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# Positive solutions for singular coupled integral boundary value problems of nonlinear Hadamard fractional differential equations

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**Abstract**

In this paper, we study the existence of positive solutions for a class of coupled integral boundary value problems of nonlinear semipositone Hadamard fractional differential equations

$$D^{\alpha}u(t)+\lambda f(t,u(t),v(t))=0,\quad D^{\beta}v(t)+\lambda g(t,u(t),v(t))=0,\quad t\in(1,e),\quad \lambda>0,$$
$$u^{(j)}(1)=v^{(j)}(1)=0,\quad 0\leq j\leq n-2,\quad u(e)=\mu\int_1^e v(s)\frac{ds}{s},\quad v(e)=\nu\int_1^e u(s)\frac{ds}{s},$$

where $\lambda,\mu,\nu$ are three parameters with $0<\mu<\beta$ and $0<\nu<\alpha$, $\alpha,\beta\in(n-1,n]$ are two real numbers and $n\geq 3$, $D^{\alpha},D^{\beta}$ are the Hadamard fractional derivative of fractional order, and $f,g$ are sign-changing continuous functions and may be singular at $t=1$ or/and $t=e$. First of all, we obtain the corresponding Green's function for the boundary value problem and some of its properties. Furthermore, by means of the nonlinear alternative of Leray-Schauder type and Krasnoselskii's fixed point theorems, we derive an interval of $\lambda$ such that the semipositone boundary value problem has one or multiple positive solutions for any $\lambda$ lying in this interval. At last, several illustrative examples were given to illustrate the main results. 

*Keywords:* Hadamard fractional differential equations, coupled integral boundary conditions, positive solutions, Green's function, fixed point theorems.
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## 1. Introduction

We consider the following coupled integral boundary value problem for systems of nonlinear semipositone Hadamard fractional differential equations

$$\begin{aligned}
&D^{\alpha}u(t)+\lambda f(t,u(t),v(t))=0, \quad D^{\beta}v(t)+\lambda g(t,u(t),v(t))=0, \quad t\in(1,e), \quad \lambda>0,\\
&u^{(j)}(1)=v^{(j)}(1)=0, \quad 0\le j\le n-2, \quad u(e)=\mu\int_1^e v(s)\frac{ds}{s}, \quad v(e)=\nu\int_1^e u(s)\frac{ds}{s},
\end{aligned} \tag{1.1}$$

where $\lambda,\mu,\nu$ are three parameters with $0<\mu<\beta$ and $0<\nu<\alpha$, $\alpha,\beta\in(n-1,n]$ are two real numbers and $n\ge 3$, $D^{\alpha},D^{\beta}$ are the Hadamard fractional derivative of fractional order, and $f,g$ are sign-changing continuous functions and may be singular at $t=1$ or/and $t=e$. To the best knowledge of the author, there are few papers which deal with the coupled integral boundary value problems for systems of nonlinear Hadamard fractional differential equations.

Coupled boundary value problems have wide applications in various fields of sciences and engineering, for example, the Sturm-Liouville problems, heat equation, reaction-diffusion equations, mathematical biology and so on. In recent years, there have been some significant developments in the study of ordinary differential equations and partial differential equations involving fractional derivatives with coupled boundary conditions, as shown by the papers [26, 27, 32, 38, 42, 43] and the references therein. For example, by mixed monotone method, Cui et al. [15] established sufficient conditions for the existence and uniqueness of positive solutions to a singular differential system with integral boundary value conditions. By using the properties of the Green's function and the Guo-Krasnosel'skii fixed point theorem, Wang et al. [35] obtained some existence results of positive solutions for higher-order singular semipositone fractional differential systems with coupled integral boundary conditions and parameters under some conditions concerning the nonlinear functions.

Due to the fact that fractional-order models are more accurate than integer-order models (that is, there are more degrees of freedom in the fractional-order models), the subject of fractional differential equations has recently developed into a interesting topic for many researchers in view of its numerous applications in the field of physics, engineering, mechanics, chemistry, and so forth. For some recent work on the topic, see [1, 4, 6, 11, 16, 19, 28, 30, 31, 37]. Specially, the study of coupled systems of fractional order differential equations has been addressed extensively by several researchers, see [3, 5, 18, 20, 21, 29, 33, 36, 40] and the references cited therein. For instance, By applying some standard fixed point theorems, Jiang et al. [23] and Yuan et al. [41] considered the existence of positive solutions to the four-point coupled boundary value problems for systems of nonlinear semipositone fractional differential equations under different conditions, respectively. In [20], Hao and Zhai studied the existence of at least one positive solution to a coupled system of fractional boundary value problems by using Schauder fixed point theorem.

However, we should point out that most of the work on the topic is based on Riemann-Liouville and Caputo type fractional differential equations in the last few years. In 1892, Hadamard introduced another kind of fractional derivatives, i.e., Hadamard type fractional differential equations, which differs from the preceding ones in the sense that the kernel of the integral and derivative contain logarithmic function of arbitrary exponent. Details and properties of Hadamard fractional derivative and integral can be found in [12, 13, 14, 17, 22, 24]. Recently, there are some results on Hadamard type fractional differential equations/inclusions, see [9, 10] and the references cited therein. For example, by applying some standard fixed point theorems, Ahmad and Ntouyas [7, 8] studied the existence and uniqueness of solutions for fractional integral boundary value problem involving Hadamard type fractional differential equations/systems with integral boundary conditions, respectively. In [34], based on some classical fixed point theorems, Thiramanus et al. investigated the existence and uniqueness of solutions for a fractional boundary value problem involving Hadamard-type fractional differential equations and nonlocal fractional integral boundary conditions. In [39], by applying some inequalities associated with Green's function and Guo-Krasnosel'skii fixed point theorems, the author showed the existence of positive solutions for a class of singular four-point coupled boundary value problem of nonlinear semipositone Hadamard fractional differential equations.

Motivated by the results mentioned above and wide applications of coupled boundary value conditions, we consider the existence of positive solutions for singular Hadamard fractional differential equations boundary value problem (1.1). In Section 2, we present some preliminaries and lemmas that will be used to prove our main results. And we obtain the corresponding Green's function for boundary value problem (1.1) and some of its properties. The main theorems are formulated and proved in Section 3. At last, several illustrative examples were given to illustrate the main results in Section 4.

## 2. Preliminaries

For the convenience of the reader, we firstly present some basic concepts of Hadamard type fractional calculus to facilitate analysis of problem (1.1).

**Definition 2.1.** [24] The Hadamard derivative of fractional order $q$ for a function $g : [1, \infty) \to \mathbb{R}$ is defined as

$$D^q g(t) = \frac{1}{\Gamma(n-q)} \left(t\frac{d}{dt}\right)^n \int_1^t \left(\log\frac{t}{s}\right)^{n-q-1} g(s)\frac{ds}{s}, \quad n-1 < q < n,$$

where $n = [q] + 1$, $[q]$ denotes the integer part of the real number $q$ and $\log(\cdot) = \log_e(\cdot)$.

**Definition 2.2.** [24] The Hadamard fractional integral of order $q$ for a function $g : [1, \infty) \to \mathbb{R}$ is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_1^t \left(\log\frac{t}{s}\right)^{q-1} g(s)\frac{ds}{s}, \quad q > 0,$$

provided the integral exists.

Now we derive the corresponding Green's function for boundary value problem (1.1), and obtain some properties of the Green's function.

**Lemma 2.3.** *Let $x, y \in C[0, 1]$ be given functions. Then the boundary value problem*

$$\begin{aligned} &D^\alpha u(t) + x(t) = 0, \quad D_q^\beta v(t) + y(t) = 0, \quad t \in (1, e), \\ &u^{(j)}(1) = v^{(j)}(1) = 0, \quad 0 \le j \le n-2, \quad u(e) = \mu \int_1^e v(s)\frac{ds}{s}, \quad v(e) = \nu \int_1^e u(s)\frac{ds}{s}, \end{aligned} \tag{2.1}$$

*has an integral representation*

$$\begin{cases} u(t) = \displaystyle\int_1^e G_1(t,s)x(s)\frac{ds}{s} + \int_1^e H_1(t,s)y(s)\frac{ds}{s}, \\ v(t) = \displaystyle\int_1^e G_2(t,s)y(s)\frac{ds}{s} + \int_1^e H_2(t,s)x(s)\frac{ds}{s}, \end{cases} \tag{2.2}$$

*where*

$$G_1(t,s) = \begin{cases} \dfrac{(\log t)^{\alpha-1}(1-\log s)^{\alpha-1}(\alpha\beta - \mu\nu + \mu\nu\log s)}{(\alpha\beta-\mu\nu)\Gamma(\alpha)} - \dfrac{(\log(t/s))^{\alpha-1}}{\Gamma(\alpha)}, & 1 \le s \le t \le e, \\ \dfrac{(\log t)^{\alpha-1}(1-\log s)^{\alpha-1}(\alpha\beta - \mu\nu + \mu\nu\log s)}{(\alpha\beta-\mu\nu)\Gamma(\alpha)}, & 1 \le t \le s \le e, \end{cases} \tag{2.3}$$

$$G_2(t,s) = \begin{cases} \dfrac{(\log t)^{\beta-1}(1-\log s)^{\beta-1}(\alpha\beta - \mu\nu + \mu\nu\log s)}{(\alpha\beta-\mu\nu)\Gamma(\beta)} - \dfrac{(\log(t/s))^{\beta-1}}{\Gamma(\beta)}, & 1 \le s \le t \le e, \\ \dfrac{(\log t)^{\beta-1}(1-\log s)^{\beta-1}(\alpha\beta - \mu\nu + \mu\nu\log s)}{(\alpha\beta-\mu\nu)\Gamma(\beta)}, & 1 \le t \le s \le e, \end{cases} \tag{2.4}$$

$$H_1(t,s) = \frac{\mu\alpha(\log t)^{\alpha-1}(1-\log s)^{\beta-1}\log s}{(\alpha\beta-\mu\nu)\Gamma(\beta)}, \quad H_2(t,s) = \frac{\nu\beta(\log t)^{\beta-1}(1-\log s)^{\alpha-1}\log s}{\alpha\Gamma(\alpha)}. \tag{2.5}$$

*Proof.* As argued in [24], the solution of Hadamard differential system in (2.1) can be written the following equivalent integral equations

$$
\begin{aligned}
u(t) &= c_{11}(\log t)^{\alpha-1} + c_{12}(\log t)^{\alpha-2} + \cdots + c_{1n}(\log t)^{\alpha-n} - \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} x(s)\frac{ds}{s}, \\
v(t) &= c_{21}(\log t)^{\beta-1} + c_{22}(\log t)^{\beta-2} + \cdots + c_{2n}(\log t)^{\beta-n} - \frac{1}{\Gamma(\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta-1} y(s)\frac{ds}{s}.
\end{aligned} \tag{2.6}
$$

From $D_q^j u(0) = D_q^j v(0) = 0$, $0 \le j \le n-2$, we have $c_{in} = c_{i(n-1)} = \cdots = c_{i2} = 0$ $(i = 1, 2)$. Thus, (2.6) reduces to

$$
\begin{aligned}
u(t) &= c_{11}(\log t)^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} x(s)\frac{ds}{s}, \\
v(t) &= c_{21}(\log t)^{\beta-1} - \frac{1}{\Gamma(\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta-1} y(s)\frac{ds}{s}.
\end{aligned} \tag{2.7}
$$

Using the boundary conditions $u(e) = \mu\int_1^e v(s)\frac{ds}{s}$ and $v(e) = \nu\int_1^e u(s)\frac{ds}{s}$, from (2.7), we obtain

$$
\begin{aligned}
c_{11} &= \mu\int_1^e v(s)\frac{ds}{s} + \int_1^e \frac{(1-\log s)^{\alpha-1}}{\Gamma(\alpha)} x(s)\frac{ds}{s}, \\
c_{21} &= \nu\int_1^e u(s)\frac{ds}{s} + \int_1^e \frac{(1-\log s)^{\beta-1}}{\Gamma(\beta)} y(s)\frac{ds}{s}.
\end{aligned} \tag{2.8}
$$

Combining (2.7) and (2.8), we have

$$
\begin{aligned}
u(t) &= (\log t)^{\alpha-1}\left(\mu\int_1^e v(s)\frac{ds}{s} + \int_1^e \frac{(1-\log s)^{\alpha-1}}{\Gamma(\alpha)} x(s)\frac{ds}{s}\right) - \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} x(s)\frac{ds}{s}, \\
v(t) &= (\log t)^{\beta-1}\left(\nu\int_1^e u(s)\frac{ds}{s} + \int_1^e \frac{(1-\log s)^{\beta-1}}{\Gamma(\beta)} y(s)\frac{ds}{s}\right) - \frac{1}{\Gamma(\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta-1} y(s)\frac{ds}{s}.
\end{aligned} \tag{2.9}
$$

Integrating the above equations (2.9) from 0 to 1, we obtain

$$
\begin{aligned}
\int_1^e u(s)\frac{ds}{s} =& \int_1^e (\log t)^{\alpha-1}\left(\mu\int_1^e v(s)\frac{ds}{s} + \int_1^e \frac{(1-\log s)^{\alpha-1}}{\Gamma(\alpha)} x(s)\frac{ds}{s}\right)\frac{dt}{t} \\
&- \int_1^e \left(\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} x(s)\frac{ds}{s}\right)\frac{dt}{t} \\
=& \frac{\mu}{\alpha}\int_1^e v(s)\frac{ds}{s} + \int_1^e \frac{(1-\log s)^{\alpha-1}}{\alpha\Gamma(\alpha)} x(s)\frac{ds}{s} - \int_1^e \frac{(1-\log s)^{\alpha}}{\alpha\Gamma(\alpha)} x(s)ds \\
=& \frac{\mu}{\alpha}\int_1^e v(s)\frac{ds}{s} + \int_1^e \frac{(1-\log s)^{\alpha-1}\log s}{\alpha\Gamma(\alpha)} x(s)\frac{ds}{s},
\end{aligned}
$$

and

$$
\begin{aligned}
\int_1^e v(s)\frac{ds}{s} =& \int_1^e (\log t)^{\beta-1}\left(\nu\int_1^e u(s)\frac{ds}{s} + \int_1^e \frac{(1-\log s)^{\beta-1}}{\Gamma(\beta)} y(s)\frac{ds}{s}\right)\frac{dt}{t} \\
&- \int_1^e \left(\frac{1}{\Gamma(\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta-1} y(s)\frac{ds}{s}\right)\frac{dt}{t} \\
=& \frac{\nu}{\beta}\int_1^e u(s)\frac{ds}{s} + \int_1^e \frac{(1-\log s)^{\beta-1}}{\beta\Gamma(\beta)} y(s)\frac{ds}{s} - \int_1^e \frac{(1-\log s)^{\beta}}{\beta\Gamma(\beta)} y(s)ds \\
=& \frac{\nu}{\beta}\int_1^e u(s)\frac{ds}{s} + \int_1^e \frac{(1-\log s)^{\beta-1}\log s}{\beta\Gamma(\beta)} y(s)\frac{ds}{s}.
\end{aligned}
$$

Solving for $\int_1^e u(s)\frac{ds}{s}$ and $\int_1^e v(s)\frac{ds}{s}$, we have

$$\begin{aligned}\int_1^e u(s)\frac{ds}{s} &= \frac{\alpha\beta}{\alpha\beta-\mu\nu}\left(\frac{\mu}{\alpha}\int_1^e \frac{(1-\log s)^{\beta-1}\log s}{\beta\Gamma(\beta)}y(s)\frac{ds}{s} + \int_1^e \frac{(1-\log s)^{\alpha-1}\log s}{\alpha\Gamma(\alpha)}x(s)\frac{ds}{s}\right),\\ \int_1^e v(s)\frac{ds}{s} &= \frac{\alpha\beta}{\alpha\beta-\mu\nu}\left(\frac{\nu}{\beta}\int_1^e \frac{(1-\log s)^{\alpha-1}\log s}{\alpha\Gamma(\alpha)}x(s)\frac{ds}{s} + \int_1^e \frac{(1-\log s)^{\beta-1}\log s}{\beta\Gamma(\beta)}y(s)\frac{ds}{s}\right).\end{aligned} \tag{2.10}$$

Combining (2.7), (2.8) and (2.10), we get

$$\begin{aligned}u(t) =& \frac{\mu\alpha\beta(\log t)^{\alpha-1}}{\alpha\beta-\mu\nu}\left(\frac{\nu}{\beta}\int_1^e \frac{(1-\log s)^{\alpha-1}\log s}{\alpha\Gamma(\alpha)}x(s)\frac{ds}{s} + \int_1^e \frac{(1-\log s)^{\beta-1}\log s}{\beta\Gamma(\beta)}y(s)\frac{ds}{s}\right)\\ &+ \int_1^e \frac{(\log t)^{\alpha-1}(1-\log s)^{\alpha-1}}{\Gamma(\alpha)}x(s)\frac{ds}{s} - \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} x(s)\frac{ds}{s}\\ =& \int_1^e \frac{\mu\nu(\log t)^{\alpha-1}(1-\log s)^{\alpha-1}\log s}{(\alpha\beta-\mu\nu)\Gamma(\alpha)}x(s)\frac{ds}{s} + \int_1^e \frac{(\log t)^{\alpha-1}(1-\log s)^{\alpha-1}}{\Gamma(\alpha)}x(s)\frac{ds}{s}\\ &- \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} x(s)\frac{ds}{s} + \int_1^e \frac{\mu\alpha(\log t)^{\alpha-1}(1-\log s)^{\beta-1}\log s}{(\alpha\beta-\mu\nu)\Gamma(\beta)}y(s)\frac{ds}{s},\end{aligned}$$

and

$$\begin{aligned}v(t) =& \frac{\nu\alpha\beta(\log t)^{\beta-1}}{\alpha\beta-\mu\nu}\left(\frac{\mu}{\alpha}\int_1^e \frac{(1-\log s)^{\beta-1}\log s}{\beta\Gamma(\beta)}y(s)\frac{ds}{s} + \int_1^e \frac{(1-\log s)^{\alpha-1}\log s}{\alpha\Gamma(\alpha)}x(s)\frac{ds}{s}\right)\\ &+ \int_1^e \frac{(\log t)^{\beta-1}(1-\log s)^{\beta-1}}{\Gamma(\beta)}y(s)\frac{ds}{s} - \frac{1}{\Gamma(\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta-1} y(s)\frac{ds}{s}\\ =& \int_1^e \frac{\mu\nu(\log t)^{\beta-1}(1-\log s)^{\beta-1}\log s}{(\alpha\beta-\mu\nu)\Gamma(\beta)}y(s)\frac{ds}{s} + \int_1^e \frac{(\log t)^{\beta-1}(1-\log s)^{\beta-1}}{\Gamma(\beta)}y(s)\frac{ds}{s}\\ &- \frac{1}{\Gamma(\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta-1} y(s)\frac{ds}{s} + \int_1^e \frac{\nu\beta(\log t)^{\beta-1}(1-\log s)^{\alpha-1}\log s}{\alpha\Gamma(\alpha)}x(s)\frac{ds}{s}.\end{aligned}$$

Hence, we have

$$\begin{cases} u(t) = \int_1^e G_1(t,s)x(s)\frac{ds}{s} + \int_1^e H_1(t,s)y(s)\frac{ds}{s},\\ v(t) = \int_1^e G_2(t,s)y(s)\frac{ds}{s} + \int_1^e H_2(t,s)x(s)\frac{ds}{s}.\end{cases}$$

This completes the proof of the lemma. □

From Lemma 2.3, the system (1.1) can be expressed the following integral form

$$\begin{cases} u(t) = \lambda\left(\int_1^e G_1(t,s)f(s,u(s),v(s))\frac{ds}{s} + \int_1^e H_1(t,s)g(s,u(s),v(s))\frac{ds}{s}\right),\\ v(t) = \lambda\left(\int_1^e G_2(t,s)g(s,u(s),v(s))\frac{ds}{s} + \int_1^e H_2(t,s)f(s,u(s),v(s))\frac{ds}{s}\right).\end{cases} \tag{2.11}$$

**Lemma 2.4.** *For $t,s\in[1,e]$, the functions $G_1(t,s)$ and $H_1(t,s)$ defined by (2.3) and (2.5) satisfy*

$$\frac{\mu\nu}{(\alpha\beta-\mu\nu)\Gamma(\alpha)}(\log t)^{\alpha-1}\rho_1(s) \le G_1(t,s) \le \frac{\max\{(\alpha\beta-\mu\nu)(\alpha-1)+\mu\nu,\alpha\beta\}}{(\alpha\beta-\mu\nu)\Gamma(\alpha)}\rho_1(s), \tag{2.12}$$

$$\frac{\mu\nu}{(\alpha\beta-\mu\nu)\Gamma(\beta)}(\log t)^{\alpha-1}\rho_2(s) \le H_1(t,s) \le \frac{\alpha\beta}{(\alpha\beta-\mu\nu)\Gamma(\beta)}\rho_2(s), \tag{2.13}$$

$$G_1(t,s) \le \frac{\max\{(\alpha\beta-\mu\nu)(\alpha-1)+\mu\nu, \alpha\beta\}}{(\alpha\beta-\mu\nu)\Gamma(\alpha)}(\log t)^{\alpha-1}, \quad H_1(t,s) \le \frac{\alpha\beta}{(\alpha\beta-\mu\nu)\Gamma(\beta)}(\log t)^{\alpha-1}, \tag{2.14}$$

*where*

$$\rho_1(s) = (1-\log s)^{\alpha-1}\log s, \quad \rho_2(s) = (1-\log s)^{\beta-1}\log s. \tag{2.15}$$

*Proof.* First, we will show that (2.12) is true. On the one hand, when $1 \le s \le t \le e$, we have

$$\begin{aligned}
G_1(t,s) =& \frac{(\log t)^{\alpha-1}(1-\log s)^{\alpha-1}(\alpha\beta-\mu\nu+\mu\nu\log s) - (\log t-\log s)^{\alpha-1}(\alpha\beta-\mu\nu)}{(\alpha\beta-\mu\nu)\Gamma(\alpha)} \\
=& \frac{(\log t)^{\alpha-1}(1-\log s)^{\alpha-1}(\alpha\beta-\mu\nu+\mu\nu\log s) - (\log t)^{\alpha-1}(1-\log s/\log t)^{\alpha-1}(\alpha\beta-\mu\nu)}{(\alpha\beta-\mu\nu)\Gamma(\alpha)} \\
\ge& \frac{(\log t)^{\alpha-1}(1-\log s)^{\alpha-1}(\alpha\beta-\mu\nu+\mu\nu\log s) - (\log t)^{\alpha-1}(1-\log s)^{\alpha-1}(\alpha\beta-\mu\nu)}{(\alpha\beta-\mu\nu)\Gamma(\alpha)} \\
=& \frac{\mu\nu(\log t)^{\alpha-1}(1-\log s)^{\alpha-1}\log s}{(\alpha\beta-\mu\nu)\Gamma(\alpha)} = \frac{\mu\nu}{(\alpha\beta-\mu\nu)\Gamma(\alpha)}(\log t)^{\alpha-1}\rho_1(s), \quad t,s\in[1,e],
\end{aligned}$$

and

$$\begin{aligned}
G_1(t,s) =& \frac{(\alpha\beta-\mu\nu)[(\log t-\log t\log s)^{\alpha-1} - (\log t-\log s)^{\alpha-1}] + \mu\nu(\log t)^{\alpha-1}(1-\log s)^{\alpha-1}\log s}{(\alpha\beta-\mu\nu)\Gamma(\alpha)} \\
\le& \frac{(\alpha\beta-\mu\nu)(\alpha-1)\int_{\log t-\log s}^{\log t-\log t\log s} x^{\alpha-2}dx + \mu\nu(\log t)^{\alpha-1}(1-\log s)^{\alpha-1}\log s}{(\alpha\beta-\mu\nu)\Gamma(\alpha)} \\
\le& \frac{(\alpha\beta-\mu\nu)(\alpha-1)(\log t-\log t\log s)^{\alpha-2}[(\log t-\log t\log s)-(\log t-\log s)]}{(\alpha\beta-\mu\nu)\Gamma(\alpha)} \\
&+ \frac{\mu\nu(\log t)^{\alpha-1}(1-\log s)^{\alpha-1}\log s}{(\alpha\beta-\mu\nu)\Gamma(\alpha)} \\
=& \frac{(\alpha\beta-\mu\nu)(\alpha-1)(\log t)^{\alpha-2}(1-\log s)^{\alpha-2}(1-\log t)\log s + \mu\nu(\log t)^{\alpha-1}(1-\log s)^{\alpha-1}\log s}{(\alpha\beta-\mu\nu)\Gamma(\alpha)} \\
=& \frac{(\alpha\beta-\mu\nu)(\alpha-1)(\log t)^{\alpha-2}(1-\log s)^{\alpha-2}(1-\log s)\log s + \mu\nu(\log t)^{\alpha-1}(1-\log s)^{\alpha-1}\log s}{(\alpha\beta-\mu\nu)\Gamma(\alpha)} \\
\le& \frac{[(\alpha\beta-\mu\nu)(\alpha-1)+\mu\nu](1-\log s)^{\alpha-1}\log s}{(\alpha\beta-\mu\nu)\Gamma(\alpha)} \\
\le& \frac{\max\{(\alpha\beta-\mu\nu)(\alpha-1)+\mu\nu, \alpha\beta\}}{(\alpha\beta-\mu\nu)\Gamma(\alpha)}\rho_1(s), \quad t,s\in[1,e],
\end{aligned}$$

On the other hand, when $1 \le t \le s \le e$, since $0 < \mu < \beta$ and $0 < \nu < \alpha$, we also have

$$G_1(t,s) \ge \frac{\mu\nu(\log t)^{\alpha-1}(1-\log s)^{\alpha-1}\log s}{(\alpha\beta-\mu\nu)\Gamma(\alpha)} = \frac{\mu\nu}{(\alpha\beta-\mu\nu)\Gamma(\alpha)}(\log t)^{\alpha-1}\rho_1(s), \quad t,s\in[1,e],$$

and

$$\begin{aligned}
G_1(t,s) =& \frac{(\log t)^{\alpha-1}(1-\log s)^{\alpha-1}(\alpha\beta-\mu\nu+\mu\nu\log s)}{(\alpha\beta-\mu\nu)\Gamma(\alpha)} \le \frac{\alpha\beta(\log s)^{\alpha-1}(1-\log s)^{\alpha-1}}{(\alpha\beta-\mu\nu)\Gamma(\alpha)} \\
\le& \frac{\alpha\beta(1-\log s)^{\alpha-1}\log s}{(\alpha\beta-\mu\nu)\Gamma(\alpha)} \le \frac{[(\alpha\beta-\mu\nu)(\alpha-1)+\mu\nu](1-\log s)^{\alpha-1}\log s}{(\alpha\beta-\mu\nu)\Gamma(\alpha)} \\
\le& \frac{\max\{(\alpha\beta-\mu\nu)(\alpha-1)+\mu\nu, \alpha\beta\}}{(\alpha\beta-\mu\nu)\Gamma(\alpha)}\rho_1(s), \quad t,s\in[1,e],
\end{aligned}$$

Next we show that (2.13) holds for $t,s\in[1,e]$. In fact, since $0<\mu<\beta$ and $0<\nu<\alpha$, we get

$$H_1(t,s) \ge \frac{\mu\nu(\log t)^{\alpha-1}(1-\log s)^{\beta-1}\log s}{(\alpha\beta-\mu\nu)\Gamma(\beta)} = \frac{\mu\nu}{(\alpha\beta-\mu\nu)\Gamma(\beta)}(\log t)^{\alpha-1}\rho_2(s), \quad t,s\in[1,e],$$

and

$$H_1(t,s) \le \frac{\alpha\beta(1-\log s)^{\beta-1}\log s}{(\alpha\beta-\mu\nu)\Gamma(\beta)} = \frac{\alpha\beta}{(\alpha\beta-\mu\nu)\Gamma(\beta)}\rho_2(s), \quad t,s\in[1,e].$$

Finally, we will prove (2.14) is valid for any $t, s \in [1, e]$. Noticing $(1-\log s)^{\alpha-2}(1-\log t) \le 1$, $(1-\log s)^{\alpha-1}\log s \le 1$, and $(1-\log s)^{\beta-1}\log s \le 1$, when $1 \le s \le t \le e$, we have

$$\begin{aligned} G_1(t,s) \le & \frac{(\alpha\beta-\mu\nu)(\alpha-1)(\log t)^{\alpha-2}(1-\log s)^{\alpha-2}(1-\log t)\log s + \mu\nu(\log t)^{\alpha-1}(1-\log s)^{\alpha-1}\log s}{(\alpha\beta-\mu\nu)\Gamma(\alpha)} \\ \le & \frac{(\alpha\beta-\mu\nu)(\alpha-1)(\log t)^{\alpha-2}(1-\log s)^{\alpha-2}(1-\log t)\log t + \mu\nu(\log t)^{\alpha-1}(1-\log s)^{\alpha-1}\log s}{(\alpha\beta-\mu\nu)\Gamma(\alpha)} \\ \le & \frac{[(\alpha\beta-\mu\nu)(\alpha-1)+\mu\nu](\log t)^{\alpha-1}}{(\alpha\beta-\mu\nu)\Gamma(\alpha)} \\ \le & \frac{\max\{(\alpha\beta-\mu\nu)(\alpha-1)+\mu\nu, \alpha\beta\}}{(\alpha\beta-\mu\nu)\Gamma(\alpha)}(\log t)^{\alpha-1}, \quad t,s\in[1,e], \end{aligned}$$

and when $1 \le t \le s \le e$, since $0 < \mu < \beta$ and $0 < \nu < \alpha$, we also have

$$G_1(t,s) \le \frac{\alpha\beta(\log t)^{\alpha-1}}{(\alpha\beta-\mu\nu)\Gamma(\alpha)} \le \frac{\max\{(\alpha\beta-\mu\nu)(\alpha-1)+\mu\nu, \alpha\beta\}}{(\alpha\beta-\mu\nu)\Gamma(\alpha)}(\log t)^{\alpha-1}, \quad t,s\in[1,e].$$

And we have

$$H_1(t,s) = \frac{\mu\alpha(\log t)^{\alpha-1}(1-\log s)^{\beta-1}\log s}{(\alpha\beta-\mu\nu)\Gamma(\beta)} \le \frac{\alpha\beta}{(\alpha\beta-\mu\nu)\Gamma(\beta)}(\log t)^{\alpha-1}, \quad t,s\in[1,e].$$

This completes the proof of the lemma. □

Similarly, we have

**Lemma 2.5.** *For $t, s \in [1, e]$, the functions $G_1(t, s)$ and $H_1(t, s)$ defined by (2.4) and (2.5) satisfy*

$$\begin{aligned} &\frac{\mu\nu}{(\alpha\beta-\mu\nu)\Gamma(\beta)}(\log t)^{\beta-1}\rho_2(s) \le G_2(t,s) \le \frac{\max\{(\alpha\beta-\mu\nu)(\beta-1)+\mu\nu, \alpha\beta\}}{(\alpha\beta-\mu\nu)\Gamma(\beta)}\rho_2(s), \\ &\frac{\mu\nu}{(\alpha\beta-\mu\nu)\Gamma(\alpha)}(\log t)^{\beta-1}\rho_1(s) \le H_2(t,s) \le \frac{\alpha\beta}{(\alpha\beta-\mu\nu)\Gamma(\alpha)}\rho_1(s), \\ &G_2(t,s) \le \frac{\max\{(\alpha\beta-\mu\nu)(\beta-1)+\mu\nu, \alpha\beta\}}{(\alpha\beta-\mu\nu)\Gamma(\beta)}(\log t)^{\beta-1}, \quad H_2(t,s) \le \frac{\alpha\beta}{(\alpha\beta-\mu\nu)\Gamma(\alpha)}(\log t)^{\beta-1}, \end{aligned}$$

*where $\rho_1(s)$ and $\rho_2(s)$ are defined in* (2.15).

*Remark* 2.6. From Lemmas 2.4 and 2.5, for $t, s \in [1, e]$, we have

$$\begin{aligned} &a(\log t)^{\alpha-1}\rho_1(s) \le G_1(t,s) \le b\rho_1(s), \quad G_1(t,s) \le b(\log t)^{\alpha-1}, \quad a(\log t)^{\alpha-1}\rho_2(s) \le H_1(t,s) \le b\rho_2(s), \\ &H_1(t,s) \le b(\log t)^{\alpha-1}, \quad a(\log t)^{\beta-1}\rho_2(s) \le G_2(t,s) \le b\rho_2(s), \quad G_2(t,s) \le b(\log t)^{\beta-1}, \\ &a(\log t)^{\beta-1}\rho_1(s) \le H_2(t,s) \le b\rho_1(s), \quad H_2(t,s) \le b(\log t)^{\beta-1}, \end{aligned}$$

where

$$\begin{aligned} a =& \frac{\mu\nu}{(\alpha\beta-\mu\nu)\max\{\Gamma(\alpha),\Gamma(\beta)\}}, \\ b =& \max\left\{\frac{\max\{(\alpha\beta-\mu\nu)(\alpha-1)+\mu\nu, \alpha\beta\}}{(\alpha\beta-\mu\nu)\Gamma(\alpha)}, \frac{\max\{(\alpha\beta-\mu\nu)(\beta-1)+\mu\nu, \alpha\beta\}}{(\alpha\beta-\mu\nu)\Gamma(\beta)}\right\}. \end{aligned}$$

In the rest of the paper, we always suppose the following assumptions hold:

($H_1$) $f(t,u,v), g(t,u,v) \in C([1,e]\times[0,+\infty)\times[0,+\infty),(-\infty,+\infty))$, moreover there exists two functions $q_1(t), q_2(t) \in L^1([1,e],(0,+\infty))$ such that $f(t,u,v) \geq -q_1(t)$ and $g(t,u,v) \geq -q_2(t)$ for any $t \in [1,e]$, $u,v \in [0,+\infty)$.

($H_1^*$) $f(t,u,v), g(t,u,v) \in C((1,e)\times[0,+\infty)\times(0,+\infty),(-\infty,+\infty))$, $f,g$ may be singular at $t=1,e$, moreover there exists two functions $q_1(t), q_2(t) \in L^1((1,e),(0,+\infty))$ such that $f(t,u,v) \geq -q_1(t)$ and $g(t,u,v) \geq -q_2(t)$ for any $t \in (1,e)$, $u,v \in [0,+\infty)$.

($H_2$) $f(t,0,0) > 0$ and $g(t,0,0) > 0$ for $t \in [1,e]$.

($H_3$) There exists $[\theta_1,\theta_2] \subset (1,e)$ such that $\liminf\limits_{u\uparrow+\infty} \min\limits_{t\in[\theta_1,\theta_2]} \frac{f(t,u,v)}{u} = +\infty$ and $\liminf\limits_{v\uparrow+\infty} \min\limits_{t\in[\theta_1,\theta_2]} \frac{g(t,u,v)}{v} = +\infty$.

($H_3^*$) There exists $[\theta_1,\theta_2] \subset (1,e)$ such that $\liminf\limits_{v\uparrow+\infty} \min\limits_{t\in[\theta_1,\theta_2]} \frac{f(t,u,v)}{v} = +\infty$ and $\liminf\limits_{u\uparrow+\infty} \min\limits_{t\in[\theta_1,\theta_2]} \frac{g(t,u,v)}{u} = +\infty$.

($H_4$) $\int_1^e \rho_i(s)q_i(s)\frac{ds}{s} < +\infty$ $(i=1,2)$, $\int_1^e \rho_1(s)f(s,u,v)\frac{ds}{s} < +\infty$, $\int_1^e \rho_2(s)g(s,u,v)\frac{ds}{s} < +\infty$ for any $u,v \in [0,m]$, $m>0$ is any constant.

**Lemma 2.7.** *Assume the condition* ($H_1$) *or* ($H_1^*$) *holds, then the boundary value problem*

$$-D^\alpha\omega_1(t) = \lambda q_1(t), \quad -D^\beta\omega_2(t) = \lambda q_2(t), \quad t\in(1,e), \quad \lambda>0,$$
$$\omega_1^{(j)}(1) = \omega_2^{(j)}(1) = 0, \quad 0\le j\le n-2, \quad \omega_1(e) = \mu\int_1^e \omega_2(s)\frac{ds}{s}, \quad \omega_2(e) = \nu\int_1^e \omega_1(s)\frac{ds}{s},$$

*have an unique solution*

$$\begin{cases} \omega_1(t) = \lambda\left(\int_1^e G_1(t,s)q_1(s)\frac{ds}{s} + \int_1^e H_1(t,s)q_2(s)\frac{ds}{s}\right), \\ \omega_2(t) = \lambda\left(\int_1^e G_2(t,s)q_2(s)\frac{ds}{s} + \int_1^e H_2(t,s)q_1(s)\frac{ds}{s}\right), \end{cases} \tag{2.16}$$

*which satisfy*

$$\begin{cases} \omega_1(t) \le \lambda b(\log t)^{\alpha-1}\int_1^e (q_1(s)+q_2(s))\frac{ds}{s}, \quad t\in[1,e], \\ \omega_2(t) \le \lambda b(\log t)^{\beta-1}\int_1^e (q_1(s)+q_2(s))\frac{ds}{s}, \quad t\in[1,e]. \end{cases} \tag{2.17}$$

*Proof.* It follows from Lemma 2.3, Remark 2.6 and the condition ($H_1$) or ($H_1^*$) that (2.16) and (2.17) hold. □

Let $E = [1,e]\times[1,e]$, then $E$ is a Banach space with the norm

$$\|(u,v)\|_1 = \|u\| + \|v\|, \quad \|u\| = \max_{t\in[1,e]}|u(t)|, \quad \|v\| = \max_{t\in[1,e]}|v(t)|$$

for any $(u,v)\in E$. Let

$$P = \{(u,v)\in E \ : \ u(t) \ge \omega(\log t)^{\alpha-1}\|u\|, \quad v(t)\ge\omega(\log t)^{\beta-1}\|v\| \text{ for } t\in[1,e]\},$$

where $0<\omega=a/b<1$. Then $P$ is a cone of $E$.

Next we only consider the following singular boundary value problem

$$\begin{aligned} &D^\alpha x(t) + \lambda(f(t,[x(t)-\omega_1(t)]^*,[y(t)-\omega_2(t)]^*) + q_1(t)) = 0, \quad t\in(1,e), \quad \lambda>0, \\ &D^\beta y(t) + \lambda(g(t,[x(t)-\omega_1(t)]^*,[y(t)-\omega_2(t)]^*) + q_2(t)) = 0, \quad t\in(1,e), \quad \lambda>0, \\ &x^{(j)}(1) = y^{(j)}(1) = 0, \quad 0\le j\le n-2, \quad x(e) = \mu\int_1^e y(s)\frac{ds}{s}, \quad y(e) = \nu\int_1^e x(s)\frac{ds}{s}, \end{aligned} \tag{2.18}$$

where a modified function $[z(t)]^*$ for any $z\in C[1,e]$ by $[z(t)]^* = z(t)$, if $z(t)\ge0$, and $[z(t)]^*=0$, if $z(t)<0$.

**Lemma 2.8.** *If $(x, y) \in C[1, e] \times C[1, e]$ with $x(t) > \omega_1(t)$ and $y(t) > \omega_2(t)$ for any $t \in (1, e)$ is a positive solution of the singular system* (2.18)*, then $(x - \omega_1, y - \omega_2)$ is a positive solution of the singular system* (1.1)*.*

*Proof.* In fact, if $(x, y) \in C[1, e] \times C[1, e]$ is a positive solution of the singular system (2.18) such that $x(t) > \omega_1(t)$ and $y(t) > \omega_2(t)$ for any $t \in (1, e]$, then from (2.18) and the definition of $[\cdot]^*$, we have

$$\begin{aligned}
&D^\alpha x(t) + \lambda(f(t, x(t) - \omega_1(t), y(t) - \omega_2(t)) + q_1(t)) = 0, \quad t \in (1, e), \quad \lambda > 0,\\
&D^\beta y(t) + \lambda(g(t, x(t) - \omega_1(t), y(t) - \omega_2(t)) + q_2(t)) = 0, \quad t \in (1, e), \quad \lambda > 0,\\
&x^{(j)}(1) = y^{(j)}(1) = 0, \quad 0 \le j \le n-2, \quad x(e) = \mu \int_1^e y(s)\frac{ds}{s}, \quad y(e) = \nu \int_1^e x(s)\frac{ds}{s}.
\end{aligned} \tag{2.19}$$

Let $u = x - \omega_1$ and $v = y - \omega_2$, then $D^\alpha u(t) = D^\alpha x(t) - D^\alpha \omega_1(t)$ and $D^\beta v(t) = D^\beta y(t) - D^\beta \omega_2(t)$ for $t \in (1, e)$, which imply that

$$\begin{aligned}
&-D^\alpha u(t) = -D^\alpha x(t) + D^\alpha \omega_1(t) = -D^\alpha x(t) - \lambda q_1(t), \quad t \in (1, e),\\
&-D^\beta v(t) = -D^\beta y(t) + D^\beta \omega_2(t) = -D^\beta y(t) - \lambda q_2(t), \quad t \in (1, e).
\end{aligned}$$

Thus (2.19) becomes

$$\begin{aligned}
&D^\alpha u(t) + \lambda f(t, u(t), v(t)) = 0, \quad D^\beta v(t) + \lambda g(t, u(t), v(t)) = 0, \quad t \in (1, e), \quad \lambda > 0,\\
&u^{(j)}(1) = v^{(j)}(1) = 0, \quad 0 \le j \le n-2, \quad u(e) = \mu \int_1^e v(s)\frac{ds}{s}, \quad v(e) = \nu \int_1^e u(s)\frac{ds}{s},
\end{aligned}$$

i.e., $(x - \omega_1, y - \omega_2)$ is a positive solution of the singular system (1.1). This proves Lemma 2.8. $\square$

Employing Lemma 2.3, the singular system (2.18) can be expressed as

$$\begin{cases}
u(t) = \lambda \int_1^e G_1(t, s)(f(s, [x(s) - \omega_1(s)]^*, [y(s) - \omega_2(s)]^*) + q_1(s))\frac{ds}{s}\\
\qquad + \lambda \int_1^e H_1(t, s)(g(s, [x(s) - \omega_1(s)]^*, [y(s) - \omega_2(s)]^*) + q_2(s))\frac{ds}{s}, \quad t \in [1, e],\\
v(t) = \lambda \int_1^e G_2(t, s)(g(s, [x(s) - \omega_1(s)]^*, [y(s) - \omega_2(s)]^*) + q_2(s))\frac{ds}{s}\\
\qquad + \lambda \int_1^e H_2(t, s)(f(s, [x(s) - \omega_1(s)]^*, [y(s) - \omega_2(s)]^*) + q_1(s))\frac{ds}{s}, \quad t \in [1, e].
\end{cases} \tag{2.20}$$

By a solution of the singular system (2.18), we mean a solution of the corresponding system of integral equation (2.20). Defined an operator $T : P \to P$ by

$$T(x, y) = (T_1(x, y), T_2(x, y)),$$

where operators $T_i : P \to C[1, e]$ $(i = 1, 2)$ are defined by

$$\begin{aligned}
T_1(x, y)(t) =& \lambda \int_1^e G_1(t, s)(f(s, [x(s) - \omega_1(s)]^*, [y(s) - \omega_2(s)]^*) + q_1(s))\frac{ds}{s}\\
&+ \lambda \int_1^e H_1(t, s)(g(s, [x(s) - \omega_1(s)]^*, [y(s) - \omega_2(s)]^*) + q_2(s))\frac{ds}{s}, \quad t \in [1, e],\\
T_2(x, y)(t) =& \lambda \int_1^e G_2(t, s)(g(s, [x(s) - \omega_1(s)]^*, [y(s) - \omega_2(s)]^*) + q_2(s))\frac{ds}{s}\\
&+ \lambda \int_1^e H_2(t, s)(f(s, [x(s) - \omega_1(s)]^*, [y(s) - \omega_2(s)]^*) + q_1(s))\frac{ds}{s}, \quad t \in [1, e].
\end{aligned} \tag{2.21}$$

Clearly, if $(x, y) \in P$ is a fixed point of $T$, then $(x, y)$ is a solution of the singular system (2.18).

**Lemma 2.9.** *Assume the condition* $(\mathrm{H}_1)$ *or* $(\mathrm{H}_1^*)$ *holds, then*

$$T : P \to P$$

*is a completely continuous operator.*

*Proof.* For any fixed $(x, y) \in P$, there exists a constant $L > 0$ such that $\|(x, y)\|_1 \le L$. And then,

$$[x(s) - \omega_1(s)]^* \le x(s) \le \|x\| \le \|(x, y)\|_1 \le L,\ \ [y(s) - \omega_2(s)]^* \le y(s) \le \|y\| \le \|(x, y)\|_1 \le L,\ \ s \in [1, e].$$

For any $t \in [1, e]$, it follows from (2.20) and Remark 2.6 that

$$\begin{aligned}
T_1(x,y)(t) =& \lambda \int_1^e G_1(t,s)\big(f(s,[x(s)-w_1(s)]^*,[y(s)-w_2(s)]^*)+q_1(s)\big)\frac{ds}{s}\\
&+ \lambda \int_1^e H_1(t,s)\big(g(s,[x(s)-w_1(s)]^*,[y(s)-w_2(s)]^*)+q_2(s)\big)\frac{ds}{s}\\
\le& \lambda \int_1^e b\rho_1(s)\big(f(s,[x(s)-w_1(s)]^*,[y(s)-w_2(s)]^*)+q_1(s)\big)\frac{ds}{s}\\
&+ \lambda \int_1^e b\rho_2(s)\big(g(s,[x(s)-w_1(s)]^*,[y(s)-w_2(s)]^*)+q_2(s)\big)\frac{ds}{s}\\
\le& \lambda Mb \int_1^e (\rho_1(s)+\rho_1(s))\frac{ds}{s} + \lambda M\mu \int_1^e (\rho_1(s)q_1(s)+\rho_2(s)q_2(s))\frac{ds}{s}\\
\le& 2\lambda Mb + \lambda M\mu \int_1^e (q_1(s)+q_2(s))\frac{ds}{s} < +\infty,
\end{aligned}$$

where

$$M = \max\left\{ \max_{t\in[1,e],u,v\in[0,L]} f(t,u,v), \max_{t\in[1,e],u,v\in[0,L]} g(t,u,v) \right\} + 1.$$

Similarly, we have

$$|T_2(x,y)(t)| \le 2\lambda Mb + \lambda M\mu \int_1^e (q_1(s)+q_2(s))\frac{ds}{s} < +\infty,$$

Thus $T : P \to E$ is well defined.

Next, we show that $T : P \to P$. For any fixed $(x, y) \in P$, $t \in [1, e]$, by (2.21) and Remark 2.6, we have

$$\begin{aligned}
T_1(x,y)(t) \le& \lambda b \int_1^e \rho_1(s)(f(s,[x(s)-\omega_1(s)]^*,[y(s)-\omega_2(s)]^*)+q_1(s))\frac{ds}{s}\\
&+ \lambda b \int_1^e \rho_2(s)(g(s,[x(s)-\omega_1(s)]^*,[y(s)-\omega_2(s)]^*)+q_2(s))\frac{ds}{s},\\
T_2(x,y)(t) \le& \lambda b \int_1^e \rho_2(s)(g(s,[x(s)-\omega_1(s)]^*,[y(s)-\omega_2(s)]^*)+q_2(s))\frac{ds}{s}\\
&+ \lambda b \int_1^e \rho_1(s)(f(s,[x(s)-\omega_1(s)]^*,[y(s)-\omega_2(s)]^*)+q_1(s))\frac{ds}{s},
\end{aligned}$$

which implies that

$$\begin{aligned}
\|T_1(x,y)\| \le& \lambda b \int_1^e \rho_1(s)(f(s,[x(s)-\omega_1(s)]^*,[y(s)-\omega_2(s)]^*)+q_1(s))\frac{ds}{s}\\
&+ \lambda b \int_1^e \rho_2(s)(g(s,[x(s)-\omega_1(s)]^*,[y(s)-\omega_2(s)]^*)+q_2(s))\frac{ds}{s},\\
\|T_2(x,y)\| \le& \lambda b \int_1^e \rho_2(s)(g(s,[x(s)-\omega_1(s)]^*,[y(s)-\omega_2(s)]^*)+q_2(s))\frac{ds}{s}\\
&+ \lambda b \int_1^e \rho_1(s)(f(s,[x(s)-\omega_1(s)]^*,[y(s)-\omega_2(s)]^*)+q_1(s))\frac{ds}{s}.
\end{aligned}$$

On the other hand, from (2.20) and Remark 2.6, we also obtain

$$\begin{aligned} T_1(x,y)(t) \geq & \lambda a(\log t)^{\alpha-1}\int_1^e \rho_1(s)(f(s,[x(s)-\omega_1(s)]^*,[y(s)-\omega_2(s)]^*)+q_1(s))\frac{ds}{s} \\ & + \lambda a(\log t)^{\beta-1}\int_1^e \rho_2(s)(g(s,[x(s)-\omega_1(s)]^*,[y(s)-\omega_2(s)]^*)+q_2(s))\frac{ds}{s}, \\ T_2(x,y)(t) \geq & \lambda a(\log t)^{\beta-1}\int_1^e \rho_2(s)(g(s,[x(s)-\omega_1(s)]^*,[y(s)-\omega_2(s)]^*)+q_2(s))\frac{ds}{s} \\ & + \lambda a(\log t)^{\alpha-1}\int_1^e \rho_1(s)(f(s,[x(s)-\omega_1(s)]^*,[y(s)-\omega_2(s)]^*)+q_1(s))\frac{ds}{s}. \end{aligned}$$

So we have

$$T_1(x,y) \geq \omega(\log t)^{\alpha-1}\|T_1(x,y)\|, \quad T_2(x,y) \geq \omega(\log t)^{\beta-1}\|T_2(x,y)\|, \quad t\in[1,e].$$

This implies that $T(P) \subset P$. According to the Ascoli-Arzela theorem, we can easily get that $T : P \to P$ is completely continuous. This completes the proof of the lemma. □

The following nonlinear alternative of Leray-Schauder type and Krasnoselskii's fixed point theorem will play major role in our next analysis.

**Theorem 2.10** (Nonlinear alternative of Leray-Schauder type, see [2])**.** *Let $X$ be a Banach space with $\Omega \in X$ closed and convex. Assume $U$ is a relatively open subset of with $0 \in U$, and let*

$$S : \overline{U} \to \Omega$$

*be a compact, continuous map. Then either*

(a) *$S$ has a fixed point in $\overline{U}$, or*
(b) *there exists $u \in \partial U$ and $v \in (0,1)$, with $u = vSu$.*

**Theorem 2.11** (Krasnoselskii's fixed point theorem, see [25])**.** *Let $X$ be a Banach space, and let $P \subset X$ be a cone in $X$. Assume $\Omega_1, \Omega_2$ are open subsets of $X$ with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $S : P \to P$ be a completely continuous operator such that, either*

(a) *$\|Sw\| \leq \|w\|$, $w \in P\cap\partial\Omega_1$, $\|Sw\| \geq \|w\|$, $w \in P\cap\partial\Omega_2$, or*
(b) *$\|Sw\| \geq \|w\|$, $w \in P\cap\partial\Omega_1$, $\|Sw\| \leq \|w\|$, $w \in P\cap\partial\Omega_2$.*

*Then $S$ has a fixed point in $P\cap(\overline{\Omega}_2\backslash\Omega_1)$.*

## 3. Main results

**Theorem 3.1.** *Suppose that* (H$_1$) *and* (H$_2$) *hold. Then there exists a constant $\lambda_* > 0$ such that the boundary value problem* (1.1) *has at least one positive solution for any $0 < \lambda \leq \lambda_*$.*

*Proof.* Fix $\delta \in (0,1)$. From (H$_2$), let $0 < \varepsilon < 1$ be such that

$$f(t,u,v) \geq \delta f(t,0,0) \text{ and } g(t,u,v) \geq \delta g(t,0,0), \quad \text{for } 1 \leq t \leq e, \quad 0 \leq u,v \leq \varepsilon. \tag{3.1}$$

Let $\overline{f}(\varepsilon) = \max\limits_{1\leq t\leq e, 0\leq u,v\leq\varepsilon}\{f(t,u,v)+q_1(t)\}$, $\overline{g}(\varepsilon) = \max\limits_{1\leq t\leq e, 0\leq u,v\leq\varepsilon}\{g(t,u,v)+q_2(t)\}$, and $c_i = \int_1^e b\rho_i(s)\frac{ds}{s}$ $(i = 1,2)$, we have

$$\lim_{z\downarrow 0}\frac{\overline{f}(z)}{z} = +\infty \text{ and } \lim_{z\downarrow 0}\frac{\overline{g}(z)}{z} = +\infty.$$

Suppose $0 < \lambda < \varepsilon/(8c\overline{h}(\varepsilon)) := \lambda_*$, where $c = \max(c_1,c_2)$ and $\overline{h}(\varepsilon) = \max(\overline{f}(\varepsilon), \overline{g}(\varepsilon))$. Since

$$\lim_{z\downarrow 0}\frac{\overline{h}(z)}{z} = +\infty \text{ and } \frac{\overline{h}(\varepsilon)}{\varepsilon} < \frac{1}{8c\lambda},$$

then exists a $R_0 \in (0, \varepsilon)$ such that

$$\frac{\overline{h}(R_0)}{R_0} = \frac{1}{8c\lambda}.$$

Let $U = \{(u,v) \in P | \|(u,v)\|_1 < R_0\}$, $(u,v) \in \partial U$ and $\theta \in (0,1)$ be such that $(u,v) = \theta T(u,v)$, i.e., $u = \theta T_1(u,v)$ and $v = \theta T_2(u,v)$. we claim that $\|(u,v)\|_1 \neq R_0$. In fact, for $(x,y) \in \partial U$ and $\|(u,v)\|_1 = R_0$, we have

$$\begin{aligned} u(t) =&\theta T_1(u,v)(t) \leq \lambda \int_1^e G_1(t,s)\big(f(s,[u(s)-w_1(s)]^*,[v(s)-w_2(s)]^*)+q_1(s)\big)\frac{ds}{s} \\ &+ \lambda \int_1^e H_1(t,s)\big(g(s,[u(s)-w_1(s)]^*,[v(s)-w_2(s)]^*)+q_2(s)\big)\frac{ds}{s} \\ \leq&\lambda \int_1^e G_1(t,s)\overline{f}(R_0)\frac{ds}{s} + \lambda \int_1^e H_1(t,s)\overline{g}(R_0)\frac{ds}{s} \leq \lambda \int_1^e b\rho_1(s)\overline{f}(R_0)\frac{ds}{s} + \lambda\int_1^e b\rho_2(s)\overline{g}(R_0)\frac{ds}{s} \\ \leq&\lambda \int_1^e b\rho_1(s)\frac{ds}{s}\overline{f}(R_0) + \lambda \int_1^e b\rho_2(s)\frac{ds}{s}\overline{g}(R_0) \leq 2c\lambda\overline{h}(R_0), \end{aligned} \tag{3.2}$$

and similarly, we also have

$$v(t) = \theta T_2(u,v)(t) \leq 2c\lambda\overline{h}(R_0). \tag{3.3}$$

It follows that $R_0 = \|(u,v)\|_1 \leq 4c\lambda\overline{h}(R_0)$, that is

$$\frac{\overline{h}(R_0)}{R_0} \geq \frac{1}{4\lambda c} > \frac{1}{8c\lambda} = \frac{\overline{h}(R_0)}{R_0},$$

which implies that $\|(u,v)\|_1 \neq R_0$. By the nonlinear alternative of Leray-Schauder type, $T$ has a fixed point $(u,v) \in \overline{U}$. Moreover, combining (3.1)-(3.3) and the fact that $R_0 < \varepsilon$, we obtain

$$\begin{aligned} u(t) =&\lambda \int_1^e G_1(t,s)\big(f(s,[u(s)-w_1(s)]^*,[v(s)-w_2(s)]^*)+q_1(s)\big)\frac{ds}{s} \\ &+ \lambda \int_1^e H_1(t,s)\big(g(s,[u(s)-w_1(s)]^*,[v(s)-w_2(s)]^*)+q_2(s)\big)\frac{ds}{s} \\ \geq&\lambda \int_1^e G_1(t,s)\big(\delta f(s,0,0)+q_1(s)\big)\frac{ds}{s} + \lambda \int_1^e H_1(t,s)\big(\delta g(s,0,0)+q_2(s)\big)\frac{ds}{s} \\ \geq&\lambda \int_1^e G_1(t,s)q_1(s)d_qs + \lambda \int_1^e H_1(t,s)q_2(s)\frac{ds}{s} = w_1(t), \quad \text{for} \quad t \in (1,e), \end{aligned}$$

and similarly, we also have

$$v(t) \geq w_2(t), \quad \text{for} \quad t \in (1,e).$$

Then $T$ has a positive fixed point $(x,y)$ and $\|(u,v)\|_1 \leq R_0 < 1$. Namely, $(u,v)$ is positive solution of the boundary value problem (3.1) with $u(t) \geq w_1(t)$ and $v(t) \geq w_2(t)$, for $t \in (1,e)$.

Let $x(t) = u(t) - w_1(t) \geq 0$ and $y(t) = u(t) - w_2(t) \geq 0$. Then $(x,y)$ is a nonnegative solution (positive on $(1,e)$) of the boundary value problem (1.1). □

**Theorem 3.2.** *Suppose that* $(\mathrm{H}_1^*)$ *and* $(\mathrm{H}_3)$*-*$(\mathrm{H}_4)$ *hold. Then there exists a constant* $\lambda^* > 0$ *such that the boundary value problem* (1.1) *has at least one positive solution for any* $0 < \lambda \leq \lambda^*$.

*Proof.* Let $\Omega_1 = \{(u,v) \in E \times E : \|u\| < R_1, \|v\| < R_1\}$, where $R_1 = \max(1,r)$, $r = \frac{b^2}{a}\int_1^e \big(q_1(s)+q_2(s)\big)\frac{ds}{s}$. Choose

$$\lambda^* = \min\left\{1, \frac{R_1}{2}(R+1)^{-1}, \frac{R_1}{2r}\right\},$$

where $R = \int_1^e b\rho_1(s)\left(\max\limits_{0\le u,v\le R_1} f(s,u,v)+q_1(s)\right)\frac{ds}{s} + \int_1^e b\rho_2(s)\left(\max\limits_{0\le u,v\le R_1} g(s,u,v)+q_2(s)\right)\frac{ds}{s}$ and $R_1 \ge 0$. Then, for any $(u,v)\in P\cap\partial\Omega_1$, we have $\|u\| = R_1$ or $\|v\| = R_1$. Moreover $u(t)-w_1(t)\le u(t)\le \|u\|\le R_1$, $v(t)-w_2(t)\le v(t)\le\|v\|\le R_1$, and it follows that

$$\begin{aligned}\|T_1(u,v)(t)\| \le&\lambda\int_1^e b\rho_1(s)\big(f(s,[u(s)-w_1(s)]^*,[v(s)-w_2(s)]^*)+q_1(s)\big)\frac{ds}{s}\\&+\lambda\int_1^e b\rho_2(s)\big(g(s,[u(s)-w_1(s)]^*,[v(s)-w_2(s)]^*)+q_2(s)\big)\frac{ds}{s}\\ \le&\lambda\int_1^e b\rho_1(s)\left(\max_{0\le u,v\le R_1} f(s,u,v)+q_1(s)\right)\frac{ds}{s}\\&+\lambda\int_1^e b\rho_2(s)\left(\max_{0\le u,v\le R_1} g(s,u,v)+q_2(s)\right)\frac{ds}{s}=\lambda R\le\frac{R_1}{2},\end{aligned}$$

and similarly, we also have $\|T_2(u,v)(t)\|\le R_1/2$. This implies

$$\|T(u,v)\|_1 = \|T_1(u,v)\|+\|T_2(u,v)\|\le R_1\le\|(u,v)\|_1,\quad\text{for}\quad (u,v)\in P\backslash\partial\Omega_1.$$

On the other hand, choose two constants $N_1, N_2>1$ such that

$$\lambda N_1\frac{a^2}{2b}\gamma\int_{\theta_1}^{\theta_2}\rho_1(s)(\log s)^{\alpha-1}\frac{ds}{s}\ge 1,\quad \lambda N_2\frac{a^2}{2b}\gamma\int_{\theta_1}^{\theta_2}\rho_2(s)(\log s)^{\beta-1}\frac{ds}{s}\ge 1,$$

where $\gamma = \min\limits_{t\in[\theta_1,\theta_2]}\{(\log t)^{\alpha-1}\}$. By assumptions ($H_3$) and ($H_4$), there exists a constant $B>R_1$ such that

$$\frac{f(t,u,v)}{u}>N_1,\quad\text{namely}\quad f(t,u,v)>N_1u,\quad\text{for}\quad t\in[\theta_1,\theta_2],\quad u>B,\quad v>0,\tag{3.4}$$

and

$$\frac{g(t,u,v)}{v}>N_2,\quad\text{namely}\quad g(t,u,v)>N_2v,\quad\text{for}\quad t\in[\theta_1,\theta_2],\quad u>0,\quad v>B.\tag{3.5}$$

Choose $R_2$, let $\Omega_1=\{(u,v)\in E\times E:\|u\|<R_2,\|v\|<R_2\}$. Then for any $(u,v)\in(P_1\times P_2)\cap\partial\Omega_2$, we have $\|u\|=R_2$ or $\|v\|=R_2$. If $\|u\|=R_2$, we can state that

$$\begin{aligned}u(t)-w_1(t)=&u(t)-\left(\lambda\int_1^e G_1(t,s)q_1(s)\frac{ds}{s}+\lambda\int_1^e H_1(t,s)q_2(s)\frac{ds}{s}\right)\\ \ge&u(t)-\left(\lambda\int_1^e b(\log t)^{\alpha-1}q_1(s)\frac{ds}{s}+\lambda\int_1^e b(\log t)^{\alpha-1}q_2(s)\frac{ds}{s}\right)\\ =&u(t)-\lambda b(\log t)^{\alpha-1}\int_1^e\big(q_1(s)+q_2(s)\big)\frac{ds}{s}\\ =&u(t)-\lambda\frac{a}{b}(\log t)^{\alpha-1}\frac{b^2}{a}\int_1^e\big(q_1(s)+q_2(s)\big)\frac{ds}{s}=u(t)-\lambda\frac{a}{b}(\log t)^{\alpha-1}r\\ \ge&u(t)-\lambda r\frac{u(t)}{\|u\|}=\left(1-\frac{\lambda r}{R_2}\right)u(t)\ge\frac{1}{2}u(t)\ge 0,\quad t\in[1,e],\end{aligned}$$

and then

$$\begin{aligned}\min_{t\in[\theta_1,\theta_2]}\{[u(t)-w_1(t)]^*\}&=\min_{t\in[\theta_1,\theta_2]}\{u(t)-w_1(t)\}\ge\min_{t\in[\theta_1,\theta_2]}\left\{\frac{1}{2}u(t)\right\}\\&\ge\min_{t\in[\theta_1,\theta_2]}\left\{\frac{\omega}{2\Delta}(\log t)^{\alpha-1}\|u\|\right\}=\frac{a}{2b}R_2\min_{t\in[\theta_1,\theta_2]}\left\{(\log t)^{\alpha-1}\right\}\ge B+1>B.\end{aligned}$$

Since $B>R_1\ge r$, from (3.4), we have

$$f(t,[u(t)-w_1(t)]^*,[v(t)-w_2(t)]^*)\ge N_1[u(t)-w_1(t)]^*\ge\frac{N_1}{2}u(t),\quad\text{for}\quad t\in[\theta_1,\theta_2].\tag{3.6}$$

It follows from (3.6) that

$$\begin{aligned}T_1(u,v)(t)=&\lambda\int_1^e G_1(t,s)\big(f(s,[u(s)-w_1(s)]^*,[v(s)-w_2(s)]^*)+q_1(s)\big)\frac{ds}{s}\\&+\lambda\int_1^e H_1(t,s)\big(g(s,[u(s)-w_1(s)]^*,[v(s)-w_2(s)]^*)+q_2(s)\big)\frac{ds}{s}\\\geq&\lambda\int_1^e G_1(t,s)\big(f(s,[u(s)-w_1(s)]^*,[v(s)-w_2(s)]^*)+q_1(s)\big)\frac{ds}{s}\\\geq&\lambda\int_{\theta_1}^{\theta_2} G_1(t,s)f(s,[u(s)-w_1(s)]^*,[v(s)-w_2(s)]^*)\frac{ds}{s}\\\geq&\lambda\int_{\theta_1}^{\theta_2} a(\log t)^{\alpha-1}\rho_1(s)\frac{N_1}{2}u(s)\frac{ds}{s}\geq\lambda a\frac{N_1}{2}(\log t)^{\alpha-1}\int_{\theta_1}^{\theta_2}\rho_1(s)\frac{a}{b}(\log s)^{\alpha-1}\|u\|\frac{ds}{s}\\\geq&\lambda N_1\frac{a^2}{2b}\min_{t\in[\theta_1,\theta_2]}\{(\log t)^{\alpha-1}\}\int_{\theta_1}^{\theta_2}\rho_1(s)(\log s)^{\alpha-1}\frac{ds}{s}R_2\\\geq&\lambda N_1\frac{a^2}{2b}\gamma\int_{\theta_1}^{\theta_2}\rho_1(s)(\log s)^{\alpha-1}\frac{ds}{s}R_2\geq R_2,\quad\text{for}\quad t\in[\theta_1,\theta_2].\end{aligned}$$

If $\|v\|=R_2$, we obtain

$$v(t)-w_2(t)=v(t)-\left(\lambda\int_1^e G_2(t,s)q_2(s)\frac{ds}{s}+\lambda\int_1^e H_2(t,s)q_1(s)\frac{ds}{s}\right)\geq\frac{1}{2}v(t)\geq 0,\quad t\in[1,e],$$

and then

$$\begin{aligned}\min_{t\in[\theta_1,\theta_2]}\{[v(t)-w_2(t)]^*\}&=\min_{t\in[\theta_1,\theta_2]}\{v(t)-w_2(t)\}\geq\min_{t\in[\theta_1,\theta_2]}\left\{\frac{1}{2}v(t)\right\}\\&\geq\min_{t\in[\theta_1,\theta_2]}\left\{\frac{a}{2b}(\log t)^{\beta-1}\|v\|\right\}=\frac{a}{2b}R_2\min_{t\in[\theta_1,\theta_2]}\left\{(\log t)^{\beta-1}\right\}\geq B+1>B.\end{aligned}$$

Since $B>R_1\geq r$, from (3.5), one verifies that

$$g(t,[u(t)-w_1(t)]^*,[v(t)-w_2(t)]^*)\geq N_2[v(t)-w_2(t)]^*\geq\frac{N_2}{2}v(t),\quad\text{for}\quad t\in[\theta_1,\theta_2].\tag{3.7}$$

It follows from (3.7) that

$$\begin{aligned}T_1(u,v)(t)=&\lambda\int_1^e G_1(t,s)\big(f(s,[u(s)-w_1(s)]^*,[v(s)-w_2(s)]^*)+q_1(s)\big)\frac{ds}{s}\\&+\lambda\int_1^e H_1(t,s)\big(g(s,[u(s)-w_1(s)]^*,[v(s)-w_2(s)]^*)+q_2(s)\big)\frac{ds}{s}\\\geq&\lambda\int_1^e H_1(t,s)\big(g(s,[u(s)-w_1(s)]^*,[v(s)-w_2(s)]^*)+q_2(s)\big)\frac{ds}{s}\\\geq&\lambda\int_{\theta_1}^{\theta_2} H_1(t,s)g(s,[u(s)-w_1(s)]^*,[v(s)-w_2(s)]^*)\frac{ds}{s}\\\geq&\lambda\int_{\theta_1}^{\theta_2} a(\log t)^{\alpha-1}\rho_2(s)\frac{N_2}{2}v(s)\frac{ds}{s}\geq\lambda a\frac{N_2}{2}(\log t)^{\alpha-1}\int_{\theta_1}^{\theta_2}\rho_2(s)\frac{a}{b}(\log s)^{\beta-1}\|v\|\frac{ds}{s}\\\geq&\lambda N_2\frac{a^2}{2b}\min_{t\in[\theta_1,\theta_2]}\{(\log t)^{\alpha-1}\}\int_{\theta_1}^{\theta_2}\rho_2(s)(\log s)^{\beta-1}\frac{ds}{s}R_2\\\geq&\lambda N_2\frac{a^2}{2b}\gamma\int_{\theta_1}^{\theta_2}\rho_2(s)(\log s)^{\beta-1}\frac{ds}{s}R_2\geq R_2,\quad\text{for}\quad t\in[\theta_1,\theta_2].\end{aligned}$$

Thus, for any $(u,v)\in(P_1\times P_2)\cap\partial\Omega_2$, we always have

$$T_1(u,v)(t)\geq R_2,\quad\text{for}\quad t\in[\theta_1,\theta_2].$$

Similarly, for any $(u,v) \in (P_1 \times P_2) \cap \partial\Omega_2$, it also holds

$$T_2(u,v)(t) \geq R_2, \quad \text{for} \quad t \in [\theta_1, \theta_2].$$

This implies

$$\|T(u,v)\|_1 = \|T_1(u,v)\| + \|T_2(u,v)\| \geq 2R_2 \geq \|(u,v)\|_1, \quad \text{for} \quad (u,v) \in (P_1 \times P_2)\backslash\partial\Omega_2.$$

Thus condition (b) of Krasnoeselskii's fixed point theorem is satisfied. As a result $T$ has a fixed point $(u,v)$ with $r \leq R_1 < \|u\| < R_2$ and $r \leq R_1 < \|v\| < R_2$.

Since $r \leq R_1 < \|u\| < R_2$ and $r \leq R_1 < \|v\| < R_2$, we get

$$\begin{aligned} u(t) - w_1(t) =& u(t) - \left(\lambda \int_1^e G_1(t,s) q_1(s) \frac{ds}{s} + \lambda \int_1^e H_1(t,s) q_2(s) \frac{ds}{s}\right) \\ \geq& \frac{a}{b} t^{\alpha-1}\|u\| - \left(\lambda \int_1^e b(\log t)^{\alpha-1} q_1(s) \frac{ds}{s} + \lambda \int_1^e b(\log t)^{\alpha-1} q_2(s) \frac{ds}{s}\right) \\ =& \frac{a}{b}(\log t)^{\alpha-1}\|u\| - \lambda b(\log t)^{\alpha-1} \int_1^e \left(q_1(s) + q_2(s)\right) \frac{ds}{s} \\ \geq& \frac{a}{b}(\log t)^{\alpha-1} r - \lambda \frac{a}{b}(\log t)^{\alpha-1} r = (1-\lambda)\frac{a}{b}(\log t)^{\alpha-1} r \geq 0, \quad t \in (1,e), \end{aligned}$$

and

$$\begin{aligned} v(t) - w_2(t) =& v(t) - \left(\lambda \int_1^e G_2(t,s) q_2(s) \frac{ds}{s} + \lambda \int_1^e H_2(t,s) q_1(s) \frac{ds}{s}\right) \\ \geq& \frac{a}{b} t^{\beta-1}\|v\| - \left(\lambda \int_1^e b(\log t)^{\beta-1} q_2(s) \frac{ds}{s} + \lambda \int_1^e b(\log t)^{\beta-1} q_1(s) \frac{ds}{s}\right) \\ =& \frac{a}{b} t^{\beta-1}\|v\| - \lambda b(\log t)^{\beta-1} \int_1^e \left(q_1(s) + q_2(s)\right) \frac{ds}{s} \\ \geq& \frac{a}{b}(\log t)^{\beta-1} r - \lambda \frac{a}{b}(\log t)^{\beta-1} r = (1-\lambda)\frac{a}{b}(\log t)^{\beta-1} r \geq 0, \quad t \in (1,e). \end{aligned}$$

Thus, $(u,v)$ is positive solution of the boundary value problem (3.1) with $u(t) > w_1(t)$ and $v(t) > w_2(t)$ for $t \in (1,e)$. Let $x(t) = u(t) - w_1(t) \geq 0$ and $y(t) = v(t) - w_2(t) \geq 0$. Then $(x,y)$ is a nonnegative solution (positive on $(1,e)$) of the boundary value problem (1.1). This concludes the proof. □

From the proof of Theorem 3.2, clearly condition ($H_3$) can be replaced by condition ($H_3^*$). So we have the following theorem.

**Theorem 3.3.** *Suppose that* ($H_1^*$), ($H_3^*$) *and* ($H_4$) *hold. Then there exists a constant* $\lambda_* > 0$ *such that the boundary value problem* (1.1) *has at least one positive solution for any* $0 < \lambda \leq \lambda_*$.

Since condition ($H_1$) implies conditions ($H_1^*$) and ($H_4$), then from the proof of Theorem 3.1 and 3.2, we immediately have the following theorem.

**Theorem 3.4.** *Suppose that* ($H_1$)-($H_3$) *hold. Then the boundary value problem* (1.1) *has at least two positive solutions for* $\lambda > 0$ *sufficiently small.*

In fact, let $0 < \lambda < \min\{\lambda_*, \lambda^*\}$, then the boundary value problem (1.1) has at least two positive solutions.

Similarly, we conclude

**Theorem 3.5.** *Suppose that* ($H_1$)-($H_2$) *and* ($H_3^*$) *hold. Then the boundary value problem* (1.1) *has at least two positive solutions for* $\lambda > 0$ *sufficiently small.*

## 4. Some examples

**Example 4.1.** Consider the following coupled integral boundary value problem

$$\begin{aligned} &D^{\alpha}u(t)+\lambda\left(u^{a}+\frac{1}{\sqrt{(1-\log t)\log t}}\cos(2\pi v)\right)=0,\quad t\in(1,e),\quad \lambda>0,\\ &D^{\beta}v(t)+\lambda\left(v^{b}+\frac{1}{\sqrt{(1-\log t)\log t}}\sin(2\pi u)\right)=0,\quad t\in(1,e),\quad \lambda>0,\\ &u^{(j)}(1)=v^{(j)}(1)=0,\quad 0\le j\le n-2,\quad u(e)=\mu\int_1^e v(s)\frac{ds}{s},\quad v(e)=\nu\int_1^e u(s)\frac{ds}{s}, \end{aligned} \tag{4.1}$$

where $a,b>1$. Then, if $\lambda>0$ is sufficiently small, (4.1) has a positive solution $(u,v)$ with $u>0$, $v>0$ for $t\in(1,e)$.

*Proof.* From (4.1), then we have

$$\begin{aligned} &f(t,u,v)=u^{a}+\frac{1}{\sqrt{(1-\log t)\log t}}\cos(2\pi v),\quad g(t,u,v)=v^{b}+\frac{1}{\sqrt{(1-\log t)\log t}}\sin(2\pi v),\\ &q_i(t)=q(t)=\frac{2}{\sqrt{(1-\log t)\log t}},\quad i=1,2. \end{aligned}$$

Clearly, for $t\in(1,e)$, we get

$$\begin{aligned} &f(t,u,v)+q(t)\ge u^{a}+\frac{1}{\sqrt{(1-\log t)\log t}}>0,\quad g(t,u,v)+q(t)\ge v^{b}+\frac{1}{\sqrt{(1-\log t)\log t}}>0,\\ &\liminf_{u\uparrow+\infty}\frac{f(t,u,v)}{u}=+\infty,\quad \liminf_{v\uparrow+\infty}\frac{g(t,u,v)}{v}=+\infty,\quad \text{for}\quad t\in[\theta_1,\theta_2]\subset(1,e), \end{aligned}$$

for $u,v\ge 0$. Thus ($H_1^*$) and ($H_3$)-($H_4$) hold.

Let $r=\frac{b^2}{a}\int_1^e\frac{2}{\sqrt{(1-\log t)\log t}}\frac{ds}{s}=\frac{2b^2\pi}{a}$ and $R_1=1+r$. We have

$$\begin{aligned} R&=\int_1^e b\rho_1(s)\left(\max_{0\le u,v\le R_1}f(s,u,v)+\frac{2}{\sqrt{(1-\log t)\log t}}\right)\frac{ds}{s}\\ &\quad+\int_1^e b\rho_2(s)\left(\max_{0\le u,v\le R_1}g(s,u,v)+\frac{2}{\sqrt{(1-\log t)\log t}}\right)\frac{ds}{s}\\ &\le\int_1^e b\left(R_1^a+\frac{3}{\sqrt{(1-\log t)\log t}}\right)\frac{ds}{s}+\int_1^e b\left(R_1^b+\frac{3}{\sqrt{(1-\log t)\log t}}\right)\frac{ds}{s}=b\big(R_1^a+R_1^b+6\pi\big). \end{aligned}$$

Let

$$\lambda^*=\min\left\{1,\frac{R_1}{2}(R+1)^{-1},\frac{R_1}{2r}\right\},$$

Now, if $\lambda<\lambda^*$, Theorem 3.2 guarantees that (4.1) has a positive solution $(u,v)$ with $\|u\|\ge 1$ and $\|v\|\ge 1$. □

**Example 4.2.** Consider the following coupled integral boundary value problem

$$\begin{aligned} &D^{\alpha}u(t)+\lambda\left(\frac{2}{1+\log t}(v-a)(v-b)+\cos\left(\frac{\pi}{2a}u\right)\right)=0,\quad t\in(1,e),\quad \lambda>0,\\ &D^{\beta}v(t)+\lambda\left(\frac{2}{1+\log t}(u-c)(u-d)+\sin\left(\frac{\pi}{2c}v\right)\right)=0,\quad t\in(1,e),\quad \lambda>0,\\ &u^{(j)}(1)=v^{(j)}(1)=0,\quad 0\le j\le n-2,\quad u(e)=\mu\int_1^e v(s)\frac{ds}{s},\quad v(e)=\nu\int_1^e u(s)\frac{ds}{s}, \end{aligned} \tag{4.2}$$

where $b > a > 0$, $d > c > 0$. Then, if $\lambda > 0$ is sufficiently small, (4.2) has two solutions $(u_1, v_1)$ and $(u_2, v_2)$ with $u_i(t) > 0$ and $v_i(t) > 0$ for $t \in (1, e)$, $i = 1, 2$.

*Proof.* From (4.2), then we can see

$$f(t,u,v) = \frac{2}{1+\log t}(v-a)(v-b) + \cos\left(\frac{\pi}{2a}u\right), \quad g(t,u,v) = \frac{2}{1+\log t}(u-c)(u-d) + \sin\left(\frac{\pi}{2c}v\right).$$

Clearly, there exists a constant $q_1(t) = q_2(t) = m_0 > 0$ such that

$$f(t,u,v) + m_0 > 0, \quad g(t,u,v) + m_0 > 0, \quad \text{for } \forall t \in (1,e).$$

Let $\delta = \frac{1}{16(ab+cd+1)}$ and $\varepsilon = \frac{1}{4}\min(1,a,c)$, $c_0 = \max\left\{\int_1^e b\rho_1(s)\frac{ds}{s}, \int_1^e b\rho_2(s)\frac{ds}{s}\right\}$, we obtain

$$f(t,u,v) \geq \delta f(t,0,0) \geq \delta(ab+1), \quad g(t,u,v) \geq \delta g(t,0,0) \geq \delta cd, \quad \text{for } t \in (1,e), \quad 0 \leq u,v \leq \varepsilon.$$

Thus (H$_1$)-(H$_2$) and (H$_3^*$) hold. Since

$$\overline{f}(\varepsilon) = \max_{1\leq t\leq e, 0\leq u,v\leq\varepsilon}\{f(t,u,v) + e_1(t)\} \leq 2(ab+cd) + m_0 + 1,$$

$$\overline{g}(\varepsilon) = \max_{1\leq t\leq e, 0\leq u,v\leq\varepsilon}\{g(t,u,v) + e_2(t)\} \leq 2(ab+cd) + m_0 + 1,$$

$$\overline{h}(\varepsilon) = \max(\overline{f}(\varepsilon), \overline{g}(\varepsilon)) \leq 2(ab+cd) + m_0 + 1.$$

We can choose

$$\lambda_* = \frac{\varepsilon}{8c_0(2(ab+cd)+m_0+1)}.$$

Now, if $\lambda < \lambda_*$, Theorem 3.1 guarantees that (4.2) has a positive solution $(u_1, v_1)$ with $\|u_1\| \leq 1/4$ and $\|v_1\| \leq 1/4$.

On the other hand, we have

$$\liminf_{u\uparrow+\infty} \frac{f(t,u,v)}{u} = +\infty, \quad \liminf_{v\uparrow+\infty} \frac{g(t,u,v)}{v} = +\infty, \quad \text{for } t \in [\theta_1, \theta_2] \subset (1,e),$$

for $u, v \geq 0$. Thus (H$_1$)-(H$_3$) also hold. Let $r = \frac{2m_0b^2}{a}$ and $R_1 = 1 + r$. We have

$$R = \int_1^e b\rho_1(s)\left(\max_{0\leq u,v\leq R_1} f(s,u,v) + m_0\right)\frac{ds}{s} + \int_1^e b\rho_2(s)\left(\max_{0\leq u,v\leq R_1} g(s,u,v) + m_0\right)\frac{ds}{s},$$

and

$$\lambda^* = \min\left\{1, \frac{R_1}{2}(R+1)^{-1}, \frac{R_1}{2r}\right\},$$

Now, if $\lambda < \lambda^*$, Theorem 3.2 guarantees that (4.2) has a positive solution $(u_2, v_2)$ with $\|u_2\| \geq 1$ and $\|v_2\| \geq 1$.

Since all the conditions of Theorem 3.4 are satisfied, if $\lambda < \min(\lambda_*, \lambda^*)$, Theorem 3.4 guarantees that (4.2) has two solutions $(u_1, v_1)$ and $(u_2, v_2)$ with $u_i(t) > 0$ and $v_i(t) > 0$ for $t \in (1, e)$, $i = 1, 2$. □

**Example 4.3.** Consider the following coupled integral boundary value problem

$$\begin{aligned}
&D^\alpha u(t) + \lambda(v^a + \cos(2\pi u)) = 0, \quad t \in (1,e), \quad \lambda > 0,\\
&D^\beta v(t) + \lambda(u^b + \cos(2\pi v)) = 0, \quad t \in (1,e), \quad \lambda > 0,\\
&u^{(j)}(1) = v^{(j)}(1) = 0, \quad 0 \leq j \leq n-2, \quad u(e) = \mu\int_1^e v(s)\frac{ds}{s}, \quad v(e) = \nu\int_1^e u(s)\frac{ds}{s},
\end{aligned} \tag{4.3}$$

where $a, b > 1$. Then, if $\lambda > 0$ is sufficiently small, (4.3) has two solutions $(u_1, v_1)$ and $(u_2, v_2)$ with $u_i(t) > 0$ and $v_i(t) > 0$ for $t \in (1, e)$, $i = 1, 2$.

*Proof.* From (4.3), then we can state that

$$f(t,u,v)=v^a+\cos(2\pi u),\quad g(t,u,v)=u^b+\cos(2\pi v),\quad q_1(t)=q_2(t)=q(t)=2.$$

Clearly, we get

$$\begin{aligned}&f(t,u,v)+q(t)\ge v^a+1>0,\quad g(t,u,v)+q(t)\ge u^b+1>0,\quad \text{for}\ \ t\in(1,e),\\ &\liminf_{v\uparrow+\infty}\frac{f(t,u,v)}{v}=+\infty,\quad \liminf_{u\uparrow+\infty}\frac{g(t,u,v)}{u}=+\infty,\quad \text{for}\ \ t\in[\theta_1,\theta_2]\subset(1,e),\end{aligned}$$

for $u,v\ge 0$. And $f(t,0,0)=g(t,0,0)=1>0$, for $t\in[1,e]$. Thus (H$_1$)-(H$_2$) hold.

Let $\delta=1/2$ and $\varepsilon=1/8$, $c_0=\max\left\{\int_1^e b\rho_1(s)\frac{ds}{s},\int_1^e b\rho_2(s)\frac{ds}{s}\right\}$, we obtain $\overline{h}(\varepsilon)=\max(\overline{f}(\varepsilon),\overline{g}(\varepsilon))$, where

$$\begin{aligned}\overline{f}(\varepsilon)&=\max_{1\le t\le e,0\le u,v\le\varepsilon}\{f(t,u,v)+e_1(t)\}\le 8^{-a}+3,\\ \overline{g}(\varepsilon)&=\max_{1\le t\le e,0\le u,v\le\varepsilon}\{g(t,u,v)+e_2(t)\}\le 8^{-a}+3.\end{aligned}$$

Then $\frac{\varepsilon}{8c_0\overline{h}(\varepsilon)}\ge\frac{1}{64c_0(1+3)}=\frac{1}{256c_0}$. Let $\lambda_*=\frac{1}{256c_0}$. Now, if $\lambda<\lambda_*$, Theorem 3.1 guarantees that (4.3) has a positive solution $(u_1,v_1)$ with $\|u_1\|\le 1/8$ and $\|v_1\|\le 1/8$.

On the other hand, let $r=4b^2/a$ and $R_1=1+r$. We have

$$R=\int_1^e b\rho_1(s)\left(\max_{0\le u,v\le R_1}f(s,u,v)+2\right)\frac{ds}{s}+\int_1^e b\rho_2(s)\left(\max_{0\le u,v\le R_1}g(s,u,v)+2\right)\frac{ds}{s},$$

and

$$\lambda^*=\min\left\{1,\frac{R_1}{2}(R+1)^{-1},\frac{R_1}{2r}\right\},$$

Now, if $\lambda<\lambda^*$, Theorem 3.3 guarantees that (4.3) has a positive solution $(u_2,v_2)$ with $\|u_2\|\ge 1$ and $\|v_2\|\ge 1$.

Since all the conditions of Theorem 3.5 are satisfied, if $\lambda<\min(\lambda_*,\lambda^*)$, Theorem 3.5 guarantees that (4.3) has two solutions $(u_1,v_1)$ and $(u_2,v_2)$ with $u_i(t)>0$ and $v_i(t)>0$ for $t\in(1,e)$, $i=1,2$. □

**Example 4.4.** Consider the following coupled integral boundary value problem

$$\begin{aligned}&D^\alpha u(t)+\lambda\left(e^u+v^2+7\cos(2\pi(t-1)u)\right)=0,\quad t\in(1,e),\quad \lambda>0,\\ &D^\beta v(t)+\lambda\left(e^v+u^2+7\cos(2\pi(t-1)v)\right)=0,\quad t\in(1,e),\quad \lambda>0,\\ &D^j u(0)=D^j v(0)=0,\quad 0\le j\le n-2,\quad u(1)=\mu\int_1^e v(s)\frac{ds}{s},\quad v(1)=\nu\int_1^e u(s)\frac{ds}{s}.\end{aligned}\tag{4.4}$$

Then, if $\lambda>0$ is sufficiently small, (4.4) has two solutions $(u_1,v_1)$ and $(u_2,v_2)$ with $u_i(t)>0$ and $v_i(t)>0$ for $t\in(1,e)$, $i=1,2$.

*Proof.* From (4.4), then we can see

$$f(t,u,v)=e^u+v^2+7\cos(2\pi(t-1)u),\quad g(t,u,v)=e^v+u^2+7\cos(2\pi(t-1)v).$$

Clearly, there exists a constant $q_1(t)=q_2(t)=8>0$ such that

$$f(t,0,0)=g(t,0,0)=8,\quad f(t,u,v)+8\ge 1>0,\quad g(t,u,v)+8\ge 1>0,\quad \text{for}\ \ \forall t\in(1,e).$$

Let $\delta=1/100$ and $\varepsilon=1/8$, we obtain

$$f(t,u,v)\ge\delta f(t,0,0),\quad g(t,u,v)\ge\delta g(t,0,0),\quad \text{for}\ \ t\in(1,e),\quad 0\le u,v\le\varepsilon.$$

Thus (H$_1$), (H$_2$), and (H$_3^*$) hold. Since

$$\overline{h}(\varepsilon)=\max\left\{\max_{1\le t\le e,0\le u,v\le\varepsilon}\{f(t,u,v)+q_1(t)\},\max_{1\le t\le e,0\le u,v\le\varepsilon}\{g(t,u,v)+q_2(t)\}\right\}\le e+16.$$

Let $c_0 = \max\left\{\int_1^e b\rho_1(s)\frac{ds}{s}, \int_1^e b\rho_2(s)\frac{ds}{s}\right\}$. We can choose

$$\lambda_* = \frac{\varepsilon}{8c_0(e+16)}.$$

Now, if $\lambda < \lambda_*$, Theorem 3.1 guarantees that (4.4) has a positive solution $(u_1, v_1)$ with $\|u_1\| \leq 1/8$ and $\|v_1\| \leq 1/8$.

On the other hand, we have

$$\liminf_{u\uparrow+\infty} \frac{f(t,u,v)}{u} = +\infty, \quad \liminf_{v\uparrow+\infty} \frac{g(t,u,v)}{v} = +\infty, \quad \text{for} \quad t \in [\theta_1, \theta_2] \subset (0,1),$$

for $u, v \geq 0$. Thus ($H_1$), ($H_2$), and ($H_3$) also hold. Let $r = \frac{16b^2}{\omega}$ and $R_1 = 1 + r$. We have

$$\begin{aligned} R &= \int_1^e b\rho_1(s)\left(\max_{0\leq u,v\leq R_1} f(s,u,v) + m_0\right)\frac{ds}{s} + \int_1^e b\rho_2(s)\left(\max_{0\leq u,v\leq R_1} g(s,u,v) + m_0\right)\frac{ds}{s} \\ &\leq \int_1^e b\rho_1(s)\left(e^{R_1} + R_1^2 + 7 + 8\right)\frac{ds}{s} + \int_1^e b\rho_2(s)\left(e^{R_1} + R_1^2 + 7 + 8\right)\frac{ds}{s} \\ &= c_0\left(e^{R_1} + R_1^2 + 15\right), \end{aligned}$$

and

$$\lambda^* = \min\left\{1, \frac{R_1}{2}(R+1)^{-1}, \frac{R_1}{2r}\right\},$$

Now, if $\lambda < \lambda^*$, Theorem 3.2 guarantees that (4.4) has a positive solution $(u_2, v_2)$ with $\|u_2\| \geq 1$ and $\|v_2\| \geq 1$.

Since all the conditions of Theorem 3.4 are satisfied, if $\lambda < \min(\lambda_*, \lambda^*)$, Theorem 3.4 guarantees that (4.4) has two solutions $(u_1, v_1)$ and $(u_2, v_2)$ with $u_i(t) > 0$ and $v_i(t) > 0$ for $t \in (1, e)$, $i = 1, 2$. □

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