Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 7 (2014), 115–125
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Existence of solutions for quasi-linear impulsive functional integrodifferential equations in Banach spaces

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Communicated by C. Park

Abstract

We study the existence of mild solutions for quasilinear impulsive integrodifferential equation in Banach spaces. The results are established by using Hausdorff's measure of noncompactness and fixed point theorem. Application is provided to illustrate the theory.

Keywords: Mild solution, nonlocal conditions, impulsive condition, Hausdorff's measure of noncompactness, fixed point theorem.
2010 MSC: 34A37, 34K05, 34K30, 47H10.

1. Introduction

Measures of noncompactness are a very useful tool in many branches of mathematics. They are used in the fixed point theory, linear operators theory, theory of differential and integral equations and others [4]. There are two measures which are the most important ones. The Kuratowski measure of noncompactness $\sigma(X)$ of a bounded set X in a metric space is defined as infimum of numbers $r > 0$ such that X can be covered with a finite number of sets of diameter smaller than r. The Hausdorff measure of noncompactness $\Psi(X)$ defined as infimum of numbers $r > 0$ such that X can be covered with a finite number of balls of radii smaller than r. The Hausdorff measure is convenient in applications. There exist many formulae on $\Psi(X)$ in various spaces [4, 5]. However, there are some differences between the Kuratowski measure and the Hausdorff measure.

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Received 2013-08-14

One of these differences is that $\sigma(X)$ depends on the set X only, while $\Psi(X)$ also depends on the space in which X is included. Let $\mathbb{E}$ be a Banach space and $\mathbb{F}$ be a subspace of $\mathbb{E}$. Let $\Psi_{\mathbb{E}}(X)$, $\Psi_{\mathbb{F}}(X)$, $\sigma_{\mathbb{E}}(X)$, $\sigma_{\mathbb{F}}(X)$ denote Hausdorff and Kuratowski measures in spaces $\mathbb{E}, \mathbb{F}$, respectively. Then, for any bounded $X \subset \mathbb{F}$ we have

$$\Psi_{\mathbb{E}}(X) \leq \Psi_{\mathbb{F}}(X) \leq \sigma_{\mathbb{F}}(X) = \sigma_{\mathbb{E}}(X) \leq 2\Psi_{\mathbb{E}}(X).$$

The notion of a measure of noncompactness turns out to be a very important and useful tool in many branches of mathematical analysis. The notion of a measure of weak compactness was introduced by De Blasi [10] and was subsequently used in numerous branches of functional analysis and the theory of differential and integral equations. Several authors have studied the measures of noncompactness in Banach spaces [4, 5, 14, 1, 13, 15, 22].

On the other hand, the study of the impulsive differential equations has attract a great deal of attention. Many evolution processes are characterized by the fact that at certain moments of time they experience a change of state abruptly. These processes are subject to short-term perturbations whose duration is negligible in comparison with the duration of the process. Consequently, it is natural to assume that these perturbations act instantaneously, that is, in the form of impulses. It is known, for example, that many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control model in economics, pharmacokinetics and frequency modulated systems, do exhibit impulsive effects. The theory of impulsive differential equations is an important branch of differential equations have studied by many authors [16, 19, 20, 21].

The existence of solution to evolution equations with nonlocal conditions in Banach space was studied first by Byszewski [7, 8]. Byszewski and Lakshmikanthan [9] proved an existence and uniqueness of solutions of a nonlocal Cauchy problem in Banach spaces. Ntouyas and Tsamatos [23] studied the existence for semilinear evolution equations with nonlocal conditions. The problem of existence of solutions of evolution equations in Banach space has been studied by several authors [3, 12, 24]. This article is motivated by the results of [4, 14, 19, 3, 12, 11].

In this paper, we consider the quasilinear integrodifferential equations with impulsive and nonlocal condition of the form

$$\begin{aligned} u'(t) + A(t, u(t))u(t) &= f(t, u(t)) + \int_0^t g(t, s, u(s))ds, \quad t \in [0, b], \quad t \neq t_i \qquad (1.1)\\ u(0) + h(u) &= u_0 \qquad (1.2)\\ \Delta u(t_i) &= I_i(u(t_i)), \quad i = 1, 2, 3, \ldots, n, \ 0 < t_1 < t_2 <, \ldots t_n < b, \qquad (1.3) \end{aligned}$$

where $A : [0, b] \times X \to X$ are continuous functions in Banach space X, $u_0 \in X$, $f : [0, b] \times X \to X$, $g : \Omega \times X \to X$, $h : \mathcal{PC}([0, b]; X) \to X$ and $\Delta u(t_i) = u(t_i^+) - u(t_i^-)$ constitutes an impulsive condition. Here $\Omega = \{(t, s) : 0 \leq s \leq t \leq b\}$.

2. Preliminaries

Let X be a Banach space with norm $||\cdot||$. Let $\mathcal{PC}([0, b]; X)$ consist of functions u from $[0, b]$ into X, such that $u(t)$ is continuous at $t \neq t_i$ and left continuous at $t = t_i$ and the right limit $u(t_i^+)$ exists for $i = 1, 2, 3, \cdots, n$. Evidently $\mathcal{PC}([0, b]; X)$ is a Banach space with the norm

$$\|u\|_{\mathcal{PC}} = \sup_{t \in [0,b]} \|u(t)\|,$$

and denoted $\mathcal{L}([0, b]; X)$ by the space of X-valued Bochner integrable functions on $[0, b]$ with the form

$$\|u\|_{\mathcal{L}} = \int_0^b \|u(t)\|dt.$$

The Hausdorff's measure of noncompactness Ψ_Y is defined by

$$\Psi(B) = inf\{r > 0, B \text{ can be covered by finite number of balls with radii r}\}$$

for bounded set B in a Banach space Y.

Lemma 2.1 ([4]). *Let Y be a real Banach space and $B, E \subseteq Y$ be bounded, with the following properties:*

(i) B *is precompact if and only if* $\Psi_X(B) = 0$.
(ii) $\Psi_Y(B) = \Psi_Y(\bar{B}) = \Psi_Y(conB)$, *where* $\bar{B}$ *and* $conB$ *mean the closure and convex hull of* B *respectively.*
(iii) $\Psi_Y(B) \leq \Psi_Y(E)$, *where* $B \subseteq E$.
(iv) $\Psi_Y(B + E) \leq \Psi_Y(B) + \Psi_Y(E)$, *where* $B + E = \{x + y : \ x \in B, \ y \in E\}$
(v) $\Psi_Y(B \cup E) \leq max\{\Psi_Y(B), \ \Psi_Y(E)\}$.
(vi) $\Psi_Y(\lambda B) \leq |\lambda|\Psi_Y(B)$ *for any* $\lambda \in R$.
(vii) *If the map* $\mathcal{F} : D(\mathcal{F}) \subseteq Y \to Z$ *is Lipschitz continuous with constant* r, *then* $\Psi_Z(\mathcal{F}B) \leq r\Psi_Y(B)$ *for any bounded subset* $B \subseteq D(\mathcal{F})$, *where* Z *be a Banach space.*
(viii) $\Psi_Y(B) = inf\{d_Y(B, E); E \subseteq Y \text{ is precompact}\} = inf\{d_Y(B, E); E \subseteq Y \text{ is finite valued}\}$, *where* $d_Y(B, E)$ *means the nonsymmetric (or symmetric) Hausdorff distance between* B *and* E *in* Y.
(ix) *If* $\{\mathbb{W}_n\}_{n=1}^{+\infty}$ *is decreasing sequence of bounded closed nonempty subsets of* Y *and* $lim_{n\to\infty}\Psi_Y(\mathbb{W}_n) = 0$, *then* $\bigcap_{n=1}^{+\infty} \mathbb{W}_n$ *is nonempty and compact in* Y.

The map $\mathcal{F} : \mathbb{W} \subseteq Y \to Y$ is said to be a Ψ_Y-contraction if there exists a positive constant $r < 1$ such that $\Psi_Y(\mathcal{F}(B)) \leq r\Psi_Y(B)$ for any bounded closed subset $B \subseteq \mathbb{W}$, where Y is a Banach space.

Lemma 2.2 (Darbo-Sadovskii [4]). *If* $\mathbb{W} \subseteq Y$ *is bounded closed and convex, the continuous map* $\mathcal{F} : \mathbb{W} \to \mathbb{W}$ *is a* Ψ_Y*-contraction, the map* $\mathcal{F}$ *has atleast one fixed point in* $\mathbb{W}$.

We denote by Ψ *the Hausdorff's measure of noncompactness of* X *and also denote* Ψ_c *by the Hausdorff's measure of noncompactness of* $\mathcal{PC}([0, b]; X)$.

Before we prove the existence results, we need the following Lemmas.

Lemma 2.3 ([4]). *If* $\mathbb{W} \subseteq \mathcal{PC}([0, b]; X)$ *is bounded, then* $\Psi(\mathbb{W}(t)) \leq \Psi_c(\mathbb{W})$ *for all* $t \in [0, b]$, *where* $\mathbb{W}(t) = \{u(t); u \in \mathbb{W}\} \subseteq X$. *Furthermore if* $\mathbb{W}$ *is equicontinuous on* $[a, b]$, *then* $\Psi(\mathbb{W}(t))$ *is continuous on* $[a, b]$ *and* $\Psi_c(\mathbb{W}) = sup\{\Psi(\mathbb{W}(t)), \ t \in [a, b]\}$.

Lemma 2.4 ([17, 18]). *If* $\{u_n\}_{n=1}^{\infty} \subset \mathcal{L}^1([a, b]; X)$ *is uniformly integrable, then the function* $\Psi(\{u_n(t)\}_{n=1}^{\infty})$ *is measurable and*

$$\Psi\Big(\Big\{\int_0^t u_n(s)ds\Big\}_{n=1}^{\infty}\Big) \leq 2\int_0^t \Psi(\{u_n(s)\}_{n=1}^{\infty})ds. \tag{2.1}$$

Lemma 2.5 ([4]). *If* $\mathbb{W} \subseteq \mathcal{PC}([0, b]; X)$ *is bounded and equicontinuous, then* $\Psi(\mathbb{W}(t))$ *is continuous and*

$$\Psi(\int_0^t \mathbb{W}(s)ds) \leq \int_0^t \Psi(\mathbb{W}(s))ds. \tag{2.2}$$

for all $t \in [0, b]$, *where* $\int_0^t \mathbb{W}(s)ds = \{\int_0^t u(s)ds; u \in \mathbb{W}\}$.

The C_0 semigroup $U_u(t, s)$ is said to be equicontinuous if $(t, s) \to \{U_u(t, s)u(s) : u \in B\}$ is equicontinuous for $t > 0$ for all bounded set B in X. The following lemma is obvious.

Lemma 2.6. *If the evolution family* $\{U_u(t, s)\}_{0\leq s\leq t\leq b}$ *is equicontinuous and* $\eta \in \mathcal{L}([0, b]; R^+)$, *then the set* $\{\int_0^t U_u(t, s)u(s)ds, \ ||u(s)|| \leq \eta(s)$ *for a.e* $s \in [0, b]$ *is equicontinuous for* $t \in [0, b]$.

From [3], we know that for any fixed $u \in \mathcal{PC}([0, b]; X)$ there exist a unique continuous function $U_u : [0, b] \times [0, b] \to B(X)$ defined on $[0, b] \times [0, b]$ such that

$$U_u(t, s) = I + \int_s^t A_u(w)U_u(w, s)dw, \tag{2.3}$$

where $B(X)$ denote the Banach space of bounded linear operators from X to X with the norm $||\mathcal{F}|| = sup\{||\mathcal{F}u|| : ||u|| = 1\}$, and I stands for the identity operator on X, $A_u(t) = A(t, u(t))$, we have

$$U_u(t, t) = I,\ U_u(t, s)U_u(s, r) = U_u(t, r), \quad (t, s, r) \in [0, b] \times [0, b] \times [0, b],$$

$$\frac{\partial U_u(t, s)}{\partial t} = A_u(t)U_u(t, s) \text{ for almost all } t, s \in [0, b]$$

3. The Existence of Mild Solution

Definition 3.1. A function $u \in \mathcal{PC}([0, b] : X)$ is said to be a mild solution of (1.1) - (1.3) if it satisfies the integral equation

$$\begin{aligned} u(t) \;&=\; U_u(t, 0)u_0 - U_u(t, 0)h(u) + \int_0^t U_u(t, s)\Big[f(s, u(s)) + \int_0^s g(s, \tau, u(\tau))d\tau\Big]ds \\ &\quad + \sum_{0<t_i<t} U_u(t, t_i)I_i(u(t_i)), \quad 0 \le t \le b. \end{aligned} \tag{3.1}$$

In this paper, we denote $M_0 = sup\{\|U_u(t, s)\| : (t, s) \in [0, b] \times [0, b]\}$ for all $u \in X$. Without loss of generality, we let $u_0 = 0$. Assume the following conditions:

(H_1) The evolution family $\{U_u(t, s)\}_{0\le s\le t\le b}$ generated by $A(t, u(t))$ is equicontinuous and $||U_u(t, s)|| \le M_0$ for almost $t, s \in [0, b]$.

(H_2) (a) The function $h : \mathcal{PC}([0, b] \times X \to X$ is continuous and compact.

(b) There exist $N_0 > 0$ such that $||h(u)|| \le N_0$ for all $u \in \mathcal{PC}([0, b]; X)$.

(H_3) (i) The nonlinear function $f : [0, b] \times X \to X$ satisfies the Caratheodory-type conditions; i.e., $f(\cdot, u)$ is measurable for all $u \in X$ and $f(t, \cdot)$ is continuous for a.e $t \in [a, b]$.

(ii) There exists a function $\alpha \in \mathcal{L}([0, b]; \mathcal{R}^+)$ such that for every $u \in X$, we have

$$\|f(t, u)\| \le \alpha(t)(1 + \|u\|)$$

a.e $t \in [0, b]$.

(iii) There exists a function $k_1 \in \mathcal{L}([0, b]; \mathcal{R}^+)$ such that, for every bounded $D \subset X$, we have

$$\Psi(f(t, D)) \le k_1(t)\Psi(D)$$

a.e $t \in [0, b]$.

(H_4) (i) The nonlinear function $g : [0, b] \times [0, b] \times X \to X$ satisfies the Caratheodory-type conditions; i.e., $g(\cdot, \cdot, u)$ is measurable for all $u \in X$ and $g(t, s, \cdot)$ is continuous for a.e $t \in [a, b]$.

(ii) There exist two functions $\beta_1 \in \mathcal{L}([0, b]; R^+)$ and $\beta_2 \in \mathcal{L}([0, b]; R^+)$ such that for every $u \in X$, we have

$$\|g(t, s, u(s))\| \le \beta_1(t)\beta_2(s)(1 + \|u(s)\|)$$

a.e $t \in [0, b]$.

(iii) There exists a function $k_2 \in \mathcal{L}([0, b]; R^+)$ such that, for every bounded $D \subset X$, we have

$$\Psi(g(t, s, D)) \le k_2(t)k_3(s)\Psi(D)$$

a.e $t \in [0, b]$. Assume that the finite bound of $\int_0^t k_2(s)ds$ is G_0.

(H_5) For every $t \in [0,b]$ and there exist positive constants N_1 and N_2, the scalar equation

$$m(t) = M_0N_1 + M_0N_2\Big[\int_0^t [\alpha(s) + C_0\beta_2(s)](1 + m(s))ds + \sum_{i=1}^n d_i\Big].$$

(H_5) $I_i : X \to X$ is continuous. There exists a constant $d_i > 0$ $i = 1, 2, 3, \ldots, n$ such that

$$\|I_i(u(t_i))\| \;\leq\; \sum_{i=1}^n d_i, \quad where,\; i = 1, 2, 3, \cdots, n.$$

For any bounded subset $D \subset X$, and there is a constant $l_i > 0$ such that

$$\Psi(I_i(D)) \leq \sum_{i=1}^n l_i\Psi(D), \quad i = 1, 2\cdots, n.$$

Theorem 3.2. *Assumptions $(H_1) - (H_5)$ holds, then the impulsive nonlocal problem (1.1) - (1.3) has at least one mild solution.*

Proof. Let $m(t)$ be a solution of the scalar equation

$$m(t) = M_0N_0 + M_0\Big[\int_0^t [\alpha(s) + C_0\beta_2(s)](1 + m(s))ds + \sum_{i=1}^n d_i\Big]. \tag{3.2}$$

Let us assume that the finite bound of $\int_0^t \beta_1(s)ds$ is C_0 for $t \in [0,b]$.
Consider the map $\mathcal{F} : \mathcal{PC}([0,b];X) \to \mathcal{PC}([0,b];X)$ defined by

$$\begin{aligned}(\mathcal{F}u)(t) \;&=\; U_u(t,0)h(u) + \int_0^t U_u(t,s)\Big[f(s,u(s)) + \int_0^s g(s,\tau,u(s))d\tau\Big]ds \\ &\quad + \sum_{0<t_i<t} U_u(t,t_i)I_i(u(t_i))\end{aligned} \tag{3.3}$$

for all $u \in \mathcal{PC}([0,b];X)$. Let us take $\mathbb{W}_0 = \{u \in \mathcal{PC}([0,b];X),\; ||u(t)|| \leq m(t)\;\; \text{for all } t \in [0,b]\}$. Then $\mathbb{W}_0 \subseteq \mathcal{PC}([0,b];X)$ is bounded and convex.

We define $\mathbb{W}_1 = \overline{con}K(\mathbb{W}_0)$, where $\overline{con}$ means the closure of the convex hull in $\mathcal{PC}([0,b];X)$. As $U_u(t,s)$ is equicontinuous, h is compact and $\mathbb{W}_0 \subseteq \mathcal{PC}([0,b];X)$ is bounded, due to Lemma 2.6 and using the assumptions, $\mathbb{W}_1 \subseteq \mathcal{PC}([0,b];X)$ is bounded closed convex nonempty and equicontinuous on $[0,b]$. For any $u \in \mathcal{F}(\mathbb{W}_0)$, we know

$$\begin{aligned}\|u(t)\| \;&\leq\; \|U_u(t,0)h(u)\| + \int_0^t \|U_u(t,s)\Big[f(s,u(s)) + \int_0^s g(s,\tau,u(s))d\tau\Big]\|ds \\ &\quad + \sum_{0<t_i<t} \|U_u(t,t_i)I_i(u(t_i))\| \\ &\leq\; M_0N_0 + M_0\Big[\int_0^t ||f(s,u(s))||ds + \int_0^t\int_0^s ||g(s,\tau,u(\tau))||d\tau ds\Big] + M_0\sum_{i=1}^n \|I_i(u(t_i))\| \\ &\leq\; M_0N_0 + M_0\int_0^t \alpha(s)(1 + \|u(s)\|)ds + M_0\int_0^t\int_0^s \beta_1(s)\beta_2(\tau)(1 + \|u(\tau)\|)d\tau ds + M_0\sum_{i=1}^n d_i \\ &\leq\; M_0N_0 + M_0\int_0^t [\alpha(s)(1 + m(s))ds + M_0C_0\int_0^t \beta_2(s)(1 + m(s))ds + M_0\sum_{i=1}^n d_i \\ &\leq\; M_0N_0 + M_0\Big[\int_0^t [\alpha(s) + C_0\beta_2(s)](1 + m(s))ds + \sum_{i=1}^n d_i\Big], \quad \text{for }\; t \in [0,b]. \\ &=\; m(t).\end{aligned}$$

It follows that $\mathbb{W}_1 \subset \mathbb{W}_0$. We define $\mathbb{W}_{n+1} = \overline{con}\mathcal{F}(\mathbb{W}_n)$, for $n = 1, 2, 3, \cdots$. From above we know that $\{\mathbb{W}_n\}_{n=1}^{\infty}$ is a decreasing sequence of bounded, closed, convex, equicontinuous on $[0, b]$ and nonempty subsets in $\mathcal{PC}([0, b]; X)$.

Now for $n \geq 1$ and $t \in [0, b]$, $\mathbb{W}_n(t)$ and $\mathcal{F}(\mathbb{W}_n(t))$ are bounded subsets of X, hence, for any $\epsilon > 0$, there is a sequence $\{u_k\}_{k=1}^{\infty} \subseteq \mathbb{W}_n$ such that (see, e.g.[6], pp125).

$$\begin{aligned}
\Psi(\mathbb{W}_{n+1}(t)) &= \Psi(\mathcal{F}\mathbb{W}_n(t)) \\
&\leq 2M_0 \int_0^t \Psi\Big(f(s, \{u_k(s)\}_{k=1}^{\infty})\Big)ds + 4M_0 \int_0^t \int_0^s \Psi\Big(g(s, \tau, \{u_k(\tau)\}_{k=1}^{\infty})\Big)d\tau ds \\
&\quad + 2M_0 \sum_{i=1}^{n} \Psi\Big(I_i(\{u_k(t_i)\}_{k=1}^{\infty})\Big) + \epsilon \\
&\leq 2M_0 \int_0^t k_1(s)\Psi\Big(\{u_k(s)\}_{k=1}^{\infty}\Big)ds + 4M_0 \int_0^t \int_0^s k_2(s)k_3(\tau)\Psi\Big(\{u_k(\tau)\}_{k=1}^{\infty}\Big)d\tau ds \\
&\quad + 2M_0 \sum_{i=1}^{n} l_i\Psi\Big(\{u_k(t_i)\}_{k=1}^{\infty}\Big) + \epsilon \\
&\leq 2M_0\Big[\int_0^t k_1(s)\Psi(\mathbb{W}_n(s))ds + 2G_0 \int_0^t k_3(s)\Psi(\mathbb{W}_n(s))ds + \sum_{i=1}^{n} l_i\Psi(\mathbb{W}_n(t_i))\Big] + \epsilon
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, it follows that from the above inequality that

$$\Psi(\mathbb{W}_{n+1}(t)) \leq 2M_0\Big[\int_0^t [k_1(s) + 2G_0k_3(s)]\Psi(\mathbb{W}_n(s))ds + \sum_{i=1}^{n} l_i\Psi(\mathbb{W}_n(t_i)\Big] \tag{3.4}$$

for all $t \in [0, b]$. Because $\mathbb{W}_n$ is decreasing for n, we have

$$\gamma(t) = lim_{n\to\infty}\Psi(\mathbb{W}_n(t))$$

for all $t \in [0, b]$. From (3.4), we have

$$\gamma(t) \leq 2M_0\Big[\int_0^t [k_1(s) + 2G_0k_3(s)]\gamma(s)ds + \sum_{i=1}^{n} l_i\gamma(t_i))\Big]$$

for $t \in [0, b]$, which implies that $\gamma(t) = 0$ for all $t_i \in [0, b]$. By Lemma 2.3, we know that $lim_{n\to\infty}\Psi(\mathbb{W}_n(t)) = 0$. Using Lemma 2.1 we know that $\mathbb{W} = \bigcap_{n=1}^{\infty} \mathbb{W}_n$ is convex compact and nonempty in $\mathcal{PC}([0, b]; X)$ and $\mathcal{F}(\mathbb{W}) \subset \mathbb{W}$. By the Schauder fixed point theorem, there exist at least one mild solution u of the intimal value problem (1.1) - (1.3), where $u \in \mathbb{W}$ is a fixed point of the continuous map $\mathcal{F}$.

Remark 3.3. If the functions f, g and I_i are compact or Lipschitz continuous (see e.g [7, 9], then $(H_3) - (H_5)$ is automatically satisfied.

In some of the early related results in references and above results, it is supposed that the map h is uniformly bounded. In fact, if h is compact, then it must be bounded on bounded set. Here we give an existence result under growth condition of f, g and I_i, when h is not uniformly bounded. Precisely, we replace the assumptions $(H_3) - (H_4)$ by

(H_6) There exists a function $p \in \mathcal{L}([0, b]; R^+)$ and a increasing function $\phi : R^+ \to R^+$ such that

$$\|f(t, u)\| \leq p(t)\phi(\|u\|),$$

for a.e $t \in [0, b]$ and forall $u \in \mathcal{PC}([0, b]; X)$

(H_7) There exist two functions $q \in \mathcal{L}([0,b]; R^+)$and $\widehat{q} \in \mathcal{L}([0,b]; R^+)$ and a increasing function $\chi : R^+ \to R^+$ such that

$$\|g(t,s,u)\| \le q(t)\widehat{q}(s)\chi(\|u\|),$$

for a.e $t \in [0,b]$ and for all $u \in \mathcal{PC}([0,b]; X)$. Assume that the finite bound of $\int_0^t q(s)ds$ is G_1.

Theorem 3.4. *Suppose that the assumptions $(H_1) - (H_2)$ and $(H_6) - (H_7)$ are satisfied, then the equation (1.1)-(1.3) has at least one mild solution if*

$$\lim_{r\to\infty} sup \frac{M_0}{r}\Big(\varphi(r) + \phi(r)\int_0^t p(s)ds + G_1\chi(r)\int_0^t \widehat{q}(s)ds + \sum_{i=1}^n d_i\Big) < 1, \tag{3.5}$$

where $\varphi(r) = sup\{||h(u)||,\ ||u|| \le r\}$.

Proof. The inequality (3.5) implies that there exist a constant $r > 0$ such that

$$M_0\Big[\varphi(r) + \phi(r)\int_0^t p(s)ds + G_1\chi(r)\int_0^t \widehat{q}(s)ds + \sum_{i=1}^n d_i\Big] < r.$$

As in the proof of Theorem 3.1, let $\mathbb{W}_0 = \{u \in \mathcal{PC}([0,b]; X),\ ||u(t)|| \le r\}$ and $\mathbb{W}_1 = \overline{con}\mathcal{F}\mathbb{W}_0$. Then for any $u \in \mathbb{W}_1$, we have

$$\begin{aligned}
\|u(t)\| &\le \|U_u(t,0)h(u)\| + \int_0^t \|U_u(t,s)\Big[f(s,u(s)) + \int_0^s g(s,\tau,u(\tau))d\tau\Big]ds \\
&\quad + \sum_{0<t_i<t} \|U_u(t,t_i)I_i(u(t_i))\| \\
&\le M_0\varphi(r) + M_0\Big[\int_0^t p(s)\phi(\|u(s)\|)ds + \int_0^t\int_0^s q(s)\widehat{q}(\tau)\chi\|(u(\tau))||d\tau ds + \sum_{i=1}^n d_i\Big] \\
&\le M_0\varphi(r) + M_0\Big[\int_0^t p(s)\phi(\|u(s)\|)ds + G_1\int_0^t \widehat{q}(s)\chi(\|u(s)\|)ds + \sum_{i=1}^n d_i\Big] \\
\|u(t)\| &\le M_0\Big[\varphi(r) + \phi(r)\int_0^t p(s)ds + G_1\chi(r)\int_0^t \widehat{q}(s)ds + \sum_{i=1}^n d_i\Big] < r
\end{aligned}$$

for $t \in [0,b]$. It means that $\mathbb{W}_1 \subset \mathbb{W}_0$. So we can complete the proof similarly to Theorem 3.1.

4. When h is Lipschitz

In this section, we discuss the equation (1.1) - (1.3) when h is Lipschitz and f, g and I_i are not Lipschitz. Assume that

(H_8) The function h is a Lipschitz continuous in X, there exist a constant $L_0 > 0$ such that

$$\|h(u) - h(v)\| \le L_0\|u - v\|, \quad u, v \in \mathcal{PC}([0,b]; X).$$

Theorem 4.1. *Suppose that the assumptions $(H_1) - (H_8)$ are satisfied, then the equation(1.1) - (1.3) has at least one mild solution provided that*

$$M_0\Big[L_0 + 2\int_0^t (k_1(s) + 2G_0)ds + \sum_{i=1}^n l_i\Big] < 1. \tag{4.1}$$

Proof. Consider the map $\mathcal{F} : \mathcal{PC}([0, B]; X) \to \mathcal{PC}([0, B]; X)$
is defined by $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$, where

$$(\mathcal{F}_1 u)(t) = U_u(t, 0)h(u),$$

$$(\mathcal{F}_2 u)(t) = \int_0^t U_u(t, s)\Big[f(s, u(s)) + \int_0^s g(s, \tau, u(\tau))d\tau\Big]ds + \sum_{0<t_i<t} U_u(t, t_i)I_i(u(t_i))$$

for $u \in \mathcal{PC}([0, B]; X)$. As defined in the proof of Theorem 3.1. We define $\mathbb{W}_0 = \{u \in \mathcal{PC}([0, B]; X) : ||u(t)|| \leq m(t)$ for all $t \in [0, b]$ and let $\mathbb{W} = \overline{con}\mathcal{F}\mathbb{W}_0$. Then from the proof of Theorem 3.1 we know that $\mathbb{W}$ is a bounded closed convex and equicontinuous subset of $\mathcal{PC}([0, b]; X)$ and $\mathcal{F}\mathbb{W} \subset \mathbb{W}$. We shall prove that $\mathcal{F}$ is Ψ_c-contraction on $\mathbb{W}$. Then Darbo-Sadovskii's fixed point theorem can be used to get a fixed point of $\mathcal{F}$ in $\mathbb{W}$, which is a mild solution of (1.1) - (1.2). First, for every bounded subset $B \subset \mathbb{W}$, from the (H_8) and Lemma 2.1 we have

$$\Psi_c(\mathcal{F}_1 B) = \Psi_c(U_B(t, 0)h(B)) \leq M_0\Psi_c(h(B)) \leq M_0L_0\Psi_c(B). \tag{4.2}$$

Next, for every bounded subset $B \subset \mathbb{W}$, for $t \in [0, b]$ and every $\epsilon > 0$, there is a sequence $\{u_k\}_{k=1}^{\infty} \subset B$, such that

$$\Psi(\mathcal{F}_2(B(t)) \leq 2\Psi(\{\mathcal{F}_2 u_k(t)\}_{n=1}^{\infty} + \epsilon.$$

Note that B and $\mathcal{F}_2 B$ are equicontinuous, we can get from Lemma 2.1, Lemma 2.4, Lemma 2.5 and using the assumptions we get

$$\begin{aligned}
\Psi(\mathcal{F}_2 B(t)) &\leq 2M_0\int_0^t \Psi\Big(f(s, \{u_k(s)\}_{k=1}^{\infty})\Big)ds + 4M_0\int_0^t\int_0^s \Psi\Big(g(s, \tau, \{u_k(\tau)\}_{k=1}^{\infty})\Big)d\tau ds \\
&\quad +2M_0\sum_{0<t_i<t}\Psi\Big(I_i(\{u_k(t_i)\}_{k=1}^{\infty})\Big) + \epsilon \\
&\leq 2M_0\int_0^t k_1(s)\Psi\Big(\{u_k(s)\}_{k=1}^{\infty}\Big)ds + 4M_0\int_0^t\int_0^s k_2(s)k_3(\tau)\Psi\Big(\{u_k(\tau)\}_{k=1}^{\infty}\Big)d\tau ds \\
&\quad +2M_0\sum_{i=1}^n l_i\Psi\Big(\{u_k(t_i)\}_{k=1}^{\infty}\Big) + \epsilon \\
&\leq 2M_0\Big[\int_0^t k_1(s)\Psi(B)ds + 2G_0\int_0^t k_3(s)\Psi(B)ds + \sum_{i=1}^n l_i\Psi(B)\Big] + \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, it follows that from the above inequality that

$$\Psi_c(\mathcal{F}_2 B(t)) \leq 2M_0\Big[\int_0^t [k_1(s) + 2G_0k_3(s)]ds + \sum_{i=1}^n l_i\Big]\Psi_c(B) \tag{4.3}$$

for any bounded $B \subset \mathbb{W}$.

Now, for any subset $B \subset \mathbb{W}$, due to Lemma 2.1, (4.2) and (4.3) we have

$$\begin{aligned}
\Psi_c(\mathcal{F}B) &\leq \Psi_c(\mathcal{F}_1 B) + \Psi_c(\mathcal{F}_2 B) \\
&\leq M_0\Big[L_0 + 2\int_0^t [k_1(s) + 2G_0k_3(s)]ds + \sum_{i=1}^n l_i\Big]\Psi_c(B).
\end{aligned} \tag{4.4}$$

By (4.4) we know that $\mathcal{F}$ is a Ψ_c-contraction on $\mathbb{W}$. By Lemma 2.2, there is a fixed point u of $\mathcal{F}$ in $\mathbb{W}$, which is a solution of (1.1) - (1.3). This completes the proof.

Theorem 4.2. *Suppose that the assumptions $(H_1) - (H_8)$ are satisfied, then the equation(1.1) - (1.3) has at least one mild solution if (4.4) and the following condition are satisfied.*

$$M_0L_0 + \lim_{r\to\infty} sup\frac{M_0}{r}\Big(\phi(r)\int_0^t p(s)ds + \chi(r)G_1\int_0^t \widehat{q}(s)ds + \sum_{i=1}^n d_i\Big) < 1. \tag{4.5}$$

Proof. From the equation (4.5) and fact that $L_0 < 1$, there exist a constant $r > 0$ such that

$$M_0\Big(rL_0 + \|h(0)\| + \phi(r)\int_0^t p(s)ds + \chi(r)G_1\int_0^t \widehat{q}(s)ds + \sum_{i=1}^n d_i\Big) < r.$$

We define $W_0 = \{u \in \mathcal{PC}([0,b];X),\ \|u(t)\| \le r, \text{ for all } \ t \in [0,b]\}$. Then for every $u \in W_0$ we have

$$\begin{aligned}
\|\mathcal{F}u(t)\| &\le \|U_u(t,0)h(u)\| + \int_0^t \|U_u(t,s)\Big[f(s,u(s)) + \int_0^s g(s,\tau,u(\tau))d\tau\Big]\|ds \\
&\quad + \sum_{0<t_i<t} \|U_u(t,t_i)I_i(u(t_i))\| \\
&\le M_0\Big(\|h(u) - h(0) + h(0)\| + M_0\Big[\int_0^t p(s)\phi(\|u(s)\|)ds \\
&\quad + \int_0^t\int_0^s q(s)\widehat{q}(\tau)\chi\|(u(\tau))\|d\tau ds\Big] + \sum_{i=1}^n d_i \\
\|\mathcal{F}u(t)\| &\le M_0\Big(rL_0 + \|h(0)\| + \phi(r)\int_0^t p(s)ds + \chi(r)G_1\int_0^t \widehat{q}(s)ds + \sum_{i=1}^n d_i\Big) < r.
\end{aligned}$$

for $t \in [0,b]$. This means that $\mathcal{F}\mathbb{W}_0 \subset \mathbb{W}_0$. Define $\mathbb{W} = \overline{con}\mathcal{F}\mathbb{W}_0$. The above proof also implies that $\mathcal{F}\mathbb{W} \subset \mathbb{W}$. So we can prove the theorem similar with Theorem 4.1 and hence we omit it.

5. Application

As an application of Theorem 3.1 we shall consider the system (1.1) - (1.3) with a control parameter such as

$$\begin{aligned}
u'(t) + A(t,u)u(t) &= f(t,u(t)) + Cv(t) + \int_0^t g(t,s,u(s))ds, \quad t \in J = [0,b],\ t \ne t_i && (5.1)\\
u(0) + h(u) &= u_0. && (5.2)\\
\Delta u(t_i) &= I_i(u(t_i)), \quad i = 1,2,\cdots,n,\ 0 < t_1 < \cdots < t_n < b,, && (5.3)
\end{aligned}$$

where A, f, g, h and I_i are as before and C is a bounded linear operator from a Banach space V into X and $v \in L^2(J,V)$. The mild solution of (5.1) - (5.3) is given by

$$\begin{aligned}
u(t) &= U_u(t,0)[u_0 - h(u)] + \int_0^t U_u(t,s)\Big[f(s,u(s)) + Cv(s) + \int_0^s g(s,\tau,u(\tau))d\tau\Big]ds \\
&\quad + \sum_{0<t_i<t} U_u(t,t_i)I_i(u(t_i)), \quad 0 \le t \le b.
\end{aligned}$$

Definition 5.1 ([2])**.** system (5.1)-(5.3) is said to be controllable on the interval J if for every $u_0, u_1 \in X$, there exists a control $v \in L^2(J,V)$ such that the mild solution $u(\cdot)$ of (5.1)-(5.3) satisfies

$$u(0) + h(u) = u_0 \text{ and } u(b) = u_1.$$

To study the controllability, we need the following additional condition

(H_9) The linear operator $W : L^2(J,V) \to X$, defined by

$$Wv = \int_0^b U_u(b,s)Cv(s)ds,$$

induces an inverse operator $\tilde{W}^{-1}$ defined an $L^2(J,V)/kerW$ and there exists a positive constant $M_1 > 0$ such that

$$\|C\tilde{W}^{-1}\| \leq M_1.$$

Theorem 5.2. *If the assumptions $(H_1) - (H_9)$ are satisfied, then the system (5.1)-(5.3) is controllable on J.*

Proof. Using the assumption (H_9), for an arbitrary function $u(\cdot)$, define the control

$$\begin{aligned} v(t) &= \tilde{W}^{-1}\Big[u_1 - U_u(b,0)[u_0 - h(u)] - \int_0^b U_u(t,s)\Big(f(s,u(s)) + \int_0^s g(s,\tau,u(\tau))d\tau\Big)ds \\ &\quad - \sum_{0<t_i<t}^{n} U_u(t,t_i)I_i(u(t_i))\Big](t). \end{aligned}$$

We shall show that when using this control, the operator $\mathcal{H} : Z \to Z$ defined by

$$\begin{aligned} (\mathcal{H}v)(t) &= U_u(t,0)[u_0 - h(u)] + \int_0^t U_u(t,s)\Big[f(s,u(s)) + Cv(s) + \int_0^s g(s,\tau,u(\tau))d\tau\Big]ds \\ &\quad + \sum_{0<t_i<t} U_u(t,t_i)I_i(u(t_i)). \end{aligned}$$

has a fixed point. This fixed point is, then a solution of (5.1)-(5.3). Clearly, $(\mathcal{H}v)(b) = u_1$, which means that the control v steers the system (5.1)-(5.3) from the initial state u_0 to u_1 in time b, provided we can obtain a fixed point of the nonlinear operator $\mathcal{H}$. The remaining part of the proof is similar to Theorem 3.1 and hence, it is omitted.

Acknowledgement

The authors are thankful to the referee for the improvements of the paper.

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