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Splitted Representation of a Crisp Relation into Dual Fuzzy Relations

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Expert knowledge representations often fail to determine compatibility levels on all objects, and these levels are represented for a certain sampling of universe. The samplings for the fuzzy terms of the linguistic variable, whose compatibility functions are aggregated according to a certain problem, may also be different. In such a case, neither L.A. Zadeh's analysis of fuzzy sets and even the dual forms of developing today R.R. Yager's q-rung orthopair fuzzy sets cannot provide the necessary aggregations. This fact, as a given, can be considered as a source of new types of information, in order to obtain different levels of compatibility according to Zadeh, presented throughout the universe. This source of information can be represented as a pair $\langle A, f_A \rangle$, where there is some crisp subset of the universe A that determines the sampling of objects from the universe, and a function f_A determines the compatibility levels of the elements of that sampling [9]. It is a notion of dual split fuzzy relations, constructed in this article, that allows for the new type of semantic representation and aggregation of complex expert information. This notion is again and again based on the notion of Zadeh fuzzy relation. In particular, the operation of splitting a crisp relation into dual fuzzy relations is introduced.

Properties of algebraic operations on split dual fuzzy-sets are studied in [9]. Properties of relations' splitting into dual fuzzy relations are studied in this work. Fuzzy relation "*no less fuzzy*" on the fuzzy sets is introduced. It is proved that extension of this relation into the splitting dual fuzzy relations induced on the sets of all fuzzy relation algebraic structure is reflexive, antisymmetric and transitive. The proofs are also presented that follow naturally from definitions and previous results. An example of multi-attribute decision making (MADM) is presented for illustration of the application of splitting operation.

Keywords: Fuzzy sets and relations, set splitting, relation splitting, duality of imperfect information, q-rung orthopair fuzzy sets, Lattices.

AMS Subject Classification: 28E10, 62A86, 60A86, 03E72.

1. Introduction

In the study of complex events analysis and synthesis problems, the use of L.A. Zadeh's theory of fuzzy sets [22, 23] has the particular importance today, when the problems of semantic representation of expert qualitative information are quite acute due to the complicated nature of the objects under study. Existing approaches to measuring the degrees of compatibility precision of studying objects are no longer satisfactory to today's researchers. This is why the two sides of imprecision – the levels of object compatibility and incompatibility – are becoming more

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and more independent in a new research [1, 2, 16–19, 21]. This independence is due to the dual representation of evaluation. Duality is becoming an important element in the presentation of incomplete information today, and study of the imprecisions and uncertainties of modeling complex events deserves a great attention. The most common direction of these issues today is to represent the dual nature of information evaluation in some independent degrees of belonging and non-belonging. This idea first came from Atanasov [2].

The intuitionistic fuzzy sets (IFS) theory by Atanasov [2] represents a new extension of Zadeh's fuzzy sets (FS) theory [4]. Since to each element of IFS, as Intuitionistic fuzzy number (IFN) (μ, ν) is assigned a membership degree (μ), a non-membership degree (v) and a hesitancy degree $(1 - \mu, 1 - \nu)$, IFS is much capable to deal with vagueness than FS. IFS theory was extensively utilized in various problems of different areas [2, 16]. Definitions of main arithmetic operations on IFN are given in [2]. The IFS theory is found to be intensively applicable in decision-making direction of research. After consideration of the huge number of existing materials, the authors of [21] presented a scientometric review on IFS studies. At the same time the IFN (μ, ν) has a serious constraint – the sum of membership and non-membership degrees must be or less than 1. Nevertheless, it may happen that a DM provides such data for certain attribute that the aforementioned sum is greater than 1 $(\mu + \nu > 1)$. To cope with such case, Yager [17, 18] introduced the concept of the Pythagorean fuzzy set (PFS) as a generalization of IFS, where a Pythagorean fuzzy number (PFN) (μ, ν) has a weaker constraint – the sum of squared degrees of membership and non-membership satisfies the inequality $\mu^2 + \nu^2 \leq 1$. But in many expert orthopair assessments, neither PFNs nor IFNs can describe fully intellectual activity, because the assessment psychology of a decision maker (DM) is too intricate for hard decision-making, and the attribute's information is still problematic to express with PFNs or IFNs. This problem was solved by Yager again [19, 20]. He introduced the notion of a q-rung orthopair fuzzy set ($q-$ROFS), where $q \geq 1$, and the sum of the qth power of the degrees of membership and non-membership cannot exceed 1. For a q-rung orthopair fuzzy number ($q-$ROFN) we have $(\mu^q + \nu^q \leq 1)$. The fundamentals of arithmetic operations on such numbers are presented in [1, 19]. Obviously, the $q-$ROFSs are generalization of IFSs and PFSs. The IFSs and PFSs represent the particular cases of the q-ROFSs for $q = 1$ and $q = 2$. Thus, q-ROFNs appear to be more suitable and capable for expressing DM's assessment information. Study of Aggregations of experts' q-rung orthopair fuzzy evaluations are actively developed in different works of authors of this paper in multi-criteria decision-making problems [7–14, 17, 22]. A completely different approach to dual representation of a fuzzy set is developed in [15]. In this paper, the concept of lower a-level sets of fuzzy sets is introduced, which is regarded as a dual concept of upper α-level sets of fuzzy sets. The authors introduce a new concept of dual fuzzy sets. Dual decomposition theorem is established. The dual arithmetic of fuzzy sets in R^1 is studied and established some interesting results based on the upper and lower α-level sets.

In practice, there are frequent cases when experts are unable to determine the levels of compatibility on all objects. In fact, these levels are represented by a certain sampling of the universe. Experts may make these samplings different. Samplings for the fuzzy terms of linguistic variables may also be different. But aggregations of such information are still needed, and the universe may not be fully represented at all. In such a case, neither the Zadeh fuzzy set analysis nor the dual forms presented here in the form of q-rung orthogonal fuzzy sets can provide the required aggregations.

Actually, it means the following. For any expert from certain universe $\Omega = \{\omega_1, \ldots, \omega_n\}$, a certain sampling of items $A = \{\omega_{i_1}, \ldots, \omega_{i_A}\}$ is available for evaluation. Suppose the compatibility levels generated by any expert are represented as some function $f_A(\omega) : A \to [0, 1]$, where the values are known only on the elements of the set $A \subset \Omega$. This data may be different for his/her other evaluations or for those of other experts. The new type of information source differs from that involved in determining the levels of compatibility according to Zadeh's point of view. In this case the source of information is presented by pairs $\langle A, f_A \rangle$. We are dealing with a source and data of a different nature. Namely the possibility of semantic representation of such information by the notion of split fuzzy set constructed in [9] by the authors of this work is studied, which is again and again based on Zadeh's concept of a fuzzy set. In particular, the operation of splitting a crisp subset into dual fuzzy sets is introduced [9]. It is this dual, split fuzzy sets lattice that will create a unified environment for aggregating expert evaluations of different samplings. In this work the operation of splitting a crisp relation into dual fuzzy relations is introduced. Algebraic properties of relations' splitting into dual fuzzy relations are studied in this work.

The second section explains the operation of splitting a crisp set indicator into dual fuzzy sets. The third section studies splitting a relation into the dual fuzzy relations. Properties of the algebraic structure induced by the splitting are also investigated. The fourth section describes the basic results obtained in this work and the perspectives of future studies in this direction.

2. Preliminary concepts: Operation of splitting of an indicator

Let's note from the beginning that the preliminary preparatory material is completely taken from the article [9] published by the authors of this study.

Consider the source of information discussed in the introduction for expert evaluations. Suppose that, for any expert from some universe $\Omega = \{\omega_1, \ldots, \omega_n\}$ a certain sampling of elements is available for evaluation. Suppose the compatibility levels generated by the expert are represented as a certain function $f_A(\omega) : A \to [0, 1]$, where the values are known only to the elements of the set $A \subset \Omega$. This data may be different for his/her other evaluations as well for other experts. The new type of information source differs from that involved in determining the levels of compatibility according to Zadeh's point of view [23]. In this case the source of information is presented by pairs $\langle A, f_A \rangle$. Let $A \subset \Omega$ and $I_A \in \{0, 1\}^{\Omega}$ be its indicator. Represent it in the following form

$$I_A(\omega) = f(\omega) I_A(\omega) + 1 - f(\omega) I_A(\omega), \quad \omega \in \Omega, \tag{1}$$

where $f(\omega) : \Omega \to [0, 1]$ is some continuation of the function $f_A(\omega) : A \to [0, 1]$ on the universe $\Omega(f(\omega) = f_A(\omega), \omega \in A)$.

Definition 2.1: ([9]) Let us call representation (1) a splitting of indicator $I_A$with respect to function f.

Introduce the notation:

$$I_{\widetilde{A}}(\omega) \equiv f(\omega) I_A(\omega) \quad \text{and} \quad I_{\widetilde{A}^D}(\omega) \equiv (1 - f(\omega)) I_A(\omega). \tag{2}$$

Indicators $I_{\widetilde{A}}, I_{\widetilde{A}^D} \in [0,1]^\Omega$ of two fuzzy subsets $\widetilde{A}, \widetilde{A}^D \subset \Omega$ are called splitting of an indicator $I_A$of a subset $A \subset \Omega$ and

$$I_A = I_{\widetilde{A}} + I_{\widetilde{A}^D}. \tag{3}$$

Definition 2.2: ([9]) Indicators $I_{\widetilde{A}}, I_{\widetilde{A}^D} \in [0,1]^\Omega$ as well as fuzzy subsets $\widetilde{A}, \widetilde{A}^D \subset \Omega$ are called dual, respectively.

According to L. Zadeh [23] $I_{\widetilde{A}}$ is an indicator or membership function (compatibility function) of some fuzzy subset $\widetilde{A}$. It is clear that splitting does not depend on the continuation of the function $f_A(\omega) : A \to [0,1]$. More exactly, the pair $\langle A, f_A \rangle$ induces a pair of splitting fuzzy sets $(\widetilde{A}, \widetilde{A}^D)$.

Example 2.3 ([9]) Let a set of digits $\Omega = \{0,1,2,3,4,5,6,7,8,9\}$ be given. Let also the sampling of universe as some subset $A \subset \Omega$ be given. For example, suppose that A is a set of odd digits, $A = \{1,3,5,7,9\}$, and let expert's evaluation only on this sampling be given by the function $f_A(\omega) : A \to [0,1]$, $f_A(\omega) = \frac{1}{\omega+1}$, $\omega \in A$. Let $f(\omega) : \Omega \to [0,1]$ be any continuation of the function $f_A(\omega)$ on Ω. Then the splitting of the I_A of the subset $A \subset \Omega$ on the universe Ω into two dual fuzzy sets (or their indicators (membership functions)) looks like this:

$$\widetilde{A} = \left\{ \frac{0}{0}, \frac{1}{(1/2)}, \frac{2}{0}, \frac{3}{(1/4)}, \frac{4}{0}, \frac{5}{(1/6)}, \frac{6}{0}, \frac{7}{(1/8)}, \frac{8}{0}, \frac{9}{(1/10)} \right\}, \tag{4}$$

$$\widetilde{A}^D = \left\{ \frac{0}{0}, \frac{1}{(1/2)}, \frac{2}{0}, \frac{3}{(3/4)}, \frac{4}{0}, \frac{5}{(5/6)}, \frac{6}{0}, \frac{7}{(7/8)}, \frac{8}{0}, \frac{9}{9/10} \right\}.$$

Practically, dual splitting fuzzy subsets $\langle \widetilde{A}, \widetilde{A}^D \rangle$ are created as fuzzy subsets on the universe Ω. Practical interpretation looks as follows: Sometimes for the description of some uncertain term of some linguistic variable on the elements of an universe we usually construct membership function. But for the extension of the information containing in the membership function to only on some elements of concrete crisp subset, we are splitting this set into dual split fuzzy subsets. So, the extended information is contained in dual fuzzy sets. Duality of this extension means that both fuzzy sets contain the same information, but codified in different ways. It is to mention that, splitting dual fuzzy subsets $\widetilde{A}$, $\widetilde{A}^D$ on Ω are induced by the subset $A, A \subset \Omega$ and some function $f_A(\omega) : A \to [0,1]$.

As mentioned earlier, the possibility of using the split operation can arise in many cases. Here is one case. Let us now consider an example on application of splitting a set into dual fuzzy sets in multi-attribute decision making (MADM).

Consider a MADM model with 5 attributes $S = \{s_1, s_2, \ldots, s_5\}$ and 3 alternatives $D = \{d_1, d_2, d_3\}$. Suppose that a decision-making matrix represents a matrix of normed ratings in $[0,1]$, where some ratings are not given:

As can be seen from this matrix, for each alternative there are attributes for which the rating evaluations are not presented. Such unusual cases can arise in

D/S	s_1	s_2	s_3	s_4	s_5
d_1	0.2	-	0.7	0.6	-
d_2	-	0.4	-	0.3	0.8
d_3	0.3	0.5	0.6	-	-

practice for many reasons. One is when the number of attributes is quite large due to the deep detailing of the task, and it is difficult for experts to make a rating assessment on all attributes. Such cases often arise when building recommendation models in collaborative filtering problems. These empty elements need to be filled somehow. This problem can be successfully implemented with a machine learning approach, if, of course, there is a large amount of prehistoric data. Otherwise, when we do not have objective data and expert evaluations are of the sparse type, the splitting operation presented here can be a way out! We see that the alternative d_1 is evaluated on a subset of attributes $S_1 \equiv \{s_1, s_3, s_4\}$, the alternative d_2 is evaluated on a subset of attributes$S_2 \equiv \{s_2, s_4, s_5\}$, and the alternative d_3 is evaluated on a subset of attributes $S_3 \equiv \{s_1, s_2, s_3\}$. Let us split these sets into dual fuzzy sets. Then the decision matrix can be written as:

D/S	s_1	s_2	s_3	s_4	s_5
$\widetilde{A}_{d_1}$	0.2	0.0	0.7	0.6	0.0
$\widetilde{A}_{d_1}^D$	0.8	0.0	0.3	0.4	0.0
$\widetilde{A}_{d_2}$	0.0	0.4	0.0	0.3	0.8
$\widetilde{A}_{d_2}^D$	0.0	0.6	0.0	0.7	0.2
$\widetilde{A}_{d_3}$	0.3	0.5	0.6	0.0	0.0
$\widetilde{A}_{d_3}^D$	0.7	0.5	0.4	0.0	0.0

Therefore, the alternative d_i, $i = 1, 2, 3$, is represented by dual split fuzzy subsets $\langle A_{d_i}, A_{d_i}^D \rangle$ on the whole universe of attributes $S = \{s_1, s_2, \ldots, s_5\}$. The creation of an aggregation instrument and the ways of constructing ranking relations can be developed in many directions, where the definitions and results presented in the following paragraphs on the operations of dual split sets will be used. Here is a simple solution. Combine the elements of split dual fuzzy sets into pairwise intuitionistic fuzzy numbers by a simple concatenation:

Here, of course, attention is drawn to the symbolic intuitionistic fuzzy number (0.0,0.0), whose attribution and non-attribution values are 0.0, which indicates the information that the evaluation is not done. As a matter of fact, if we gave a formula-quantitative value to ratings that are not evaluated by such representations, it is natural to replace it with zero intuitionistic fuzzy numerical rating (0.0,1.0). Then the decision-making matrix takes the following form.

Suppose that the vector of attribute weights in this model is $W = \{w_1, w_2, \ldots, w_5\} = \{0.1, 0.2, 0.4, 0.1, 0.2\}$. For ranking of alternatives let us use the intuitionistic fuzzy weighted averaging (IFWA) operator:

$$d_i \sim IFWA(a_1, \ldots, a_5). \tag{5}$$

D/S	s_1	s_2	s_3	s_4	s_5
d_1	(0.2,0.8)	(0.0,1.0)	(0.7,0.3)	(0.6,0.4)	(0.0,1.0)
d_2	(0.0,1.0)	(0.4,0.6)	(0.0,1.0)	(0.7,0.3)	(0.8,0.2)
d_3	(0.3,0.7)	(0.5,0.5)	(0.6,0.4)	(0.0,1.0)	(0.0,1.0)

For example, for d_1 we will have

$$d_1 \sim 0.1 \otimes (0.2, 0.8) \oplus 0.2 \otimes (0.0, 1.0) \oplus 0.4 \otimes (0.7, 0.3) \oplus 0.1 \otimes (0.6, 0.4) \oplus 0.2 \otimes (0.0, 1.0), \tag{6}$$

where $\oplus$ and $\otimes$ denote addition and multiplication operations on intuitionistic fuzzy numbers, respectively.

Definition 2.4: ([9]) Splitting of a crisp set is equivalent to the splitting of the corresponding indicator and is formally represented as follows

$$(I_A = I_{\widetilde{A}} + I_{\widetilde{A}^D}) \rightleftarrows (A = \widetilde{A} \oplus \widetilde{A}^D), \tag{7}$$

where $\oplus$ is a splitting operation of a set.

As it was mentioned, from Definition 2.1 for the splitting of a crisp set A, the building materials are an indicator I_A and some function$f_A(\omega) : A \to [0, 1]$. We interpret the split set on $\widetilde{A}$ into and its dual split set $\widetilde{A}^D$ as a fuzzy subset in the sense of Zadeh [22]. With this interpretation, $I_{\widetilde{A}}$ is considered as an indicator (membership function) of a fuzzy split subset $\widetilde{A}$.

Properties of splitting of sets indicators, or Cartesian product's splitting, also the Lattice of split elements of the Boolean lattice of indicators i are studied in [9].

3. Splitting of a relation into the dual fuzzy relations. Properties of the algebraic structure induced by the splitting

The notion of dual fuzzy set is introduced into the lattice of fuzzy subsets. On the basis of this notion the relationship "x is no less fuzzy than y" is considered. In the following the properties of relation splitting will be presented.

Consider the lattice of fuzzy subsets $P^{\sim}(\Omega)$ [9]. A dual element (a dual subset) in [9] is defined based on the procedure for representation of a crisp set as a pair of dual fuzzy sets, which is called the splitting procedure of a crisp set

$$A = \widetilde{A} \oplus \widetilde{A}^D, \tag{8}$$

where $A \subseteq \Omega$. $\widetilde{A}$ is a fuzzy subset, $\widetilde{A}^D$ is a dual fuzzy subset. The relationship of the dual subset with the fuzzy complement is as follows:

$$\neg\widetilde{A} = \widetilde{A}^D \cup A^*, \tag{9}$$

where $A^* \equiv (\varnothing : \widetilde{A})$ is the pseudocompletion of the element $\widetilde{A}$ [3].

The dual element indicates the following: if it can be argued that the elements are endowed to some extent with the certain property [4, 5], then to some extent they are not endowed with the same property. The "closer" [6] $\widetilde{A}$ and $\widetilde{A}^D$, the "more" fuzzy is contained in the statement "the elements of A are endowed with property P". Similar ideas underlie the concept of fuzziness in works [4, 5], where instead of $\widetilde{A}$ and $\widetilde{A}^D$ there are compared $\widetilde{A}$and $\neg\widetilde{A}$. There is also a generalization of the concept of "metric" collation of the degrees of fuzziness of two subsets to "non-metric", which is being reduced to a certain relation f between the corresponding subsets. Below we will show that it is possible to introduce an equivalent relation φ, which in our opinion better emphasizes the fact that fuzziness is an internal property of a fuzzy subset and does not depend on pseudocompletion.

The basis of our approach, as well as in [6], is the concept of "between" in the distributive lattice [3].

Definition 3.1: Let $\widetilde{x}$ and $\widetilde{y} \in P^{\sim}(\Omega)$. $\widetilde{x}$ is no less fuzzy than $\widetilde{y}$, $(\widetilde{x}\varphi\widetilde{y})$, if $\widetilde{x} \odot y$ and $(\widetilde{x} \odot y)^D = \widetilde{x}^D \odot y$ is between $\widetilde{y}$ and $\widetilde{y}^D$ in $P^{\sim}(\Omega)$, where $\odot$ is the operation of sequential splitting the intersection of two subsets [9].

$$(\widetilde{x}\varphi\widetilde{y}) \Leftrightarrow \begin{cases} (\widetilde{y}, \widetilde{x} \odot y, \widetilde{y}^D) \\ (\widetilde{y}, \widetilde{x}^D \odot y, \widetilde{y}^D) \end{cases} \Leftrightarrow \widetilde{y} \cap \widetilde{y}^D \subseteq \widetilde{x} \odot y \subseteq \widetilde{y} \cup \widetilde{y}^D. \tag{10}$$

It is clear that $(\widetilde{x}\varphi x)\ \forall \widetilde{x} \in P^{\sim}(\Omega)$.

Proposition 3.2: *The relations f and φ are equivalent on $P^{\sim}(\Omega)$.*

Proof: Let $(\widetilde{x}f\widetilde{y})$. This means that

$$\begin{cases} (\widetilde{y}, \widetilde{x}, \neg\widetilde{y}) \\ (\widetilde{y}, \neg\widetilde{x}, \neg\widetilde{y}) \end{cases} \Leftrightarrow \begin{cases} (I_{\widetilde{y}} \wedge I_{\neg\widetilde{y}}) \le I_{\widetilde{x}} \le (I_{\widetilde{y}} \vee I_{\neg\widetilde{y}}) \\ (I_{\widetilde{y}} \wedge I_{\neg\widetilde{y}}) \le I_{\neg\widetilde{x}} \le (I_{\widetilde{y}} \vee I_{\neg\widetilde{y}}) \end{cases}$$

$$\Rightarrow \begin{cases} (I_{\widetilde{y}} \wedge I_{\neg\widetilde{y}})I_y \le I_{\widetilde{x}}I_y \le (I_{\widetilde{y}} \vee I_{\neg\widetilde{y}})I_y \\ (I_{\widetilde{y}} \wedge I_{\neg\widetilde{y}})I_y \le I_{\neg\widetilde{x}}I_y \le (I_{\widetilde{y}} \vee I_{\neg\widetilde{y}})I_y \end{cases}$$

$$\Rightarrow \begin{cases} (I_{\widetilde{y}} \wedge I_{\widetilde{y}^D}) \le I_{\widetilde{x}}I_y \le (I_{\widetilde{y}} \vee I_{\widetilde{y}^D}) \\ (I_{\widetilde{y}} \wedge I_{\widetilde{y}^D}) \le I_{\widetilde{x}^D}I_y \le (I_{\widetilde{y}} \vee I_{\widetilde{y}^D}) \end{cases}$$

$$\Leftrightarrow \begin{cases} (\widetilde{y}, \widetilde{x} \odot y, \widetilde{y}^D) \\ (\widetilde{y}, \widetilde{x}^D \odot y, \widetilde{y}^D) \end{cases} \Leftrightarrow (\widetilde{x}\varphi\widetilde{y}).$$

Let now, vice versa, $(\widetilde{x}\varphi\widetilde{y})$, i.e.,

$$\begin{cases} (\widetilde{y}, \widetilde{x} \odot y, \widetilde{y}^D) \\ (\widetilde{y}, \widetilde{x}^D \odot y, \widetilde{y}^D) \end{cases} \Leftrightarrow \begin{cases} (I_{\widetilde{y}} \wedge I_{\widetilde{y}^D}) \le I_{\widetilde{x}}I_y \le (I_{\widetilde{y}} \vee I_{\widetilde{y}^D}) \\ (I_{\widetilde{y}} \wedge I_{\widetilde{y}^D}) \le I_{\widetilde{x}^D}I_y \le (I_{\widetilde{y}} \vee I_{\widetilde{y}^D}) \end{cases}$$

$$\Rightarrow \begin{cases} (I_{\widetilde{y}} \wedge I_{\neg\widetilde{y}}) \le I_{\widetilde{x}}I_y + I_{\widetilde{x}}I_{y^C} \le (I_{\widetilde{y}} \vee I_{\widetilde{y}^D}) + (I_{\widetilde{y}} \vee I_{y^C}) \\ (I_{\widetilde{y}} \wedge I_{\neg\widetilde{y}}) \le I_{\widetilde{x}^D}I_y + I_{x^C}I_y + I_{\neg\widetilde{x}}I_{\widetilde{y}^C} \le (I_{\widetilde{y}} \vee I_{\widetilde{y}^D}) + (I_{\widetilde{y}} \vee I_{y^C}). \end{cases}$$

The transition that we made is correct for the following reasons. Firstly, we strengthened the first right inequality, adding to it $I_{\widetilde{x}}I_{y^C} \le I_{\widetilde{y}} \vee I_{y^C}$. Secondly, in order for $(\widetilde{x}\varphi\widetilde{y})$ to be fulfilled, it is necessary to have $y \subseteq x$, therefore $I_{x^C}I_y = 0$, and adding to the second right side inequality $I_{\neg\widetilde{x}}I_{y^C} \leq I_{\widetilde{y}} \vee I_{y^C}$, we also strengthen it.

Thus,

$$\begin{cases} (\widetilde{y}, \widetilde{x} \odot y, \widetilde{y}^D) \\ (\widetilde{y}, \widetilde{x}^D \odot y, \widetilde{y}^D) \end{cases}$$
$$\Rightarrow \begin{cases} (I_{\widetilde{y}} \wedge I_{\neg\widetilde{y}}) \leq I_{\widetilde{x}} \leq (I_{\widetilde{y}} \vee I_{\neg\widetilde{y}}) \\ (I_{\widetilde{y}} \wedge I_{\neg\widetilde{y}}) \leq I_{\neg\widetilde{x}} \leq (I_{\widetilde{y}} \vee I_{\neg\widetilde{y}}) \end{cases}$$
$$\Rightarrow \begin{cases} (\widetilde{y}, \widetilde{x}, \neg\widetilde{y}) \\ (\widetilde{y}, \neg\widetilde{x}, \neg\widetilde{y}) \end{cases} \Leftrightarrow (\widetilde{x} f \widetilde{y}).$$

□

Lemma 3.3: *If $z \subseteq y$, then from $(\widetilde{y}, \widetilde{x} \odot y, \widetilde{y}^D)$ it follows $(\widetilde{y} \odot z, \widetilde{x} \odot z, \widetilde{y}^D \odot z)$.*

Proof:

$$\begin{aligned} (\widetilde{y}, \widetilde{x} \odot y, \widetilde{y}^D) &\Leftrightarrow (I_{\widetilde{y}} \wedge I_{\widetilde{y}^D}) \leq I_{\widetilde{x}} I_y \leq (I_{\widetilde{y}} \vee I_{\widetilde{y}^D}) \Rightarrow (I_{\widetilde{y}} \wedge I_{\widetilde{y}^D}) I_z \leq I_{\widetilde{x}} I_y I_z \\ &\leq (I_{\widetilde{y}} \vee I_{\widetilde{y}^D}) I_z \Rightarrow (I_{\widetilde{y}} \odot I_{\widetilde{z}} \wedge I_{\widetilde{y}^D} I_z) \leq I_{\widetilde{x}} \odot I_z \\ &\leq (I_{\widetilde{y}} \odot I_z \vee I_{\widetilde{y}^D} I_z) \Leftrightarrow (I_{\widetilde{y} \odot z} \wedge I_{\widetilde{y}^D \odot z}) \\ &\leq (I_{\widetilde{y} \odot z} \vee I_{\widetilde{y}^D \odot z}) \Leftrightarrow (\widetilde{y} \odot z, \widetilde{x} \odot z, \widetilde{y}^D \odot z). \end{aligned}$$

□

Proposition 3.4: *φ is transitive on $P^{\sim}(\Omega)$, i.e., $(\widetilde{x}\varphi\widetilde{y})$ and $(\widetilde{y}\varphi\widetilde{z}) \Rightarrow (\widetilde{x}\varphi\widetilde{z})$.*

Before the proof, we will give some useful properties of the relationship (a, b, c) that R. Yager gave in his work [5] and which we use in our reasoning:

$$\begin{aligned} &P_1 : (abc) \Leftrightarrow (cba), \\ &P_2 : (abc) \text{ and } (acb) \Leftrightarrow b = c, \\ &P_3 : (abc) \text{ and } (axb) \Rightarrow (axc), \\ &P_4 : (abc) \text{ and } (bcd) \text{ and } b \neq c \Rightarrow (abd), \\ &P_5 : (abc) \text{ and } (acd) \Rightarrow (bcd), \\ &P_6 : (aba) \Leftrightarrow a = b, \\ &P_7 : (aab) \Leftrightarrow (baa) \Leftrightarrow (abb) \Leftrightarrow (bba), \\ &P_8 : (abc) \Rightarrow (aab), \\ &P_9 : \text{if the lattice is distributive, then} \\ &\quad (a) \;\; (qbc) \text{ and } (qdc) \text{ and } (bxd) \Rightarrow (qxc) \\ &\quad (b) \;\; (qbc) \text{ and } (qdb) \text{ and } (cxd) \Rightarrow (qbx). \end{aligned} \tag{11}$$

Proof:

$$
\left.\begin{array}{l}
(\widetilde{x}\varphi\widetilde{y}) \Leftrightarrow \begin{cases} (\widetilde{y}, \widetilde{x}\odot y, \widetilde{y}^D) \\ (\widetilde{y}, \widetilde{x}^D\odot y, \widetilde{y}^D) \end{cases} \\
(\widetilde{y}\varphi z) \Leftrightarrow \begin{cases} (\widetilde{z}, \widetilde{y}\odot z, \widetilde{z}^D) \\ (\widetilde{z}, \widetilde{y}^D\odot z, \widetilde{z}^D) \end{cases}
\end{array}\right\}
$$

$$
\begin{aligned}
&\Rightarrow \left\{(\widetilde{z}, y\odot z, \widetilde{z}^D) \text{ and } (\widetilde{z}, (\widetilde{y}\odot z)^D, \widetilde{z}^D) \text{ and } (\widetilde{y}, \widetilde{x}\odot y, \widetilde{y}^D)\right\} \\
&\Rightarrow \left\{(\widetilde{z}, \widetilde{y}\odot z, \widetilde{z}^D) \text{ and } (\widetilde{z}, (\widetilde{y}\odot z)^D, \widetilde{z}^D) \text{ and } (\widetilde{y}\odot z, \widetilde{x}\odot z, \widetilde{y}^D\odot z)\right\} \\
&\Rightarrow (\widetilde{z}, \widetilde{x}\odot z, x).
\end{aligned}
$$

The last transition is made on the basis of the property of the "between" relationship P_3 indicated in [5].

Similarly, it can be proved that

$$
\begin{cases} (\widetilde{x}\varphi\widetilde{y}) \\ (\widetilde{y}\varphi\widetilde{z}) \end{cases} \Rightarrow (z, (\widetilde{x}\odot z)^D, \widetilde{z}^D).
$$

Thus,

$$
\begin{cases} (\widetilde{z}, (\widetilde{x}\odot z), \widetilde{z}^D) \\ (\widetilde{z}, (\widetilde{x}\odot z)^D, \widetilde{z}^D) \end{cases} \Leftrightarrow (\widetilde{x}\varphi\widetilde{z}).
$$

□

Proposition 3.5: *φ is reflexive on $P^{\sim}(\Omega) : (\widetilde{x}\varphi\widetilde{x})$.*

Proof: We have

$$
\begin{aligned}
&\begin{cases} (I_{\widetilde{x}} \wedge I_{\widetilde{x}^D}) \le I_{\widetilde{x}} \le (I_{\widetilde{x}} \vee I_{\widetilde{x}^D}) \\ (I_{\widetilde{x}} \wedge I_{\widetilde{x}^D}) \le I_{\widetilde{x}^D} \le (I_{\widetilde{x}} \vee I_{\widetilde{x}^D}) \end{cases} \\
\Rightarrow &\begin{cases} (I_{\widetilde{x}} \wedge I_{\widetilde{x}^D}) \le I_{\widetilde{x}} I_x \le (I_{\widetilde{x}} \vee I_{\widetilde{x}^D}) \\ (I_{\widetilde{x}} \wedge I_{\widetilde{x}^D}) \le I_{\widetilde{x}^D} I_x \le (I_{\widetilde{x}} \vee I_{\widetilde{x}^D}) \end{cases} \\
\Rightarrow &\begin{cases} (\widetilde{x}, (\widetilde{x}\odot x), \widetilde{x}^D) \\ (\widetilde{x}, (\widetilde{x}^D\odot x), \widetilde{x}^D) \end{cases} \Leftrightarrow (\widetilde{x}\varphi\widetilde{x}).
\end{aligned}
$$

□

Proposition 3.6: *If $(\widetilde{x}\varphi\widetilde{y})$ and $(\widetilde{y}\varphi\widetilde{x})$, then either $\widetilde{x}\odot y = \widetilde{y}\odot x$ and $\widetilde{x}^D\odot y = y^D\odot x$, or $\widetilde{x}\odot y = \widetilde{y}^D\odot x$ and $\widetilde{x}^D\odot y = \widetilde{y}^D\odot x$.*

Proof:

$$
(\widetilde{x}\varphi\widetilde{y}) \Leftrightarrow \begin{cases} (\widetilde{y}, (\widetilde{x}\odot y), \widetilde{y}^D) \\ (\widetilde{y}, (\widetilde{x}\odot y)^D, \widetilde{y}^D) \end{cases} \quad \text{and} \quad (\widetilde{y}\varphi\widetilde{x}) \Leftrightarrow \begin{cases} (\widetilde{x}, (\widetilde{y}\odot x), \widetilde{x}^D) \\ (\widetilde{x}, (\widetilde{y}\odot x)^D, \widetilde{x}^D) \end{cases}.
$$

Suppose that $\widetilde{x}\odot y \neq \widetilde{y}\odot x$, then, according to Lemma 3.3, P_1 and P_4, we have

$$\begin{cases}(\widetilde{y},\widetilde{x}\odot y,\widetilde{y}^D)=(\widetilde{y}^D,\widetilde{x}\odot y,\widetilde{y}) \\ (\widetilde{x},\widetilde{y}\odot x,\widetilde{x}^D)\end{cases} \Rightarrow \begin{cases}(\widetilde{y}^D\odot x,\widetilde{x}\odot y,\widetilde{y}\odot x) \\ (\widetilde{x}\odot y,\widetilde{y}\odot x,\widetilde{x}^D\odot y)\end{cases}$$
$$\Rightarrow (\widetilde{y}^D\odot x,\widetilde{x}\odot y,\widetilde{x}^D\odot y)=(\widetilde{x}^D\odot y,\widetilde{x}\odot y,,\widetilde{y}^D\odot x).$$

Due to P_2, we have

$$\begin{cases}(\widetilde{x}^D\odot y,\widetilde{x}\odot y,\widetilde{y}^D\odot x) \\ (\widetilde{x},(\widetilde{y}\odot x)^D,\widetilde{x}^D)\end{cases} \Rightarrow \begin{cases}(\widetilde{x}^D\odot y,\widetilde{x}\odot y,\widetilde{y}^D\odot x) \\ (\widetilde{x}\odot y,\widetilde{y}^D\odot x,\widetilde{x}^D\odot y)=(\widetilde{x}\odot y,\widetilde{y}^D\odot x,\widetilde{x}\odot y)\end{cases}$$
$$\Rightarrow \widetilde{x}\odot y=\widetilde{y}^D\odot x.$$

Analogously, on the basis of P_1, P_2 and P_4 and Lemma 3.3,

$$\begin{cases}(\widetilde{y},(\widetilde{x}\odot y)^D,\widetilde{y}^D)=(\widetilde{y}^D,(\widetilde{x}\odot y)^D,\widetilde{y}) \\ (\widetilde{x},(\widetilde{y}\odot x)^D,\widetilde{x}^D)\end{cases} \Rightarrow \begin{cases}(\widetilde{y}^D\odot x,\widetilde{x}^D\odot y,\widetilde{y}\odot x) \\ (\widetilde{x}\odot y,\widetilde{y}^D\odot x,\widetilde{x}^D\odot y)\end{cases}$$
$$\Rightarrow (\widetilde{x}\odot y,\widetilde{x}^D\odot y,\widetilde{y}\odot x)$$

Let now $\widetilde{x}\odot y=\widetilde{y}\odot x$, then

$$(\widetilde{x}\odot y,\widetilde{y}^D\odot x,\widetilde{x}^D\odot y)=(\widetilde{y}\odot x,\widetilde{y}^D\odot x,\widetilde{x}^D\odot y).$$

Due to P_1, we can write

$$\begin{cases}(\widetilde{y}\odot x,\widetilde{y}^D\odot x,\widetilde{x}^D\odot y) \\ (\widetilde{y}\odot x,\widetilde{x}^D\odot y,\widetilde{y}^D\odot x)\end{cases} \Rightarrow \widetilde{x}^D\odot y=\widetilde{y}^D\odot x.$$

We see that φ on $P^{\sim}(\Omega)$ is not antisymmetric and, therefore, is not a partial order relation. □

Proposition 3.7: *The relation φ on $P^{\sim}(\Omega)$ is such that*

(1) $(\widetilde{x}\varphi\widetilde{x}^D)$ *and* $(\widetilde{x}^D\varphi\widetilde{x})$,
(2) $(\widetilde{x}\varphi\widetilde{y}) \Leftrightarrow (\widetilde{x}^D\varphi\widetilde{y}) \Leftrightarrow (\widetilde{x}^D\varphi\widetilde{y}^D)$.

Proof: (1) Since the operation $(\ )^D$ is an involution, then

$$(\widetilde{x}\varphi\widetilde{x}^D) \Leftrightarrow \begin{cases}(\widetilde{x}^D,\widetilde{x}\odot x,(\widetilde{x}^D)^D) \\ (\widetilde{x}^D,\widetilde{x}^D\odot x,(\widetilde{x}^D)^D)\end{cases} \Rightarrow \begin{cases}(\widetilde{x}^D,\widetilde{x},\widetilde{x}) \\ (\widetilde{x}^D,\widetilde{x}^D,\widetilde{x})\end{cases}.$$

To show that $(\widetilde{x}^D,\widetilde{x},\widetilde{x})$ takes place, note that $(\widetilde{x}^D\cap\widetilde{x})\cup\widetilde{x}=\widetilde{x}$ and $(\widetilde{x}^D\cup\widetilde{x})\cap\widetilde{x}=\widetilde{x}$. Due to $P_7(\widetilde{x}^D,\widetilde{x},\widetilde{x}) \Leftrightarrow (\widetilde{x}^D,\widetilde{x}^D,x)$. Thus, we have $(\widetilde{x}\varphi\widetilde{x}^D)$. To show that $(\widetilde{x}^D\varphi\widetilde{x})$, we should have $(\widetilde{x},\widetilde{x}^D\odot x,\widetilde{x}^D)=(\widetilde{x},\widetilde{x}^D,\widetilde{x}^D)$ and $(\widetilde{x},(\widetilde{x}^D\odot x)^D,\widetilde{x}^D)=$

$(\widetilde{x}, (\widetilde{x}^D)^D, \widetilde{x}^D)$, which again follows from P_7;

$$(\widetilde{x}\varphi\widetilde{y}) = \begin{cases} (\widetilde{y}, \widetilde{x} \odot y, \widetilde{y}^D) \\ (\widetilde{y}, \widetilde{x}^D \odot y, \widetilde{y}^D) \end{cases}.$$

Since $(\widetilde{x}^D)^D = \widetilde{x}$, then $(\widetilde{y}, \widetilde{x} \odot y, \widetilde{y}^D) \Rightarrow (\widetilde{y}, (\widetilde{x}^D)^D, \widetilde{y}^D)$. However,

$$\begin{cases} (\widetilde{y}, \widetilde{x}^D \odot y, \widetilde{y}^D) \\ (\widetilde{y}, (\widetilde{x}^D)^D \odot y, \widetilde{y}^D) \end{cases} \Leftrightarrow (\widetilde{x}^D\varphi\widetilde{y}).$$

To show that $(\widetilde{x}^D\varphi\widetilde{y}^D)$ takes place, we should have $(\widetilde{y}^D, \widetilde{x}^D \odot y, (\widetilde{y}^D)^D)and(\widetilde{y}^D, (\widetilde{x}^D)^D \odot y, (\widetilde{y}^D)^D)$, which follow from $(\widetilde{x}\varphi\widetilde{y})$:

$$\begin{aligned} &(\widetilde{y}, \widetilde{x}^D \odot y, \widetilde{y}^D) \Rightarrow (\widetilde{y}^D, \widetilde{x}^D \odot y, \widetilde{y}) \Rightarrow (\widetilde{y}^D, \widetilde{x}^D \odot y, (\widetilde{y}^D)^D) \\ \text{and } &(\widetilde{y}, \widetilde{x}^D \odot y, \widetilde{y}^D) \Rightarrow (\widetilde{y}^D, \widetilde{x} \odot y, \widetilde{y}) \Rightarrow (\widetilde{y}^D, (\widetilde{x}^D)^D \odot y, (\widetilde{y}^D)^D). \end{aligned}$$

□

Proposition 3.8: *If $\widetilde{x}, \widetilde{y} \in P^{\sim}(\Omega)$ and $(\widetilde{x}\varphi\widetilde{y})$, then $(\widetilde{x}\varphi(\widetilde{y}^D : \widetilde{y}))$ and $(\widetilde{x}\varphi(\widetilde{y} : \widetilde{y}^D))$.*

Proof: We have [1]

$$(\widetilde{y}^D : \widetilde{y}) = y^C_{>1/2} \cup \widetilde{y}^D, \quad (\widetilde{y} : \widetilde{y}^D) = y^C_{<1/2} \cup \widetilde{y}.$$

Further,

$$(\widetilde{x}\varphi(\widetilde{y}^D : \widetilde{y})) = \begin{cases} ((y^C_{>1/2} \cup \widetilde{y}^D, \widetilde{x} \odot y^C_{>1/2} \cup \widetilde{y}^D, (y^C_{>1/2} \cup \widetilde{y}^D)^D) \\ ((y^C_{>1/2} \cup \widetilde{y}^D, \widetilde{x}^D \odot y^C_{>1/2} \cup \widetilde{y}^D, (y^C_{>1/2} \cup \widetilde{y}^D)^D) \end{cases}$$

$$\Rightarrow \begin{cases} ((y^C_{>1/2} \cup \widetilde{y}^D), \widetilde{x}, \widetilde{y}_{>1/2}) \\ ((y^C_{>1/2} \cup \widetilde{y}^D), \widetilde{x},^D \widetilde{y}_{>1/2}) \end{cases}$$

$$\Rightarrow \begin{cases} \widetilde{y}_{>1/2} \cap (\widetilde{y}_{>1/2} \cup y^D) \subseteq \widetilde{x} \subseteq \widetilde{y}_{>1/2} \cup (y^C_{>1/2} \cup \widetilde{y}^D) \\ \widetilde{y}_{>1/2} \cap (\widetilde{y}_{>1/2} \cup y^D) \subseteq \widetilde{x}^D \subseteq \widetilde{y}_{>1/2} \cup (y^C_{>1/2} \cup \widetilde{y}^D) \end{cases}$$

$$\Rightarrow \begin{cases} \widetilde{y}_{>1/2} \cap y^D \subseteq \widetilde{x} \subseteq \widetilde{y}_{>1/2} \cup y^D \cup y^C_{>1/2} \\ \widetilde{y}_{>1/2} \cap y^D \subseteq \widetilde{x}^D \subseteq \widetilde{y}_{>1/2} \cup y^D \cup y^C_{>1/2} \end{cases}$$

$$\Rightarrow \begin{cases} \widetilde{y}^D_{>1/2} \subseteq \widetilde{x} \subseteq \widetilde{y}_{>1/2} \cup y^C_{>1/2} \\ \widetilde{y}^D_{>1/2} \subseteq \widetilde{x}^D \subseteq \widetilde{y}_{>1/2} \cup y^C_{>1/2} \end{cases}$$

This expression can be obtained from $(\widetilde{x}\varphi\widetilde{y})$:

$$\begin{cases} \widetilde{y} \cap y^D \subseteq \widetilde{x} \odot y \subseteq \widetilde{y} \cup y^D \\ \widetilde{y} \cap y^D \subseteq \widetilde{x}^D \odot y \subseteq \widetilde{y} \cup y^D \end{cases}$$

$$\Rightarrow \begin{cases} \widetilde{y}_{>1/2} \cap \widetilde{y}^D \subseteq \widetilde{x} \odot y \cup \widetilde{x} \odot y^D \subseteq \widetilde{y}_{>1/2} \cup \widetilde{y}^D \cup y^C_{>1/2} \\ \widetilde{y}_{>1/2} \cap \widetilde{y}^D \subseteq \widetilde{x}^D \odot y \cup \widetilde{x}^D \odot y^D \subseteq \widetilde{y}_{>1/2} \cup \widetilde{y}^D \cup y^C_{>1/2} \end{cases}$$

Since $\widetilde{y}_{>1/2} \cap \widetilde{y}^D \subseteq \widetilde{y} \cap y^D$, $\widetilde{y} \cup \widetilde{y}^D = \widetilde{y}_{>1/2} \cup \widetilde{y}^D$ and $y^C \subseteq y^D_{>1/2}$ (and, furthermore, $\widetilde{x} \odot y^C$ and $\widetilde{x}^D \odot y^C$ are subsets of $y^C_{>1/2}$). Thus, $(\widetilde{x}\varphi(\widetilde{y}^D : \widetilde{y})$.

$$(\widetilde{x}\varphi(\widetilde{y} : \widetilde{y}^D) = \begin{cases} ((y^C_{\leq 1/2} \cup \widetilde{y}), \widetilde{x} \odot (y \cup y^C_{\leq 1/2}), (y^C_{\leq 1/2} \cup \widetilde{y})^D) \\ ((y^C_{\leq 1/2} \cup \widetilde{y}), \widetilde{x}^D \odot (y \cup y^C_{\leq 1/2}), (y^C_{\leq 1/2} \cup \widetilde{y})^D) \end{cases}$$

$$\Rightarrow \begin{cases} ((y^C_{\leq 1/2} \cup \widetilde{y}), \widetilde{x}, \widetilde{y}^D_{>1/2}) \\ ((y^C_{\leq 1/2} \cup \widetilde{y}), \widetilde{x},^D \widetilde{y}^D_{>1/2}) \end{cases}$$

$$\Rightarrow \begin{cases} \widetilde{y}_{\geq 1/2} \cap (y^C_{\leq 1/2} \cup \widetilde{y}) \subseteq \widetilde{x} \subseteq \widetilde{y}^D_{\geq 1/2} \cup (y^C_{\leq 1/2} \cup \widetilde{y}) \\ \widetilde{y}_{\geq 1/2} \cap (y^C_{\leq 1/2} \cup \widetilde{y}) \subseteq \widetilde{x}^D \subseteq \widetilde{y}^D_{\geq 1/2} \cup (y^C_{\leq 1/2} \cup \widetilde{y}) \end{cases}$$

$$\Rightarrow \begin{cases} \widetilde{y}^D_{\geq 1/2} \cap \widetilde{y} \subseteq \widetilde{x} \subseteq \widetilde{y} \cup \widetilde{y}^D_{\geq 1/2} \cup y^C_{\leq 1/2} \\ \widetilde{y}^D_{\geq 1/2} \cap \widetilde{y} \subseteq \widetilde{x}^D \subseteq \widetilde{y} \cup \widetilde{y}^D_{\geq 1/2} \cup y^C_{\leq 1/2} \end{cases}$$

$$\Rightarrow \begin{cases} \widetilde{y}_{\leq 1/2} \subseteq \widetilde{x} \subseteq \widetilde{y}^D_{\geq 1/2} \cup y^C_{\leq 1/2} \\ \widetilde{y}_{\leq 1/2} \subseteq \widetilde{x}^D \subseteq \widetilde{y}^D_{\geq 1/2} \cup y^C_{\leq 1/2} \end{cases}$$

Let us return to the expression for $(\widetilde{x}\varphi\widetilde{y})$:

$$\begin{cases} \widetilde{y} \cap \widetilde{y}^D \subseteq \widetilde{x} \odot y \subseteq \widetilde{y} \cup \widetilde{y}^D \\ \widetilde{y} \cap \widetilde{y}^D \subseteq \widetilde{x}^D \odot y \subseteq \widetilde{y} \cup \widetilde{y}^D \end{cases}$$

$$\Rightarrow \begin{cases} \widetilde{y} \cap \widetilde{y}^D_{\geq 1/2} \subseteq \widetilde{x} \odot y \cup \widetilde{x} \odot y^C \subseteq \widetilde{y} \cup \widetilde{y}^D_{\geq 1/2} \cup y^C_{\leq 1/2} \\ \widetilde{y} \cap \widetilde{y}^D_{\geq 1/2} \subseteq \widetilde{x}^D \odot y \cup \widetilde{x}^D \odot y^C \subseteq \widetilde{y} \cup \widetilde{y}^D_{\geq 1/2} \cup y^C_{\leq 1/2} \end{cases}.$$

Since $\widetilde{y} \cup \widetilde{y}^D_{\geq 1/2} \subseteq \widetilde{y} \cap \widetilde{y}^D$, $\widetilde{y} \cup \widetilde{y}^D = \widetilde{y} \cup \widetilde{y}^D_{\geq 1/2}, y^C \subseteq y^C_{\leq 1/2}$ (and therefore $\widetilde{x} \odot \widetilde{y}^C \subseteq y^C_{\leq 1/2}, \widetilde{x}^D \odot y^C \subseteq y^C_{\leq 1/2}$). Thus, $(\widetilde{x}\varphi\widetilde{y}) \Rightarrow (\widetilde{x}\varphi(\widetilde{y} : \widetilde{y}^D))$.

On $L^\sim$, one can define a relation E such that $(\widetilde{x}E\widetilde{y})$ if $\widetilde{x} = \widetilde{y}$ or $\widetilde{x}^D = \widetilde{y}$, or$\widetilde{x} = \widetilde{y}^D$. The relation E on the split degree of the universal set is the equivalence relation. Each equivalence class consists of a fuzzy subset and a corresponding dual subset. If $\widetilde{x} = \widetilde{x}^D$, then the equivalence class consists of one element. Let us verify that E is actually an equivalence relation. Reflexivity and symmetry are evident from the above. Assume that $(\widetilde{x}E\widetilde{y})$ and $(\widetilde{y}E\widetilde{z})$, and suppose that from the first follows $\widetilde{x} = \widetilde{y}^D$; if $(\widetilde{y}E\widetilde{z})$ means that $\widetilde{x} = \widetilde{z}$, then $\widetilde{x} = \widetilde{z}^D$ (while $\widetilde{x}^D = \widetilde{z}$); if $\widetilde{y}^D = \widetilde{z}$ (or $\widetilde{y} = \widetilde{z}^D$), then $\widetilde{x} = \widetilde{z}$. Other cases are verified similarly. Thus $\{\{(\widetilde{x}E\widetilde{y}\ )$ and $\{(\widetilde{y}E\widetilde{z}\ )\} \Rightarrow \left\{\widetilde{x} = \widetilde{z},\ \text{or}\ \widetilde{x} = z^D,\ \text{or}\ \widetilde{x}^D = \widetilde{z}\right\} \Rightarrow \{(\widetilde{x}E\widetilde{z})\}$, which proves the transitivity of E. □

A subset of $P^{\sim}(\Omega)$, consisting of some element and the corresponding dual element will be called a dual pair. According to Proposition 3.7, if one element of the dual pair is fuzzier than the element of the other dual pair, then any element of the first pair is fuzzier than any element of the second pair. Therefore, it makes sense to introduce a definition of the fuzziness of the dual pair.

Definition 3.9: Let Mbe the set of dual pairs with components from $P^{\sim}(\Omega)$. Let us define on M relation Φ such that if for $u^{\sim}$ and $v^{\sim} \in M(u^{\sim}\Phi v^{\sim})$, then $(\widetilde{x}\varphi\widetilde{y})$ $\forall\widetilde{x} \in u^{\sim}$ and $\forall\widetilde{y} \in v^{\sim}$. If $(u^{\sim}\Phi v^{\sim})$, then we say that the dual pair $u^{\sim}$ is fuzzier than the pair $v^{\sim}$.

Proposition 3.10: *The relation Φon the set M of dual pairs, which components belong to $P^{\sim}(\Omega)$, is a partial order.*

Proof: We should show that Φ is reflexive, antisymmetric and transitive.

(1) Reflexivity means that $(u^{\sim}\Phi u^{\sim})$. Consider the element $\widetilde{x} \in u^{\sim}$, since $(\widetilde{x}\varphi\widetilde{x})$, then $(u^{\sim}\Phi u^{\sim})$.
(2) Transitivity means that $(u^{\sim}\Phi v^{\sim})$ and $(v^{\sim}\Phi w^{\sim}) \Rightarrow (u^{\sim}\Phi w^{\sim})$. $(u^{\sim}\Phi v^{\sim})$ indicates that $(\widetilde{x}\varphi\widetilde{y})$ $\forall\widetilde{x} \in u^{\sim}$ and $\forall\widetilde{y} \in v^{\sim}$; $(v^{\sim}\Phi w^{\sim})$ indicates that $(\widetilde{y}\varphi\widetilde{z})$ $\forall\widetilde{y} \in v^{\sim}$ and $\forall\widetilde{z} \in w^{\sim}$. Due to Proposition 3.4 $(\widetilde{x}\varphi\widetilde{y})and(\widetilde{y}\varphi\widetilde{z}) \Rightarrow (\widetilde{x}\varphi\widetilde{z})$, therefore, $(u^{\sim}\Phi w^{\sim})$.
(3) Anti-symmetry means that $(u^{\sim}\Phi v^{\sim})$ and $(v^{\sim}\Phi u^{\sim}) \Rightarrow u^{\sim} = v^{\sim}$. We have

$$\begin{aligned}(u^{\sim}\Phi v^{\sim}) &\Rightarrow \text{if } \widetilde{x} \in u^{\sim} \text{ and } \widetilde{y} \in v^{\sim}, \text{ then } (\widetilde{x}\varphi\widetilde{y}),\\ (v^{\sim}\Phi u^{\sim}) &\Rightarrow \text{if } \widetilde{x} \in u^{\sim} \text{ and } \widetilde{y} \in v^{\sim}, \text{ then } (\widetilde{y}\varphi\widetilde{x}).\end{aligned} \tag{12}$$

According to the Proposition 3.2 we have $(\widetilde{x}f\widetilde{y})$ and $(\widetilde{y}f\widetilde{x})$, which means that $\widetilde{x} = \widetilde{y}$ or $\widetilde{x} = \widetilde{y}^D$ [3, 4]; therefore $u^{\sim} = v^{\sim}$.

Thus, based on the previously introduced [9] concept of a dual fuzzy set for any fuzzy set, in this work we determined the relation "no less fuzzy than" and studied the properties of this important relations. □

4. Conclusion

The article deals with the operation of splitting a crisp relation in the dual fuzzy relations. The representations of some algebraic operations on split indicators are also given. The lattice of split elements of the Boolean lattice L^{-} is studied, where it is proved that the lattice of all split elements of this lattice $\widetilde{i}$ is a Brewer lattice. A number of facts are given on the properties of this lattice. Splitting operation of a crisp relation and set are defined, which is equivalent to splitting operation of its indicator. The main properties of this operation are given, with some proofs. The concept of the generalized degree of the universe is defined, which is the lattice of the elements obtained by splitting all the subsets of the universe. A simple example of MADM is presented for illustration of the application of splitting operation. Future studies, aimed at multi-criteria decision-making problems, will use the results presented in this article. It is also planned to generalize the splitting operation from the fuzzy sets and relations of Zadeh for the dual q-rung orthopair fuzzy sets and relations.

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