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SHORT COMMUNICATIONS

Set-theoretical Aspects of Absolute Nonmeasurability of the Union of Two Uniform Subsets of the Euclidean Plane

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It is shown that the Continuum Hypothesis is equivalent to the existence of two uniform subsets X and Y of the plane $\mathbf{R}^2$ such that $X \cup Y$ is an absolutely nonmeasurable set with respect to the class of all nonzero σ-finite translation invariant (quasi-invariant) measures on $\mathbf{R}^2$.

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Let E be a ground set equipped with a group G of transformations of E.

We say that a subset X of E is G-absolutely nonmeasurable in E if, for any nonzero σ-finite G-quasi-invariant measure μ on E, the relation $X \notin \mathrm{dom}(\mu)$ holds true (see [4], [5], [6]).

We say that a subset Y of E is G-negligible in E (see again [4], [5], [6]) if the following two conditions are satisfied:

(*) there exists at least one nonzero σ-finite G-quasi-invariant measure ν on E such that $Y \in \mathrm{dom}(\nu)$;

(**) for every σ-finite G-quasi-invariant measure μ on E, the relation $Y \in \mathrm{dom}(\mu)$ implies the relation $\mu(Y) = 0$.

It was proved that under some natural assumptions on the space (E, G), the union of two G-negligible sets can be a G-absolutely nonmeasurable set (cf. [4], [6]).

Let us consider a more concrete situation when E is the Euclidean plane $\mathbf{R}^2$ and G is an uncountable subgroup of the group of all translations of $\mathbf{R}^2$. In this case we have a natural class of G-negligible subsets of $\mathbf{R}^2$.

To describe this class, take a nonzero vector z in $\mathbf{R}^2$.

A subset Z of $\mathbf{R}^2$ is called uniform in direction z if, for any straight line $l \subset \mathbf{R}^2$ parallel to z, the set $l \cap Z$ is either empty or singleton (cf. [8], [9]).

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A subset U of $\mathbf{R}^2$ will be called uniform (in $\mathbf{R}^2$) if there exists a nonzero vector $z \in \mathbf{R}^2$ such that U turns out to be uniform in direction z.

Example 1 : It is not hard to verify that a uniform in direction z subset of $\mathbf{R}^2$ is G-negligible, where G is any uncountable subgroup of the one-dimensional group $\mathbf{R}z$. Taking $z = (0, 1)$, one can see that a set $Z \subset \mathbf{R}^2$ is uniform in direction z if and only if Z coincides with the graph of some partial function acting from $\mathbf{R}$ into itself.

Example 2 : According to the profound result of Davies (see [1], [2]), there exists a uniform subset U of $\mathbf{R}^2$ such that $\cup\{g_i(U) : i \in I\} = \mathbf{R}^2$ for some countable family $\{g_i : i \in I\} \subset Is_2$, where Is_2 denotes the group of all isometries (motions) of $\mathbf{R}^2$. In particular, the set U is Is_2-absolutely nonmeasurable.

The main goal of this short note is to demonstrate that the existence of two uniform sets U_1 and U_2 in $\mathbf{R}^2$ such that $U_1 \cup U_2$ is $\mathbf{R}^2$-absolutely nonmeasurable is equivalent to the Continuum Hypothesis (**CH**).

Lemma 1 : *If a subset X of (E, G) is G-absolutely nonmeasurable, then there exists a countable family $\{g_i : i \in I\}$ of elements of G such that $\cup\{g_i(X) : i \in I\} = E$.*

Using Sierpiński's classical result (see, e.g., [8], [9]), we get the following statement.

Lemma 2 : *If, for some two uniform subsets U_1 and U_2 of $\mathbf{R}^2$, there exists a countable family $\{h_i : i \in I\}$ of translations of $\mathbf{R}^2$ such that*

$$\cup\{h_i + (U_1 \cup U_2) : i \in I\} = \mathbf{R}^2,$$

then the Continuum Hypothesis holds true.

Taking into account Lemmas 1 and 2, we obtain

Theorem 1 : *If there exist two uniform subsets U_1 and U_2 of $\mathbf{R}^2$ such that $U_1 \cup U_2$ is an $\mathbf{R}^2$-absolutely nonmeasurable set in $\mathbf{R}^2$, then the Continuum Hypothesis is valid.*

Lemma 3 : *Suppose that the Continuum Hypothesis is true.*

Then there exist two subsets X and Y of $\mathbf{R}^2$ satisfying the following three conditions:

(1) the set X is uniform in direction $(0, 1)$ and the set Y is uniform in direction $(1, 0)$;

(2) for some countable family $\{h_i : i \in I\}$ of translations of $\mathbf{R}^2$, the equality

$$\cup\{h_i + (X \cup Y) : i \in I\} = \mathbf{R}^2$$

takes place;

(3) for each point $x \in X$ and for each point $y \in Y$, the distance between x and y is strictly greater than 2.

Lemma 4 : *Let X and Y be two subsets of $\mathbf{R}^2$ described in Lemma 3.*

Then the set $X \cup Y$ is $\mathbf{R}^2$-absolutely nonmeasurable in $\mathbf{R}^2$.

Proof : Suppose on the contrary that there exists a nonzero σ-finite translation quasi-invariant measure μ on the plane $\mathbf{R}^2$ such that $X \cup Y$ turns out to be μ-

measurable.

Denote by H the set of all those vectors in $\mathbf{R}^2$ whose lengths do not exceed 1 and which are parallel to the line $\{0\} \times \mathbf{R}$. Clearly, the cardinality of H is equal to the cardinality continuum. In the sequel, we only need the uncountability of H. Let h_1 and h_2 be any two distinct vectors from H. It is not difficult to verify that

$$(h_1 + (X \cup Y)) \cap (h_2 + (X \cup Y)) = (h_1 + Y) \cap (h_2 + Y).$$

The set $(h_1 + Y) \cap (h_2 + Y)$ is uniform in direction $(1, 0)$ and simultaneously is μ-measurable. Therefore,

$$\mu((h_1 + Y) \cap (h_2 + Y)) = 0.$$

So we infer that the uncountable family $\{h + (X \cup Y) : h \in H\}$ of μ-measurable sets is almost disjoint with respect to μ. This circumstance implies that

$$\mu(X \cup Y) = 0.$$

On the other hand, by virtue of condition (2) of Lemma 3, we must have $\mu(X \cup Y) > 0$.

The obtained contradiction yields the required result. □

Theorem 2 : *If the Continuum Hypothesis holds true, then there exist two uniform subsets X and Y of $\mathbf{R}^2$ such that the set $X \cup Y$ is $\mathbf{R}^2$-absolutely nonmeasurable in $\mathbf{R}^2$.*

Remark 1 : It can be proved (within **ZFC** theory) that there are two $\mathbf{R}^2$-negligible subsets of $\mathbf{R}^2$ such that their union is $\mathbf{R}^2$-absolutely nonmeasurable (in this connection, see [4], [6]).

Remark 2 : It should be noticed that non-measurability properties of uniform subsets of the plane $\mathbf{R}^2$ were considered in several works (see, for instance, [3], [4], [5], [6], [7], [8]). In particular, the nonmeasurability of special uniform sets, with respect to the standard two-dimensional Lebesgue measure on $\mathbf{R}^2$, was first demonstrated by Sierpiński (see [8], [9]).

In [4], assuming **CH**, it was shown that the absolute nonmeasurability of the union of certain two uniform subsets of $\mathbf{R}^2$ is somehow connected with Banach's widely known theorem stating the existence of a finitely additive translation invariant measure which is defined on the ring $\mathcal{S}$ of all bounded sets in $\mathbf{R}^2$ and which extends the restriction of the two-dimensional Lebesgue measure to $\mathcal{S}$ (see, e.g., [5], [10]). The argument presented in this short communication does not rely on the above-mentioned Banach theorem.

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