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# A Controlled Integral Equation and Properties of its Kernel

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The nonlinear controlled integral equation corresponding to the quasi-linear controlled neutral differential equation is constructed. The structure and properties of kernel of the integral equation are established. For the neutral and integral equations theorems on the existence and uniqueness of solution are provided. The equivalence of the integral and neutral differential equations is established.

**Keywords:** Controlled integral equation, Properties of the integral kernel, Neutral differential equation; Existence and uniqueness, Equivalence.

**AMS Subject Classification**: 34K40, 45G15.

## 1. Introduction

In the present paper for the controlled quasi-linear neutral differential equation

$$\dot{x}(t) = A(t, x(t), x(t-\tau), u(t))\dot{x}(t-\tau) + f(t, x(t), x(t-\tau), u(t)), t \in [t_0, t_1] \quad (1.1)$$

with the initial condition

$$x(t) = \varphi(t), t < t_0, x(t_0) = x_0 \quad (1.2)$$

the corresponding nonlinear controlled integral equation is constructed

$$y(t) = x_0 + \int_{t_0}^{t_0+\tau} Y(\xi; t, y(\cdot), u(\cdot))A(\xi, y(\xi), y(\xi-\tau), u(\xi))\dot{\varphi}(\xi-\tau)d\xi$$

$$+ \int_{t_0}^{t} Y(\xi; t, y(\cdot), u(\cdot))f(\xi, y(\xi), y(\xi-\tau), u(\xi))d\xi, y(\xi) = \varphi(\xi), \xi < t_0. \quad (1.3)$$

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The matrix-function $Y(\xi; t, y(\cdot), u(\cdot))$ is called the integral kernel. The essential novelty here is that equations (1.1) and (1.3) contain a control function $u(t)$ and for this case structure and properties of the kernel are established. Besides, theorems on the existence and uniqueness of solution are provided and equivalence of equations (1.1) and (1.3) is proved. We note that the above mentioned questions play the principal role in the study of well-posedness of Cauchy's problem. The details, about this investigations are given in [1-3] for the quasi-linear neutral differential equations without control. The results obtained in the paper are generalization of the assertions given in [1-3].

## 2. The controlled integral equation

Let $R_x^n$ be the $n$-dimensional vector space of points $x = (x^1, \ldots, x^n)^T$, where $T$ is the sign of transposition; let $I = [t_0, t_1]$ be a fixed interval and let $\tau > 0$ be a given number, with $t_0 + \tau < t_1$; the $n \times n$-dimensional matrix-function $A(t, x, y, u)$ and the $n$-dimensional vector-function $f(t, x, y, u)$ are continuous and bounded on the set $I \times R_x^n \times R_x^n \times R_u^r$ and satisfy the Lipschptz condition with respect to $(x, y, u)$, i. e. there exist $L_A > 0$ and $L_f > 0$ such that we have

$$|A(t, x_1, y_1, u_1) - A(t, x_2, y_2, u_2)| \leq L_A\Big(|x_1 - x_2| + |y_1 - y_2| + |u_1 - u_2|\Big)$$

$$\forall t \in I, (x_i, y_i, u_i) \in R_x^n \times R_x^n \times R_u^r, i = 1, 2,$$

and

$$|f(t, x_1, y_1, u_1) - f(t, x_2, y_2, u_2)| \leq L_f\Big(|x_1 - x_2| + |y_1 - y_2| + |u_1 - u_2|\Big)$$

$$\forall t \in I, (x_i, y_i, u_i) \in R_x^n \times R_x^n \times R_u^r, i = 1, 2.$$

Further, denote by $\Omega$ the set of piecewise-continuous control functions $u(t) \in R_u^r$ with finitely many discontinuous of the first kind equipped with the norm $\|u\| = sup\{|u(t)| : t \in I\}; \varphi(t) \in R_x^n, t \in [t_0 - \tau, t_0]$ is a given continuously differentiable initial function; $x_0 \in R_x^n$ is a given initial vector.

Let us consider the quasi-linear controlled neutral differential equation

$$\dot{x}(t) = A(t, x(t), x(t-\tau), u(t))\dot{x}(t-\tau) + f(t, x(t), x(t-\tau), u(t)), t \in I \tag{2.1}$$

with the initial condition

$$x(t) = \varphi(t), t \in [\hat{\tau}, t_0), x(t_0) = x_0, \tag{2.2}$$

where $\hat{\tau} = t_0 - \tau$.

**Definition 2.1:** Let $u(t) \in \Omega$. A function $x(t) = x(t; u), t \in I_1 = [\hat{\tau}, t_1]$, is called a solution of equation (2.1) with the initial condition (2.2), if it satisfies condition (2.2) and is absolutely continuous on the interval $I$ and satisfies equation (2.1) almost everywhere on $I$.

**Theorem 2.2:** For any $u(t) \in \Omega$ there exists the unique solution $x(t) = x(t; u), t \in I_1$.

**Proof:** The existence of the global solution will be proved by the step method from left to right.
**Step 1.** Let $t \in [t_0, t_0 + \tau]$ then we have the ordinary differential equation

$$\dot{x}(t) = A(t, x(t), \varphi(t-\tau), u(t))\dot{\varphi}(t-\tau) + f(t, x(t), \varphi(t-\tau), u(t)) \tag{2.3}$$

with the initial condition

$$x(t_0) = x_0. \tag{2.4}$$

It is clear that the function

$$A(t, x, \varphi(t-\tau), u(t))\dot{\varphi}(t-\tau) + f(t, x, \varphi(t-\tau), u(t))$$

satisfies the Lipschptz condition with resect to $x$ on the set $R^n_x$. Therefore, there exists the global unique solution $x_1(t), t \in [t_0, t_0 + \tau]$ for the problem (2.3)-(2.4).
**Step 2.** Let $[t_0 + \tau, t_0 + 2\tau] \subset I$ and $t \in [t_0 + \tau, t_0 + 2\tau]$ then we consider the problem

$$\begin{cases} \dot{x}(t) = A(t, x(t), x_1(t-\tau), u(t))\dot{x}_1(t-\tau) \\ +f(t, x(t), x_1(t-\tau), u(t)), t \in [t_0 + \tau, t_0 + 2\tau], \\ x(t_0 + \tau) = x_1(t_0 + \tau). \end{cases} \tag{2.5}$$

The problem (2.5) on the interval $t \in [t_0 + \tau, t_0 + 2\tau]$ has the global unique solution $x_2(t)$. Thus, the function

$$x(t) = \begin{cases} \varphi(t), t \in [\hat{\tau}, t_0), \\ x_1(t), t \in [t_0, t_0 + \tau], \\ x_2(t), t \in (t_0 + \tau, t_0 + 2\tau] \end{cases}$$

is the solution of problem (2.1)-(2.2) on the interval $[\hat{\tau}, t_0 + 2\tau]$. Continuing this process we establish existence of the unique solution $x(t)$ on the interval $I_1$. □

**Theorem 2.3:** *The solution $x(t), t \in I_1$ of problem (2.1)-(2.2) can be represented* *on the interval $I$ in the following form:*

$$x(t) = x_0 + \int_{t_0-\tau}^{t_0} Y(\xi+\tau;t,x(\cdot),u(\cdot))A(\xi+\tau,x(\xi+\tau),x(\xi),u(\xi+\tau))\dot{\varphi}(\xi)d\xi$$

$$+\int_{t_0}^{t} Y(\xi;t,x(\cdot),u(\cdot))f(\xi,x(\xi),x(\xi-\tau),u(\xi))d\xi, \quad (2.6)$$

*where*

$$x(\xi)=\varphi(\xi),\xi\in[\hat{\tau},t_0) \quad (2.7)$$

*and $Y(\xi,t,x(\cdot),u(\cdot))$ is the matrix-function satisfying the difference equation*

$$Y(\xi;t,x(\cdot),u(\cdot)) = E + Y(\xi+\tau;t,x(\cdot),u(\cdot))$$

$$\times A(\xi+\tau,x(\xi+\tau),x(\xi),u(\xi+\tau)) \quad (2.8)$$

*on $(t_0,t)$ for any fixed $t\in(t_0,t_1]$ and the condition*

$$Y(\xi;t,x(\cdot),u(\cdot)) = \begin{cases} E,\xi=t, \\ \Theta,\xi>t. \end{cases} \quad (2.9)$$

*Here, $E$ is the identity matrix and $\Theta$ is the zero matrix.*

**Proof:** On the interval $(t_0,t)$, where $t\in(t_0,t_1]$, consider the equation

$$\dot{x}(\xi)=A(\xi,x(\xi),x(\xi-\tau),u(\xi))\dot{x}(\xi-\tau)$$

$$+f(\xi,x(\xi),x(\xi-\tau),u(\xi)),\xi\in(t_0,t) \quad (2.10)$$

with the initial condition

$$x(\xi)=\varphi(\xi),\xi\in[\hat{\tau},t_0),x(t_0)=x_0.$$

Multiplying equation (2.10) on the matrix-function $Y(\xi;t,x(\cdot),u(\cdot))$ and integrating in $\xi\in[t_0,t]$, we obtain

$$\int_{t_0}^{t} Y(\xi;t,x(\cdot),u(\cdot))\dot{x}(\xi)d\xi = \int_{t_0}^{t} Y(\xi;t,x(\cdot),u(\cdot))A(\xi,x(\xi),x(\xi-\tau),u(\xi))\dot{x}(\xi-\tau)d\xi$$

$$+\int_{t_0}^{t} Y(\xi;t,x(\cdot),u(\cdot))f(\xi,x(\xi),x(\xi-\tau),u(\xi))d\xi. \tag{2.11}$$

Further,

$$\int_{t_0}^{t} Y(\xi;t,x(\cdot),u(\cdot))A(\xi,x(\xi),x(\xi-\tau),u(\xi))\dot{x}(\xi-\tau)d\xi$$

$$=\int_{t_0-\tau}^{t-\tau} Y(\xi+\tau;t,x(\cdot),u(\cdot))A(\xi+\tau,x(\xi+\tau),x(\xi),u(\xi+\tau))\dot{x}(\xi)d\xi$$

$$=\int_{t_0-\tau}^{t_0} Y(\xi+\tau;t,x(\cdot),u(\cdot))A(\xi+\tau,x(\xi+\tau),x(\xi),u(\xi+\tau))\dot{\varphi}(\xi)d\xi$$

$$+\int_{t_0}^{t} Y(\xi+\tau;t,x(\cdot),u(\cdot))A(\xi+\tau,x(\xi+\tau),x(\xi),u(\xi+\tau))\dot{x}(\xi)d\xi \tag{2.12}$$

(see (2.9)). Taking into account (2.12), from (2.11) we obtain

$$\int_{t_0}^{t}\Big[Y(\xi;t,x(\cdot),u(\cdot))-Y(\xi+\tau;t,x(\cdot),u(\cdot))A(\xi+\tau,x(\xi+\tau),x(\xi),u(\xi+\tau))\Big]\dot{x}(\xi)d\xi$$

$$=\int_{t_0-\tau}^{t_0} Y[\xi+\tau;t,x(\cdot),u(\cdot))A(\xi+\tau,x(\xi+\tau),x(\xi),u(\xi+\tau))\dot{\varphi}(\xi)d\xi$$

$$+\int_{t_0}^{t} Y(\xi,x(\cdot),u(\cdot))f(\xi,x(\xi),x(\xi-\tau),u(\xi))d\xi.$$

$Y(\xi;t,x(\cdot),u(\cdot))$ satisfies equation (2.8) and, therefore, the latter relation implies formula (2.6). □

The expression

$$y(t)=x_0+\int_{t_0}^{t_0+\tau} Y(\xi;t,y(\cdot),u(\cdot))A(\xi,y(\xi),y(\xi-\tau),u(\xi))\dot{\varphi}(\xi-\tau)d\xi$$

$$+\int_{t_0}^{t} Y(\xi;t,y(\cdot),u(\cdot))f(\xi,y(\xi),y(\xi-\tau),u(\xi))d\xi \tag{2.13}$$

with the condition

$$y(\xi) = \varphi(\xi), \xi \in [\hat{\tau}, t_0) \tag{2.14}$$

is called the integral equation corresponding to problem (2.1)-(2.2).

**Definition 2.4:** Let $u(t) \in \Omega$. A function $y(t) = y(t;u), t \in I_1$, is called a solution of equation (2.13) with condition (2.14), if it satisfies condition (2.14) and is continuous on the interval $I$ and satisfies equation (2.13) everywhere on $I$.

## 3. Properties of the Integral Kernel. Existence and uniqueness. Equivalence

**Theorem 3.1:** Let $t \in (t_0, t_1]$ be a fixed point. The solution of the difference equation (2.8) can be represented by the following formula:

$$Y(\xi; t, x(\cdot), u(\cdot)) = \chi(\xi;t)E + \sum_{i=1}^{k} \chi(\xi + i\tau; t) \prod_{q=i}^{1} A(\xi + q\tau, x(\xi + q\tau),$$

$$x(\xi + (q-1)\tau), u(\xi + q\tau)),$$

where

$$\chi(\xi;t) = \begin{cases} 1, t_0 \le \xi \le t, \\ 0, \xi > t \end{cases}$$

and $k$ is a minimal natural number satisfying the condition

$$t_1 - k\tau < t_0.$$

**Theorem 3.2:** Let $s_1, s_2 \in (t_0, t_1]$ and $0 < s_2 - s_1 < \tau$. Let $y(t), t \in I$ be a continuous function. Then there exist subintervals $I_1(s_1, s_2) \subset I$ *and* $I_2(s_1, s_2) \subset I$ such that

$$\begin{cases} Y(\xi; s_1, y(\cdot), u(\cdot)) = Y(\xi; s_2, y(\cdot), u(\cdot)), \xi \in I_1(s_1; s_2), \\ Y(\xi; s_1, y(\cdot), u(\cdot)) \ne Y(\xi; s_2, y(\cdot), u(\cdot)), \xi \in I_2(s_1; s_2), \end{cases}$$

with

$$\lim_{s_2 - s_1 \to 0} mes I_2(s_1, s_2) \to 0.$$

**Theorem 3.3:** Let $y(t) \in R^n, t \in I_1$ be a given piecewise-continuous function, with $y(\xi)=\varphi(\xi), \xi \in [\hat{\tau}, t_0)$ *and* $u(t) \in \Omega$. Then the function

$$z(t) = x_0 + \int_{t_0-\tau}^{t_0} Y(\xi+\tau; t, y(\cdot), u(\cdot)) A(\xi+\tau, y(\xi+\tau), y(\xi), u(\xi+\tau))\dot{\varphi}(\xi)d\xi$$

$$+ \int_{t_0}^{t} Y(\xi; t, y(\cdot), u(\cdot)) f(\xi, y(\xi), y(\xi-\tau), u(\xi))d\xi$$

is continuous on the interval $I$.

**Theorem 3.4 :** Let $y_i(t) \in R_x^n, t \in I, i = 1,2$ be continuous functions and $u_i(t) \in \Omega, i = 1,2$. Then for $\forall\ (\xi, t) \in I^2$

$$|Y(\xi; t, y_1(\cdot), u_1(\cdot)) - Y(\xi; t, y_2(\cdot), u_2(\cdot))|$$

$$\leq L_A \sum_{i=1}^{k} \chi(\xi + i\tau; t) \|A\|^{i-1} \Big( \sum_{q=i}^{1} \Big[ |y_1(\xi + q\tau) - y_2(\xi + q\tau)|$$

$$+|y_1(\xi+(q-1)\tau) - y_2(\xi+(q-1)\tau)| + |u_1(\xi+q\tau) - u_2(\xi+q\tau)| \Big] \Big),$$

*where*

$$\|A\| = \sup\{|A(t,x,y,u)| : (t,x,y,u) \in I \times R_x^n \times R_x^n \times R_u^r\}.$$

**Theorem 3.5 :** *Let* $y_m(t) \in R_x^n, t \in I, m = 0,1,...$ be continuous functions and $u_m(t) \in \Omega, m = 0,1,...,$ with

$$\|y_m - y_0\| \to 0, \ \ \|u_m - u_0\| \to 0,$$

where

$$\|y\| = \{|y(t)| : t \in I\}.$$

Then

$$\int_{t_0}^{t} Y(\xi; t, y_m(\cdot), u_m(\cdot))d\xi \to \int_{t_0}^{t} Y(\xi; t, y_0(\cdot), u_0(\cdot))d\xi$$

uniformly for $t \in I$.

**Theorem 3.6 :** The integral equation (2.13) with condition (2.14) has the unique solution.

**Theorem 3.7 :** The quasi-linear neutral differential equation (2.1) and the integral equations (2.13) are equivalent.

**Proof:** It is clear that if $x(t), t \in I_1$ is a solution of equation (2.1) with the initial condition (2.2), then it is a solution of the integral equation (2.13) with the initial condition (2.14) also (see Theorem 1.2). Let $y(t), t \in I_1$ be a solution of the integral equation (2.13) with the initial condition (2.14) and it is not solution of equation (2.1). By Theorem 2.1 equation (2.1) has the unique solution $\hat{x}(t)$, which is a solution of the integral equation (2.13) also. But equation (2.13) has the unique solution, i. e. $\hat{x}(t) = y(t)$. □

**Remark 1:** The analogous theorems for the quasi-linear differential equations, where $A(t, x(t), x(t-\tau), u(t)) \equiv A(t)$ are proved in [1-3].

## 4. Conclusion

On the basis of the given theorems continuous dependence of a solution of the quasi-linear controlled neutral differential equation (2.1) can be investigated with respect to perturbations of the initial data. In future work will consider the case when a controlled integral equation contains several variable delays.

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