumtion 2.4 and F_2 is upper semi-continuous in the first argument. Assume that $\Theta = F(T) \cap \Omega \neq \emptyset$, where $\Omega = \{z \in C : z \in EP(F_1)$ and $Az \in EP(F_2)\}$. For an initial point $x_1 \in H_1$ with $C_1 = C$, let $\{u_n\}$, $\{y_n\}$ and $\{x_n\}$ be sequences defined by*

$$\begin{cases} u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n, \\ y_n \in \alpha_n + (1-\alpha_n)P_T u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \ \forall n \geq 1 \end{cases} \tag{3.20}$$

where $\{\alpha_n\} \subset (0,1)$, $r_n \subset (0,\infty)$ and $\gamma \in (0, 1/L]$ such that L is the spectral radius of A^*A and A^* is the adjoint of A. Assume that the following conditions hold:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$*;*

(ii) $\liminf_{n\to\infty} r_n > 0$.

If P_T is nonspreading multivalued mapping, then the sequences $\{x_n\}$, $\{y_n\}$ and $\{x_n\}$ converge strongly to $P_\Theta x_1$.

Proof. By the same proof as in Theorem 3.5, we have

$$\lim_{n\to\infty} \|u_n - z_n\| = 0$$

where $z_n \in P_T u_n$. This implies that

$$d(u_n, Tu_n) \le d(u_n, P_T u_n) \le \|u_n - z_n\| \to 0$$

as $n \to \infty$. From $I - T$ is demiclosed at 0, so we obtain the result. □

4 Examples and Numerical Results

In this section, we give examples and numerical results for supporting our main theorem.

Example 4.1. Let $H_1 = H_2 = \mathbb{R}$, $C = [-3, 0]$ and $Q = [0, \infty)$. Let $F_1(u, v) = 2u(v - u)$ for all $u, v \in C$ and $F_2(x, y) = x(y - x)$ for all $x, y \in Q$. Define two mappings $A : \mathbb{R} \to \mathbb{R}$ and $T : C \to K(C)$ by $Ax = 3x$ for all $x \in \mathbb{R}$ and

$$Tx = \begin{cases} \{0\}, & x \in [-2, 2]; \\ \big[-\exp\{x + 2\}, 0\big], & x \notin [-2, 2]. \end{cases}$$

Choose $\alpha_n = r_n = \frac{n}{100n+1}$ and $\gamma = \frac{1}{100}$. It is easy to check that F_1 and F_2 satisfy all conditions in Theorem 3.5 and T satisfies Condition (A). For each $r > 0$ and $x \in C$, we divide the process of our iteration into 6 Steps as follows:

Step 1. Find $z \in Q$ such that $F_2(z, y) + \frac{1}{r}\langle y - z, z - Ax\rangle \ge 0$ for all $y \in Q$. Noting that $Ax = 3x$, we have

$$\begin{aligned} F_2(z, y) + \frac{1}{r}\langle y - z, z - Ax\rangle \ge 0 \quad &\iff \quad z(y - z) + \frac{1}{r}\langle y - z, z - 3x\rangle \ge 0 \\ &\iff \quad rz(y - z) + (y - z)(z - 3x) \ge 0 \\ &\iff \quad (y - z)((1 + r)z - 3x) \ge 0. \end{aligned}$$

By Lemma 2.5, we know that $T_r^{F_2} Ax$ is single-valued. Hence $z = \frac{3x}{1+r}$.

Step 2. Find $s \in C$ such that $s = x - \gamma A^*(I - T_r^{F_2})Ax$. From Step 1, we have

$$\begin{aligned} s = x - \gamma A^*(I - T_r^{F_2})Ax \quad &= \quad x - \gamma A^*(Ax - T_r^{F_2} Ax) \\ &= \quad x - \gamma\Big(9x - \frac{3(3x)}{1 + r}\Big) \\ &= \quad (1 - 9\gamma)x + \frac{3\gamma}{1 + r}(3x). \end{aligned}$$

Step 3. Find $u \in C$ such that $F_1(u, v) + \frac{1}{r}\langle v - u, u - s\rangle \ge 0$ for all $v \in C$. From Step 2, we have

$$\begin{aligned} F_1(u, v) + \frac{1}{r}\langle v - u, u - s\rangle \ge 0 \quad &\iff \quad (2u)(v - u) + \frac{1}{r}\langle v - u, u - s\rangle \ge 0 \\ &\iff \quad r(2u)(v - u) + (v - u)(u - s) \ge 0 \\ &\iff \quad (v - u)((1 + 2r)u - s) \ge 0. \end{aligned}$$

Similarly, by Lemma 2.5, we obtain $u = \frac{s}{1+2r} = \frac{(1-9\gamma)x}{1+2r} + \frac{9\gamma x}{(1+r)(1+2r)}$.

Step 4. Find $y_n \in \alpha_n u_n + (1-\alpha_n)Tu_n$, where $u_n = \frac{(1-9\gamma)x_n}{1+2r_n} + \frac{9\gamma x_n}{(1+r_n)(1+2r_n)}$. Then, we have $y_n = \alpha_n u_n + (1-\alpha_n)z_n$, where

$$z_n \in \begin{cases} \{0\}, & u_n \in [-2,2]; \\ \big[-\exp\{u_n+2\}, 0\big], & u_n \notin [-2,2]. \end{cases}$$

Step 5. Find $C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}$ where $C_1 = [-3,0]$. Since $\|y_n - z\| \le \|x_n - z\|$, we have

$$(2z - (y_n + x_n))(x_n - y_n) \le 0.$$

We observe the following cases:
Case 1: If $x_n - y_n \ge 0$, then

$$z \le \frac{y_n + x_n}{2}.$$

This implies that $C_2 = [-3, (y_1+x_1)/2] \cap [-3,0]$ and $C_{n+1} = [-3, (y_n+x_n)/2] \cap [-3, (y_{n-1}+x_{n-1})/2]$ for all $n \ge 2$.
Case 2: If $x_n - y_n \le 0$, then

$$z \ge \frac{y_n + x_n}{2}.$$

This implies that $C_2 = [(y_1+x_1)/2, 0] \cap [-3,0]$ and $C_{n+1} = [(y_n+x_n)/2, 0] \cap [(y_{n-1}+x_{n-1})/2, 0]$ for all $n \ge 2$.

Step 6. Compute the numerical results of $x_{n+1} = P_{C_{n+1}} x_1$. Choosing $x_1 = -3$, we obtain

n	u_n	y_n	C_n	x_n
1	-2.41483E+00	-1.29552E-01	[-3.00000E+00,0]	-3.00000E+00
2	-1.25944E+00	-1.25317E-02	[-1.56478E+00,0]	-1.56478E+00
3	-6.34744E-01	-6.32635E-03	[-7.88654E-01,0]	-7.88654E-01
4	-3.19913E-01	-3.19115E-03	[-3.97490E-01,0]	-3.97490E-01
5	-1.61239E-01	-1.60917E-03	[-2.00341E-01,0]	-2.00341E-01
6	-8.12666E-02	-8.11314E-04	[-1.00975E-01,0]	-1.00975E-01
7	-4.09596E-02	-4.09012E-04	[-5.08931E-02,0]	-5.08931E-02
8	-2.06443E-02	-2.06186E-04	[-2.56511E-02,0]	-2.56511E-02
9	-1.04051E-02	-1.03936E-04	[-1.29286E-02,0]	-1.29286E-02
10	-5.24437E-03	-5.23913E-05	[-6.51628E-03,0]	-6.51628E-03
⋮	⋮	⋮	⋮	⋮
50	-6.57171E-15	-6.57040E-17	[-8.16566E-15,0]	-8.16566E-15

Table 1. Numerical results of Example 4.1 being randomized in the first time.

n	u_n	y_n	C_n	x_n
1	-2.41483E+00	-4.58971E-01	[-3.00000E+00,0]	-3.00000E+00
2	-1.39201E+00	-1.38508E-02	[-1.72949E+00,0]	-1.72949E+00
3	-7.01558E-01	-6.99227E-03	[-8.71668E-01,0]	-8.71668E-01
4	-3.53587E-01	-3.52705E-03	[-4.39330E-01,0]	-4.39330E-01
5	-1.78211E-01	-1.77856E-03	[-2.21429E-01,0]	-2.21429E-01
6	-8.98208E-02	-8.96713E-04	[-1.11604E-01,0]	-1.11604E-01
7	-4.52710E-02	-4.52065E-04	[-5.62502E-02,0]	-5.62502E-02
8	-2.28174E-02	-2.27889E-04	[-2.83511E-02,0]	-2.83511E-02
9	-1.15004E-02	-1.14876E-04	[-1.42895E-02,0]	-1.42895E-02
10	-5.79639E-03	-5.79060E-05	[-7.20219E-03,0]	-7.20219E-03
⋮	⋮	⋮	⋮	⋮
50	-7.26345E-15	-7.26200E-17	[-9.02519E-15,0]	-9.02519E-15

Table 2. Numerical results of Example 4.1 being randomized in the second time.

From Table 1 and Table 2, we see that 0 is the solution in Example 4.1.

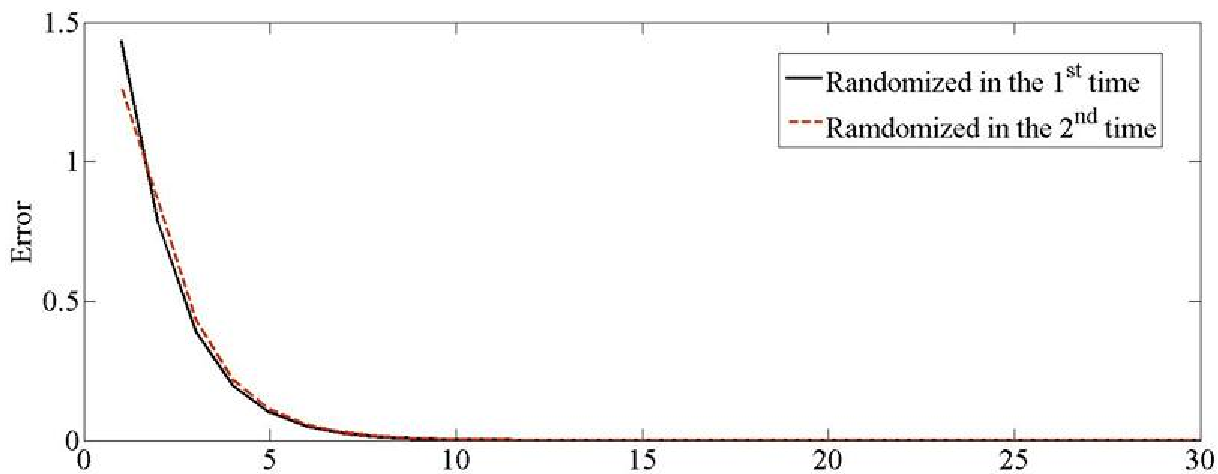

Figure 1. Error plots for all sequences $\{x_n\}$ in Table 1 and Table 2.

Acknowledgement. The author would like to thank University of Phayao.

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