BENDING OF A CUSPED PLATE ON AN ELASTIC FOUNDATION

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Abstract. The present paper deals with the bending of a cusped Kirchoff-Love plate on an elastic foundation. For cusped plates see surveys in [1], [2], [3].

Key words: cusped plates, elastic foundation, bending, existence and uniqueness theorems, weighted Sobolev spaces.

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1. Introduction

The bending equation of isotropic Kirchhoff-Love plates on an elastic foundation has the following form (see, e.g., [4]):

$$\begin{aligned} J_k w \;&:=\; (Dw_{,11})_{,11} + (Dw_{,22})_{,22} + \nu(Dw_{,22})_{,11} \\ &+\; \nu(Dw_{,11})_{,22} + 2(1-\nu)(Dw_{,12})_{,12} \\ &+\; kw = f(x_1, x_2) \ in \ \Omega \subset \mathbb{R}^2, \end{aligned} \tag{1}$$

where $w = w(x_1, x_2)$ is the deflection, $k_1 \geq k \geq k_0 > 0$, k_0, $k_1 = const$, k is the modulus of a foundation, $k(x_1, x_2) \in C(\overline{\Omega})$, f is the intensity of the lateral load, Ω is a bounded plane domain with Lipschitz boundary $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$ with Γ_1 lying on the x_1 axis and Γ_2 lying in the upper half-plane $\{x_2 > 0\}$, $D \in C^2(\Omega) \cap C(\overline{\Omega})$ is the flexural rigidity of the plate,

$$D := \frac{2Eh^3}{3(1-\nu^2)}, \tag{2}$$

$2h(x_1, x_2)$ is the thickness of the plate, $E(x_1, x_2)$ is the Young's modulus, ν is the Poisson's ratio, $0 < \nu < 1$, and indices after comma mean differentiation with respect to the corresponding variables.

Throughout this paper we assume once and for all that

$$D(x_1, x_2) > 0 \ on \ \Omega \cup \Gamma_2, \ D(x_1, 0) \geq 0 \quad for \quad (x_1, 0) \in \Gamma_1. \tag{3}$$

If

$$2h(x_1, x_2)|_{\Gamma_1} = 0 \tag{4}$$

(i.e., $2h(x_1,0)=0$ for $(x_1,0)\in\Gamma_1$). In this case

$$D(x_1,x_2)|_{\Gamma_1}=0. \tag{5}$$

But (5) may also occur if $2h|_{\Gamma_1}>0$ but $E|_{\Gamma_1}=0$ (or if both quantities vanish). In all these cases, the plate will be called a cusped one, although it can be even of constant thickness but with properties of cusped plate caused by vanishing of the inhomogeneous Young's modulus E on Γ_1.

We recall that for the *bending moments* $M_\alpha w$, $\alpha=1,2$, the *twisting moments* $M_{12}w$, $M_{21}w$, the *shearing forces* $Q_\alpha w$, $\alpha=1,2$, and the *generalized shearing forces* $Q^*_\alpha w$, $\alpha=1,2$, we have the following expressions:

$$\begin{aligned} M_\alpha w &= -D(w_{,\alpha\alpha}+\nu w_{,\beta\beta}),\ \alpha,\beta=1,2;\ \alpha\neq\beta, && (6)\\ M_{12}w &= -M_{21}w=2(1-\nu)Dw_{,12}, && (7)\\ Q_\alpha w &= (M_\alpha w)_{,\alpha}+(M_{21}w)_{,\beta},\ \alpha=1,2;\ \alpha\neq\beta, && (8)\\ Q^*_\alpha w &= Q_\alpha w+(M_{21}w)_{,\beta},\ \alpha=1,2;\ \alpha\neq\beta. && (9)\end{aligned}$$

At points of the boundary $\partial\Omega$ where D vanishes, all the above quantities will be defined as limits from the inside of Ω.

Problem 1 *Let us consider for equation (1) the following inhomogeneous BCs:*

– *on* Γ_2

$$w=g_1,\quad \frac{\partial w}{\partial n}=g_2, \tag{10}$$

– *on* Γ_1

$$\text{either } w=w_0(x_1),\ w_{,2}=w_0^1(x_1)\quad iff\ I_{02}<+\infty, \tag{11}$$

$$\text{or } w_{,2}=w_0^1(x_1),\ Q_2^*=Q_2^0(x_1)\quad iff\ I_{02}<+\infty, \tag{12}$$

$$\text{or } w=w_0(x_1),\ M_2=M_2^0(x_1)\begin{cases}\not\equiv 0 \text{ when } I_{02}<+\infty,\\ \equiv 0 \text{ when } I_{02}=+\infty,\end{cases}\quad iff\ \ I_{22}<+\infty, \tag{13}$$

or

$$\begin{aligned} M_2=M_2^0(x_1)\begin{cases}\not\equiv 0 \text{ when } I_{02}<+\infty,\\ \equiv 0 \text{ when } I_{02}=+\infty,\end{cases}\\ Q_2^*=Q_2^0(x_1)\begin{cases}\not\equiv 0 \text{ when } I_{12}<+\infty,\\ \equiv 0 \text{ when } I_{12}=+\infty,\end{cases}\end{aligned} \tag{14}$$

where g_1,g_2 *and* w_0,w_0^1,Q_2^0,M_2^0 *are prescribed functions on* Γ_2 *and* Γ_1, *respectively,*

$$I_{k2}\equiv I_{k2}(x_1):=\int_0^{l(x_1)}\tau^k D^{-1}(x_1,\tau)d\tau,\ k\in\{0,1,\ldots\},\ (x_1,0)\in\Gamma_2, \tag{15}$$

where $(x_1,l(x_1))\in\Omega$ *for* $(x_1,0)\in\Gamma_1$.

2. Function Spaces

Definition 2 *Let*

$$W^{2,2}(\Omega, p) \tag{16}$$

be the sets of all measurable functions $w(x_1, x_2)$ defined on Ω which have on Ω locally summable generalized derivatives $\partial^{(\alpha_1,\alpha_2)}_{x_1,x_2} w$ for $\alpha_1 + \alpha_2 \leq 2, \quad \alpha_1, \alpha_2 \in \{0, 1, 2\}$, such that

$$\int\limits_{\Omega} \rho_{\alpha_1,\alpha_2}(x_1, x_2)|\partial^{(\alpha_1,\alpha_2)}_{x_1,x_2} w|^2 d\Omega < +\infty, \ \partial^{(0,0)}_{x_1,x_2} w = w, \tag{17}$$

for

$$\rho_{0,0} := 1, \ \rho_{2,0} = \rho_{1,1} = \rho_{0,2} := p(x_1, x_2)$$

with a bounded measurable on Ω function $p(x_1, x_2)$ and the following norm

$$\|w\|^2_{W^{2,2}(\Omega,p)} := \int\limits_{\Omega} \{w^2 + p[(w_{,11})^2 + (w_{,12})^2 + (w_{,22})^2]\} d\Omega \tag{18}$$

Let us further consider the following sets for different cases of the function $p(x_1, x_2)$:

$$W^{2,2}(\Omega, D) \tag{19}$$

with $p(x_1, x_2) = D(x_1, x_2)$ satisfying (3), and

$$W^{2,2}(\Omega, x_2^{\varkappa}) \quad (p(x_1, x_2) = x_2^{\varkappa}), \tag{20}$$

$$W^{2,2}(\Omega, d^{\varkappa}) \quad (p(x_1, x_2) = d(x_1, x_2)), \tag{21}$$

where

$$d(x_1, x_2) := dist\{(x_1, x_2) \in \Omega, \ \partial\Omega\}.$$

Further, in (19) let us introduce another norm:

$$\begin{aligned} \|w\|^2_{\widetilde{W}^{2,2}(\Omega,D)} &:= \int\limits_{\Omega} [w^2 + \nu D(w_{,11} + w_{,22})^2 + (1-\nu) D(w_{,11})^2 \\ &+ 2(1-\nu) D(w_{,12})^2 + (1-\nu) D(w_{,22})^2] d\Omega, \end{aligned} \tag{22}$$

From (3) it is clear that in the case under consideration if $D \in C(\overline{\Omega})$, then

$$\rho^{-1}_{\alpha_1,\alpha_2} \in L_1^{loc}(\Omega).$$

Therefore, according to [5], the spaces (19)–(21) with the norms (18)–(22), respectively, will be Banach spaces, and moreover, Hilbert spaces under the appropriate scalar products.

Lemma 3

$$W^{2,2}(\Omega, x_2^{\varkappa}) \subset W^{2,2}(\Omega, d^{\varkappa}(x_1, x_2)) \quad \forall\ \varkappa \geq 0. \tag{23}$$

Proof follows from the obvious inequality

$$d(x_1, x_2) \leq x_2 \quad for \quad (x_1, x_2) \in \Omega \tag{24}$$

(if $d(x_1, x_2)$ is a regularized distance, then in the inequality (24) arises a constant factor). □

Lemma 4

$$V^{2,2}(\Omega, x_2^{\varkappa}) \ \subset \ V^{2,2}(\Omega, x_2^4) \ for\ 0 \leq \varkappa < 4. \tag{25}$$

Proof of (25) follows from $l^{4-\varkappa} x_2^{\varkappa} \geq x_2^4$ for $0 \leq x_2 < l$, where

$$l = const > \max_{(x_1,x_2)\in\bar{\Omega}} \{x_2\}.$$

□

Let

$$\Omega_\delta := \{(x_1, x_2) \in \Omega : x_2 > \delta,\ \delta = const\ > 0\}.$$

Evidently,

$$W^{2,2}(\Omega_\delta, D) \subset W^{2,2}(\Omega_\delta), \tag{26}$$

where $W^{2,2}(\Omega_\delta) \equiv W_2^2(\Omega_\delta)$ is the usual (i.e., non-weighted) Sobolev space. Hence, we get the following

Lemma 5 *There exist the traces*

$$w|_{\Gamma_2} \in W^{\frac{3}{2},2}(\Gamma_2),\ \left.\frac{\partial w}{\partial n}\right|_{\Gamma_2} \in W^{\frac{1}{2},2}(\Gamma_2) \quad \forall\ v \in W^{2,2}(\Omega, D).$$

Lemma 6 *The norms* $\|w\|^2_{W^{2,2}(\Omega,D)}$ *and* $\|w\|^2_{\widetilde{W}^{2,2}(\Omega,D)}$ *are equivalent.*

Proof. Evidently,

$$\|w\|^2_{\widetilde{W}^{2,2}(\Omega,D)} \leq \int_\Omega \left\{w^2 + 2\nu D\left[(w,_{11})^2 + (w,_{11})^2\right]\right.$$
$$\left.+(1-\nu)D\left[(w,_{11})^2 + (w,_{11})^2\right] + 2(1-\nu)D(w,_{12})^2\right\} d\Omega$$
$$\leq \int_\Omega \left\{2w^2 + (1+\nu)D\left[(w,_{11})^2 + (w,_{11})^2\right] + 2D(w,_{12})^2\right\} d\Omega \leq 2\|w\|^2_{W^{2,2}(\Omega,D)}$$

and

$$\|w\|^2_{W^{2,2}(\Omega,D)} = \frac{1}{1-\nu}\int\limits_{\Omega} \left\{(1-\nu)w^2 + (1-\nu)D\left[(w,_{11})^2 + (w,_{22})^2\right]\right.$$
$$\left. +(1-\nu)D(w,_{11})^2\right\} d\Omega \le \frac{1}{1-\nu}\int\limits_{\Omega} \left\{w^2 + (1-\nu)D\left[(w,_{11})^2 + (w,_{22})^2\right]\right.$$
$$\left. +2(1-\nu)D(w,_{12})^2\right\} d\Omega \le \frac{1}{1-\nu}\|w\|^2_{W^{2,2}(\Omega,D)}$$

□

3. Existence and Uniqueness Theorems

Now, we constitute the space V from the space $W^{2,2}(\Omega, D)$ as follows:

$$\begin{aligned} V \;:=\; & \Big\{ v \in W^{2,2}(\Omega, D) : v|_{\Gamma_2} = 0,\ \frac{\partial v}{\partial n}\Big|_{\Gamma_2} = 0, \\ & \textit{and additionally} \\ & \textit{either } v|_{\Gamma_1} = 0,\ v_{,2}|_{\Gamma_1} = 0 \ (\textit{if we consider BCs } (11)) \\ & \textit{or } v_{,2}|_{\Gamma_1} = 0 \ (\textit{if we consider BCs } (12)) \\ & \textit{or } v|_{\Gamma_1} = 0 \ (\textit{if we consider BCs } (13)) \\ & \textit{in the sense of traces} \Big\} \end{aligned} \qquad (27)$$

and

$$\widetilde{V} := \left\{ v \in \widetilde{W}^{2,2}(\Omega, D) : v|_{\Gamma_2} = 0,\ \frac{\partial v}{\partial n}\Big|_{\Gamma_2} = 0 \textit{ in the sense of traces} \right\}. \qquad (28)$$

Using the trace theorem, it is not difficult to prove the completeness of V.

We suppose that the functions g_1, g_2, w_0, w_0^1 from Problem 1 are traces of a prescribed function

$$u \in W^{2,2}(\Omega, D). \qquad (29)$$

Let further $Q_2^0, M_2^0 \in L_2(\Gamma_1)$.

Definition 7 *Let $f \in L_2(\Omega)$. A function $w \in W^{2,2}(\Omega, D)$ will be called a weak solution of Problem 1, in the space $W^{2,2}(\Omega, D)$ if it satisfies the following conditions:*

$$w - u \in V \qquad (30)$$

and

$$J_k(w,v) := \int\limits_{\Omega} B_k(w,v)d\Omega = \int\limits_{\Omega} fvd\Omega + \gamma_2 \int\limits_{\Gamma_1} Q_2^0 v dx_1 - \gamma_1 \int\limits_{\Gamma_1} M_2^0 v_{,2} dx_1 \quad \forall\, v \in V, \tag{31}$$

where

$$\begin{aligned} &\gamma_1 = 0,\ \gamma_2 = 0 \ for \ the \ BCs \ (11),\\ &\gamma_1 = 0,\ \gamma_2 = 1 \ for \ the \ BCs \ (12),\\ &\gamma_1 = 1,\ \gamma_2 = 0 \ for \ the \ BCs \ (13),\\ &\gamma_1 = 1,\ \gamma_2 = 1 \ for \ the \ BCs \ (14),\end{aligned}$$

and

$$\begin{aligned} B_k(w,v) &:= \nu D(w_{,11}+w_{,22})(v_{,11}+v_{,22}) + (1-\nu)Dw_{,11}v_{,11}\\ &+ 2(1-\nu)Dw_{,12}v_{,12} + (1-\nu)Dw_{,22}v_{,22} + kwv. \end{aligned} \tag{32}$$

Theorem 8 *Let*

$$T := \max\{1, k_1\} \ \ and \ \ m := \min\{1, k_0\}. \tag{33}$$

There exists a unique weak solution of Problem 1 (more precisely, of each of all the four BVPs stated in Problem 1). This solution is such that

$$\|w\|_{W^{2,2}(\Omega,D)} \le C[\|f\|_{L_2(\Omega)} + \|u\|_{W^{2,2}(\Omega,D)} + \gamma_1 \|M_2^0\|_{L_2(\Gamma_1)} + \gamma_2 \|Q_2^0\|_{L_2(\Gamma_1)}], \tag{34}$$

where the constant C is independent of f, u, M_2^0, and Q_2^0.

Proof of Theorem 8 is based on the Lax-Milgram theorem. It is easy to show the following three inequalities (see (37), (40), (41) below which imply the proof).

In view of (31), (32), (22), and Lemma 5, we have

$$\begin{aligned} |J_k(w,v)| \le & \int\limits_{\Omega} (\nu D)^{\frac{1}{2}} |w_{,11}+w_{,22}| \cdot (\nu D)^{\frac{1}{2}} |v_{,11}+v_{,22}| d\Omega \\ &+ \int\limits_{\Omega} [(1-\nu)D]^{\frac{1}{2}} |w_{,11}| \cdot [(1-\nu)D]^{\frac{1}{2}} |v_{,11}| d\Omega \\ &+ \int\limits_{\Omega} [2(1-\nu)D]^{\frac{1}{2}} |w_{,12}| \cdot [2(1-\nu)D]^{\frac{1}{2}} |v_{,12}| d\Omega \end{aligned}$$

$$
\begin{aligned}
&+ \int_\Omega [(1-\nu)D]^{\frac{1}{2}}|w_{,22}| \cdot [(1-\nu)D]^{\frac{1}{2}}|v_{,22}|d\Omega + T\int_\Omega |w||v|d\Omega \\
&\le \left[\int_\Omega \nu D(w_{,11}+w_{,22})^2 d\Omega\right]^{\frac{1}{2}} \left[\int_\Omega \nu D(v_{,11}+v_{,22})^2 d\Omega\right]^{\frac{1}{2}} \\
&+ \left[\int_\Omega (1-\nu)D(w_{,11})^2 d\Omega\right]^{\frac{1}{2}} \left[\int_\Omega (1-\nu)D(v_{,11})^2 d\Omega\right]^{\frac{1}{2}} \\
&+ \left[\int_\Omega 2(1-\nu)D(w_{,12})^2 d\Omega\right]^{\frac{1}{2}} \left[\int_\Omega 2(1-\nu)D(w_{,12})^2 d\Omega\right]^{\frac{1}{2}} \\
&+ \left[\int_\Omega (1-\nu)D(w_{,22})^2 d\Omega\right]^{\frac{1}{2}} \left[\int_\Omega (1-\nu)D(v_{,22})^2 d\Omega\right]^{\frac{1}{2}} \\
&+ T\left[\int_\Omega w^2 d\Omega\right]^{\frac{1}{2}} \left[\int_\Omega v^2 d\Omega\right]^{\frac{1}{2}} \\
&\le (4+T)\|w\|_{\widetilde{W}^{2,2}(\Omega,D)}\|v\|_{\widetilde{W}^{2,2}(\Omega,D)} \\
&\le 4(4+T)\|w\|_{W^{2,2}(\Omega,D)}\|v\|_{W^{2,2}(\Omega,D)}. \qquad (35)
\end{aligned}
$$

In particular,

$$
|J_k(w,v)| \le 4(4+T)\|w\|_{W^{2,2}(\Omega,D)}\|v\|_V \quad \forall\, w \in W^{2,2}(\Omega,D) \text{ and } \forall\, v \in V \qquad (36)
$$

and

$$
|J_k(w,v)| \le 4(4+T)\|w\|_V\|v\|_V \quad \forall\, w,v \in V. \qquad (37)
$$

Taking into account (36), and

$$
\left|\int_{\Gamma_1} vQ_2^0 dx_1\right| \le \|v\|_{L_2(\Gamma_1)}\|Q_2^0\|_{L_2(\Gamma_1)} \le C_0\|v\|_V\|Q_2^0\|_{L_2(\Gamma_1)}, \qquad (38)
$$

$$
\begin{aligned}
\left|\int_{\Gamma_1} v_{,2}M_2^0 dx_1\right| &\le \|v_{,2}\|_{L_2(\Gamma_1)}\|M_2^0\|_{L_2(\Gamma_1)} \qquad (39)\\
&\le C_0\|v\|_V\|M_2^0\|_{L_2(\Gamma_1)}
\end{aligned}
$$

with the positive constant C_0 from the trace theorem, it is not difficult to see, that the functional

$$
F_k v := \int_\Omega f v d\Omega - J_\omega(u,v) + \gamma_2\int_{\Gamma_1} Q_2^0 v dx_1 - \gamma_1\int_{\Gamma_1} M_2^0 v_{,2} dx_1,\ v \in V,
$$

is bounded in V. Indeed,

$$|F_k v| \leq \{\|f\|_{L_2(\Omega)} + 4(4+T)\|u\|_{W^{2,2}(\Omega,D)} \\ +C_0[\gamma_2\|Q_2^0\|_{L_2(\Gamma_1)} + \gamma_1\|M_2^0\|_{L_2(\Gamma_1)}]\}\|v\|_V. \quad (40)$$

Besides, since $m \leq 1$, $0 < \nu < 1$,

$$\|v\|_V^2 = \\ \frac{1}{(1-\nu)m}\int_\Omega \left\{(1-\nu)mv^2 + (1-\nu)mD\left[(v,_{11})^2 + (v,_{22})^2 + (v,_{12})^2\right]\right\} d\Omega \\ \leq \frac{1}{(1-\nu)m}\int_\Omega \left\{mv^2 + (1-\nu)D\left[(v,_{11})^2 + (v,_{22})^2\right] + 2(1-\nu)D(v,_{12})^2\right\} d\Omega \\ \leq \frac{1}{(1-\nu)m}\int_\Omega \{kv^2 + D[\nu(v_{,11}+v_{,22})^2 + (1-\nu)(v_{,11})^2 + 2(1-\nu)(v_{,12})^2 \\ +(1-\nu)(v_{,22})^2]\}d\Omega = \frac{1}{(1-\nu)m}J_k(v,v).$$

Whence,

$$J_k(v,v) \geq (1-\nu)m\|v\|_V^2. \quad (41)$$

□

In view of (38), (40), (41), according to Lax-Milgram theorem there exists a unique $z \in V$ such that,

$$J_k(z,v) = Fv := (v,f) - J_k(u,v) + \gamma_2\int_{\Gamma_1} vQ_2^0 dx_1 - \gamma_1\int_{\Gamma_1} v,_2 M_2^0 dx_1 \; \forall v \in V,$$

i.e.,

$$J_k(w,v) = (v,f) + \gamma_2\int_{\Gamma_1} vQ_2^0 dx_1 - \gamma_1\int_{\Gamma_1} v,_2 M_2^0 dx_1 \quad \forall v \in V, \quad (42)$$

where

$$w := u + z \in W^{2,2}(\Omega, D). \quad (43)$$

So,

$$w - u = z \in V,$$

and (30) is fulfilled. (42) coincides with (31). Thus, the existence of a unique weak solution $w \in W^{2,2}(\Omega, D)$ of the Problem 1 has been proved.

From (40) it follows that

$$||F_k||_{V^*} \leq ||f||_{L_2(\Omega)} + 4(4+T)||u||_{W^{2,2}(\Omega,D)} \\ +C_0(\gamma_2||Q_2^0||_{L_2(\Gamma_1)} + \gamma_1||M_2^0||_{L_2(\Gamma_1)}), \quad (44)$$

where V^* is the conjugate to V space.

By virtue of (43), (44),

$$||w||_{W^{2,2}(\Omega,D)} \le ||u||_{W^{2,2}(\Omega,D)} + ||z||_V \le ||u||_{W^{2,2}(\Omega,D)}$$
$$+\alpha^{-1}[||f||_{L_2(\Omega)} + 4(4+T)||u||_{W^{2,2}(\Omega,D)} + C_0(\gamma_2||Q_2^0||_{L_2(\Gamma_1)} + \gamma_1||M_2^0||_{L_2(\Gamma_1)})]$$
$$\le C[||f||_{L_2(\Omega)} + ||u||_{W^{2,2}(\Omega,D)} + \gamma_1||M_2^0||_{L_2(\Gamma_1)} + \gamma_2||Q_2^0||_{L_2(\Gamma_1)}],$$

where

$$C := max\{4(4+T)\alpha^{-1}+1, \ \ \alpha^{-1}C_0\},$$

where the coefficient α comes from the Lax-Milgram theorem (see, e.g. [2]).

□

Let us note, that the existence of traces g_1, g_2, w_0, w_0^1 of (29) has been supposed but the traces g_1 and g_2 on Γ_2 exist, by virtue of Lemma 5. The traces w_0 and w_0^1 exist in the corresponding cases under some additional restrictions on D.

Let

$$D(x_1,x_2) \ge D_\varkappa x_4^\varkappa \quad \forall \ (x_1,x_2) \in \Omega, \ i.e., \ 0 < D_\varkappa := \inf_\Omega \frac{D(x_1,x_2)}{x_2^\varkappa}. \tag{45}$$

If $\varkappa \ge 1$, then by $\varkappa$ we denote the minimal among all the exponents $\delta \ge \varkappa \ge 1$ for which (45) holds. It means that if we have the inequality (45) for $\varkappa \ge 1$, we have to check whether there exists the less exponent for which (45) is valid. If it does exist, then we have to continue this procedure until we arrive at the minimal one.

If (45) holds for $\varkappa < 1$, then we need no additional revision since for all the $\varkappa < 1$ we have the same result concerning the traces.

The condition (45) is essential in a right neighbourhood of Γ_1. Then it can be easily extended for the whole domain Ω.

Let us note, that when $k \equiv 0$ or $k \le 0$, Problem 1 was considered in [2].

Lemma 9 *If (45) takes place, then*

$$W^{2,2}(\Omega,D) \subset W^{2,2}(\Omega,x_2^\varkappa) \subset W^{2,2}(\Omega,d^\varkappa(x_1,x_2)) \quad \forall \ \varkappa \ge 0, \tag{46}$$

Proof of (46) follows from (45), (23). □

Lemma 10 *If $w \in W^{2,2}(\Omega,D)$ and (45) is valid, then there exist traces*

$$w|_{\Gamma_1} \ \in B_2^{\frac{3-\varkappa}{2}}(\Gamma_1) \subset L_2(\Gamma_1) \ \ if \ \ 0 \le \varkappa < 3 \ (i.e., \ I_{22}|_{\Gamma_1} < +\infty), \tag{47}$$
$$w_{,2}|_{\Gamma_1} \ \in B_2^{\frac{1-\varkappa}{2}}(\Gamma_1) \subset L_2(\Gamma_1) \ \ if \ \ 0 \le \varkappa < 1 \ (i.e., \ I_{02}|_{\Gamma_1} < +\infty), \tag{48}$$

where $B_2^{\frac{3-\varkappa}{2}}(\Gamma_1)$ and $B_2^{\frac{1-\varkappa}{2}}(\Gamma_1)$ are Besov spaces.

Proof. Since (45) is valid, according to Lemma 9 (see (46)), $w \in W^{2,2}(\Omega, D)$ implies

$$w \in W^{2,2}(\Omega, d^{\varkappa}(x_1, x_2)).$$

But functions from this space (see [6], Theorem 1.1.2) have properties (47) and (48) if $\partial\Omega \in C^{1+\varepsilon}$ and $\partial\Omega \in C^{2+\varepsilon}$ (which means that the boundary is locally described by functions whose first and second derivatives, satisfy the Hölder condition with a Hölder exponent $\varepsilon \in]0,1[$, respectively). Since in our case Γ_1 is a part of a straight line, these local conditions are fulfilled all the more.

□

Similarly can be investigated the bending of cusped Euler-Bernoulli beams [2] on an elastic foundation.

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