MAIN ARTICLES

ON A PROBLEM $\Phi^+ + \Phi^- = \varphi$ FOR THE DOUBLY QUASI-PERIODIC FUNCTIONS

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Abstract: The Problem $\Phi^+ + \Phi^- = \varphi$ for the doubly quasi-periodic functions is studied.

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In a complex z-plane, $z = x + iy$, consider the line L which is a union of countable number of smooth non-intersected open arcs L^j_{mn} doubly-periodically distributed with periods $2\omega_1$ and $2i\omega_2$ (ω_1 and ω_2 are given positive constants):

$$\begin{aligned} &L = \bigcup_{m,n=-\infty}^{\infty} L_{mn}, \\ &L_{mn} = \bigcup_{j=1}^{k} L^j_{mn}, \;\; L^{j_1}_{mn} \cap L^{j_2}_{mn} = \oslash; \;\;\; j_1 \neq j_2, \;\; j_1, j_2 = 1, 2, \ldots, k, \end{aligned} \tag{1}$$

The line L defined by (1) is called *the doubly-periodic line*, the ends of the line L_{00} we denote by $c_1, c_2, \ldots, c_{2k}$. z plane cut along L is denoted by S.

Definition 1. A function $F_0(z)$ is called *doubly quasi-periodic* in z plane with periods $2\omega_1$ and $2i\omega_2$ if

$$F_0(z + 2m\omega_1 + 2ni\omega_2) = F_0(z) + m\gamma_1 + n\gamma_2, \quad m, n = 0, \pm 1, \pm 2, \ldots \tag{2}$$

γ_1 and γ_2 are the definite constants,called the *addends*. If $\gamma_1 = \gamma_2 = 0$ the function $F_0(z)$ is called the *doubly-periodic function*.

Definition 2. The function defined by the series

$$\begin{aligned} &\zeta(z) = \tfrac{1}{z} + \sum_{m,n=-\infty}^{\infty} \Big(\frac{1}{z - T_{mn}} + \frac{1}{T_{mn}} + \frac{z}{T^2_{mn}} \Big), \;\; |m| + |n| \neq 0, \\ &T_{mn} = 2m\omega_1 + 2ni\omega_2, \end{aligned} \tag{3}$$

is called the Weierstrass "ζ-function" [1] . The Weierstrass ζ function has the following properties:

1. It is meromorphic function with simple poles T_{mn}, $m, n = 0, \pm 1, \pm 2, \ldots$;
2. $\zeta(z)$ is doubly quasi-periodic, i.e.,

$$\zeta(z + 2\omega_1) = \zeta(z) + \delta_1, \quad \zeta(z + 2i\omega_2) = \zeta(z) + \delta_2,$$

where δ_1 and δ_2 are the addends of this function satisfying the condition

$$i\omega_2\delta_1 - \omega_1\delta_2 = \pi i.$$

Definition 3. The function $\Phi_0(z)$ is called sectionally holomorphic doubly quasi- periodic if it has the following properties:

1. It is holomorphic in each finite region not containing points of the line L;

2. $\Phi_0(z)$ is continuous on L from the left and from the right, with the possible exception of the ends, near which the following condition is fulfilled

$$|\Phi_0(z)| \leq \frac{C}{|z-c|^\alpha},$$

where c is the corresponding end, and C and α are the certain real constants, $\alpha < 1$;

3. $$\Phi_0(z + 2m\omega_1 + 2ni\omega_2) = \Phi_0(z) + m\gamma_1 + n\gamma_2, \quad m, n = 0, \pm 1, \pm 2, \dots .$$

Problem 1. Find sectionally holomorphic doubly quasi-periodic function $\Phi(z)$ satisfying the boundary condition

$$\Phi^+(t_0) + \Phi^-(t_0) = \varphi(t_0), \quad t_0 \in L, \tag{4}$$

where $\varphi(t)$ is the given doubly quasi-periodic function of H class on L_{00}, with the addends $2\gamma_1, 2\gamma_2$. $\Phi^+(t_0)$ and $\Phi^(t_0)$ are the limiting values from the left and from the right of L, respectively.

We classify the solutions of the Problem 1 with respect to the ends of the line L and find the solutions in the Muskhelishvili-Kveselava classes h_q (the class of solutions bounded at the end-points $c_1, c_2, \dots, c_q$ and having singularities less then 1 at others, $q \leq 2k$).

Let us consider the function:

1)

$$\Psi(z) = \Phi(z) - A_0\zeta(z - a_0) - A_1\zeta(z - a_1) - Bz, \tag{5}$$

in the case $q \leq k + 1$, where a_0, a_1 are an arbitrary points of z-plane not belonging to the line L , the constants A_0, A_1, B satisfy the conditions

$$\gamma_1 = (A_0 + A_1)\delta_1 + 2B\omega_1, \quad \gamma_2 = (A_0 + A_1)\delta_2 + 2Bi\omega_2,$$

and $A_0 = 0$ in the case of $q < k + 1$.

2)

$$\Psi(z) = \Phi(z) - A_1\zeta(z - a_1) - A_2\zeta(z - a_2) - \dots - A_{q-k}\zeta(z - a_{q-k}) - Bz, \tag{6}$$

in the case $q > k+1$, where $a_1, a_2, , a_{q-k}$ are an arbitrary points of z-plane not belonging to the line L and the constants $A_1, A_2, , A_{q-k}, B$ satisfy the conditions

$$\gamma_1 = (A_1 + A_2 + \dots + A_{q-k})\delta_1 + 2B\omega_1, \quad \gamma_2 = (A_1 + A_2 + \dots + A_{q-k})\delta_2 + 2Bi\omega_2.$$

The function $\Psi(z)$ is sectionally holomorphic doubly- periodic satisfying the boundary condition

$$\Psi^{+}(t_0) + \Psi^{-}(t_0) = f_0(t_0),\ t_0 \in L, \tag{7}$$

where

$$\begin{aligned} f_0(t_0) &= \varphi(t_0) - 2A_0\zeta(t_0 - a_0) - 2A_1\zeta(z - a_1) - 2A_2\zeta(z - a_2) \\ &\quad - \ldots - 2A_{q-k}\zeta(z - a_{q-k}) - 2Bz, \end{aligned}$$

$A_0 = A_2 = \ldots = A_{q-k} = 0$ in the case when $q < k+1$ and $A_0 = 0$ in the case when $q > k+1$. Having find $\Psi(z)$ the function $\Phi(z)$ will be given by

$$\Phi(z) = \Psi(z) + A_0\zeta(t_0 - a_0) + A_1\zeta(z-a_1) + A_2\zeta(z-a_2) + \ldots + A_{q-k}\zeta(z-a_{q-k}) + Bz.$$

Let us consider the homogeneous problem (7), i.e.,

$$\Psi_0^{+}(t_0) + \Psi_0^{-}(t_0) = 0,\ t_0 \in L, \tag{8}$$

The solutions of the problem (8) are given by:

1)

$$\begin{aligned} \Psi_0(z) &= \frac{C\sigma(z - c_1') \cdots \sigma(z - c_{k-q+1}')}{\sigma(z-a_1)} \\ &\quad \times \sqrt{\frac{\sigma(z - c_1)\sigma(z - c_2) \cdots \sigma(z - c_q)}{\sigma(z - c_{q+1}) \cdots \sigma(z - c_{2k})}}, \end{aligned} \tag{9}$$

in the case $q < k+1$, where $c_1', \ldots, c_{k-q+1}'$ are the constants satisfying the conditions

$$2(c_1' + c_2' + \ldots + c_{k-q+1}') = 2a_1 + c_{q+1} + c_{q+2} + \ldots + c_{2k} - (c_1 + \ldots + c_q),$$

$$c_i' \neq c_j',\ c_i' \notin L,\ \ i, j = 1, 2, \ldots, k - q + 1.$$

C is an arbitrary fixed non-zero constant.

2)

$$\Psi_0(z) = \frac{C}{\sigma(z-a_1)\sigma(z-a_2) \cdots \sigma(z-a_{q-k})} \cdot \sqrt{\frac{\sigma(z-c_1) \cdots \sigma(z - c_q)}{\sigma(z - c_{q+1}) \cdots \sigma(z-c_{2k})}}, \tag{10}$$

in the case $q > k+1$,, where

$$c_1 + \ldots + c_q = c_{q+1} + \ldots + c_{2k} + 2a_1 + 2a_2 + \ldots + 2a_{q-k},$$

with an arbitrary fixed non-zero constant C.

3)

$$\Psi_0(z) = \frac{C\,\sigma(z - c_1')}{\sigma(z - a_0)\sigma(z - a_1)} \sqrt{\frac{\sigma(z - c_1) \cdots \sigma(z - c_q)}{\sigma(z - c_{q+1}) \cdots \sigma(z - c_{2k})}}, \tag{11}$$

in the case $q = k+1$, where

$$2a_0 + 2a_1 + c_{k+2} + \ldots + c_{2k} = 2c_1' + c_1 + \ldots + c_{k+1}, \;\; c_1' \notin L.$$

In the formulaes (9)-(11) $\sigma(z)$ is the Weierstrass "σ-function" [1] and the quantity

$$\sqrt{\frac{\sigma(z-c_1)\cdots\sigma(z-c_q)}{\sigma(z-c_{q+1})\cdots\sigma(z-c_{2k})}} \tag{12}$$

is understood as the branch which is holomorphic in S. The boundary value taken by the root (12) on L from the left will be denoted by

$$\Big[\sqrt{\frac{\sigma(z-c_1)\cdots\sigma(z-c_q)}{\sigma(z-c_{q+1})\cdots\sigma(z-c_{2k})}}\,\Big]^+ = \sqrt{\frac{\sigma(z-c_1)\cdots\sigma(z-c_q)}{\sigma(z-c_{q+1})\cdots\sigma(z-c_{2k})}}.$$

Taking into account (7) and (8), it is easy to see that the function $\frac{\Psi(z)}{\Psi_0(z)}$ satisfies the boundary condition

$$\frac{\Psi^+(t)}{\Psi_0^+(t)} - \frac{\Psi^-(t)}{\Psi_0^-(t)} = \frac{f_0(t)}{\Psi_0^+(t)}\,, \;\; t \in L.$$

The problem of this type was solved by the author in [3].So we conclude:

For $q < k+1$ a solution of the Problem 1 exists and is given by

$$\Phi(z) = \frac{\Psi_0(z)}{2\pi i}\int_L \frac{f_0(t)}{\Psi_0^+(t)}[\zeta(t-z)+\zeta(z-c_1')]dt + C\frac{\Psi_0(z)}{2\pi i} + A_1\zeta(z-a_1) + Bz,$$

where $\Psi_0(z)$ is given by (9), and the constant C is defined from the condition

$$\frac{\Psi_0^1}{2\pi i}\Big[\int_{L_{00}} \frac{f_0(t)}{\Psi_0^+(t)}[\zeta(t-a_1)+\zeta(a_1-c_1')]\,dt + C\Big] + A_1 = 0, \;\; \Psi_0^1 = \lim_{z\to a_1}(z-a_1)\Psi_0(z).$$

For $q > k+1$ a unique solution exists if and only if $f_0(t)$ satisfies the condition

$$\int_{L_{00}} \frac{f_0(t)}{\Psi_0^+(t)}\,dt = 0,$$

and is given by

$$\Phi(z) = \frac{\Psi_0(z)}{2\pi i}\int_{L_{00}} \frac{f_0(t)}{\Psi_0^+(t)}\,\zeta(t-z)\,dt + C\frac{\Psi_0(z)}{2\pi i} + A_1\zeta(z-a_1) + A_2\zeta(z-a_2) + \\ + \ldots + A_{q-k}\zeta(z-a_{q-k}) + Bz,$$

$\Psi_0(z)$ is given by (10),the constants C and $A_i; i = 1,2,,q-k$, are defined from the conditions

$$\frac{\Psi_0^i}{2\pi i}\Big[\int_{L_{00}} \frac{f_0(t)}{\Psi_0^+(t)}\zeta(t-a_i)\,dt + C\Big] + A_i = 0,$$

$$\Psi_0^i = \lim_{z\to a_i}(z-a_i)\Psi_0(z),\ i=1,2,\dots,q-k.$$

For $q=k+1$, the solution exists and is given by

$$\Phi(z) = \frac{\Psi_0(z)}{2\pi i}\int_{L_{00}}\frac{f_0(t)}{\Psi_0^+(t)}[\zeta(t-z)+\zeta(z-c_1')]\,dt + C\frac{\Psi_0(z)}{2\pi i}$$

$$+A_0\zeta(z-a_0)+A_1\zeta(z-a_1)+Bz,$$

where $\Psi_0(z)$ is given by (11) and the constants C, A_0, A_1 are defined from the conditions

$$\frac{\Psi_0^i}{2\pi i}\Big[\int_{L_{00}}\frac{f_0(t)}{\Psi_0^+(t)}[\zeta(t-a_i)+\zeta(a_i-c_1')]\,dt + C\Big] + A_i = 0,$$

$$\Psi_0^i = \lim_{z\to a_i}(z-a_i)\Psi_0(z),\ i=0,1.$$

References

[1] Akhiezer, N.I., Elements of the Theory of Elliptic Functions. Moscow, Nauka, 1974.

[2] Muskhelishvili, N. I., Singular Integral Equations. Groningen, P. Noordhoff, 1953.

[3] Khatiashvili, N., On linear conjugation problem with the doubly periodic jump line. Proc. of A. Razmadze Math. Inst., 136 (2004), 63-84.

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