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On an Exponential Inequality

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Exponential function is one of the most important functions in mathematics and is helpful in theoretical investigations and practical applications. For example, exponential functions are the solutions to the simplest types of dynamical systems. In particular, an exponential function arises in simple models of bacterial growth, it can describe growth or decay, etc. The main purpose of this paper is one exponential inequality, which arose in the study of the properties of exponential functions.

Keywords: Inequality, exponential function, Taylor series, mathematical induction, hyperbolic cosine.

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1. Introduction

The exponential function plays an important role in theoretical investigations and practical applications. If we look at the scientific literature, it is easy to see that in recent times many papers are devoted to further specifying the properties of the exponential function, proving new inequalities involving exponential and hyperbolic functions; it can be said that this direction has become a subject of intensive investigation. There exists a vast literature on such inequalities, for more information on this subject and related topics, one may refer to [1], [2], [3], [4], [5], [6]], [7], [8], [9] and the references therein.

Using various mathematical software, we observe that the inequality

$$\frac{e^{\sqrt[4]{2}\,t}+e^{-\sqrt[4]{2}\,t}}{2}+t-e^{t}\geq 0$$

holds for any numerical value of the parameter t (see Fig. 1).

However, it is clear that we cannot consider this circumstance as an objective proof of this fact, since the current practical calculations in mathematical software are carried out with a certain accuracy, in many cases the values are rounded, which, of course, causes certain errors that can distort the final result. Therefore, if

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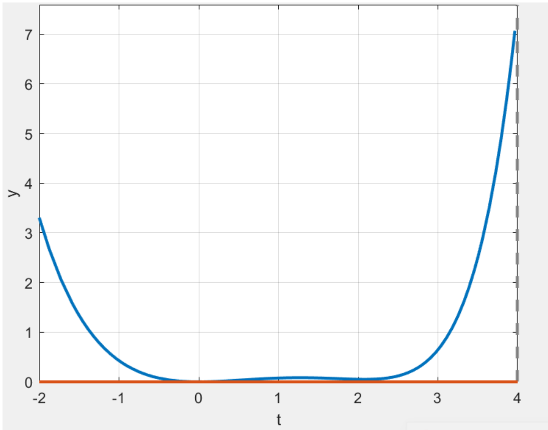

Fig. 1. Graph of the function $y = \frac{e^{\sqrt[4]{2}\,t}+e^{-\sqrt[4]{2}\,t}}{2} + t - e^t$

we want to be sure that a fact is true for any value of the variable t, it is necessary to find its analytical proof, which is what we do in this paper.

2. Main result

Let us formulate and prove the main result of the paper.

Theorem 2.1: *For every number $t \in \mathcal{R}^1$ the following inequality is valid*

$$\frac{e^{\sqrt[4]{2}\,t}+e^{-\sqrt[4]{2}\,t}}{2} + t \geq e^t. \tag{1}$$

Proof: Let us expand the function e^t into a Taylor series. As this series converges absolutely we can write:

$$e^t = 1 + t + \sum_{n=1}^{\infty} \frac{t^{2n+1}}{(2n+1)!} + \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} = 1 + t + \sum_{n=1}^{\infty} \left(1 + \frac{t}{2n+1}\right) \frac{t^{2n}}{(2n)!}. \tag{2}$$

Let's choose positive numbers p_n and q_n so that for every $t \in \mathcal{R}^1$ the following inequalities are fulfilled:

$$\frac{t}{2n+1} \leq p_n + q_n t^2, \quad n = 1, 2, \ldots. \tag{3}$$

Obviously, (3) will always hold as soon as the numbers p_n and q_n satisfy the inequalities:

$$p_n \cdot q_n \geq \frac{1}{4(2n+1)^2}, \quad n = 1, 2, \ldots.$$

In particular, suppose that

$$q_n = \frac{1}{4(2n+1)^2 p_n}, \quad n = 1, 2, \ldots. \tag{4}$$

Then, taking into account (3) and (4) from (2) we obtain

$$\begin{aligned}
e^t &\le 1+t+\sum_{n=1}^{\infty}\left(1+p_n+q_n t^2\right)\frac{t^{2n}}{(2n)!}\\
&= 1+t+\sum_{n=1}^{\infty}(1+p_n)\frac{t^{2n}}{(2n)!}+\sum_{n=1}^{\infty}\frac{1}{4(2n+1)^2p_n}\frac{t^{2n+2}}{(2n)!}\\
&= 1+t+(1+p_1)\frac{t^2}{2}+\sum_{n=2}^{\infty}\left[1+p_n+\frac{n}{2(2n-1)p_{n-1}}\right]\frac{t^{2n}}{(2n)!}. \qquad (5)
\end{aligned}$$

Taking into account (5), our goal is to choose a sequence of positive numbers (p_n) such that

$$e^t \le t+\sum_{n=0}^{\infty}\frac{2^{n/2}t^{2n}}{(2n)!}, \qquad (6)$$

which obviously implies (1).

According to (5), (6) holds if the numbers (p_n) satisfy the following conditions:

$$1+p_1 \le \sqrt{2}, \qquad (7)$$

$$1+p_n+\frac{n}{2(2n-1)p_{n-1}} \le 2^{n/2}, \quad n=2,3,\dots. \qquad (8)$$

Let's select the sequence (p_n) as follows

$$p_n=\frac{2.15}{\sqrt{(2n+1)^3}}, \quad n=1,2,\dots,$$

and let's check that the conditions (7) and (8) hold.

Condition (7)*:*

$$1+p_1=\frac{2.15}{\sqrt{27}}<\sqrt{2},$$

i.e. condition (7) is fulfilled.

Condition (8)*:*

$$1+p_2+\frac{1}{3p_1}=1+\frac{2.15}{\sqrt{125}}+\frac{1}{3\cdot\frac{2.15}{\sqrt{27}}}<2. \qquad (9)$$

Hence condition (8) for $n=2$ is fulfilled.

Now for every integer $n > 2$ we have to show that

$$1 + \frac{2.15}{\sqrt{(2n+1)^3}} + \frac{n\sqrt{2n-1}}{4.3} \le 2^{n/2}. \tag{10}$$

Using the method of mathematical induction, suppose that (10) is valid for a fixed n and show its validity for $(n+1)$. That is, we have to prove the validity of the following inequality

$$1 + \frac{2.15}{\sqrt{(2n+3)^3}} + \frac{(n+1)\sqrt{2n+1}}{4.3} \le 2^{(n+1)/2}, \quad n = 2, 3, \dots.$$

Indeed, we have

$$1 + \frac{2.15}{\sqrt{(2n+3)^3}} + \frac{(n+1)\sqrt{2n+1}}{4.3} \le$$
$$\le 2^{n/2} + \frac{(n+1)\sqrt{2n+1} - n\sqrt{2n-1}}{4.3} - 2.15\left[\frac{1}{\sqrt{(2n+1)^3}} - \frac{1}{\sqrt{(2n+3)^3}}\right]. \tag{11}$$

Our goal is to show that for every $n \ge 2$ the expression (11) is less than $2^{\frac{n+1}{2}}$, that is:

$$\frac{(n+1)\sqrt{2n+1} - n\sqrt{2n-1}}{4.3} - 2.15\left[\frac{1}{\sqrt{(2n+1)^3}} - \frac{1}{\sqrt{(2n+3)^3}}\right] \le (\sqrt{2}-1)2^{n/2}. \tag{12}$$

Obviously, (12) will be proved if we prove

$$\frac{(n+1)\sqrt{2n+1} - n\sqrt{2n-1}}{4.3} \le (\sqrt{2}-1)2^{n/2}. \tag{13}$$

It is not difficult to verify validity of the following inequality

$$\frac{(n+1)\sqrt{2n+1} - n\sqrt{2n-1}}{4.3} \le \frac{6n+1}{8.6\sqrt[4]{4n^2-1}}.$$

Therefore, (13) is proved if we show that

$$\frac{6n+1}{\sqrt[4]{4n^2-1}} \le 8.6(\sqrt{2}-1)2^{n/2}, \quad n = 2, 3, \dots. \tag{14}$$

Let's prove this inequality by induction. It is elementary to verify the validity of the inequality

$$\frac{13}{\sqrt[4]{15}} < 17.2(\sqrt{2}-1),$$

which shows the validity of (14) for $n = 2$

Now suppose that (14) is valid for a fixed $n\,(n \geq 2)$, and show its validity for $(n+1)$, i.e. let's prove that

$$\frac{6n+7}{\sqrt[4]{4n^2+8n+3}} \leq 8.6(\sqrt{2}-1)2^{\frac{n+1}{2}}, \quad n = 2, 3, \ldots.$$

Indeed, we have

$$\frac{6n+7}{\sqrt[4]{4n^2+8n+3}} \leq 8.6(\sqrt{2}-1)2^{n/2} \cdot \frac{6n+7}{6n+1} \cdot \sqrt[4]{\frac{2n+1}{2n+3}}.$$

Based on the last expression, we conclude that the proof ends if for every $n \geq 2$ we show that

$$\frac{6n+7}{6n+1} \cdot \sqrt[4]{\frac{2n+1}{2n+3}} \leq \sqrt{2}. \tag{15}$$

Indeed, consider the following function f of x:

$$f(x) = \frac{(6x+7)^4 \cdot (2x+1)}{(6x+1)^4 \cdot (2x+3)}, \quad x \geq 2,$$

and let's investigate its behavior in the interval $x \geq 2$. To do this, we find the derivative:

$$f'(x) = -\frac{8(6x+7)^3 \cdot (36x^2+24x-61)}{(6x+1)^5 \cdot (2x+3)^2}.$$

It is easy to see that $f'(x) < 0$ for every $x \geq 2$, which means that f is a decreasing function on the interval $x \geq 2$ and therefore, in this interval, its maximum is attained at the point $x = 2$.

Therefore, if we return to (15) we get

$$\frac{6n+7}{6n+1}\sqrt[4]{\frac{2n+1}{2n+3}} \leq \frac{6 \cdot 2+7}{6 \cdot 2+1}\sqrt[4]{\frac{2 \cdot 2+1}{2 \cdot 2+3}} < 1.36 < \sqrt{2}$$

for any n, which proves (15) and, therefore, completes the proof of Theorem 2.1. □

Remark 1: Inequality (1) can be obviously rewritten in terms of a hyperbolic cosine as follows

$$\cosh\left(\sqrt[4]{2}\,t\right) + t \geq e^t.$$

Remark 2: Let us formulate the proved theorem as follows: for $a = \sqrt[4]{2}$ and every number $t \in \mathcal{R}^1$ the following inequality is valid

$$\frac{e^{at}+e^{-at}}{2} + t \geq e^t. \tag{16}$$

A natural question arises: for which other values of the parameter a the given inequality is valid? Applying various computer programs to check the validity of (16) we see that it seems to be true for all real numbers t as $a \geq \sqrt[4]{2}$, but we do not know yet the analytical proof of this fact. Moreover, elementary calculations show that (16) is false when $a < \sqrt[4]{2}$. In particular, this can be seen from the graph of the function $\varphi_a(t) = \frac{e^{at}+e^{-at}}{2} + t - e^t$ as $a = 1.185 < \sqrt[4]{2}$ (see Fig. 2).

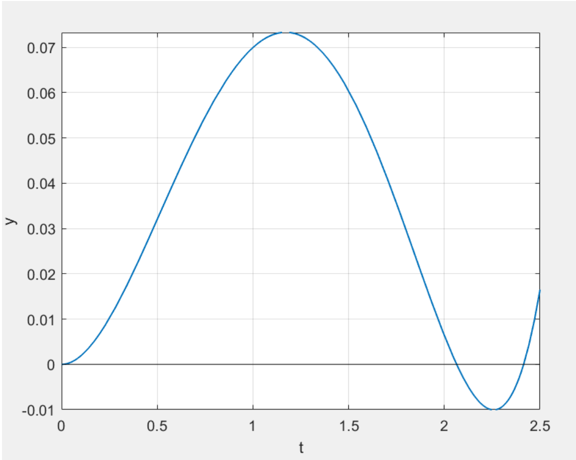

Fig. 2. Graph of the function $y = \frac{e^{at}+e^{-at}}{2} + t - e^t$ at $a = 1.185$

The same mathematical programs show that (16) seems to be true for $a = 1.189 < \sqrt[4]{2}$. In this regard, we can pose the following problem: find the minimum positive value of the parameter a, for which (16) is valid.

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